6320-001 - SPRING 2021 - WEEK 9 (3/16, 3/18)

1. This week in 6320

3/16 – Fields, roots of polynomials, and extensions
3/18 – Splitting fields, algebraic closures, finite fields

2. Comments and suggested reading

Dummit and Foote, 13.1-2, 13.4.

3. Preview of next week

(This is subject to change depending on how far we get this week!)
Separability (13.5), field automorphisms (14.1), Galois extensions and the fundamental theorem (statements only) (part of 14.2), Galois theory of finite fields (14.3), primitive element theorem (14.4), cyclotomic fields (parts of 13.3, 13.6, 14.5).

4. Homework

Due Tuesday, *March 23*, at 11:59pm on Gradescope
All solutions must be typeset using TeX and submitted via Gradescope; handwritten or late submissions will not be accepted. All exercises and problems submitted must start with the statement of the exercise or problem.

You may work in groups, but you must write up your final solutions individually. Any instances of academic misconduct will be taken very seriously.

Justify your answers carefully!

4.1. **Exercises.** *Complete and turn in ALL exercises*:
Grading scale (for each part of an exercise):
3 points – A correct, clearly written solution
2 points – Right idea, but a minor mistake or not clearly argued
1 point – Some progress but multiple minor mistakes or a major mistake
0 points – Nothing written, totally incorrect, or no substantive progress made towards a solution.

Exercise 1.

(1) If V is a vector space over $\mathbb{F}_p$ of dimension $n \in \mathbb{N}$, what is $|V|$? (Here for any set X we denote by $|X|$ the cardinality of X).
(2) Deduce that if k is a field, then $|k|$ is either infinite or p^n for a prime number p

Exercise 2. (DF 13.2 - Exercise 5). Show that $x^3 - 2$ and $x^3 - 3$ are irreducible over $\mathbb{Q}(i)$.

Exercise 3 (cf. DF 13.2 - Exercise 7)

(1) Show that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.

(2) Show that $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$.
(3) Deduce that $[\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}] = 4$, then find the minimal polynomial of $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$.

Exercise 4. (cf. DF 13.4, Exercises 1-4) For each of the following polynomials $p(x) \in \mathbb{Q}[x]$, express all of the roots of $p(x)$ in $\mathbb{C}$ in the form $e^{2\pi ik/n}a^{1/b}$ for non-negative integers $0 \le k \le n$, a, and b, and compute the degree over $\mathbb{Q}$ of the splitting field of $p(x)$ in $\mathbb{C}$. (Here $a^{1/b}$ denotes the unique non-negative bth root of a in $\mathbb{R}$).

(1) $x^4 - 2$
(2) $x^4 + 2$
(3) $x^4 + x^2 + 1$
(4) $x^6 - 4$

Exercise 5.

(1) Show $x^2 - t$ is irreducible over $\mathbb{F}_p(t)$.
(2) How many distinct roots does $x^2 - t$ have in an algebraic closure of $\mathbb{F}_p(t)$? (Hint: the obvious answer is right when $p \neq 2$. What happens when $p = 2$?).

4.2. **Problems.** *Attempt as many as you have time for, but only turn in one (of your choice).*
Grading scale (for the problem you turn in):
10 points - A correct, complete, and clearly written solution.
8 points - Right idea, but one or two minor mistakes or not clearly argued.
5 points - Some progress but several minor mistakes or a major mistake.
0 points - Nothing written, totally incorrect, or no substantive progress made.
Revision policy: *If you score at least 5 points on the problem you turn in then you will be allowed to submit* **one** *revision to your solution before the final exam (May 3, 10:30am). If the revision is correct, complete, and clearly written then your mark will change to 9 points. This policy only applies to the problem you submit, not to the exercises in the previous section.*

Problem 1 (cf. DF 13.2 - Exercises 19-21.) Let K/F be an extension of degree n.

(1) For any $\alpha \in K$, show that multiplication by α induces a linear transformation on K, thought of as an n-dimensional F-vector space.
(2) If $f(x)$ is the character polynomial of the linear transformation from (1), show that $f(\alpha) = 0$.
(3) Use (2) to compute the minimal polynomial of $1 + 2^{1/3} + 4^{1/3}$ over $\mathbb{Q}$ (here $a^{1/3}$ denotes the unique real cube root of a real number a).

Problem 2 (DF 13.2 - Exercise 16). Let K/F be an algebraic extension and let R be a *ring* contained in K and containing F. Show that R is a field.