6370-001 - FALL 2021 - WEEK 9 (10/26, 10/28)

Exercise 1. More about finding roots.

- (1) If you did not do the exercise about Hensel's lemma last week, go back and do it!
- (2) Read the section of Milne about Newton's polygon, then do Milne's exercise 7-8.

Exercise 2. $\overline{\mathbb{Q}}_p$ and \mathbb{C}_p . Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of \mathbb{Q}_p , and recall that the absolute value on \mathbb{Q}_p extends uniquely to an absolute value on $\overline{\mathbb{Q}}_p$.

- (1) Show that $\overline{\mathbb{Q}}_p$ is not complete (hint: for each $n \ge 1$ let ζ_n be a primitive *n*th root of unity in $\overline{\mathbb{Q}}_p$, and consider $\sum_{(n,p)=1} \zeta_n p^n$).
- (2) Let \mathbb{C}_p be the completion of $\overline{\mathbb{Q}}_p$. Show \mathbb{C}_p is algebraically closed (hint: use one of the versions of Krasner's lemma from the lecture).
- (3) Show that \mathbb{C} , \mathbb{C}_p , and $\overline{\mathbb{Q}}_p$ are all isomorphic as *fields*.

Exercise 3. Rings of integers and absolute values.

- (1) Let K be a field that is complete for a non-archimedean absolute value $|\cdot|$. Write $\mathcal{O}_K := \{k \in K, |k| \leq 1\}$ (last week you showed this was a subring if you didn't, then do it now!). If L/K is a finite extension, show that \mathcal{O}_L is equal to the integral closure of \mathcal{O}_K in L (where here we define \mathcal{O}_L to be the elements of absolute value ≤ 1 for the extension of $|\cdot|$ to L).
- (2) Let A be a Dedekind domain, and K = Frac(A). Show that

 $A = \{k \in K, |k|_{\mathfrak{p}} \leq 1 \text{ for all nonzero primes } \mathfrak{p} \text{ of } A.\}$

When $K = \mathbb{C}(t)$, explain what this means geometrically.

(3) Let A be a Dedekind domain, and $K = \operatorname{Frac}(A)$. For any prime \mathfrak{p} of A, show that the topological closure of A in $K_{\mathfrak{p}}$ (the completion of K for $|\cdot|_{\mathfrak{p}}$), is $\mathcal{O}_{K_{\mathfrak{p}}}$.

Exercise 4. Rings of integers in extensions of \mathbb{Q}_p are monogenic. Let K/\mathbb{Q}_p be a finite extension. We will show that $\mathcal{O}_K = \mathbb{Z}_p[\alpha]$ for some α , i.e. that \mathcal{O}_K is monogenic.

- (1) Let K_0 be the maximal unramified subextension. Show that $\mathcal{O}_{K_0} = \mathbb{Z}_p[\zeta]$ for some root of unity $\zeta \in K$.
- (2) Let π be a uniformizer, i.e. a generator of the maximal ideal in \mathcal{O}_K . Show that $\mathcal{O}_K = \mathcal{O}_{K_0}[\pi]$.
- (3) For ζ, π as above, show that $\mathcal{O}_K = \mathbb{Z}_p[\zeta + \pi]$ (hint: find a simple polynomial f with coefficients in \mathbb{Z}_p such that $f(\zeta + \pi)$ is also a uniformizer, then apply (1) and (2)).

Exercise 5. Compare with Week 4 - Exercise 1.

- (1) Show that $f(x) = x^3 + 3x + 12$ is irreducible in $\mathbb{Q}[x]$.
- (2) Use the strong Hensel's lemma for roots (see last week's worksheet) to show that f(x) has three distinct roots in \mathbb{Q}_2 .
- (3) Let $K = \mathbb{Q}[x]/f(x)$. Deduce that \mathcal{O}_K is unramified at (2), and thus that there are three distinct maps $\mathcal{O}_K \to \mathbb{F}_2$.
- (4) Conclude that the integral closure of $\mathbb{Z}_{(2)}$ in K is not monogenic over $\mathbb{Z}_{(2)}$.
- (5) Why does this not contradict the previous exercise?
- (6) Give an algorithm that, for any $k \ge 1$, finds an irreducible polynomial g(x) such that, for $K = \mathbb{Q}[x]/g(x)$, \mathcal{O}_K cannot be generated by k elements over \mathbb{Z} .

Exercise 6. Local fields.

(1) Suppose K is complete for a discrete non-archimedean absolute value and the residue field of \mathcal{O}_K is finite. Show that \mathcal{O}_K is a compact and open subset of K.

It turns out that any valued field as in (1) is isomorphic to a finite extension of \mathbb{Q}_p with the *p*-adic absolute value or to $F_q((t))$ with the *t*-adic absolute value. More generally, a field *K* complete with respect to an absolute value is called a *local field* if it admits an open neighborhood *U* of 0 such that the closure \overline{U} is compact (this implies *K* is locally compact). It can be shown that the non-archimedean local fields are exactly those considered above; the only two archimedean complete fields, \mathbb{R} and \mathbb{C} , are also local.

Exercise 7. Existence and uniqueness of absolute values on field extension.

- (1) In the last paragraph of the proof of Theorem 7.38, Milne claims a certain sequence is a Cauchy sequence. Verify this claim.
- (2) Prove that, for K a field complete for an absolute value $|\cdot|$, any finite dimensional normed K-vector space is equivalent to K^n with the sup norm, thus, in particular complete (two normed vector spaces are equivalent if there is an linear homeomorphism between them). Hint: argue by induction. At each step of the induction, show that any linear functional is continuous by applying the inductive hypothesis to deduce that the kernel is closed.
- (3) What is the relation between (1) and (2)?
- (4) For K a complete field, show that any two norms on K^n are equivalent, i.e. $\exists m, M > 0$ s.t.

$$m \| \cdot \|_1 \le \| \cdot \|_2 \le M \| \cdot \|_1$$

Hint: Use the previous exercise and the open mapping theorem.

- (5) Conclude that for L/K a finite extension, there is at most one absolute value extending the absolute value on K.
- (6) In the lecture we gave a simple algebraic argument using the theory of Dedekind domains to show that if the absolute value on K is discretely valued then there exists an extension of the absolute value to any finite separable extension L. Use uniqueness to deduce that

$$|\cdot|_L = |\operatorname{Nm}_{L/K}(\cdot)|^{1/[L:K]}$$

(7) One can show in general that this formula always defines an absolute value, thus an extension of an absolute value always exist. The archimedean case is simple because the only possibilities are \mathbb{R} and \mathbb{C} which can be done explicitly, so the tricky part is the triangle inequality in the general non-archimedean case. To prove this, first reduce to showing that if L = K(c)and $|N_{L/K}(c)|_K \leq 1$ then also $|N_{L/K}(1+c)| \leq 1$. To establish this latter fact, it suffices to show that *all* of the coefficients of the minimal polynomial of c have absolute value ≤ 1 ; to establish this, argue by contradiction – to get the contradiction, you'll need to show that any polynomial whose largest coefficient is not the leading or constant term is reducible (careful not to use cyclic reasoning here; this can be argued directly like proving Hensel's lemma, and is in fact a special case of a version of Hensel's lemma for factorization that we did not state).