6370-001 - FALL 2021 - WEEK 10 (11/2, 11/4)

This week's worksheet is a bit shorter than normal, but I think most of you have plenty of exercises from the last two weeks left to work on if you finish up with these!

Exercise 1 (The product formula).

(1) Show that, for any $a \in \mathbb{Q}^\times$, $|a|_\infty \cdot \prod_p |a|_p = 1$. Here $|\cdot|_\infty$ is the archimedean absolute value.

(2) This result generalizes as follows: Let $K/\mathbb{Q}$ be a number field. For any p, we write $|\cdot|_p$ for the natural extension of the p-adic absolute value to $\mathbb{C}_p$ (the completion of an algebraic closure of $\mathbb{Q}_p$). For any $\alpha \in K^\times$, show

$$\prod_{\iota:K\hookrightarrow\mathbb{C}} |\iota(\alpha)|_{\mathbb{C}} \cdot \prod_p \prod_{\iota:K\hookrightarrow\mathbb{C}_p} |\iota(\alpha)|_p = 1$$

Hint: Use the product formula for $\mathbb{Q}$ along with the identity

$$N_{K/\mathbb{Q}}(\alpha) = \prod_{\iota:K\hookrightarrow C} \iota(\alpha)$$

where C is any algebraically closed field containing $\mathbb{Q}$.

(3) Why is the formula in (2) equivalent to the formula given in Milne, Theorem 8.8?

(4) Prove a product formula for $\mathbb{C}(t)$. What does this have to do with the residue theorem in complex analysis?

(5) (Milne 8-4). Let $K = \kappa(t)$, where κ is a finite field. Assuming that every absolute value of K comes from a prime ideal of $\kappa[t]$ or $\kappa[t]^{-1}$. Prove a product formula for K. Under the same assumption, also prove a product formula for any finite separable extension of K.

Exercise 2 (Milne 8-1). Let $K = \mathbb{Q}[\alpha]$, where α is a root of $x^3 - x^2 - 2x - 8$. Show that there are three extensions of the 2-adic absolute value to K. Deduce that $2 \mid \mathrm{disc}(Z[\alpha]/\mathbb{Z})$ but $2 \nmid \mathrm{disc}(\mathcal{O}_K/\mathbb{Z})$.

Exercise 3 (A fundamental finiteness theorem).
Hermite's Theorem says that for any integer D there are only finitely many number fields with discriminant D.

(1) Assuming Hermite's theorem, show that for any integer N and any finite set of primes S, up to isomorphism there are only finitely many number fields of degree $n < N$ that are unramified outside of the primes in S. (This is one of the fundamental finiteness theorems in algebraic number theory and arithmetic geometry, and it comes up everywhere!)

(2) Read the proof of Hermite's theorem in Milne (Theorem 8.4.3).

Remark: In some of the deepest and most important results and conjectures in number theory, the key step is to prove some set of things with "bounded size" and "bounded ramification" is finite – the theorem above is the prototypical example and plays a role, e.g., in the proof of the Mordell-Weil theorem that the rational points on an abelian variety are a finitely generated abelian group. A much deeper result of this kind occurs in Faltings' proof of the Mordell conjecture (that a curve of genus ≥ 2 with rational coefficients has at most finitely many rational points – this implies, in particular, that Fermat's equation $x^d + y^d = z^d$ has at most finitely many primitive integer solutions for $d \geq 4$).