6370-001 - FALL 2021 - WEEK 12 (11/16, 11/18)

A week's holiday with the Riemann zeta function... the following exercises will looks like they have nothing to do with algebraic number theory, but we'll spend the rest of the semester connecting the ideas below to the behavior of class groups. Everything below is elementary complex analysis or calculus. If you know how, justify convergence carefully! Remember that the limit of a sequence of analytic functions that converges uniformly on compact sets is itself an analytic function....

Exercise 1 – *The Riemann zeta function, a first look.*
The Riemann Zeta function is the function defined on the complex half-plane $\mathrm{Re} s > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(1) Use the integral test to prove this sum converges absolutelyx for $\mathrm{Re} s > 1$, so that this defines a function. Prove it is analytic on this half-plane.

It turns out that ζ admits a meromorphic continuation to $\mathbb{C}$ with a single pole at $s = 1$, which is simple with residue 1 (i.e. $\zeta(s) - \frac{1}{s-1}$ is an entire function). We will now see one way to extend it to $\mathrm{Re} s > 0$ and verify this claim about the poles.

(2) For $\mathrm{Re} s > 1$, show

$$(1 - 2^{1-s})\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots.$$

(3) Show that the sum above defines a function $f_2(s)$ on $\mathrm{Re} s > 0$. *Hint: group neighborhing terms then express each pair as an integral.* Show that it is analytic.

Thus $\zeta(s) := \frac{f_2(s)}{1-2^{1-s}}$ defines a meromorphic continuation of $\zeta(s)$ to $\mathrm{Re} s > 0$.

(4) Show $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.

Since $1 - 2^{1-s}$ has more zeroes (where are they?), a priori $\zeta(s)$ may have more poles.

(5) Define $f_3(s) = (1 - 3^{1-s})\zeta(s)$ for $\mathrm{Re} s > 1$. Repeat an analysis similar to the above to get another formula for the meromorphic continuation of $\zeta(s)$ to $\mathrm{Re} s > 0$. Comparing the two, deduce that the only pole of $\zeta(s)$ for $\mathrm{Re} s > 0$ is at $s = 1$.

Here's another way to carry out this meromorphic continuation:

(6) Consider $g(s) = \zeta(s) - \frac{1}{s-1}$ for $\mathrm{Re} s > 1$. Note $\frac{1}{s-1} = \int_1^\infty x^{-s}$. Break this into chunks and mix it with the sum defining $\zeta(s)$ to get something that converges for $\mathrm{Re} s > 0$.

Exercise 2 – *Some special values.*
The first two parts are a warm-up:

(1) Compute the Fourier series for x^2, viewed as a function on the circle by restriction to $[-\pi, \pi]$.
(2) (Basel problem) Use this computation to deduce $\zeta(2) = \pi^2/6$.

We now compute $\zeta(m)$ for all positive even[1] integers m. We will need two definitions:

- For $R > 2\pi > r > 0$, the contour $H_{r,R}$ comes in from R on the bottom side of the positive real axis (i.e. the determination of log on the real numbers with argument $2\pi i$) to r, follows a circle clockwise around the origin, and then goes back out to R along the top side of the positive real axis (the determination of log with argument 0). We write $\overline{H}_{r,R}$ for the contour that is closed by taking a circle of radius R.
- The Bernoulli numbers B_n are defined $\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_k \frac{z^k}{n!}$.

[1] The values $\zeta(m)$ for m odd are much more mysterious!

(3) For $k \geq 0$ an integer, show

$$\int_{H_{r,R}} \frac{z^{-k}}{e^z - 1} dz = -2\pi i \frac{B_k}{k!}.$$

(Hint: the contributions along the real axis will cancel).

(4) For $k > 1$ an integer, show

$$\lim_{R \to \infty, R \neq 2\pi k} \int_{\overline{H}_{r,R}} \frac{z^{-k}}{e^z - 1} dz = (1 + (-1)^k)(2\pi i)^{1-k} \zeta(k)$$

(5) For m a positive integer, deduce

$$\zeta(2m) = (-1)^{m+1} (2\pi)^{2m} \frac{B_{2m}}{2 \cdot (2m)!}$$

Hint: take a sequence of $R \to \infty$ such that integral along the circle of radius R goes to 0.

(6) For m a positive integer, deduce $B_{2m+1} = 0$. Give an elementary argument for this from the series definition of the Bernoulli numbers. Note that here we learn nothing about the odd special values $\zeta(2m+1)$, which are very mysterious!

Exercise 3 – ***Analytic continuation and functional equation via the Hankel contour.*** The contour integration from the previous exercise can be used to give a meromorphic continuation and functional equation relation $\zeta(s)$ and $\zeta(1-s)$. We need the Γ function, defined for $\mathrm{Re} s > 0$ by:

$$\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx \text{ (this is a Mellin transform).}$$

(1) Use integration by parts to show and $s\Gamma(s) = \Gamma(s+1)$.

(2) Deduce $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}_{\geq 1}$.

(3) Use (3) to give a meromorphic continuation of Γ to $\mathbb{C}$. Find all of the poles of Γ, show that they are simple, and compute their residues.

(4) Show that for $\mathrm{Re} s > 1$, $\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$. *(another Mellin transform).*

(5) For any $2\pi > \epsilon > 0$, consider the contour $H_\epsilon := H_{\epsilon, +\infty}$ (see previous exercise). Show that

$$F(s) = \int_{H_\epsilon} \frac{z^{s-1}}{e^z - 1}$$

defines an entire function of s, that is independent of our choice of ϵ.

(6) For $\mathrm{Re} s > 1$, show that

$$F(s) = (1 - e^{2\pi i (s-1)}) \Gamma(s) \zeta(s).$$

Hint: take $\epsilon \to 0$ and show the contribution from the circle vanishes.

(7) For $\mathrm{Re} s < 0$, show that

$$F(s) = (1 + e^{\pi i (s-1)}) (2\pi i)^s \zeta(1-s)$$

Hint: Argue as in part (4) of the previous exercise.

(8) Deduce from either formula that $\zeta(s)$ admits a meromorphic continuation to $\mathbb{C}$.

(9) Using Euler's reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

and the formulas above, deduce the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s).$$

Exercise 4 – *Functional equation from the Jacobi θ function.* We define, for $\mathrm{Re}s > 1$,

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

We will show $\xi(s)$ admits a meromorphic continuation to $\mathbb{C}$ with simple poles only at 0 and 1, satisfying the simple functional equation $\xi(s) = \xi(1-s)$ which is equivalent to the functional equation described in the previous exercise for the ζ function (though it's some work to show it!) **Everywhere below where you need it you may use without proof that the Γ function has no zeroes**.

(1) Before we prove the functional equation, assume the claim is true and use it to deduce that $\zeta(s)$, as a meromorphic function on $\mathbb{C}$, has a simple pole only at $s = 1$, has simple zeroes at the negative even integers, and that any other zero is in the critical strip $0 \le \mathrm{Re}s \le 1$.

(2) Show that

$$2\zeta(s) = \int_0^\infty (\theta(x) - 1)x^{s/2-1}dx \textit{ (yet another Mellin transform!)}$$

where $\theta(x) = \sum_{n\in\mathbb{Z}}^\infty e^{-\pi n^2 x}$ is the Jacobi θ function (we'll see this function again later!).

(3) Deduce a meromorphic continuation of ζ to $\mathrm{Re}s > 0$.

(4) Show $\theta(1/x) = x^{1/2}\theta(x)$ by using the *Poisson summation formula*, which says, for nice enough functions f (including θ),

$$\sum_{k\in\mathbb{Z}} f(k) = \sum_{k\in\mathbb{Z}} \hat{f}(k),$$

where $\hat{f}$ is the Fourier transformation

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{2\pi i x y} f(x)dx$$

(5) Deduce the functional equation and the meromorphic continuation of $\xi(s)$.

(6) Prove the Poisson summation formula for f a C^2-function with $(|x|^r + 1)(|f(x)| + |f''(x)|)$ bounded for some $r > 1$. *Hint: apply Fourier inversion to the periodic function obtained by summing up f over $\mathbb{Z}$.*