6320-001 - SPRING 2021 - WEEK 2 (1/26, 1/28)

1. This week in 6320

We'll start the week with a quick review of group actions – that is, the theory of how an abstract group can be realized concretely via the symmetries of a set. A (left[1]) action of a group G on a set X is just a map of groups

$$G \to \mathrm{Aut}_{\mathrm{Set}}(X),$$

though it is often axiomatized separately and written with the notation $G \circlearrowright X$, with the map induced by $g \in G$ written $x \mapsto g \cdot x$ and the restrictions $g \cdot (h \cdot x) = (gh)x, e \cdot x = x$.

After a basic study, we'll use group actions to prove some basic counting theorems for finite groups. The main elementary results are Lagrange's theorem and the orbit stabilizer theorem. These provide fundamental tools for treating the classification and the structure theory of finite groups – given an abstract group G, one constructs from thin air (read: the abstract theory itself) some natural sets on which G acts, and then relates the numerics of these actions to the structure of the group.

1/26 – Group actions.

1/28 – Some counting results.

Key concepts: *group actions, orbits, stabilizers, cosets, orbit-stabilizer theorem, Lagrange's theorem, Cayley's theorem, class equation, center, centralizer, normalizer*

2. Comments and suggested reading

- **Dummit and Foote** – 1.7, 2.2-2.5, 3.2-3.3, 4.1-4.3.
- In addition to the homework below, you are always encouraged (but not required) to work as many supplementary exercises as you have time for from the suggested reading sections!

3. Preview of next week

(This is subject to change depending on how far we get this week!)
Simple groups, Sylow Theorems, simplicity of $A_n, n \geq 5$ (3.5, 4.4-4.6).

4. Homework

Due Tuesday, February 2 at 11:59pm on Gradescope
All solutions must be typeset using TeX and submitted via Gradescope; handwritten or late submissions will not be accepted. All exercises and problems submitted must start with the statement of the exercise or problem.

You may work in groups, but you must write up your final solutions individually. Any instances of academic misconduct will be taken very seriously.

Justify your answers carefully!

[1]A right action is axiomatized in the obvious way but with $X \circlearrowleft G$, i.e. with the action written as $x \cdot g$, so $(x \cdot h) \cdot g = x \cdot (hg)$. It is equivalent to a map $G^{\mathrm{op}} \to \mathrm{Aut}_{\mathrm{Set}}(X)$, where G^{op} is the *opposite* group obtained by reversing the order of multiplication on G. One can move back and forth between left and right actions using the isomorphism $G \xrightarrow{\sim} G^{\mathrm{op}}$, $g \mapsto g^{-1}$.

4.1. **Exercises.** *Complete and turn in ALL exercises*:
Grading scale (for each part of an exercise):
3 points – A correct, clearly written solution
2 points – Right idea, but a minor mistake or not clearly argued
1 point – Some progress but multiple minor mistakes or a major mistake
0 points – Nothing written, totally incorrect, or no substantive progress made towards a solution.

Exercise 1. How many elements of order 6 are there in S_6?

Exercise 2. For $k \leq n$, what is the average number of k-cycles in a permutation in S_n?

Exercise 3. Draw the subgroup lattice for the quaterion group, Q (cf. DF1.5 or Problem 3 from last week). For each subgroup appearing, give generators and the order; indicate which subgroups are conjugate to eachother (and, in particular, which are normal). **You do not need to prove anything for this exercise; it will suffice to give a single clean diagram containing all of the above information.** It may be helpful to look up the packages xypic or tikzcd for drawing the digram in TeX.

Exercise 4. For Q the quaternion group as above, show there is no faithful permutation representation of Q on a set of size < 8.

Exercise 5. Let $T \subset \mathrm{GL}_n(\mathbb{C})$ be the group of invertible diagonal matrices. Describe the orbits of T acting on $\mathbb{C}^n$. How many are there?

4.2. **Problems.** *Attempt as many as you have time for, but only turn in one (of your choice).*
Grading scale (for the problem you turn in):
10 points - A correct, complete, and clearly written solution.
8 points - Right idea, but one or two minor mistakes or not clearly argued.
5 points - Some progress but several minor mistakes or a major mistake.
0 points - Nothing written, totally incorrect, or no substantive progress made.
Revision policy: *If you score at least 5 points on the problem you turn in then you will be allowed to submit* **one** *revision to your solution before March 4th (the day of the midterm). If the revision is correct, complete, and clearly written then your mark will change to 9 points. This policy only applies to the problem you submit, not to the exercises in the previous section.*

Problem 1. For G a finite group, show that if $H < G$ is a proper subgroup ($H \neq G$) then
$$G \neq \bigcup_{g \in G} gHg^{-1}.$$

Problem 2.

1. Give representatives for all of the conjugacy classes in $\mathrm{GL}_2(\mathbb{C})$.
2. Give representatives for all of the conjugacy classes in $\mathrm{GL}_2(\mathbb{R})$.
3. Which of the conjugacy classes in $\mathrm{GL}_2(\mathbb{R})$ become the same in $\mathrm{GL}_2(\mathbb{C})$?