4.4 Undetermined Coefficients

The **method of undetermined coefficients** applies to solve differential equations

$$ay'' + by' + cy = f(x). \tag{1}$$

Restrictions: The symbols a, b, c are constant, $a \neq 0$. The nonhomogeneous term $f(x)$ is a sum of terms, each of which is one the following forms, called **atoms**:

$$\begin{array}{ll} p(x) & \textbf{polynomial}, \\ p(x)e^{kx} & \textbf{polynomial} \times \textbf{exponential}, \\ p(x)e^{kx}\cos(mx) & \textbf{polynomial} \times \textbf{exponential} \times \textbf{cosine}, \\ p(x)e^{kx}\sin(mx) & \textbf{polynomial} \times \textbf{exponential} \times \textbf{sine}. \end{array} \tag{2}$$

The polynomial $p(x)$ can be a constant. Symbols k and $m \geq 0$ are constants. The trigonometric terms may appear without an exponential, e.g., $(1+2x)e^{0x}\sin 3x$ is normally written $(1+2x)\sin 3x$. The method's importance is argued from its direct applicability to second order differential equations in mechanics and circuit theory.

Included as possible functions f in (1) are $\sinh x$ and $\cos^3 x$, due to identities from algebra and trigonometry. *Specifically excluded* are $\ln|x|$, $|x|$, e^{x^2} and fractions like $x/(1+x^2)$.

Superposition $y = y_h + y_p$ allows us to solve equation (1) in two stages: (a) Apply the linear equation **recipe** to find y_h; (b) Apply **the method of undetermined coefficients** to find y_p. We expect to find two arbitrary constants c_1, c_2 in the solution y_h, but in contrast, no arbitrary constants appear in y_p.

The **basic trial solution method**, which requires linear algebra, is presented on page 174. Readers should make an effort to learn this method, because literature normally omits details of the method, referencing only *the method of undetermined coefficients*. To enrich this basic method, we add a **library of special methods** for finding y_p, which includes Kümmer's method; see page **??**. The library uses only college algebra and polynomial calculus. The trademark of the library method is the *absence of linear algebra, tables or special cases*, that can be found in other literature on the subject.

The Algorithm for Undetermined Coefficients

A particular solution y_p of (1) will be expressed as a sum

$$y_p = y_1 + \cdots + y_n$$

where each y_k solves a related easily-solved differential equation.

The idea can be quickly communicated for $n = 3$. The superposition principle applied to the three equations

$$\begin{aligned} ay_1'' + by_1' + cy_1 &= f_1(x), \\ ay_2'' + by_2' + cy_2 &= f_2(x), \\ ay_3'' + by_3' + cy_3 &= f_3(x) \end{aligned} \tag{3}$$

shows that $y = y_1 + y_2 + y_3$ is a solution of

$$ay'' + by' + cy = f_1 + f_2 + f_3. \tag{4}$$

If each equation in (3) is easily solved, then solving equation (4) is also easy: *add the three answers for the easily solved problems.*

To use the idea, it is necessary to start with $f(x)$ and determine a decomposition $f = f_1 + f_2 + f_3$ so that equations (3) are easily solved.

The process is called the **method of undetermined coefficients**. This method requires decomposing (1) into a number of **easily-solved equations**. For instance, if an easily-solved equation has forcing term $f(x)$ equal to a polynomial, then a particular solution is found by substituting a polynomial **trial solution**

$$y = d_0 + d_1 x + \cdots + d_m \frac{x^m}{m!}$$

with **undetermined coefficients** $d_0, \ldots, d_m$. Undetermined coefficients are found by calculus and college algebra back–substitution.

The Easily Solved Equations. Each easily-solved equation is engineered to have right side in one of the four forms below, each of which is called an **atom**:

$$\begin{array}{ll} p(x) & \textbf{polynomial}, \\ p(x)e^{kx} & \textbf{polynomial} \times \textbf{exponential}, \\ p(x)e^{kx}\cos mx & \textbf{polynomial} \times \textbf{exponential} \times \textbf{cosine}, \\ p(x)e^{kx}\sin mx & \textbf{polynomial} \times \textbf{exponential} \times \textbf{sine}. \end{array} \tag{5}$$

To illustrate, consider

$$ay'' + by' + cy = x + xe^x + x^2 \sin x - \pi e^{2x}\cos x + x^3. \tag{6}$$

The right side is decomposed as follows, in order to define the easily solved equations (also called the **atomic equations**):

$$\begin{array}{ll} ay_1'' + by_1' + cy_1 = x + x^3 & \text{Polynomial.} \\ ay_2'' + by_2' + cy_2 = xe^x & \text{Polynomial} \times \text{exponential.} \\ ay_3'' + by_3' + cy_3 = x^2 \sin x & \text{Polynomial} \times \text{exponential} \times \text{sine.} \\ ay_4'' + by_4' + cy_4 = -\pi e^{2x}\cos x & \text{Polynomial} \times \text{exponential} \times \text{cosine.} \end{array}$$

There are $n = 4$ equations. In the illustration, x^3 is included with x, but it could have caused creation of a fifth equation. To decrease effort, minimize the number n of easily solved equations. *One final checkpoint*: the right sides of the n equations must add to the right side of (6).

The Basic Trial Solution Method

Literature referencing *the method of undetermined coefficients* usually means the basic trial solution method. Readers are asked to spend enough time to understand the method's mechanics and intricacies.

Assume given a constant-coefficient second order differential equation $ay'' + by' + cy = f(x)$, with f an **atom**, as in (2). The method:

Homogeneous solution. Solve the homogeneous equation for y_h by the *recipe*. It contains arbitrary constants c_1, c_2.

Initial trial solution. Differentiate the atom $f(x)$ repeatedly. Isolate independent functions whose linear combinations are the derivatives. Multiply them by **undetermined coefficients** d_1, d_2, ..., d_k to define an initial trial solution.

Fixup rule. If the initial trial solution duplicates terms found in y_h, then multiply the trial solution by x repeatedly until it doesn't. The **final trial solution** is the modified expression.

Substitute and evaluate. Substitute the final trial solution into the nonhomogeneous differential equation. Match coefficients of the independent functions to obtain equations for the undetermined coefficients d_1, d_2, ..., d_k. Solve the system.

Report $y = y_h + y_p$. Homogeneous solution y_h was reported above. Particular solution y_p is the final trial solution with the evaluated coefficients. Add to obtain the general solution y.

The algorithm actually works for sums of atoms, provided the fixup rule is applied to each individual atom. Linear algebra techniques are used to solve the system of equations in the 4th step. An answer check is prudent, because of many opportunities for arithmetic errors.

Illustration. Let's solve $y'' - y = x + xe^x$, verifying that $y_h = c_1e^x + c_2e^{-x}$ and $y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$.

- **Homogeneous solution**. The characteristic equation $r^2 - 1 = 0$ has roots $r = \pm 1$. Recipe **case 1** implies $y_h = c_1e^x + c_2e^{-x}$.
- **Initial trial solution**. The atoms of $f = x + xe^x$ are $f_1 = x$, $f_2 = xe^x$. Then $f = f_1 + f_2$ and the easily-solved problems are $y_1'' - y_1 = x$ and $y_2'' - y_2 = xe^x$. A particular solution is $y_p = y_1 + y_2$. Initial trial solutions, found by differentiation, involve the independent terms 1, x for y_1 and e^x, xe^x for y_2. Then $y_1 = d_1 + d_2x$, $y_2 = d_3e^x + d_4e^x$. The undetermined coefficients are d_1, d_2, d_3, d_4.

- **Fixup rule**. No terms of y_1 match those of y_h, so y_1 is the final trial solution for $y_1'' - y_1 = x$. Terms of y_2 match in y_h. Multiplication once by x is required to eliminate duplicates. Then $y_2 = x(d_3e^x + d_4xe^x)$ is the final trial solution.

- **Substitute and evaluate**. The details for y_1:

$x = y_1'' - y_1$	Reverse sides in the equation.
$= 0 - (d_1 + d_2x)$	Substitute $y_1 = d_1 + d_2x$.
$= (-d_1) + (-d_2)x$	Collect on powers of x.

Equating matching powers in the last equation gives the system of equations

$$\begin{aligned} 0 &= -d_1, \\ 1 &= -d_2. \end{aligned}$$

Therefore, $d_1 = 0$, $d_2 = -1$ and $y_1 = -x$.

- **Substitute and evaluate**. The details for y_2:

$xe^x = y_2'' - y_2$	Reverse equation sides.
$= (d_3xe^x + d_4x^2e^x)'' - (d_3xe^x + d_4x^2e^x)$	Use $y_2 = d_3xe^x + d_4x^2e^x$.
$= 2d_4e^x + 2d_3e^x + 4d_4xe^x$	Differentiate and simplify.

Matching like terms left and right gives the system of equations

$$\begin{aligned} 0 &= 2d_3 + 2d_4, \\ 1 &= 4d_4. \end{aligned}$$

Then $d_4 = 1/4$, $d_3 = -1/4$ and $y_2 = -\frac{1}{4}xe^x + \frac{1}{4}x^2e^x$.

- **Report** $y = y_h + y_p$. From above, $y_h = c_1e^x + c_2e^{-x}$ and $y_p = y_1 + y_2 = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$. Then $y = y_h + y_p$ is given by

$$y = c_1e^x + c^2e^{-x} - x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$

Answer check. Computer algebra system `maple` is used.

```
yh:=c1*exp(x)+c2*exp(-x);
y1:=-x;
y2:=-(1/4)*x*exp(x)+(1/4)*x^2*exp(x);
de:=diff(y(x),x,x)-y(x)=x*exp(x):
odetest(y(x)=yh+y1+y2,de);
```

5 Example (Sine–Cosine Trial solution) Verify for $y'' + 4y = \sin x - \cos x$ that $y_p(x) = 5\cos x + 3\sin x$, by using the trial solution $y = A\cos x + B\sin x$.

Solution: Substitute $y = A\cos x + B\sin x$ into the differential equation and use $u'' = -u$ for $u = \sin x$ or $u = \cos x$ to obtain the relation

$$\begin{aligned}\sin x - \cos x &= y'' + 4y \\ &= (-A+4)\cos x + (-B+4)\sin x.\end{aligned}$$

Comparing sides, matching sine and cosine terms, gives

$$\begin{aligned}-A+4 &= -1, \\ -B+4 &= 1.\end{aligned}$$

Solving, $A = 5$ and $B = 3$. The trial solution $y = A\cos x + B\sin x$ becomes $y_p(x) = 5\cos x + 3\sin x$. Generally, this method produces linear algebraic equations that must be solved by linear algebra techniques.

6 Example (Basic Trial Solution Method: I)

Solve for y_p in $y'' = 2 - x + x^3$ using the basic trial solution method, verifying $y_p = x^2 - x^3/6 + x^5/20$.

Solution:

Homogeneous solution. The equation $y'' = 0$ has characteristic equation $r^2 = 0$ and therefore $y_h = c_1 + c_2 x$.

Initial trial solution. The various derivatives of $f(x) = 2 - x + x^3$ are linear combinations of the independent terms 1, x, x^2, x^3. Then the initial trial solution is $y = d_1 + d_2 x + d_3 x^2 + d_4 x^3$.

Fixup rule and final trial solution. The homogeneous solution $y_h = c_1 + c_2 x$ duplicates terms d_1 and $d_2 x$ in the initial trial solution. Multiply y by x two times to eliminate duplications. Then $y = x^2(d_1 + d_2 x + d_3 x^2 + d_4 x^3)$ is the final trial solution.

Substitute and evaluate. The details:

$$\begin{aligned}2 - x + x^3 &= y'' && \text{Reverse sides.} \\ &= 2d_1 + 6d_2 x + 12 d_3 x^2 + 20 d_4 x^3 && \text{Substitute the final trial solution.}\end{aligned}$$

Equate like terms on each side of the equal sign to obtain the system of equations

$$\begin{aligned}2d_1 &= 2, \\ 6d_2 &= -1, \\ 12d_3 &= 0, \\ 20d_4 &= 1.\end{aligned}$$

This is a triangular system of linear equations for unknowns d_1, d_2, d_3, d_4. Solving gives $d_1 = 1$, $d_2 = -1/6$, $d_3 = 0$, $d_4 = 1/20$.

Report y_p. The expression $y = x^2(d_1 + d_2 x + d_3 x^2 + d_4 x^3)$ after substitution of the values found gives

$$y = x^2(1 - x/6 + x^3/20).$$

7 Example (Basic Trial Solution Method: II)

Solve $y'' - y' + y = 2 + e^x + \sin(x)$ by the basic trial solution method, verifying $y = c_1 e^{x/2}\cos(\sqrt{3}x/2) + c_2 e^{x/2}\sin(\sqrt{3}x/2) + 2 + e^x + \cos(x)$.

Solution:

Summary. There are three atoms: 2, e^x and $\sin x$. The easily solved equations are $y_1'' - y_1' + y_1 = 2$, $y_2'' - y_2' + y_2 = e^x$ and $y_3'' - y_3' + y_3 = \sin(x)$. Each such equation is solvable by trial solution methods, giving $y_1 = 2$, $y_2 = e^x$ and $y_3 = \cos x$. Then $y_p = y_1 + y_2 + y_3$ is the particular solution $y_p = 2 + e^x + \cos(x)$.

Homogeneous solution. The characteristic equation $r^2 - r + 1 = 0$ has roots $(1 \pm i\sqrt{3})/2$. The recipe implies $y_h = (c_1 \cos\sqrt{3}x/2 + c_2 \sin\sqrt{3}x/2)e^{x/2}$, where c_1 and c_2 are arbitrary constants.

Equation 1: $y_1'' - y_1' + y_1 = 2$. An initial trial solution is $y_1 = d_1 \cdot 1$, because 1 is the only independent function obtained by differentiation of the RHS. The fixup rule changes nothing, due to no duplications in y_h. Then $y_1 = d_1$ is the final trial solution. Substitution into $y_1'' - y_1' + y_1 = 2$ gives $d_1 = 2$. Then $y_1 = 2$.

Equation 2: $y_2'' - y_2' + y_2 = e^x$. Differentiation of the RHS gives one independent function e^x. Then $y_2 = d_2 e^x$ is the initial trial solution. The fixup rule changes nothing, due to no duplications. Then $y_2 = d_2 e^x$ is the corrected trial solution. Substitution into $y_2'' - y_2' + y_2 = e^x$ gives $(d_2 - d_2 + d_2)e^x = e^x$. Hence $d_2 = 1$. Then $y_2 = e^x$.

Equation 3: $y_3'' - y_3' + y_3 = \sin(x)$. Differentiation of the RHS gives independent functions $\cos x$, $\sin x$. The initial trial solution is $y_3 = d_3 \cos x + d_4 \sin x$. No terms of y_h are duplicated in y_3, therefore the fixup rule implies y_3 is the final trial solution. Substitution into $y_3'' - y_3' + y_3 = \sin x$ gives $-d_3 \cos x - d_4 \sin x - (-d_3 \sin x + d_4 \cos x) + (d_3 \cos x + d_4 \sin x) = \sin x$. Matching sine and cosine terms left and right implies $-d_3 = 0$, $d_4 = 1$. Then $y_3 = \cos x$.

Solution y_p. The particular solution is given by addition, $y_p = y_1 + y_2 + y_3$, with result $y_p = 2 + e^x + \cos(x)$.

General Solution. Add y_h and y_p to obtain the general solution

$$y = c_1 e^{x/2} \cos(\sqrt{3}x/2) + c_2 e^{x/2} \sin(\sqrt{3}x/2) + 2 + e^x + \cos(x).$$

8 Example (Basic Trial Solution Method: III)

Solve for y_p in $y'' - 2y' + y = (1 + x - x^2)e^x$ by the basic trial solution method, verifying that $y_p = (x^2/2 + x^3/6 - x^4/12)e^x$.

Solution: The right side is an atom, so there is no need to decompose it into easily-solved problems.

Homogeneous solution. The characteristic equation $r^2 - 2r + 1 = 0$ for $y'' - 2y' + y = 0$ has roots $r = 1$, $r = 1$. The recipe implies $y_h = c_1 e^x + c_2 x e^x$, where c_1 and c_2 are arbitrary constants.

Final Trial solution. The derivatives of the RHS give independent functions e^x, xe^x, x^2e^x. An initial trial solution is $y = (d_1 + d_2 x + d_3 x^2)e^x$. There are duplications of y_h terms in y. The fixup rule implies y should be multiplied twice by x to obtain the final trial solution $y = x^2(d_1 + d_2 x + d_3 x^2)e^x$.

Evaluate. Substitute the final trial solution into $y'' - 2y' + y = (1 + x - x^2)e^x$, in order to find the undetermined coefficients d_1, d_2, d_3. To present the details, let $q(x) = x^2(d_1 + d_2x + d_3x^2)$, then $y = q(x)e^x$ implies

$$\begin{aligned}
\text{LHS} &= y'' - 2y' + y \\
&= [q(x)e^x]'' - 2[q(x)e^x]' + q(x)e^x \\
&= q(x)e^x + 2q'(x)e^x + q''(x)e^x - 2q'(x)e^x - 2q(x)e^x + q(x)e^x \\
&= q''(x)e^x \\
&= [2d_1 + 6d_2x + 12d_2x^2]e^x.
\end{aligned}$$

Because LHS = RHS = $(1+x-x^2)e^x$, then e^x cancels and $2d_1+6d_2x+12d_3x^2 = 1 + x - x^2$. Matching powers of x gives $2d_1 = 1$, $6d_2 = 1$, $12d_3 = -1$. Then $y = x^2(1/2 + x/6 - x^2/12)e^x$.

Exercises 4.4

Polynomial Solutions. Determine a polynomial solution y_p for the given differential equation.

1. $y'' = x$

2. $y'' = x - 1$

3. $y'' = x^2 - x$

4. $y'' = x^2 + x - 1$

5. $y'' - y' = 1$

6. $y'' - 5y' = 10$

7. $y'' - y' = x$

8. $y'' - y' = x - 1$

9. $y'' - y' + y = 1$

10. $y'' - y' + y = -2$

11. $y'' + y = 1 - x$

12. $y'' + y = 2 + x$

13. $y'' - y = x^2$

14. $y'' - y = x^3$

Polynomial-Exponential Solutions. Determine a solution y_p for the given differential equation.

15. $y'' + y = e^x$

16. $y'' + y = e^{-x}$

17. $y'' = e^{2x}$

18. $y'' = e^{-2x}$

19. $y'' - y = (x + 1)e^{2x}$

20. $y'' - y = (x - 1)e^{-2x}$

21. $y'' - y' = (x + 3)e^{2x}$

22. $y'' - y' = (x - 2)e^{-2x}$

23. $y'' - 3y' + 2y = (x^2 + 3)e^{3x}$

24. $y'' - 3y' + 2y = (x^2 - 2)e^{-3x}$

Sine and Cosine Solutions. Determine a solution y_p for the given differential equation.

25. $y'' = \sin(x)$

26. $y'' = \cos(x)$

27. $y'' + y = \sin(x)$

28. $y'' + y = \cos(x)$

29. $y'' = (x + 1)\sin(x)$

30. $y'' = (x + 1)\cos(x)$

31. $y'' - y = (x + 1)e^x \sin(2x)$

32. $y'' - y = (x + 1)e^x \cos(2x)$

33. $y'' - y' - y = (x^2 + x)e^x \sin(2x)$

34. $y'' - y' - y = (x^2 + x)e^x \cos(2x)$

Undetermined Coefficients Algorithm. Determine a solution y_p for the given differential equation. These exercises require decomposition into easily-solved equations.

35. $y'' = x + \sin(x)$

36. $y'' = 1 + x + \cos(x)$

37. $y'' + y = x + \sin(x)$

38. $y'' + y = 1 + x + \cos(x)$

39. $y'' + y = \sin(x) + \cos(x)$

40. $y'' + y = \sin(x) - \cos(x)$

41. $y'' = x + xe^x + \sin(x)$

42. $y'' = x - xe^x + \cos(x)$

43. $y'' - y = \sinh(x) + \cos^2(x)$

44. $y'' - y = \cosh(x) + \sin^2(x)$

45. $y'' + y' - y = x^2e^x + xe^x \cos(2x)$

46. $y'' + y' - y = x^2e^{-x} + xe^x \sin(2x)$

Additional Proofs. The exercises below fill in details in the text.

47. **(Superposition)** Let Ly denote $ay'' + by' + cy$. Show that solutions of $Lu = f(x)$ and $Lv = g(x)$ add to give $y = u+v$ as a solution of $Ly = f(x) + g(x)$.

48. **(Easily Solved Equations)** Let Ly denote $ay'' + by' + cy$. Let $Ly_k = f_k(x)$ for $k = 1, \ldots, n$ and define $y = y_1 + \cdots + y_n$, $f = f_1 + \cdots + f_n$. Show that $Ly = f(x)$.