Extensions and absolute values.

Example: $K/\mathbb{Q}$ finite extension. $K = \mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/f(x)$
f minimal polynomial of α.

$$K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$$

because $f(x)$ factors in $\mathbb{R}[x]$

$$\prod_{i=1}^{r}(x-\lambda_i) \prod_{j=1}^{s}\underbrace{(x^2+a_jx+b_j)}_{\text{irreducible.}}$$

$$\mathbb{Q}[x]/f(x) \otimes \mathbb{R} \cong \mathbb{R}[x]/f(x) \cong \prod_{i=1}^{r} \underbrace{\mathbb{R}[x]/(x-\lambda_i)}_{\text{each factor } \cong \mathbb{R}} \times \prod_{j=1}^{s} \underbrace{\mathbb{R}[x]/x^2+a_jx+b_j}_{\text{each factor } \cong \mathbb{C}}$$

Example: $K/\mathbb{Q}$ finite extension

$$K \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{\substack{\mathfrak{p} \mid p \\ \mathfrak{p} \text{ in } \mathcal{O}_K}} K_{\mathfrak{p}}$$

completion for $|\cdot|_{\mathfrak{p}}$.

to get $K_{\mathfrak{p}}$ factor $f(x)$ in $\mathbb{Q}_p[x]$.

Motivates the study of extensions of $\mathbb{Q}_p$, or more generally of nonarchimedean fields $\leftarrow$ complete w.r.t. a nonarch. absolute value (e.g. $\mathbb{Q}_p$, $\mathbb{F}_p((t))$, ...).

Theorem: If K is a non-archimedean field and L/K is a finite extension, then

$$|\cdot|_L := |N_{L/K}(\cdot)|_K^{1/[L:K]}$$

is an absolute value on L, L is complete for $|\cdot|_L$, and this is the unique absolute value on L extending the absolute value on K.

- Says algebra over a non-arch field automatically upgrades to analysis.
- Also true for K archimedean field because $K = \mathbb{R}$ or $\mathbb{C}$, $L = \mathbb{R}$ or $\mathbb{C}$

$\mathbb{C}/\mathbb{R}$ $\quad |z|_{\mathbb{C}} = |z\bar{z}|_{\mathbb{R}}$

Remarks on proof: Easier if $|\cdot|_K$ is discrete ($\Leftrightarrow \mathcal{O}_K := \{x \in K \mid |x| \le 1\}$ is a complete dvr.)

Theorem 7.38 in Milne in this case if L/K is separable.
Ex. 7 on worksheet in general case.

Uniqueness – really a general fact about topological vector spaces / completeness (proof of 7.38 skips an important point).

Existence ~ manipulations to get down to a Hensel's lemma.

In DVR case just use $\mathcal{O}_L$ has a prime above $\mathfrak{m} \in \mathcal{O}_K$

Uniqueness $\Rightarrow$ this is the only possible formula.

if $\sigma \in \mathrm{Aut}(L/K)$ $\quad$ if $|\ |_1$ is a abs value extending $|\ |_K$, then
$$|\alpha|_2 := |\sigma(\alpha)|_1 \text{ is too.}$$

Uniqueness $\Rightarrow |\alpha|_2 = |\alpha|_1$

i.e. $|\alpha| = |\sigma(\alpha)|$.

if Galois then
$$N_{L/K}(\alpha) = \prod_{\sigma \in \mathrm{Aut}(L/K)} \sigma(\alpha)$$
$$|N_{L/K}(\alpha)|_K = \prod |\sigma(\alpha)| = |\alpha|^{|\mathrm{Aut}(L/K)|} = |\alpha|^{[L:K]}$$

Corollary: If K a nonarchimedean field and L/K is an algebraic extension, there is a unique absolute value on L extending the absolute value on K.

e.g. $\bar{K}$

Useful because it tells us it makes sense to talk about the absolute values of roots of $f \in K[x]$.

Krasner's Lemma (v1): If K is nonarchimedean, $\bar{K}$ alg. closure of K and $f \in K[x]$ is separable, then for any $g \in K[x]$ sufficiently close to f (for top. on coefficients), for each root α_i of f there is a unique closest root β_i of g and $K(\alpha_i) = K(\beta_i)$.

Proof: Use Newton method / Hensel's lemma. Key point: $f'(\alpha_i)$

$= \prod$

$$= \prod_{j \neq i} |\alpha_i - \alpha_j| .$$

Corollary: If $L/\mathbb{Q}_p$ is a finite extension, $\exists$ a fin. extension $K/\mathbb{Q}$ and a prime $\mathfrak{p}|p$ in $\mathcal{O}_K$ s.t.

$$K_{\mathfrak{p}} \cong L .$$

Pf: Because $\mathbb{Q}$ dense in $\mathbb{Q}_p$, so for $L = \mathbb{Q}_p[x]/f(x)$ can find $g \in \mathbb{Q}[x]$ close enough to f to apply Krasner's lemma.

$$K = \mathbb{Q}[x]/g(x).$$

Traditional Krasner's lemma: K non-arch, $\overline{K}$ alg. closure of K, $\alpha, \beta \in \overline{K}$, α closer to β then to any conjugate of itself, and α separable over $K(\beta)$. Then

$$K(\alpha) \subseteq K(\beta).$$

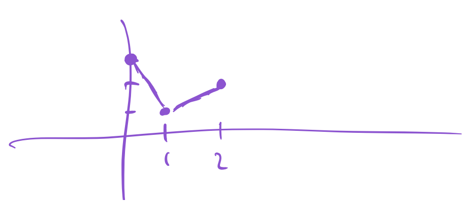
Pf: Consider $\sigma : K(\alpha, \beta) \hookrightarrow \overline{K}$ that fix $K(\beta)$.

Newton polygons. K non-arch, $v := \log |\cdot|$. (if $|\cdot|$ discrete $\sim$ order of vanishing.)

If $f(x) = a_0 x^n + a_{n-1} x^{n-1} + \dots + a_n$.

The NP of f is the lower convex hull of points $(k, v(a_k))$

Example: $K = \mathbb{Q}_2$ $\quad 2^3 x^2 + 2x + 2^2$

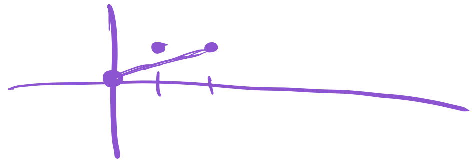


$\rightsquigarrow$ 1 root α $v(\alpha) = -2$

1 root β $v(\beta) = 1$

$v = \log_2 |\cdot|_2$

$x^2 + 2x + 2.$



2 roots α
$v(\alpha) = \frac{1}{2}.$

Theorem: Segments of $NP(f) \longleftrightarrow$ roots of same absolute value
x-length of segment = # of roots
slope = additive valuation of those roots

Separable fin. extensions of K non-arch. discretely valued. (e.g. $K = \mathbb{Q}_p$)

L/K fin. sep. **Lemma** (Exercise). $\mathcal{O}_L$ = integral closure of $\mathcal{O}_K$.

$\mathcal{O}_K$ is a complete DVR, so general theory of DD's applies.

$\mathfrak{p}$ = unique nonzero prime ideal in $\mathcal{O}_K$ $(\{x \mid |x| < 1\})$.

$\mathfrak{p}$ factors in $\mathcal{O}_L$ $\mathfrak{q}^e$ $\mathfrak{q} = (\{\alpha \mid |\alpha|_L < 1\})$

also a complete DVR (e.g. from formula for $|\ |_L$).

$f = [\mathcal{O}_L/\mathfrak{q} : \mathcal{O}_K/\mathfrak{p}]$. (assume residue field ext'n is separable)

$[L:K] = fe.$

Theorem L/K as above, there is a unique maximal unramified extension $L_0 \subseteq L$ L_0/K is unramified.

$[L_0 : K] = f.$

$[L : L_0] = e$ totally ramified.

So can always split into an unramified ext'n followed by a totally ramified extension.

Theorem Unramified extensions $\longleftrightarrow$ (separable) extensions of the residue field.

Pf: Hensel's lemma.

Theorem If L/K is totally ramified then $L = K(\pi)$ for any uniformizer π (i.e. $(\pi) = \mathfrak{q}$), $M_\pi(x)$ is Eisenstein, and conversely if $f \in \mathcal{O}_K[x]$ Eisenstein then $K[x]/f(x)$ is totally ramified and $\bar{x}$ is a uniformizer.

Pf: Newton polygons.

Corollary: Let $\overline{\mathbb{Q}_p}$ be an a.c. of $\mathbb{Q}_p$. Let $N > 0$. There are only finitely many $\mathbb{Q}_p \subseteq L \subseteq \overline{\mathbb{Q}_p}$ with $[L : \mathbb{Q}_p] \leq N$.

Proof: Only finitely many unramified ✓ ($\leftrightarrow$ extensions of $\mathbb{F}_p$).
+ Krasner's lemma + Eisenstein polynomials are in $\mathcal{O}_K[x]$

$$\cong$$

$$\mathfrak{p} \times \mathfrak{p} \times \dots \times \mathfrak{p} \setminus \mathfrak{p}^2.$$

compact!