6320-001 - SPRING 2021 - WEEK 5 (2/16, 2/18)

1. This week in 6320

2/16 – p-groups, upper central series, nilpotent groups, commutators, lower central series, derived series, solvable groups (6.1)

2/18 – groups of medium order (6.2)

Key concepts: *Commutators, upper and lower central series, nilpotent groups, derived series*

2. Comments and suggested reading

- **Dummit and Foote** – 6.1-6.2
- In addition to the homework below, you are always encouraged (but not required) to work as many supplementary exercises as you have time for from the suggested reading sections!

3. Preview of next week

(This is subject to change depending on how far we get this week!)
Characters of abelian groups – in particular, Fourier theory and linear independence of characters.

The latter is Theorem 7 in **DF**-14.2, but the classes this week will stand on their own without any reading. Treating this topic now serves two purposes going forward – it gives a preview of representation theory, which we will study in more detail in the last few weeks of the class, and also introduces the main technical lemma that is used to establish the fundamental theorem of Galois theory (which we will come to a couple of weeks after the spring pause).

4. Homework

Due Tuesday, February 23 at 11:59pm on Gradescope
All solutions must be typeset using TeX and submitted via Gradescope; handwritten or late submissions will not be accepted. All exercises and problems submitted must start with the statement of the exercise or problem.

You may work in groups, but you must write up your final solutions individually. Any instances of academic misconduct will be taken very seriously.

Justify your answers carefully!

4.1. **Exercises.** *Complete and turn in ALL exercises*:

Grading scale (for each part of an exercise):
3 points – A correct, clearly written solution
2 points – Right idea, but a minor mistake or not clearly argued
1 point – Some progress but multiple minor mistakes or a major mistake
0 points – Nothing written, totally incorrect, or no substantive progress made towards a solution.

Exercise 1. A sequence of group homomorphisms

$$\dots G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} \dots$$

is called exact if $\mathrm{Im} f_i = \mathrm{Ker} f_{i+1}$ for all i. In particular, a sequence

$$1 \to A \xrightarrow{i} B \xrightarrow{p} C \to 1$$

is exact if and only if p is a surjection and i identifies A with the kernel of p. Such an exact sequence is called a *short exact sequence*. A *splitting* of a short exact sequence is a map $s : C \to B$ such that $p \circ s = \mathrm{Id}$. If a splitting exists, the sequence is called split; otherwise it is non-split.

(1) Show that a splitting gives rise to a natural presentation as a semi-direct product

$$B = A \rtimes C.$$

(2) Give any homomorphism $\phi : C \to \mathrm{Aut}(A)$, find a splitting of the natural short exact sequence

$$1 \to A \to A \rtimes_\phi C \to C \to 1.$$

(3) Give an example of a non-split short exact sequence with B abelian.
(4) Give an example of a non-split short exact sequence with B nonabelian.

Exercise 2. For G a group, recall that a subgroup $H \le G$ is *characteristic* if any automorphism $\phi : G \to G$ satisfies $\phi(H) = H$. In particular, taking ϕ to be conjugation by an element of G, we find that a characteristic subgroup is normal. **Pick one of the following to write up and turn in:**

(1) (Upper central series) Show that $Z_i(G)$ is a characteristic subgroup of G.
(2) (Lower central series) Show that G^i is a characteristic subgroup of G.
(3) (Derived/commutator series) Show that $G^{(i)}$ is a characteristic subgroup of G.

Exercise 3. This exercise shows the different definitions of a solvable group given in the book and class are equivalent.

(1) Show the following two conditions on a group G are equivalent:
   (a) $G^{(i)} = \{1\}$ for i sufficiently large (i.e. the derived series terminates at $\{1\}$).
   (b) There exists a chain of subgroups

   $$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

   such that G_i/G_{i-1} is abelian for $1 \le i \le n$.

   A group G satisfying either of these conditions is called solvable.
(2) If G is finite, show that G is solvable if and only if there exists a chain of subgroups

   $$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

   such that G_i/G_{i-1} is cyclic for $1 \le i \le n$.

Exercise 4. Let G be the group of 3×3 invertible upper-triangular matrices with real entries.

(1) Compute the derived subgroup $[G, G]$.
(2) Show $[G, G]$ is nilpotent.
(3) Deduce that G is solvable.
(4) Is G nilpotent?

4.2. **Problems.** *Attempt as many as you have time for, but only turn in one (of your choice).*
Grading scale (for the problem you turn in):
10 points - A correct, complete, and clearly written solution.
8 points - Right idea, but one or two minor mistakes or not clearly argued.
5 points - Some progress but several minor mistakes or a major mistake.
0 points - Nothing written, totally incorrect, or no substantive progress made.
Revision policy: *If you score at least 5 points on the problem you turn in then you will be allowed to submit* **one** *revision to your solution before March 4th (the day of the midterm). If the revision is correct, complete, and clearly written then your mark will change to 9 points. This policy only applies to the problem you submit, not to the exercises in the previous section.*

Problem 1. Show that A_n is the commutator subgroup of S_n for all n.

Problem 2. Show that the commutator subgroup of $\mathrm{GL}_2(\mathbb{R})$ is $\mathrm{SL}_2(\mathbb{R})$.

Problem 3. Classify the groups of order 105.