6370-001 - FALL 2021 - WEEK 14 (11/30, 12/2)

Exercise 1 – Modular forms (first definitions)
A *weakly modular form of weight* k is a function F on the set of lattices $\Lambda \subset \mathbb{C}$ such that $F(t\Lambda) = t^{-k}F(\Lambda)$ for all $t \in \mathbb{C}^\times$. It is *holomorphic* if the function $f(\tau) := F(\langle 1, \tau\rangle)$ on $\mathbb{H} := \mathrm{Im}(z) > 0$ is.

(1) Show that if F is a holomorphic weakly modular form with associated function f on $\mathbb{H}$, then $f(\tau) = f(\tau+1)$. Deduce that f admits a Laurent expansion (called the q-expansion)

$$\sum_{n \in \mathbb{Z}} a_n q^n, \ q := e^{2\pi i \tau}.$$

that converges for $0 < |q| < 1$. We say F or f is *a holomorphic modular form if* $a_n = 0 \forall n < 0$.

(2) More is true: show (or admit for now and move on) that a function f on $\mathbb{H}$ defines a modular form of weight k if and only if the identity

$$(c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau)$$

is satisfied for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, or equivalently, for just the two matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that the first matrix corresponds to part (1)! *Hint: show that a modular form is a fixed point for an action of* $\mathrm{SL}_2(\mathbb{Z})$ *and use that these two matrices generate* $\mathrm{SL}_2(\mathbb{Z})$.

Exercise 2 – Eisenstein series

(1) For $k > 3$, show that

$$G_k(\Lambda) := \sum_{z \in \Lambda \setminus \{0\}} \frac{1}{z^k}$$

defines a holomorphic weakly modular form of weight k. Show it is zero if k is odd.

(2) For a lattice Λ, write $\Lambda^\star$ for the set of primitive vectors in Λ (i.e. those not of the form $n\lambda$ for $n > 1$ and $\lambda \in \Lambda$). For $k \geq 3$, let $E_k(\Lambda) := \frac{1}{2}\sum_{z \in \Lambda^\star} \frac{1}{z^k}$. Show $G_k = 2\zeta(k)E_k$.

(3) For $k > 2$ even, we are now going to compute the the q-expansion $G_k(q)$.

(a) Break up

$$G_k(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(m + n\tau)^k} = 2\zeta(k) + 2\sum_{n > 0} H_k(n\tau)$$

where for $\eta \neq 0$,

$$H_k(\eta) := \sum_{m \in \mathbb{Z}} \frac{1}{(m + \eta)^k}.$$

Note that $H_k(n\tau)$ is a holomorphic function of q!

(b) For $\eta \in \mathbb{H}$, let $h_{k,\eta}$ be the complex valued function on $\mathbb{R}$ defined by $h_{k,\eta}(t) = \frac{1}{(t+\eta)^k}$. Since

$$H_k(\eta) = \sum_{m \in \mathbb{Z}} g_{k,\eta}(t),$$

we can compute $H_k(\eta)$ using Poisson summation! Do this. Hint: use contour integration to compute the Fourier transform of $h_{k,\eta}$.

(c) Deduce that the q-expansion of G_k is given by

$$2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

where $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$.

Exercise 3 – Sums of four squares via Minkowski (Milne Remark 4.20) We will prove:

Theorem (Lagrange)**.** *Every positive integer can be represented as the sum of 4 squares.*

I.e. for any integer $n > 0$, $n = a^2 + b^2 + c^2 + d^2$ for integers a, b, c, d.

(1) Find the first positive positive integer that cannot be expressed as a sum of three squares. Can you show that there are infinitely many?

(2) Using the identity

$$(a^2 + b^2 + c^2 + d^2)(A^2 + B^2 + C^2 + D^2) =$$
$$(aA - bB - cC - dD)^2 + (aB + bA + cD - dC)^2 + (aC - bD + cA + dB)^2 + (aD + bC - cB + dA)^2$$

(which can be deduced painlessly if one accepts basic properties of the norm and multiplication in the Hamiltonian quaternions!), reduce Lagrange's theorem to showing that every prime p can be represented as a sum of four squares.

(3) Verify the theorem for $p = 2$. We assume now that p is an odd prime.

(4) For p odd, show that $m^2 + n^2 + 1 \equiv 0 \bmod p$ has a solution.

(5) Fix a solution m, n to the congruence and consider the lattice $\Lambda\mathbb{Z}^4$ of (a, b, c, d) such that

$$c \equiv ma + nb, d \equiv mb - na \bmod p.$$

(6) Show that if $(a, b, c, d) \in \Lambda\backslash\{0\}$ and $a^2 + b^2 + c^2 + d^2 < 2p$ then $a^2 + b^2 + c^2 + d^2 = p$. *Hint: reduce mod p.*

(7) Show that the volume of Λ is p^2. *Hint: consider the $\mathbb{F}_p$-vector space $\Lambda/p\mathbb{Z}^4$.*

(8) Conclude using Minkowski's lattice point theorem (Milne Theorem 4.19) with Λ and a ball of suitable radius.

Exercise 4 – E_8 lattice. Let $E_8 \subseteq \mathbb{R}^8$ denote the lattice generated by the vectors $(a_1, \ldots, a_8) \in \mathbb{Z}^8$ such that $\sum a_i$ is even and the vector $(1/2, \ldots, 1/2)$.

(1) Show that Λ has volume 1 (i.e. is unimodular) and that $x \cdot x$ is an even integer for any $x \in \Lambda$.

(2) Compute the number of elements in Λ with $x \cdot x = 2$.

(3) Let $N_\Lambda(\tau) = \sum_{\lambda\in\Lambda} e^{\pi i(\lambda\cdot\lambda)}$. Using Poisson summation , one can show $N_\Lambda(\frac{-1}{\tau}) = \tau^4 N_\Lambda(\tau)$ (the computation is very similar to the formula for the Jacobi θ function in Week 12, Exercise 4). Assuming this, use Exercise 1, part (2), to show $N_\lambda(\tau)$ is a weight 4 modular form.

(4) Let $r_\Lambda(n)$ denote the number of $\lambda \in \Lambda$ such that $\lambda \cdot \lambda = n$. What is the relation between $r_\Lambda(n)$ and the q-expansion of $N_\lambda(\tau)$?

(5) It can be shown using the elementary theory of Riemann surfaces that the set of weight 4 holomorphic modular forms is 1-dimensional complex vector space. Admitting this fact, deduce a very simple formula for $r_n(\tau)$ in terms of $\sigma_3(\tau)$ (*hint: you just need one coefficient).*

Remark: This method can be generalized to many other lattices/quadratic forms – for example, the sums of 8 squares by taking the integer lattice $\mathbb{Z}^8$; in that case one needs to use modular forms for a proper subgroup of $\mathrm{SL}_2(\mathbb{Z})$, however.