6320-001 - SPRING 2021 - WEEK 4 (2/09, 2/11)

1. This week in 6320

2/09 – Finish the proof of the Sylow theorems, simplicity of $A_n, n \geq 5$
2/11 – Direct and semi-direct products, exact and split-exact sequences
Key concepts: *Simplicity of A_n, semi-direct products*

2. Comments and suggested reading

- **Dummit and Foote** – 4.5-4.6, 5.1–5.5 (5.1-5.4 are review!)
- In addition to the homework below, you are always encouraged (but not required) to work as many supplementary exercises as you have time for from the suggested reading sections!

3. Preview of next week

(This is subject to change depending on how far we get this week!)
Classification of groups of small order, commutator subgroups, solvable groups, nilpotent groups.

Note that next week is the last week we will be spending on abstract group theory – in the following week we'll treat character theory as a preparation for/introduction to Galois theory and the representation theory of finite groups, then we'll have our exam week followed by "Spring Pause".

4. Homework

Due Tuesday, February 16 at 11:59pm on Gradescope
All solutions must be typeset using TeX and submitted via Gradescope; handwritten or late submissions will not be accepted. All exercises and problems submitted must start with the statement of the exercise or problem.

You may work in groups, but you must write up your final solutions individually. Any instances of academic misconduct will be taken very seriously.

Justify your answers carefully!

4.1. **Exercises.** *Complete and turn in ALL exercises*:
Grading scale (for each part of an exercise):
3 points – A correct, clearly written solution
2 points – Right idea, but a minor mistake or not clearly argued
1 point – Some progress but multiple minor mistakes or a major mistake
0 points – Nothing written, totally incorrect, or no substantive progress made towards a solution.

Exercise 1. (**DF**-4.5.6) Exhibit all Sylow 3-subgroups of A_4 and S_4.

Exercise 2. For p prime, let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ denote the field with p elements.

(1) Find $|\mathrm{GL}_2(\mathbb{F}_p)|$ and $|\mathrm{SL}_2(\mathbb{F}_p)|$.
(2) $\mathrm{SL}_2(\mathbb{F}_p)$ acts on $\mathbb{F}_p^2\backslash\{0\}$. Show that the Sylow p-subgroups are precisely the stabilizer groups of points for this action. How many Sylow p-subgroups are there?
(3) Express the stabilizer in $\mathrm{GL}_2(\mathbb{F}_p)$ of a point in $\mathbb{F}_p^2\backslash\{0\}$ as a semidirect product.

Exercise 3.

(1) Give an explicit description of $Z(\mathrm{GL}_2(\mathbb{F}_p))$ in terms of matrices and show that
$$Z(\mathrm{SL}_2(\mathbb{F}_p)) = Z(\mathrm{GL}_2(\mathbb{F}_p)) \cap \mathrm{SL}_2(\mathbb{F}_p).$$

(2) We write
$$\mathrm{PGL}_2(\mathbb{F}_p) := \mathrm{GL}_2(\mathbb{F}_p)/Z(\mathrm{GL}_2(\mathbb{F}_p)) \text{ and } \mathrm{PSL}_2(\mathbb{F}_p) := \mathrm{SL}_2(\mathbb{F}_p)/Z(\mathrm{SL}_2(\mathbb{F}_p)).$$
By (1), $\mathrm{PSL}_2(\mathbb{F}_p)$ is canonically identified with a normal subgroup of $\mathrm{PGL}_2(\mathbb{F}_p)$. Compute
$$[\mathrm{GL}_2(\mathbb{F}_p) : \mathrm{SL}_2(\mathbb{F}_p)] \text{ and } [\mathrm{PGL}_2(\mathbb{F}_p) : \mathrm{PSL}_2(\mathbb{F}_p)].$$

(3) For any prime p, show that the kernel of the action of $\mathrm{GL}_2(\mathbb{F}_p)$ on the on the set $\mathbb{P}^1(\mathbb{F}_p)$ of lines through the origin (i.e., one dimensional subspaces) in $\mathbb{F}_p^2$ is equal to $Z(\mathrm{GL}_2(\mathbb{F}_p))$. Deduce that there is a faithful action of $\mathrm{PGL}_2(\mathbb{F}_p)$ on $\mathbb{P}^1(\mathbb{F}_p)$.

(4) For $p = 3$, show the action above induces an isomorphism
$$\mathrm{PGL}_2(\mathbb{F}_3) \xrightarrow{\sim} S_4.$$
What is the image of $\mathrm{PSL}_2(\mathbb{F}_3)$ under this isomorphism?

Exercise 4. (**DF**-5.5.6-7).

(1) Suppose K is a cyclic group, H is an arbitrary group, and $\phi_i : K \to \mathrm{Aut}(H)$, $i = 1, 2$ are homomorphisms from K to $\mathrm{Aut}(H)$ such that $\phi_1(K)$ and $\phi_2(K)$ are conjugate subgroups of $\mathrm{Aut}(H)$. If K is infinite, assume furthermore that the maps ϕ_i are both injective. Construct an explicit isomorphism
$$H \rtimes_{\phi_1} K \cong H \rtimes_{\phi_2} K.$$

(2) Construct a non-abelian group of order 75.

(3) Classify the groups of order 75.

4.2. **Problems.** *Attempt as many as you have time for, but only turn in one (of your choice).*
Grading scale (for the problem you turn in):
10 points - A correct, complete, and clearly written solution.
8 points - Right idea, but one or two minor mistakes or not clearly argued.
5 points - Some progress but several minor mistakes or a major mistake.
0 points - Nothing written, totally incorrect, or no substantive progress made.
Revision policy: *If you score at least 5 points on the problem you turn in then you will be allowed to submit* **one** *revision to your solution before March 4th (the day of the midterm). If the revision is correct, complete, and clearly written then your mark will change to 9 points. This policy only applies to the problem you submit, not to the exercises in the previous section.*

Problem 1. (**DF**-4.5.30) How many elements of order 7 must there be in a simple group of order 168?

Problem 2. Show $\mathrm{PSL}_2(\mathbb{F}_5)$ is a simple group.

Problem 3.

(1) Give representatives for all of the conjugacy classes in $\mathrm{SL}_2(\mathbb{C})$.

(2) Give representatives for all of the conjugacy classes in $\mathrm{SL}_2(\mathbb{R})$.

(3) Which of the representatives you gave for $\mathrm{SL}_2(\mathbb{R})$ become conjugate in $\mathrm{SL}_2(\mathbb{C})$?