$G$ is simple if it has no normal subgroups except $\{e\}$ and $G$.

**Example:** $\mathbb{Z}/p\mathbb{Z}$, $p$ prime, $A_n$ for $n \geq 5$.

Only simple abelian groups (every subgroup of an abelian group is normal).

We'll talk about $A_n$ a lot!

Recall: $S_n \hookrightarrow \text{GL}_n(\mathbb{C}^\times)$ (permuting the coordinates).

$\text{sgn} \rightarrow \text{det} \rightarrow \mathbb{C}^\times$

$A_n = \ker \text{sgn}$.

**Note:** Depending on how you define $\text{det}$ this may be cyclic reasoning!

**Def'n of $\text{det}$:** $\text{GL}_n(\mathbb{C}) \cong \mathbb{C}^n$ linear

$\text{GL}_n(\mathbb{C}) \cong \Lambda^n \mathbb{C}^n$ linear

Fixing a basis for $\Lambda^n \mathbb{C}^n$ 1-dim'l

Fixing a nonzero alternating $\Lambda^n$-linear form $\langle \cdot, \cdot, \cdot \rangle$ so $\text{GL}_n(\mathbb{C}) \rightarrow \text{Aut}(\Lambda^n \mathbb{C}^n)$

$\text{det} \langle \cdot, \cdot, \cdot \rangle = \text{det}g \langle \cdot, \cdot, \cdot \rangle$.

$e_1 \wedge \ldots \wedge e_n$ is a basis for $\Lambda^n \mathbb{C}^n$

Express in the basis then get the expression in terms of columns $/ \text{sgn}$
\[ \sigma = t_1 t_2 \ldots t_m \quad \text{\(t_i\) are transpositions (2-cycles).} \]
\[ e.g. \quad t_i = (12) \]

\[ \text{\(\text{sgn}(\sigma) = \prod_{i=1}^{m} \text{sgn}(t_i) = (-1)^m\)} \]

\[ t_i \text{ swaps } 2 \text{ coordinates} \]
\[ \text{in } \mathbb{C}^n \]
\[ \text{Reflection, so det } -1. \]

\[ \text{Transpositions generate } S_n - \text{so every permutation is a product of transpositions.} \]

\[ \text{\(\text{sgn}(\sigma) = (-1)^{\text{number of 2-cycles}}\)} \]

\[ \begin{array}{c}
1 \quad 2 \quad 3 \quad y \quad \sigma \\
\times \quad \times \quad \times \\
\end{array} \]

\[ \text{Def: } A_n = \text{Ker} \text{sgn} \subseteq S_n \]

\[ \text{for } n \geq 2 \quad \Rightarrow S_n / A_n = \{ \pm 1 \} \leq \mathbb{Z}_2, \]

\[ [S_n : A_n] = 2. \]

\[ \text{= "even permutations"} \]

\[ \text{If written as product of transpositions, there are an even #.} \]

**Example** \( A_5 \) is simple by a counting argument.

2 Facts

1. Any normal subgroup is a union of conjugacy classes.

2. Lagrange: order divides \(|A_5| = 60\).
Conjugacy classes have sizes:

1, 15, 20, 12, 12

Let $C(x)$ denote the conjugacy class of $x$.

Try to add these up and get a divisor of 60

...can't, the group is simple.

Sylow's Theorem(s):

A $p$-group is a group of order a power of $p$, a prime number.

$H \leq G$ is a $p$-subgroup if it's a $p$-group.

It's a Sylow $p$-subgroup if it's a $p$-subgroup.

$pX \subseteq H$.

I.e. if $|G| = p^n K$.

Then $H$ is a Sylow $p$-subgroup.

If $|H| = p^n$.

Theorem: If $G$ is a finite group.

1. $G$ has a Sylow $p$-subgroup and they are all conjugate.

2. If $H \leq G$ is a $p$-subgroup then it is contained in a Sylow $p$-subgroup.
(3) \( n_p \) denotes the \# of Sylow \( p \)-subgroups, then \( n_p \mid \text{K} \) \((K = p^n \times K, p \times K)\), \( n_p = 1 \) mod \( p \).

**Observation:** \( \Rightarrow \) a Sylow \( p \)-subgroup is normal \((\Rightarrow n_p = 1)\).

Some control over the \( \mathfrak{A} \). \( \mapsto 3 \).

**Example** There are no simple groups of order \( 448 = 2^{6} \times 7 \).

Suppose \( |G| = 448 \).

\[ n_2 \mid 2^6 \quad \Rightarrow 1 \text{ mod } 7 \]
\[ n_2 = 1 \quad n_7 = 8 \quad n_7 = 64 \]
\[ n_2 \mid 7 \quad n_2 \leq 1 \text{ mod } 2 \quad \Rightarrow \text{no infa.} \]

\[ n_2 = 1 \quad n_2 = 7. \]

If \( n_2 = 1 \) then Sylow 2-subgroup is normal.

If \( n_2 = 7 \) then 

\[ G \leq \text{ sylow 2-subgroup, } R \not\geq \text{ elema.}\]

\[ \begin{array}{c}
G \not\leq S_7 \\
\text{action non-trivial because taut.}\end{array} \]

\[ \text{Ker } \phi \neq G \]

Want to show \( \text{Ker } \phi \neq \{e\} \).

(Then it's a non-trivial normal subgroup).
\[ \text{If } \ker \phi = \{ e \}, \]
\[ \phi : G \to S_7. \]
\[ \therefore |G| = |S_7| = 7! \]
\[ 2^6 \cdot 7 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \]
\[ = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 8 \]
\[ 2^6 \times 2^4 = 2^{10} \]
So \( \ker \phi = \{ e \}, G \) is a minimal normal subgroup.

---

**Proof**

**Part 1**

**Existence:** Induction on the order of the group, \( |G| \).

**Base case:** \( |G| = p \) prime.
\[ G = \mathbb{Z}/p\mathbb{Z}, \quad G \text{ is itself a } \mathbb{Z} \text{-group}. \]

**Inductive step:** \( |G| = p^n \cdot K, \quad n \geq 1 \).

Use class equation.

Fix representatives \( g_1, \ldots, g_m \) for the non-central conjugacy classes.

\[ |G| = p^n |K| \left( \sum_{i=1}^{m} |C_G(g_i)| \right). \]

Suppose \( p \nmid |C_G(g_i)| \), then \( |C_G(g_i)| = \frac{|G|}{p^n} \).
If \( p \mid [G:C_G(g_i)] \), \( \forall \ i \),

since \( p \mid |G| \).

so \( p \mid |Z(G)| \).

\( Z(G) \) is a finite abelian group.

So it contains a subgroup of order \( p \).

\( H \leq Z(G) \).

\( H \) is a normal subgroup of \( G \).

of order \( p \).

\( H \rightarrow G \xrightarrow{\pi} G/H \)

\( |G/H| = p^{a-1}K < |G| \).

So it has a Sylow \( p \)-subgroup \( M \).

\[ |\pi^{-1}(M)| = |H||M| \]

\[ \leq p \cdot p^{a-1} = p^a. \]

Discussion after class:

Average \# of \( K \)-cycles in a permutation in \( S_n \).
\[
\begin{array}{l}
\sum_{\sigma \in S_n} \# \text{K-cycles in } \sigma \ni 6 \\
\sum_{\sigma \in S_n} \# \text{K-cycles in } \sigma \ni 6 \\
= \sum_{\text{K-cycles } \tau \ni 6} \text{ in } S_n \text{ containing } \tau \text{.} \\
= \sum_{\text{K-cycles } \tau} (n-K)! \\
= \binom{\#	ext{K-cycles}}{\tau} (n-K)! \\
= \left( \binom{n}{k} / k \right) (n-K)! \\
= \frac{n!}{k}.
\end{array}
\]