6320-001 - SPRING 2021 - FINAL (5/03)

1. Instructions

The final is open-book (Dummit and Foote) and open-notes, but no other sources are allowed. The exam must be taken individually while logged on to Zoom. You will be put in individual breakout rooms so that you can call me in to ask any questions.

You have **1 hour and 45 minutes** to work on the exam; after that period, you will upload your ANSWERS ONLY to Gradescope. After that time, no more work is to be done, but you can copy your answers onto a new piece of paper to take a picture of, or type them on your computer in a word processor or TeX, etc.; whatever works best for you to upload. Please stay in the Zoom session while doing this so that we can troubleshoot if you have any difficulties.

NOTE: All group representations below are over $\mathbb{C}$!

2. True/False (15 pts total)

For each of the following, please answer TRUE or FALSE (1 pt each).

1. Every group of order $155 = 5 \cdot 31$ is abelian. **F**
2. S_4 is solvable. **T**
3. There is a simple group of order $132 = 2^2 \cdot 3 \cdot 11$. **F**
4. There is a simple group of order $360 = 2^3 \cdot 3^2 \cdot 5$. **T**
5. There is a simple group of order $512 = 2^9$. **F**
6. If L/K is Galois and M/L is Galois then M/K is Galois. **F**
7. The polynomial $x^{p^2} + x^p + x - t \in \mathbb{F}_p(t)[x]$ is separable. **T**
8. There are exactly two monic irreducible polynomials of degree 3 in $\mathbb{F}_2[x]$. **T**
9. The Galois group of $x^3 - 7 \in \mathbb{Q}[x]$ is abelian. **F**
10. There is a finite extension of $K/\mathbb{Q}$ such that, for every $\alpha \in K$, $\mathbb{Q}(\alpha) \neq K$. **F**
11. There is a degree 3 Galois extension of $\mathbb{R}$. **F**
12. If $f(x) \in K[x]$ is irreducible and separable with Galois group G then $|G| = \deg f$. **F**
13. For H the subgroup of D_{24} (the dihedral group with 24 elements) generated by the 180 degree rotation, the restriction map on class functions $C(G) \to C(H)$ is surjective. **T**
14. $\mathbb{Z}/17\mathbb{Z}$ has a 2-dimensional irreducible representation. **F**
15. If ρ is an n-dimensional representation of a group G and $g \in G$, then $|\chi_\rho(g)| \leq n$. **T or F;**
I meant for G to be finite, in which case the answer is true, but it can be false if G is not finite! I accepted either answer.

3. Short answer (10 pts total)

The answer to each of the following is a single positive integer (2 pts each).

1. How many Sylow 7-subgroups are there in $\mathrm{GL}_3(\mathbb{F}_2)$? **8**
2. What is the degree over $\mathbb{Q}$ of a splitting field of $x^4 + 2x^2 + 1 \in \mathbb{Q}[x]$? **2**
3. For $f(x) = x^{27} - x \in \mathbb{F}_3[x]$, how many orbits are there for the action of the Galois group of f on the roots of f in a splitting field? **11**
4. Up to isomorphism, how many irreducible representations does S_5 have? **7**
5. What is the largest dimension of an irreducible representation of D_{10}, the dihedral group with 10

elements? **2**

4. Answer with proof (5 pts total)

Choose *one* of the following questions to answer with a full written justification. If you turn in more than one then I will pick one to grade at random.

1. (Group theory). Suppose $\sigma \in S_n$ is a permutation whose cycle type consists of distinct cycles of odd length (note – even though we don't usually write down 1-cycles, the cycle type includes 1-cycles, so this conditions means also that there is at most one 1-cycle in the decomposition!). Show that the conjugacy class of σ in S_n splits into two distinct conjugacy classes of equal size in A_n.

2. (Galois theory). Show that if L/K is Galois and $f \in K[x]$ is monic irreducible, then every irreducible factor of f in $L[x]$ has the same degree.

3. (Representation theory). Compute, with full justification, the character table of A_4.