Exercise 0.
(1) Show any \( x \in \mathbb{Q}_p \) can be expressed uniquely as \( \sum_{k=-N}^{\infty} a_k p^k \) for \( a_k \in \{0,1,\ldots,p-1\} \).
(2) Express \(-1\) in this way.

Exercise 1. Let \( K \) be a field with an absolute value \( |\cdot| \). Recall that \( |\cdot| \) is called non-archimedean if there exists \( C > 0 \) such that \( |m| \leq C \) for all \( m \in \mathbb{Z} \) (using the natural map \( \mathbb{Z} \to K \)).
(1) Show that \( |\cdot| \) is non-archimedean if and only if the strong triangle inequality holds:
\[
|x + y| \leq \max(|x|,|y|)
\]
(2) Show that \( |\cdot| \) is non-archimedean and \( |x| \neq |y| \) then
\[
|x + y| = \max(|x|,|y|).
\]
(3) Generalize (1) and (2) to \( \sum_{i=1}^{N} x_i |\).

Exercise 2 (Similar to Milne 7-2). Let \( K \) be a field with a non-archimedean absolute value \( |\cdot| \).
(1) Show that the set of elements in \( K \) of absolute value \( \leq 1 \) is a subring (called the valuation ring of \( |\cdot| \)). Why doesn’t this hold for an archimedean absolute value?
(2) We can define a norm on the vector space \( K^n \) by \( ||(a_1,\ldots,a_n)|| = \max(|a_1|,\ldots,|a_n|) \). Show that “any point in a ball in \( K^n \) is its center.” (part of the exercise is to make sense what this means! This is already interesting when \( n = 1 \), so feel free to treat just that case).
(3) The freshman’s dream. If \( K \) is complete, then for \( a_n \) a sequence in \( K \), show that the series \( \sum_{n=0}^{\infty} a_n \) converges if and only if \( \lim_{n \to \infty} a_n = 0 \).

Exercise 3. Let \( K \) be complete with respect to a non-archimedean absolute value \( |\cdot| \) and \( \text{char} K = 0 \).
(1) What are the possible restrictions of \( |\cdot| \) to \( \mathbb{Q} \subseteq K \) (Hint: Ostrowski’s theorem).
(2) For which \( x \in K \) does \( \log(1 + x) \) converge, where
\[
\log(1 + x) := x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots
\]
(3) For which \( x \in K \) does \( \exp(x) \) converge, where
\[
\exp(x) := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\]
(4) Show that \( \exp \) and \( \log \) are inverse functions when they are defined, i.e.
\[
\exp(\log(s)) = s \quad \text{and} \quad \log(\exp(t)) = t.
\]
for values of \( s \) and \( t \) where these make sense (what values are these?)

Exercise 4. If you know a little bit of functional analysis, prove that if \( \mathbb{C} \subseteq K \) and \( K \) is complete for an absolute value extending the standard absolute value on \( \mathbb{C} \), then \( \mathbb{C} = K \). Hint: \( K \) is a Banach space; what do you know about the spectrum of a bounded operator on a complex Banach space?.
(No analogous statement holds for non-archimedean absolute values – in particular, there is no “biggest” complete algebraically closed field containing \( \mathbb{Q}_p \). This fact is one reason that \( p \)-adic analytic geometry behaves like a mixture of complex analytic geometry and algebraic geometry).
Exercise 5. Consider the following result:

**Theorem (Weak Approximation).** Let $\cdot |_{1}, \cdot |_{2}, \ldots, \cdot |_{n}$ be nontrivial inequivalent absolute values on a field $K$, and let $a_{1}, \ldots, a_{n}$ be elements of $K$. For any $\varepsilon > 0$, there is an element $a \in K$ such that $|a - a_{i}| < \varepsilon$ for all $1 \leq i \leq n$.

1. (Similar to Milne 7-1) Suppose $A$ is a Dedekind domain, $K = \text{Frac}(A)$, and $|\cdot |_{i}$ are all absolute values that come from distinct primes of $A$. Prove the weak approximation theorem in this case by using the Chinese Remainder Theorem.
2. Prove the Weak Approximation theorem (if you get stuck this follows a section in Milne):
   (a) First show there is an element $a$ such that $|a|_{1} > 1$ and $|a|_{i} < 1$ for $i \neq 1$.
   (b) Use this to construct an element $a$ with $|a - 1|_{1}$ close to 0 and $|a|_{i}$ close to zero for $i \neq 1$.
   (c) Conclude.

Exercise 6. We have the following important results on roots and factorization:

**Theorem (Simple Hensel's lemma for roots).** Let $A$ be a complete DVR with residue field $\kappa$ (e.g. $A = \mathbb{Z}_{p}$ or $A = \kappa[[t]]$). For $f \in A[x]$, write $\bar{f}$ for the image in $\kappa[x]$ by reducing all the coefficients modulo the maximal ideal. Show that if there is an $\bar{a} \in \kappa$ such that $\bar{f}(\bar{a}) = 0$ and $\bar{f}'(\bar{a}) \neq 0$, then there is a unique $a \in A$ with reduction $\bar{a}$ such that $f(a) = 0$. In other words, simple roots in $\kappa$ lift uniquely to simple roots in $A$.

**Theorem (Strong Hensel's lemma for roots).** Let $K$ be complete for a non-archimedean absolute value $|\cdot |$, and let $A \subset K$ be the valuation subring / unit ball consisting of $k \in K$ with $|k| \leq 1$. Suppose $f(x) \in A[x]$ and $a_{0} \in K$ is such that $|f(a_{0})| < |f'(a_{0})|^{2}$. Show there is a unique root $a$ of $f(x)$ with $|a - a_{0}| \leq |f(a_{0})|/|f'(a_{0})|$.

**Theorem (Hensel's lemma for factorization).** Let $A$ be a complete DVR with residue field $\kappa$. Suppose $f \in A[x]$ is monic and $\bar{f}(x) = \bar{g}_{1}(x) \ldots \bar{g}_{m}(x)$ where the $\bar{g}_{i}(x)$ are pairwise coprime in $\kappa[x]$. Then the factorization lifts uniquely to a factorization $f(x) = g_{1}(x) \ldots g_{m}(x)$ in $A[x]$.

1. What is the relation between these three results? (I.e. which imply which?)
2. Compute $\mu(\mathbb{Q}_{p})$, the group of roots of unity in $\mathbb{Q}_{p}$. Hint: Hensel's lemma will do most of the job, but you'll also need an earlier computation for $p$th roots. Pay attention when $p = 2$!
3. Show that $(x^{2} - 2)(x^{2} - 17)(x^{2} - 34)$ has a root in $\mathbb{Z}_{p}$ for all $p$ and in $\mathbb{R}$, but has no root in $\mathbb{Q}$.
4. Prove strong Hensel's lemma for roots (Hint: use Newton's method.).
5. Show that $5x^{3} - 7x^{2} + 3x + 6$ has a root $\alpha \in \mathbb{Z}_{7}$ with $|\alpha - 1|_{7} < 1$. Find $\alpha \in \mathbb{Z}$ such that $|\alpha - a|_{7} < 7^{-4}$.
6. Prove Hensel's lemma for factorization, or read the proof in Milne (Theorem 7.33).

Exercise 7.

1. Show that for $p$ odd, $\mathbb{Q}_{p}^{\times} \cong \mathbb{Z}_{p}^{\times} \times p^{\mathbb{Z}} \cong (1 + p\mathbb{Z}_{p}) \times \mu(\mathbb{Q}_{p}) \times p^{\mathbb{Z}}$.
2. What happens for $p = 2$? Hint: the first identity still holds, but what about the second one?
3. Compute $\mathbb{Q}_{p}^{\times}/(\mathbb{Q}_{p}^{\times})^{2}$. Hint: use Hensel's lemma or the exponential/logarithm, but pay attention when $p = 2$!
4. How many quadratic extensions of $\mathbb{Q}_{p}$ are there?
5. Show $\mathbb{Q}_{p}^{\times}/(\mathbb{Q}_{p}^{\times})^{n}$ is finite for any $p$ and $n$ – in particular, if $\mathbb{Q}_{p}$ contains the $n$th roots of unity, deduced that there are only finitely many cyclic degree $n$ extensions of $\mathbb{Q}_{p}$. Next week we will see that there are only finitely many extensions of any fixed degree of $\mathbb{Q}_{p}$.