Group actions.

A (left) action of $G$ on a set $X$ is a homomorphism

$$G \to \text{Aut}_{\text{set}}(X) \subset \text{bijective maps } \frac{X \times X}{f \circ g = f \circ (g \circ h)}$$

i.e. $g \cdot x \iff gh \cdot x = g \cdot (h \cdot x)$
and $e \cdot x = x$.

right action $G' \to \text{Aut}_{\text{set}}(X)$
the same underlying set with multiplication $g \ast h = h g$.

If $G \subseteq X \leftarrow$ left action
$x \cdot g := g^{-1} x$.
... also to go from right to left.

Examples: $\pi_1(X, x_0) \subseteq$ covering space of $X$.
$S_n \subseteq \{1, \ldots, n\}$. (here is $S_n = \text{Aut}(\{1, \ldots, n\})$

If $\gamma = (1, 1, 1)$
then the action of $(\mathbb{R}, +)$ on $\mathbb{R}^3
\gamma \cdot x = t x + \tilde{x}$.

group set (acting by translation)
$G \subseteq G \leftarrow$ left mul. action
$\gamma \cdot h = \gamma h$. 

(e.g. if $K = \mathbb{R}^n$)

$S_n \subset \mathbb{R}^n$ by permuting coordinates.

This gives an action on smooth functions

above is restriction to polynomial functions.

Example: $D_8 \leq O(2)$ $\triangleleft$ rigid transformations of $\mathbb{R}^2$

$\triangleleft$ subgroup preserving the square

\[ \cdot \circ \cdot \]

- rotation
- reflection

$D_8 \cong$ All points in the square.

Set of 4 corners of the square.

Set of 4 sides of the square.

Set of 2 diagonals of the square.

Orbits: If $G \times \mathcal{X}$

an orbit is an equivalence class for the relation
Example: What are the orbits for $D_8$ acting on the vertices/edges of a square?

If $G \times X$, $X = \bigcup \sigma$ of orbits of $G \times X$ (maybe $\text{Mallat's HSS}$).

"Main result of group actions."
If $X$ is a finite set, $G \times X$

$|X| = \sum |\sigma|$

Lagrange's Theorem: If $G$ is a finite group and $H \leq G$ then $|H| | \underline{|G|}$.

Proof: $H \times G$ via left multiplication.

$G = \bigcup \sigma$ of orbits

$|G| = \sum |\sigma|$

If $g \in \sigma$ then $\sigma = \{ag \mid a \in H \}$ for $a \in H$.
Defn: The right casts of \( H \) in \( G \) are
the orbits of \( H \cdot G \) via left multiplication
\( = H \backslash G \)

The left casts of \( H \) in \( G \) are
the orbits of \( G \cdot H \) via right multiplication
\( = G / H \).

\( G \cong G / H \) \hspace{1cm} \( H \backslash G \cong G \)

by left multiplication \hspace{1cm} by right multiplication

Defn \( G \cong X \) transitively if there is a single orbit.

Theorem (Classification Theorem for \( G \)-sets = set \( \times \) \( \cdot \) (at \ 6\-action))

\( 1 \) Any transitive \( G \)-set is isomorphic to \( G / H \) for some \( H \).

\( 2 \) Any \( G \)-set \( X \) has a unique decomposition
\( X = \bigsqcup_{i \in I} X_i \) for \( X \subseteq X \)

Proof: \( 2 \) - \( X_i \) are the orbits!

\( 1 \) Exercise: If \( G \cong X \) transitively,
Fix $v \in X$. \[ \text{Stab}(v) \leq G \]
\[ \{ g \in G \mid g \cdot v = v \} \]
The map \[ G/\text{Stab}(v) \to X \]
sending \[ g \mapsto g \cdot v \]
is well-defined and an isomorphism of \( G \)-sets.

**Example:** \( D_8 \times G \)

Corners of square are a simple \( D_8 \)-set (i.e., \( D_8 \) acts transitively).

\[ \text{Stab}(p) = \{ \text{Reflection about \( \overline{PS} \)} \} \]

Get an isomorphism of \( D_8 \)-sets:

\[ D_8 / \text{Stab}(p) \to \text{Corners of the square} \]
\[ g \mapsto g \cdot p \]

\[ \text{Stab}(q) = \{ \text{Reflection about \( \overline{PQ} \)} \} \]

\[ D_8 / \text{Stab}(q) \to \text{Corners of the square} \]
\[ g \mapsto g \cdot q \]

\[ \text{Stab}(r) = \text{Stab}(s) = \{ \text{Reflection about \( \overline{PR} \)} \} \]
Fact/Exercise: If \( G \subseteq \text{Aut}(X) \) and \( x \in X \) then \( \text{stab}(G \cdot x) = G \cdot \text{stab}(x) \).

**Observation:** In \( G = S_n \) the identification \( G_H = \{ h \in G : h(1) = 1 \} \) is not canonical. But different choices lead to different subgroups \( G_H \).