6320-001 - SPRING 2021 - WEEK 1 (1/19, 1/21)

1. This week in 6320

1/19 – Introduction and motivation. Groups and symmetry; examples.
1/21 – Basic constructions and their universal properties
Key concepts: *automorphism groups, matrix groups, symmetric groups, subgroups, normal subgroups, quotient groups, homomorphisms, (direct) products, coproducts, free groups*

2. Comments and suggested reading

The first day, 1/19, will give a sweeping overview of the course and some related mathematics; we will return to many (but not all) topics in more detail later. The course begins in earnest on 1/21 when we will cover some basic notions and constructions with an emphasis on their universal properties; but much of what is discussed should be familiar already from your undergraduate algebra and we will move quickly.

- **Dummit and Foote** – Preliminaries and 1.1-1.6, 2.1, 3.1, 6.3. *In the first couple of weeks much of the should be familiar from your undergraduate algebra course so we will move fast and the suggested reading will cover a lot of sections. If you don't have time to read through carefully you might start instead by skimming to see what you need brushing up on first and then going back to those parts for a closer look!*
- In addition to the homework below, you are always encouraged (but not required) to work as many supplementary exercises as you have time for from the suggested reading sections!

3. Preview of next week

(This is subject to change depending on how far we get this week!)
Subgroups, group actions, and basic numerics (DF 1.7, 2.2-2.5, 3.2-3.3, 4.1-4.3).

4. Homework

Due Tuesday, January 26 at 11:59pm on Gradescope
All solutions must be typeset using TeX and submitted via Gradescope; handwritten or late submissions will not be accepted. All exercises and problems submitted must start with the statement of the exercise or problem.

You may work in groups, but you must write up your final solutions individually. Any instances of academic misconduct will be taken very seriously.

Justify your answers carefully!

4.1. **Exercises.** *Complete and turn in ALL exercises*:

Grading scale (for each part of an exercise):
3 points – A correct, clearly written solution
2 points – Right idea, but a minor mistake or not clearly argued
1 point – Some progress but multiple minor mistakes or a major mistake
0 points – Nothing written, totally incorrect, or no substantive progress made towards a solution.

Exercise 1. Find an n such that $(\mathbb{Z}/n\mathbb{Z})^\times$ is not cyclic.

Exercise 2. Find a finite subgroup of $\mathrm{GL}_2(\mathbb{R})$ which is not abelian, and write down the matrices of the group elements explicitly.

Exercise 3.

(1) Let R be a commutative ring. Show that $r \in R^\times$ if and only if r is not contained in any maximal ideal.

(2) If $R = \mathbb{C}[x_1, x_2, ..., x_n]/I$ for a radical ideal I, give a geometric intuition for part (1). You do not have to prove anything, but make your explanation short and precise. *Hint: what is the geometric interpretation of the maximal ideals in R? (You hopefully covered this last term!).*

4.2. **Problems.** *Attempt as many as you have time for, but only turn in one (of your choice).*
Grading scale (for the problem you turn in):
10 points - A correct, complete, and clearly written solution.
8 points - Right idea, but one or two minor mistakes or not clearly argued.
5 points - Some progress but several minor mistakes or a major mistake.
0 points - Nothing written, totally incorrect, or no substantive progress made.
Revision policy: *If you score at least 5 points on the problem you turn in then you will be allowed to submit* **one** *revision to your solution before March 4th (the day of the midterm). If the revision is correct, complete, and clearly written then your mark will change to 9 points. This policy only applies to the problem you submit, not to the exercises in the previous section.*

Problem 1.

(1) Suppose $g \in \mathrm{SL}_2(\mathbb{Z})$ has finite order. Show that $|g| = 1, 2, 3, 4,$ or 6. *Hint: First show that the eigenvalues of g in $\mathbb{C}$ are a pair of complex conjugate roots of unity whose sum is in $\mathbb{Z}$.*

(2) Find examples of elements $g \in \mathrm{SL}_2(\mathbb{Z})$ of orders 1, 2, 3, 4, and 6.

Problem 2.

(1) Show that all continuous group homomorphisms $\mathbb{R} \to \mathbb{C}^\times$ are of the form
$$t \mapsto e^{\lambda t}$$
for $t \in \mathbb{C}$ (you may admit the identity $e^{a+b} = e^a e^b$ and the continuity of $t \mapsto e^t$.)

(2) Describe all continuous group homomorphisms
$$\mathbb{C}^\times \to \mathbb{C}^\times$$
Hint: Construct a map $\mathbb{C}^\times \to \mathbb{R} \times \mathbb{R}/\mathbb{Z}$ which is a group isomorphism and a homemomorphism.

Problem 3. Let H denote the Hamiltonian quaternions:
$$H = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$$
with multiplication determined by
$$i^2 = j^2 = k^2 = -1, ij = -ji, ik = -ki, jk = -kj.$$
You may assume this defines a ring structure on H with multiplicative identity 1.

(1) Show H is a *division algebra*; that is, every non-zero element in H is invertible. *Hint: write*
$$\overline{a + bi + cj + dk} = a - bi - cj - dk$$
What is $h\overline{h}$?

(2) Consider the subring
$$H_{\mathbb{Z}} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k \subset \mathbb{H}$$
and its group of units $Q = H_{\mathbb{Z}}^\times$. What is $|Q|$? What is its center, $Z(Q)$?

(3) Find an inclusion of rings $H \hookrightarrow \mathrm{M}_2(\mathbb{C})$, and compute the images of i, j, and k. *Hint: equip H with the structure of a $\mathbb{C}$-vector space basis by considering *right* multiplication by $\mathbb{C} = \mathbb{R} + \mathbb{R}i \subset \mathbb{H}$. Then show that any element $h \in H$ induces a linear transformation by *left* multiplication.*