Basic Theory of Linear Differential Equations

• Picard-Lindelöf Existence-Uniqueness
  – Vector \( n \)th Order Theorem
  – Second Order Linear Theorem
  – Higher Order Linear Theorem

• Homogeneous Structure

• Recipe for Constant-Coefficient Linear Homogeneous Differential Equations
  – First Order
  – Second Order
  – \( n \)th Order

• Superposition

• Non-Homogeneous Structure
Theorem 1 (Picard-Lindelöf Existence-Uniqueness)
Let the \( n \)-vector function \( \vec{f}(x, \vec{y}) \) be continuous for real \( x \) satisfying \(|x - x_0| \leq a\) and for all vectors \( \vec{y} \) in \( \mathbb{R}^n \) satisfying \(|\vec{y} - \vec{y}_0| \leq b\). Additionally, assume that \( \partial \vec{f}/\partial \vec{y} \) is continuous on this domain. Then the initial value problem

\[
\begin{align*}
\vec{y}' &= \vec{f}(x, \vec{y}), \\
\vec{y}(x_0) &= \vec{y}_0
\end{align*}
\]

has a unique solution \( \vec{y}(x) \) defined on \(|x - x_0| \leq h\), satisfying \(|\vec{y} - \vec{y}_0| \leq b\), for some constant \( h, 0 < h < a \).

The unique solution can be written in terms of the Picard Iterates

\[
\vec{y}_{n+1}(x) = \vec{y}_0 + \int_{x_0}^{x} \vec{f}(t, \vec{y}_n(t))dt, \quad \vec{y}_0(x) \equiv \vec{y}_0,
\]

as the formula

\[
\vec{y}(x) = \vec{y}_n(x) + R_n(x), \quad \lim_{n \to \infty} R_n(x) = 0.
\]

The formula means \( \vec{y}(x) \) can be computed as the iterate \( \vec{y}_n(x) \) for large \( n \).
Theorem 2 (Second Order Linear Picard-Lindelöf Existence-Uniqueness)

Let the coefficients $a(x)$, $b(x)$, $c(x)$, $f(x)$ be continuous on an interval $J$ containing $x = x_0$. Assume $a(x) \neq 0$ on $J$. Let $g_1$ and $g_2$ be real constants. The initial value problem

$$
\begin{align*}
\begin{cases}
    a(x)y'' + b(x)y' + c(x)y &= f(x), \\
    y(x_0) &= g_1, \\
    y'(x_0) &= g_2
\end{cases}
\end{align*}
$$

has a unique solution $y(x)$ defined on $J$. 

Theorem 3 (Higher Order Linear Picard-Lindelöf Existence-Uniqueness)
Let the coefficients $a_0(x), \ldots, a_n(x), f(x)$ be continuous on an interval $J$ containing $x = x_0$. Assume $a_n(x) \neq 0$ on $J$. Let $g_1, \ldots, g_n$ be constants. Then the initial value problem

$$\begin{aligned}
& a_n(x)y^{(n)}(x) + \cdots + a_0(x)y = f(x), \\
& y(x_0) = g_1, \\
& y'(x_0) = g_2, \\
& \vdots \\
& y^{(n-1)}(x_0) = g_n
\end{aligned}$$

has a unique solution $y(x)$ defined on $J$. 
Theorem 4 (Homogeneous Structure 2nd Order)
The homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ has a general solution of the form

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

where $c_1, c_2$ are arbitrary constants and $y_1(x), y_2(x)$ are independent solutions.

Theorem 5 (Homogeneous Structure n-th Order)
The homogeneous equation $a_n(x)y^{(n)} + \cdots + a_0(x)y = 0$ has a general solution of the form

$$y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x),$$

where $c_1, \ldots, c_n$ are arbitrary constants and $y_1(x), \ldots, y_n(x)$ are independent solutions.
Theorem 6 (First Order Recipe)

Let \( a \) and \( b \) be constants, \( a \neq 0 \). Let \( r_1 \) denote the root of \( ar + b = 0 \) and construct its corresponding atom \( e^{r_1x} \). Multiply the atom by arbitrary constant \( c_1 \). Then \( y = c_1 e^{r_1x} \) is the general solution of the first order equation

\[
ay' + by = 0.
\]

The equation \( ar + b = 0 \), called the characteristic equation, is found by the formal replacements \( y' \to r, y \to 1 \) in the differential equation \( ay' + by = 0 \).
Theorem 7 (Second Order Recipe)
Let \( a \neq 0 \), \( b \) and \( c \) be real constant. Then the general solution of

\[
ay'' + by' + cy = 0
\]

is given by the expression \( y = c_1y_1 + c_2y_2 \), where \( c_1, c_2 \) are arbitrary constants and \( y_1, y_2 \) are two atoms constructed as outlined below from the roots of the characteristic equation

\[
ar^2 + br + c = 0.
\]

The characteristic equation \( ar^2 + br + c = 0 \) is found by the formal replacements \( y'' \rightarrow r^2, y' \rightarrow r, y \rightarrow 1 \) in the differential equation \( ay'' + by' + cy = 0 \).
Construction of Atoms for Second Order

The atom construction from the roots \( r_1, r_2 \) of \( ar^2 + br + c = 0 \) is based on Euler’s theorem below, organized by the sign of the discriminant \( D = b^2 - 4ac \).

\[
\begin{align*}
D > 0 & \quad (\text{Real distinct roots } r_1 \neq r_2) \\
D = 0 & \quad (\text{Real equal roots } r_1 = r_2) \\
D < 0 & \quad (\text{Conjugate roots } r_1 = \bar{r}_2 = A + iB)
\end{align*}
\]

\[
\begin{align*}
y_1 &= e^{r_1x}, \quad y_2 = e^{r_2x}. \\
y_1 &= e^{r_1x}, \quad y_2 = xe^{r_1x}. \\
y_1 &= e^{Ax} \cos(Bx), \quad y_2 = e^{Ax} \sin(Bx).
\end{align*}
\]

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**Theorem 8 (Euler’s Theorem)**

The atom \( y = x^k e^{Ax} \cos(Bx) \) is a solution of \( ay'' + by' + cy = 0 \) if and only if \( r_1 = A + iB \) is a root of the characteristic equation \( ar^2 + br + c = 0 \) and \( (r - r_1)^k \) divides \( ar^2 + br + c \).

Valid also for \( \sin(Bx) \) when \( B > 0 \). Always, \( B \geq 0 \). For second order, only \( k = 1, 2 \) are possible.

Euler’s theorem is valid for any order differential equation: replace the equation by \( a_n y^{(n)} + \cdots + a_0 y = 0 \) and the characteristic equation by \( a_n r^n + \cdots + a_0 = 0 \).
Theorem 9 (Recipe for $n$th Order)
Let $a_n \neq 0, \ldots, a_0$ be real constants. Let $y_1, \ldots, y_n$ be the list of $n$ distinct atoms constructed by Euler’s Theorem from the $n$ roots of the characteristic equation

$$a_n r^n + \cdots + a_0 = 0.$$  

Then $y_1, \ldots, y_n$ are independent solutions of

$$a_n y^{(n)} + \cdots + a_0 y = 0$$

and all solutions are given by the general solution formula

$$y = c_1 y_1 + \cdots + c_n y_n,$$

where $c_1, \ldots, c_n$ are arbitrary constants.

The characteristic equation is found by the formal replacements $y^{(n)} \to r^n, \ldots, y' \to r, y \to 1$ in the differential equation.
Theorem 10 (Superposition)
The homogeneous equation \( a(x)y'' + b(x)y' + c(x)y = 0 \) has the superposition property:

If \( y_1, y_2 \) are solutions and \( c_1, c_2 \) are constants, then the combination \( y(x) = c_1 y_1(x) + c_2 y_2(x) \) is a solution.

The result implies that linear combinations of solutions are also solutions.

The theorem applies as well to an \( n \)th order linear homogeneous differential equation with continuous coefficients \( a_0(x), \ldots, a_n(x) \).

The result can be extended to more than two solutions. If \( y_1, \ldots, y_k \) are solutions of the differential equation, then all linear combinations of these solutions are also solutions.

The solution space of a linear homogeneous \( n \)th order linear differential equation is a subspace \( S \) of the vector space \( V \) of all functions on the common domain \( J \) of continuity of the coefficients.
**Theorem 11 (Non-Homogeneous Structure 2nd Order)**

The non-homogeneous equation \( a(x)y'' + b(x)y' + c(x)y = f(x) \) has general solution

\[
y(x) = y_h(x) + y_p(x),
\]

where

- \( y_h(x) \) is the general solution of the homogeneous equation
  \[
a(x)y'' + b(x)y' + c(x)y = 0,
  \]
- \( y_p(x) \) is a particular solution of the nonhomogeneous equation
  \[
a(x)y'' + b(x)y' + c(x)y = f(x).
  \]

The theorem is valid for higher order equations: the general solution of the non-homogeneous equation is \( y = y_h + y_p \), where \( y_h \) is the general solution of the homogeneous equation and \( y_p \) is any particular solution of the non-homogeneous equation.

**An Example**

For equation \( y'' - y = 10 \), the homogeneous equation \( y'' - y = 0 \) has general solution \( y_h = c_1 e^x + c_2 e^{-x} \). Select \( y_p = -10 \), an equilibrium solution. Then \( y = y_h + y_p = c_1 e^x + c_2 e^{-x} - 10 \).
Theorem 12 (Non-Homogeneous Structure \(n\)th Order)
The non-homogeneous equation \(a_n(x)y^{(n)} + \cdots + a_0(x)y = f(x)\) has general solution
\[
y(x) = y_h(x) + y_p(x),
\]
where

- \(y_h(x)\) is the general solution of the homogeneous equation \(a_n(x)y^{(n)} + \cdots + a_0(x)y = 0\), and
- \(y_p(x)\) is a particular solution of the nonhomogeneous equation \(a_n(x)y^{(n)} + \cdots + a_0(x)y = f(x)\).