6320-001 - SPRING 2021 - WEEK 3 (2/02, 2/04)

1. This week in 6320

2/02 – After quickly hitting Cayley's theorem (which we didn't get to last week), we'll discuss normal subgroups and quotient groups in a bit more detail, summarize the Jordan-Holder program, and then give a counting argument with conjugacy classes to see that A_5 is a simple group.

2/04 – The Sylow theorem(s)

Key concepts: *Cayley's theorem, normalizer, normal subgroups, simple groups, the alternating group, Sylow theorems*

2. Comments and suggested reading

- **Dummit and Foote** – 3.4-3.5, 4.2, 4.4-4.5.
- In addition to the homework below , you are always encouraged (but not required) to work as many supplementary exercises as you have time for from the suggested reading sections!

3. Preview of next week

(This is subject to change depending on how far we get this week!)
Simplicity of $A_n, n \geq 5$ (4.6) and semi-direct products (5.5).

4. Homework

Due Tuesday, February 9 at 11:59pm on Gradescope
All solutions must be typeset using TeX and submitted via Gradescope; handwritten or late submissions will not be accepted. All exercises and problems submitted must start with the statement of the exercise or problem.

You may work in groups, but you must write up your final solutions individually. Any instances of academic misconduct will be taken very seriously.

Justify your answers carefully!

4.1. **Exercises.** *Complete and turn in ALL exercises*:
Grading scale (for each part of an exercise):
3 points – A correct, clearly written solution
2 points – Right idea, but a minor mistake or not clearly argued
1 point – Some progress but multiple minor mistakes or a major mistake
0 points – Nothing written, totally incorrect, or no substantive progress made towards a solution.

Exercise 1. (**DF** Exercise 3.3.7) Let M and N be normal subgroups of G such that $G = MN$. Show that the natural map

$$G/(M \cap N) \to G/M \times G/N$$

is an isomorphism.

Exercise 2. (**DF** Exercises 3.5.3-4)

(1) Show that $S_n = \langle (i, i+1) \mid 1 \leq i \leq n-1 \rangle$.

(2) Show that $S_n = \langle (1\,2), (1\,2\,3\,\ldots\,n) \rangle$.

Exercise 3.

(1) Let G be a finite group and p the smallest prime dividing $|G|$. Show that if $H \leq G$ is such that $[G : H] = p$ then $H \triangleleft G$.

(2) Give an example of a non-abelian G and a subgroup H where part (1) applies in each of the following cases:

  (a) $[G : H] = p = 2$

  (b) $[G : H] = p > 2$

Exercise 4. Find a formula for the size of each conjugacy class in S_n. (Recall that a conjugacy class in S_n is uniquely determined by the number of k-cycles m_k for each $1 \leq k \leq n$ in its cycle decomposition). (Hint: If you get stuck or want to check your formula, see **DF** Exercise 4.3.33).

Exercise 5.

(1) Show that if $|G| = p^n$ for p prime then $Z(G) \neq \{e\}$.

(2) Deduce Cauchy's theorem (if p prime divides $|G|$ then G has an element of order p) from the above and Sylow's theorem.

4.2. **Problems.** *Attempt as many as you have time for, but only turn in one (of your choice).*
Grading scale (for the problem you turn in):
10 points - A correct, complete, and clearly written solution.
8 points - Right idea, but one or two minor mistakes or not clearly argued.
5 points - Some progress but several minor mistakes or a major mistake.
0 points - Nothing written, totally incorrect, or no substantive progress made.
Revision policy: *If you score at least 5 points on the problem you turn in then you will be allowed to submit* **one** *revision to your solution before March 4th (the day of the midterm). If the revision is correct, complete, and clearly written then your mark will change to 9 points. This policy only applies to the problem you submit, not to the exercises in the previous section.*

Problem 1. Let X be a conjugacy class of even permutations in S_n. Then, X is a disjoint union of conjugacy classes in A_n. How many are there? Your answer should break into two cases based on a simple condition on the cycle decomposition. (*Hint*: This is outlined in **DF** Exercises 4.3.19-21, but try thinking about it on your own first!)

Problem 2. Show that for $n \neq 6$ every automorphism of S_n is inner. (*Hint*: You can use the steps outlined in **DF** Exercise 4.4.18, but try thinking about it on your own first!)

Problem 3. (**DF 4.5.19-23**). Show there is no simple group of order n for

(1) $n = 6545$

(2) $n = 1365$

(3) $n = 2907$

(4) $n = 132$

(5) $n = 462$