Recall: If $K/\mathbb{Q}$ a number field, $\mathcal{O}_K \subseteq K$ ring of integers

$\mathfrak{p} \subseteq \mathcal{O}_K$ nonzero, $\mathcal{O}_{K,\mathfrak{p}} \leftarrow$ localization at $\mathfrak{p}$.

DVR

$$\mathfrak{p}\mathcal{O}_{K,\mathfrak{p}} = (\pi)$$

$$K = \operatorname{Frac}\mathcal{O}_K = \operatorname{Frac}\mathcal{O}_{K,\mathfrak{p}} = \mathcal{O}_{K,\mathfrak{p}}[\tfrac{1}{\pi}]$$

$a \in K$ is written uniquely as $a = \pi^n \cdot u$ $n \in \mathbb{Z}$, $u \in \mathcal{O}_{K,\mathfrak{p}}^\times$.

$$|a|_{\mathfrak{p}} = c^{-n} \quad \text{for } c = |\mathcal{O}_K/\mathfrak{p}| = N(\mathfrak{p}).$$

any $c > 1$ gives an equivalent absolute value.

Observation: $a \in \mathcal{O}_K$ $|a|_{\mathfrak{p}} < 1 \iff a \in \mathfrak{p}$.

in particular if $\mathfrak{p} \mid (p)$ (i.e. $\mathfrak{p} \cap \mathbb{Z} = (p)$),

then $|p|_{\mathfrak{p}} < 1$. $\Rightarrow$ $\left. |\ |_{\mathfrak{p}} \right|_{\mathbb{Q}} = |\ |_p$.

Theorem: Let $K/\mathbb{Q}$ be a number field, let p be an integer prime such that $(p) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m}$ in $\mathcal{O}_K$.

Then $K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{i=1}^m K_{\mathfrak{p}_i}$ ($K_{\mathfrak{p}_i}$ = completion of K w.r.t. $|\ |_{\mathfrak{p}_i}$).

Moreover $K_{\mathfrak{p}_i}/\mathbb{Q}_p$ has ramification degree $e = e_i$ and inertial degree $f = f_i$

Recall Like $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$.

Corollary: If $\alpha \in K$, $\operatorname{Tr}_{K/\mathbb{Q}}(\alpha) = \sum_{i=1}^m \operatorname{Tr}_{K_{\mathfrak{p}}/\mathbb{Q}_p}(\alpha)$,

$N_{K/\mathbb{Q}}(\alpha) = \prod_{i=1}^m N_{K_{\mathfrak{p}}/\mathbb{Q}_p}(\alpha)$.

In particular, the power of p dividing $\operatorname{disc} K/\mathbb{Q}$ ($\operatorname{disc}\mathcal{O}_K/\mathbb{Z}$) = the power of p dividing $\prod \operatorname{disc} K_{\mathfrak{p}}/\mathbb{Q}_p$ ($= \prod \operatorname{disc}\mathcal{O}_{K_{\mathfrak{p}}}/\mathbb{Z}_p$).

Application of Theorem: Reduce factoring (p) into prime ideals in $\mathcal{O}_K$ to factoring polynomials (in some sense) over $\mathbb{Q}_p$.

Why? Theorem says $K \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i=1}^{m} K_{\mathfrak{p}}$

$K = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/f(x)$ $\quad$ $f(x)$ minimal polynomial of α

$f(x) = \prod_{i=1}^{s} f_i(x)$ $\quad$ $f_i(x)$ irred. in $\mathbb{Q}_p[x]$.

$$K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathbb{Q}_p[x]/f(x) = \mathbb{Q}_p[x]\Big/\prod_{i=1}^{s} f_i(x) = \prod_{i=1}^{s} \mathbb{Q}_p[x]/f_i(x)$$

each of these is a field.

Must have $s = m$ and up to reordering,

$$\mathbb{Q}_p[x]/f_i(x) \cong K_{\mathfrak{p}_i}$$

Note ring of intergers is monogenic over $\mathbb{Z}_p$. so can get e_i and f_i.

Lemma 1: Suppose $L/\mathbb{Q}_p$ is a finite extension and K a number field, $\iota: K \longrightarrow L$. Then the absolute value $|\iota(\cdot)|_p$ on K is equivalent to one of the form $|\ |_{\mathfrak{p}}$ for $\mathfrak{p} \mid (p)$ in $\mathcal{O}_K$. and the topological closure of $\iota(K)$ in L is $K_{\mathfrak{p}}$.

__Proof:__ Note that for $a \in \mathcal{O}_K$ $|\iota(a)|_p \leq 1$.

Because a integral over $\mathbb{Z}$ and $|x|_p \leq 1$ for any $x \in \mathbb{Z}$. (exercise last week).

Let $\mathfrak{p} = \{a \in \mathcal{O}_K \mid |\iota(a)|_p < 1\}$.

$\mathfrak{p} \cap \mathbb{Z} = (p)$ $\quad$ (because $|p|_p = \frac{1}{p} < 1$).

$a \in K$ $\quad$ $a = \pi^n u$ $\quad$ $\mathfrak{p}\mathcal{O}_K = (\pi)$ and $u \in \mathcal{O}_K^\times$

$$|\iota(a)|_p = |\iota(\pi)|_p^n \cancel{|\iota(u)|_p}$$

$\hookleftarrow$ Need to check $|\iota(\pi)|_p < 1$.

$$\pi^e \mathcal{O}_{K,\mathfrak{p}} = p\,\mathcal{O}_{K,\mathfrak{p}}$$

$$|\iota(\pi^e)|_p = |p|_p = \frac{1}{p}$$

$$|\iota(\pi)|_p^e = \frac{1}{p}$$

$$|\iota(\pi)|_p = \frac{1}{p^{1/e}}$$

Remark Proof basically gives Ostrowski's theorem for K.

Lemma 2: If $(p) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_m^{e_m}$ in $\mathcal{O}_K$

then $p\,\mathcal{O}_{K_{\mathfrak{p}_i}} = \mathfrak{p}_i^{e_i}\mathcal{O}_{K_{\mathfrak{p}_i}} = \pi_i^{e_i}\mathcal{O}_{K_{\mathfrak{p}_i}}$ $\quad (\mathfrak{p}_i\mathcal{O}_{K,\mathfrak{p}_i} = \pi_i \mathcal{O}_{K,\mathfrak{p}_i})$

and $\mathcal{O}_{K_{\mathfrak{p}_i}}/\mathfrak{p}_i\mathcal{O}_{K_{\mathfrak{p}_i}} \cong \mathcal{O}_K/\mathfrak{p}$

$e_i = e$ $\qquad$ $f_i = f$.

Proof: True if we replace $\mathcal{O}_{K_{\mathfrak{p}_i}}$ with $\mathcal{O}_{K,\mathfrak{p}_i}$, nothing about either changes after completing.

Proof of theorem:

$K \otimes_{\mathbb{Q}} \mathbb{Q}_p$. $\quad$ Let $K = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/f(x)$

$f(x) = \prod_{i=1}^{s} f_i(x)$ in $\mathbb{Q}_p[x]$

$$K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{i=1}^{s} \mathbb{Q}_p[x]/f_i(x).$$

For each i get a map $K \hookrightarrow \mathbb{Q}_p[x]/f_i(x)$

image is dense $(\mathbb{Q}_p[x]/f_i(x)$

is a finite extension of $\mathbb{Q}_p$!

By Lemma 1, $\mathbb{Q}_p[x]/f_i(x) = K_{\mathfrak{p}_j}$ for some j

Claim: distinct i's give distinct j's.

Need to show the induced topologies on K for i_1, i_2 are different.

Take polynomials $\underset{\mathbb{Q}[x]}{\underset{\cap}{g_{i,j}(x)}} \xrightarrow[j\to\infty]{} f_i(x) \in \mathbb{Q}_p[x]$)

Exercise: $g_{i,j}(\alpha) \xrightarrow[j\to\infty]{} 0$ in $\mathbb{Q}_p[x]/f_i(x)$

but not in any other $\mathbb{Q}_p[x]/f_{i'}(x)$. ✓

To see we get all primes dividing $\mathfrak{p}$, use

$$[K:\mathbb{Q}] = \dim_{\mathbb{Q}} K = \dim_{\mathbb{Q}_p} K \otimes_{\mathbb{Q}} \mathbb{Q}_p = \sum_{i=1}^{s} \dim_{\mathbb{Q}_p} \mathbb{Q}_p[x]/f_i(x)$$

$$\| \quad \sum_{i=1}^{n} e_i f_i$$

$$= \sum_{i=1}^{s} \dim_{\mathbb{Q}_p} K_{\mathfrak{p}_{j_i}}$$

$$= \sum_{i=1}^{s} e_{j_i} f_{j_i} \leftarrow \text{(Lemma 2)}.$$

Need $s=m$ and hitting everything!

$\boxtimes$