Goal this week: Sylow theorems.
Structure of subgroups of prime power order in a finite group.

A few more adjectives for group actions:

$G \not\to X$ faithfully if $G \to \text{Aut}(X)$ is injective,
i.e., if Kernel = $\{e\}$.

Q: If $H \leq G$, $G \not\to G/H$ when is this faithful?

A: If and only if the largest normal subgroup of $G$ contained in $H$ is $\langle e \rangle$.

$\ker = \bigcap_{gH} \text{stab}(gH) = \bigcap_{g \in G} \text{stab}(gH) = \bigcap_{g \in G} gHg^{-1}$

the largest normal subgroup of $G$ contained in $H$.

Example $G \not\to G$ by left multiplication $\sigma \not\to \{1, \ldots, n\}$ is faithful.

$\text{stab}(e) \equiv S_{n-1}$ and intersection of all is trivial.

$G \not\to G$ by conjugation is faithful (iff $\mathbb{Z}(G) = \{e\}$).

$D_8 \not\to G$ action comes is faithful!
12:50 pm

Cayley's Theorem/Observer: Any finite group can be realized as a subgroup of $S_n$ for $n$ sufficiently large.

Proof: $G \to A_{set}(G) \cong S_n$

left multiplication action for $n = 161$.

(somewhat optimal sometimes not)

$q_g$

Last week we talked a lot about $G \cong G$ by conjugation.

$G \cong \text{Subsets of } G$

by conjugation.

$A \in G$

$gA = gAg^{-1} = (gag^{-1}) \cap A$

(Note: If $G \cong X$ then $G \cong \text{Subsets of } X$.)

If $A \leq G$ is a subset

$N_G(A) = \text{stabilizer of } A \text{ under conjugation}.$

$N_G(A)$
\[ N_G(A) \leq A. \]
\[ \text{by conjugation.} \]
\[ C_G(A) = \text{kernel of the action.} = \text{all elements of } G \text{ that commute with everything in } A. \]

3. Subgroup of \( H \leq \{ \text{subsets of } G \}. \)

\[ \text{Sub } G \text{-set for the conjugation action.} \]

(Why?): \[ g a b g^{-1} = g \cdot ((g b g^{-1}) a (g b g^{-1}))^{-1}. \]

If \( H \leq G \) then for \( a \in G \)
\[ H \to g H g^{-1} \]
\[ h \mapsto g h g^{-1} \]
is an isomorphism of groups.

(Conjugate subgroups of G are isomorphic as groups).

If \( H \leq G \) then \( N_G(H) \leq H \)
\[ N_G(H) \rightarrow \text{Aut}_G(H), \]
\[ N_G(H) \leq \text{Aut}_G(H). \]

Normal subgroups.

Definition/theorem: A subgroup \( H \leq G \) is normal if any of the following equivalent conditions hold:

1. \( H \) is the kernel of a homomorphism \( G \to G'. \)
2. \( N_G(H) = G \) (i.e., \( g H g^{-1} = H \) for all \( g \in G \)).
3. \( H \) is a union of conjugates of itself.
(3) $G/H = H ackslash G$ \hspace{1cm} (4) $gH = Hg$ \hspace{0.5cm} for \hspace{0.5cm} any \hspace{0.5cm} $g \in G$.

\hspace{1cm} \text{Subsets of $G$.} \hspace{1cm} \Leftrightarrow \text{any left coset is a right coset}.

\hspace{1cm} \text{H is a union of conjugacy classes in $G$.}

To show (2)- (4) $\Rightarrow$ (1)

Show \hspace{0.5cm} \text{$G/H \trianglelefteq H ackslash G$}

\hspace{1cm} \text{has group structure}

\hspace{1.5cm} \text{(aH)(bZH) = a1b2H,}

\hspace{1cm} \text{work for a normal subgroup}

Just multiplication of subsets \hspace{0.5cm} \text{a1H a2H}

\hspace{1cm} \text{AB = \{ab \mid a \in A, b \in B\}} \hspace{0.5cm} \text{g1g2 Hg2 \text{-}1H}.

\hspace{1cm} \text{For $H$ normal:}

\hspace{1cm} \text{H = ker $G \rightarrow G/H$.}

\hspace{1cm} \text{g1g2 H H}

\hspace{1cm} \text{g1 \not\mid \text{g2 H}}.

\text{Exercise: If $H_1, H_2$ are subgroups of $G$}

\hspace{1cm} \text{then $H_1 H_2$ is a subgroup of $G$.}

\hspace{1cm} \Rightarrow \hspace{0.5cm} H_1 H_2 = H_2 H_1.

\text{Example: If $H_1 \leq G$ and $H_2 \leq N_6(H_1)$}

\hspace{1cm} \text{then $H_1 H_2 = H_2 H_1$}

\hspace{1cm} \text{(if $a \in H_1, b \in H_2$ \hspace{0.5cm} ab = b b^{-1} a b = b (b^{-1} a b) \hspace{0.5cm} H_2 H_1$.}
Second isomorphism theorem: If \( H \leq G \), \( M \leq N_G(H) \) then:

1. \( HM \leq G \)
2. \( H \triangleleft HM \).
3. \( \frac{M}{M \cap H} \cong \frac{HM}{H} \)

(Jurgen von Robin: Classify finite groups.

Find building blocks: Simple groups.

How to put them together:

- Extensions

\[ G \text{ is simple if the only normal subgroups are } \{e\} \text{ and } G. \]

\[ \text{If } G \text{ is not simple, take } N \text{ to be a normal subgroup of } G. \]
Let $N < G \rightarrow G/N \rightarrow 1$.

**Jordan-Holder Theorem**: $G$ has a filtration by subgroups $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n = 1$.

$G_i / G_{i+1}$ single.

The isomorphism classes of these don't depend on the choice.