6370-001 - FALL 2021 - WEEK 11 (11/9, 11/11)

Exercise 1.

(1) Let $K = \mathbb{Q}_3(\zeta_3)$, where ζ_3 is a primitive 3rd root of unity. Find the Galois group of $K/\mathbb{Q}_3$, and compute its ramification groups.
(2) Do the same for $\mathbb{Q}_3(\zeta_9)$.

Exercise 2.

(1) Compute the Galois group of $x^n - t \in \mathbb{C}((t))[x]$.
(2) Show that every Galois extension of $\mathbb{C}((t))$ is abelian.
(3) Is the same true for $\mathbb{R}((t))$?
(4) Compute the Galois group of $x^n - p \in \mathbb{Q}_p[x]$, for $p \nmid n$.
(5) Show every tamely ramified extension of $\mathbb{Q}_p$ is contained in $\mathbb{Q}_p(\zeta_n, p^{1/n})$ for some $p \nmid n$.

Exercise 3.

(1) Show every polynomial over $\mathbb{R}$ can be solved in radicals.
(2) Can every polynomial over $\mathbb{Q}_p$ be solved in radicals?

Exercise 4.

(1) Let $K/\mathbb{Q}$ be a Galois extension. Show that the inertia groups $I(\mathfrak{p})$ as $\mathfrak{p}$ runs over all prime ideals in $\mathcal{O}_K$ generate $\mathrm{Gal}(K/\mathbb{Q})$.
(2) Use basic algebraic topology to explain an analogous result if $\mathbb{Q}$ is replaced with $\mathbb{C}(t)$.

Exercise 5. If L/K is a Galois extension of number fields and β_1 and β_2 are prime ideals in $\mathcal{O}_L$ that lie above the same prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$, show the Frobenius elements for β_1 and β_2 are conjugate.

Exercise 6 (Milne 8-3). Read the section "computing Galois groups the easy way" in Milne, then try compute the Galois group over $\mathbb{Q}$ of

$$X^6 + 2 * X^5 + 3 * X^4 + 4 * X^3 + 5 * X^2 + 6 * X + 7.$$

Exercise 7. Do the practice exam in Milne - Appendix B (p. 162). Questions 5 and 6 from the practice exam are exercises 1-(1) and 4-(1) above, so you're almost done! The other questions will be review of earlier topics, but it's a good time to go back and revisit some earlier ideas.