Group actions.

A (left) action of G on a set X is a homomorphism

$$G \to \mathrm{Aut}_{set}(X) \leftarrow$$ bijective maps $X \to X$, $f_g(x) = f(g(x))$

i.e. $g \cdot x$ s.t. $gh \cdot x = g \cdot (h \cdot x)$ and $e \cdot x = x$.

right action $G^{op} \to \mathrm{Aut}_{set}(X)$

G^{op}: same underlying set with multiplication $g * h = hg$.

$$G \cong G^{op}$$
$$g \mapsto g^{-1}.$$

If $G \circlearrowright X$ ← left action

$x \cdot g := g^{-1}x$.

... also to go from right to left.

Examples: $\pi_1(X, x_0) \circlearrowright$ Covering space of X.

$S_n \circlearrowright \{1, \ldots, n\}$.

$(S_n = \mathrm{Aut}(\{1, \ldots, n\}))$

If $\vec{v} = (1,1,1)$

then get action of $(\mathbb{R}, +)$ on $\mathbb{R}^3$

$t \cdot \vec{x} = t\vec{v} + \vec{x}$.

(action by translation).

group → $G \circlearrowright G$ ← set   left mult. action

$g \cdot h = gh$.

$$\overset{\text{set}}{G} \circlearrowleft \overset{\text{group}}{G} \qquad \text{right mult. action}$$

$$h \cdot g = hg.$$

viewed as a left action.

$$G \circlearrowright G$$

$$g \cdot h = hg^{-1}$$

If X is a topological space.

$$\mathrm{Homeo}(X) \circlearrowright X$$

$$\mathrm{Aut}_{Top}(X) = \mathrm{Homeo}(X) \subseteq \mathrm{Aut}_{set}(X)$$

$$G \circlearrowright G \qquad \text{conjugation action.}$$

$$g \cdot h = ghg^{-1}$$

If $\vec{v}$ is a complete vector field on manifold $X \rightsquigarrow \mathbb{R}$-action on X.

$$\int \vec{v} : \mathbb{R} \longrightarrow \mathrm{Diffeo}(X) \circlearrowright X$$

$$\mathrm{Diffeo}(X) \circlearrowright C^{\infty}(X, \mathbb{R})$$

smooth $\mathbb{R}$-valued on X.

$$\psi \qquad\qquad \phi \cdot f = f(\phi^{-1}(x)).$$

$$\phi \qquad\qquad \left(= (\phi^{-1})^{*}(f) \right).$$

$$S_n \circlearrowright K[x_1, \dots, x_n]$$

$$\sigma \cdot f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

(e.g. if $K = \mathbb{R}$

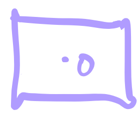
$S_n \circlearrowright \mathbb{R}^n$ by permuting coordinates.

this gives an action on smooth functions

above is restriction to polynomial functions.)

Example $D_8 \subseteq O(2) \leftarrow$ rigid linear transformations of $\mathbb{R}^2$

$\quad =$ subgroup preserving the square.



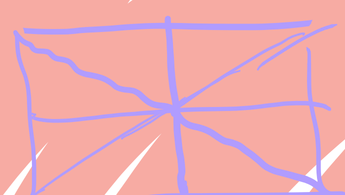
4 rotation
4 reflection

$D_8 \circlearrowright$ All points in the square.

set of 4 corners of the square.

set of 4 sides of the square.

set of 2 diagonals of the square

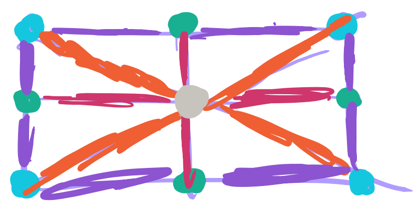
~~$D_8 \circlearrowright$~~  ~~$\subseteq \mathbb{R}^2$.~~

Orbits: If $G \circlearrowright X$

an orbit is an equivalence class for the relation

$x_1 \sim x_2 \iff \exists\, g \in G$ s.t. $gx_1 = x_2$.

Example: What are the orbits for D_8 acting on the vertices/edges of



?

If $G \circlearrowright X$ $\quad X = \bigsqcup_{\substack{\mathcal{O} \text{ orbits} \\ \text{of } G \circlearrowright X}} \mathcal{O}$ $\quad$ (maybe \mathfrak{so})

"Main result of group actions"

If X is a finite set. $G \circlearrowright X$

$$|X| = \sum_{\mathcal{O} \text{ orbit}} |\mathcal{O}|.$$

Lagrange's Theorem: If G is a finite group and $H \leq G$ then $|H| \mid |G|$.

Proof: $H \circlearrowright G$ via left multiplication.

$$G = \bigsqcup_{\text{orbits}} \mathcal{O}$$

$$|G| = \sum_{\text{orbits}} \mathcal{O}$$

if $g \in \mathcal{O}$ then $\mathcal{O} = \{hg \mid h \in H\}$.

for $h_1 \neq h_2$

$h_1 g \neq h_2 g$.

so $|O| = |H|$.

thus
$$|G| = (\#\text{ of orbits})\,|H|.$$

Def'n: The right cosets of H in G are the orbits of $H \circlearrowright G$ via left multiplication
$= H \backslash G$

The left cosets of H in G are the orbits of $G \circlearrowleft H$ via right multiplication
$= G/H$.

$G \circlearrowright G/H$ by left multiplication. $H\backslash G \circlearrowleft G$ by right multiplication

Def'n $G \circlearrowright X$ transitively if there is a single orbit.

Theorem (Classification theorem for G-sets = set w/a left G-action)

① Any transitive G-set is isomorphic to G/H for some H.

② Any G-set X has a unique decomposition
$$X = \bigsqcup_{i \in I} X_i \quad \text{for } X_i \subseteq X$$
subsets on which G acts transitively.

Proof. ② – X_i are the orbits!

① Exercise: If $G \circlearrowright X$ transitively.

Fix $x \in X$. $\mathrm{Stab}(x) \leq G$

$$\mathrm{Stab}(x) = \{g \in G \mid gx = x\}.$$

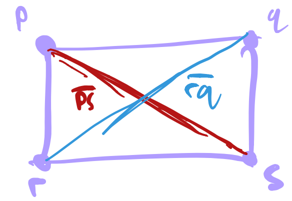
The map $G/_{\mathrm{Stab}(x)} \to X$

sending

$$g \longmapsto gx$$

is well-defined and an isomorphism of G-sets.

Example: $D_8 \circlearrowright$



Corners of square are a simple D_8-set (i.e. D_8 acts transitively).

$$\mathrm{Stab}(p) = \left\{ \begin{matrix} \text{Reflection about } \overline{ps} \\ e \end{matrix} \right\}$$

Get an isomorphism of D_8-sets.

$$D_8/_{\mathrm{Stab}(p)} \xrightarrow{\sim} \text{Corners of the square.}$$
$$g \longmapsto g \cdot p.$$

$$\mathrm{Stab}(q) = \left\{ \begin{matrix} \text{reflection about } \overline{rq} \\ e \end{matrix} \right\}.$$

$$D_8/\mathrm{Stab}(q) \xrightarrow{\sim} \text{Corners of the square}$$
$$g \longmapsto g \cdot q.$$

$$\mathrm{Stab}(r) = \mathrm{Stab}(q) = \{\text{reflection about } \overline{rq}, e\}.$$

$$D_8/\mathrm{Stab}(r) \cong \text{Corners of the square}$$
$$g \mapsto g\cdot r.$$

Observation: in ① the identification
$$G/H \cong X$$
really is not canonical —
what's canonical is if you fix $x \in X$
$$G/\mathrm{Stab}(x) \cong X.$$
But different choices lead to different isomorphisms and possibly different stabilizers.

Fact/Exercise: If $G \circlearrowright X$ and $x \in X$ then $\mathrm{Stab}(g\cdot x) = g\,\mathrm{Stab}(x)\,g^{-1}$.

Corollary: If $G \circlearrowright X$ transitively then the stabilizers are a conjugacy class of subgroups.

(If X is a G-set.
Get a map of G-sets
$G \circlearrowright$ by conjugation
$$X \longrightarrow \{\text{Subgroups of } G\}$$
$$x \longmapsto \mathrm{Stab}(x) \quad)$$