6370-001 - FALL 2021 - WEEK 13 (11/23)

This week we study and prove:

Theorem (Analytic class number formula for quadratic imaginary fields). *Let $K/\mathbb{Q}$ be a quadratic imaginary field, and let χ_K be the function on $\mathbb{N}$ defined on primes by*

$$\chi_K(p) = \begin{cases} 1 & \text{if } p \text{ splits in } K \\ -1 & \text{if } p \text{ is inert in } K \\ 0 & \text{if } p \text{ ramifies in } K. \end{cases}$$

and extended multiplicatively to composites (and set $\chi_K(1) = 1$). Then, for h_K the class number of K, D_K the discriminant of K, and $\mu(K)$ the roots of unity in K,

$$2\pi \cdot h_K = |\mu(K)| \cdot \sqrt{|D_K|} \cdot \sum_{k=1}^{\infty} \frac{\chi_K(n)}{n}.$$

Exercise 1.

(1) Use the analytic class number formula when $K = \mathbb{Q}(i)$ to evaluate

$$1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + \dots$$

Confirm your answer by using a Taylor series to compute $\arctan(1)$.

(2) Repeat for $K = \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$ (except for the arctan part!).

(3) Using quadratic reciprocity explain why, in general, χ_K is a character of $(\mathbb{Z}/D_K\mathbb{Z})^\times$.

(4) Explain why a computer can use this formula to quickly compute the class number of an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. If yo're up for it, try implementing this computation in PARI (or any computer algebra system), then compare with the answers you worked out using the Minkowski bound in Week 5.

In the remaining exercises (flip the page over!) we will prove the theorem. The strategy is to introduce the *Dedekind zeta function* of a number field K

$$\zeta_K(s) := \sum_{I \subseteq \mathcal{O}_K \text{ a nonzero ideal}} \frac{1}{N(I)^s}$$

When $K = \mathbb{Q}$ this is just the Riemann zeta function (why?). Like the Riemann zeta function, the sum can be shown to converge for $\mathrm{Re}\, s > 1$ and admit a meromorphic continuation with a simple pole at $s = 1$, a functional equation, etc.... We will work through just the parts of this we need in the case when K is imaginary quadratic, and then the theorem will be obtained by computing the residue at $s = 1$ in two different ways.

For a more comprehensive treatment and complements, see Marcus - Number Fields, Ch. 5-7 (but I recommend working through the problems below first!).

From last week we need only the results of Exercise 1, which gave a meromorphic continuation of $\zeta(s) = \zeta_{\mathbb{Q}}(s)$ to $\mathrm{Re}\, s > 0$ with a simple pole at $s = 1$ of residue 1.

Gobble Gobble, Happy Thanksigiving!

Exercise 2 – An infinite product expansion for ζ_K.

(1) For any $K/\mathbb{Q}$, explain why, as formal series (i.e. ignoring all convergence questions)

$$\zeta_K(s) = \prod_{\mathfrak{p} \text{ a nonzero prime of } \mathcal{O}_K} \frac{1}{1 - N(\mathfrak{p})^{-s}}$$

In the rest of this exercise, we assume $K/\mathbb{Q}$ is a quadratic field.

(2) Show that for

$$L_{\chi_K}(s) := \prod_p \frac{1}{1 - \chi_K(p)p^{-s}} = \sum_{n \geq 1} \frac{\chi_K(n)}{n^s},$$

we have a formal identity

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s) L_{\chi_K}(s).$$

(3) Deduce that the series describing $\zeta_K(s)$ converges to an analytic function on $\mathrm{Re}(s) > 1$.

(4) Show the series describing $L_{\chi_K}(s)$ converges to an analytic function on $\mathrm{Re} s > 0$.
Hint: $\sum_{n=0}^{D_K - 1} \chi_K(n) = 0$ *because* χ_K *is a non-trivial character of* $(\mathbb{Z}/D_K\mathbb{Z})^\times$.

(5) Continue $\zeta_K(s)$ to $\mathrm{Re} s > 0$ and show it has a simple pole at $s = 1$ with residue $\sum_{n=1}^{\infty} \frac{\chi_K(n)}{n}$.

Exercise 3 – A lemma on Dirichlet series. Prove (or admit for later exercises):

Lemma. *If* $\sum_{k=1}^{n} a_k = O(n^r)$ *for* $r \in \mathbb{R}_{>0}$, *then* $\sum_{n=1}^{\infty} \frac{a_n}{n^{-s}}$ *defines an analytic function on* $\mathrm{Re} s > r$.

Exercise 4 – Counting ideals in an ideal class. Let K be an imaginary quadratic field, and let C be an ideal class of $\mathcal{O}_K$. Let $I_C(M)$ denote the number of ideals in C of norm $\leq M$. We now show

Lemma. $I_C(M) = \kappa M + O(\sqrt{M})$, *where* $\kappa := \frac{2\pi}{|\mu(K)|\sqrt{|D_K|}}$.

(1) Let J be an ideal in the class C^{-1}. Show that $I \mapsto IJ$ defines a bijection between ideals I in the class C with $N(I) \leq M$ and principal ideals $(\alpha) \subseteq I$ with $N(\alpha) \leq M \cdot N(J)$.

(2) Deduce that $|\mu(K)| I_C(M)$ is the number of elements $\alpha \in J$ with $N(\alpha) \leq M \cdot N(J)$.

(3) Estimate the number of these elements to deduce the lemma.

Exercise 5 – A second meromorphic continuation of ζ_K

(1) Explain why we can rewrite

$$\zeta_K(s) = \sum_{C \in \mathrm{Cl}(\mathcal{O}_K)} \sum_{n \geq 1} \frac{\#\{I \in C | N(I) = n\}}{n^s}$$

Then, apply the results of the previous two exercises to show $\zeta_K(s)$ has a meromorphic continuation to $\mathrm{Re} s > 1/2$ with a simple pole at $s = 1$ of residue $h_K \kappa$.
Hint: subtract the series for $h_k \cdot \kappa \cdot \zeta(s)$ *from the expression above for* $\zeta_K(s)$.

(2) Combine with the computation of Exercise 2 to deduce the analytic class number formula.

Remarks. For any number field K there is an analytic class number formula computing h_K in terms of the residue of $\zeta_K(s)$ at $s = 1$, the discriminant of K, the number of roots of unity, a simple factor depending on the number of real and complex places (2π in the above), and the *regulator*, which measures the size of the logarithm lattice of the units (so is only interesting when the units have positive rank, which is why this didn't appear in the imaginary quadratic case). The proof follows the same lines as Exercise 4 and 5, but the estimate of the number of principal ideals of bounded size contained in a given ideal is more difficult when there are infinitely many units (try doing the real quadratic case without looking up a proof!). This residue can be matched with simple character sums like those coming from Exercise 2 only for *abelian* extensions of $\mathbb{Q}$; for example this gives a method to compute the class numbers of cyclotomic fields! (compare Week 5 - Exercise 2).