1 Coloring of $A_N$

Let $A_N$ be the arc graph of a polygonal disc with $N$ sides, $N > 4$.

**Problem 1.** What is the chromatic number of the arc graph of a polygonal disc of $N$ sides?

Or we would like to know how the chromatic number of arc graphs grows as $N$ grows.

**Definition 2** (Arc Graph of a N-polygonal disc). The vertices correspond to diagonals of a regular $N$-gon. Two vertices are joined by an edge if the corresponding diagonals are disjoint in the interior of the polygonal disk.

**Definition 3** (Proper coloring of a graph). A coloring of vertices of a graph such that adjacent vertices get different colors is called a proper coloring of a graph.

**Definition 4** (Maximal independent set in a graph). A set of vertices is called independent if they are pairwise disjoint in the graph. A maximal independent set is an independent set that is not a subset of any other independent set.

For a finite graph $G$, the following inequalities for chromatic number $\chi(G)$ are well known.

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq \Delta(G) + 1$$

where $n = \text{number of vertices in } G$, $\alpha(G)$ is the cardinality of maximum independent set in $G$ and $\Delta(G)$ is the maximum valence of a vertex in $G$.

So for Arc graph of a N-polygonal disc, denoted $A_N$, we get

$$\frac{\binom{N}{2} - N}{\lceil N/2 \rceil} \leq \chi(A_N) \leq f(N)$$

where $f(N)$ is quadratic in $N$.

We also have

$$N - 3 \leq \chi(A_N)$$

where $N - 3$ is the size of the biggest clique on $A_N$. 

1

Radhika Gupta
1.1 Main Result

**Theorem 5.** *The chromatic number of* $A_N$ *is less that equal to* $N \ln(N) + N$.

To prove the above theorem we will specify process of coloring $A_N$. This coloring will be done by finding maximal independent sets and choosing distinct color for each MIS. Thus the number of colors required will be the total number of mutually disjoint maximal independent sets we find. Note that we don’t know if this coloring is minimal or not. We will show that the number of colors used by this coloring is bounded by $N \ln(N) + N$.

1.2 Coloring

Given a polygonal disc with $N$ vertices, label the vertices with numbers from 0 to $N - 1$. Now label a diagonal by the set of ordered integers $i_1 i_2 \ldots i_j$, where $i_{k+1} = i_k + 1$ such that the diagonal joins the vertices labelled $i_1$ and $i_j$. Now order the diagonals lexicographically and arrange them in columns as shown in the examples below.

*For example* : In a pentagon, we label the diagonal joining 0 and 2 as 1, diagonal joining 0 and 3 as 12 and so on. So we get the labelled and ordered diagonals as 1, 12, 2, 23, 3. (The letters correspond to colors as explained later.)

For $A_6$ we get the following arrangement :

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1(a)</td>
<td>12(b)</td>
<td>123(d)</td>
</tr>
<tr>
<td>2(a)</td>
<td>23(b)</td>
<td>234(d)</td>
</tr>
<tr>
<td>3(c)</td>
<td>34(b)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>4(c)</td>
</tr>
</tbody>
</table>

For $A_7$ we get the following arrangement :

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1(a)</td>
<td>12(b)</td>
<td>123(d)</td>
<td>1234(e)</td>
</tr>
<tr>
<td>2(a)</td>
<td>23(b)</td>
<td>234(d)</td>
<td>2345(e)</td>
</tr>
<tr>
<td>3(c)</td>
<td>34(b)</td>
<td>345(d)</td>
<td></td>
</tr>
<tr>
<td>4(c)</td>
<td></td>
<td>45(f)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>5(g)</td>
<td></td>
</tr>
</tbody>
</table>

For $A_N$ we get the following arrangement :

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>$\ldots$</td>
<td>12$\ldots$N-3</td>
</tr>
<tr>
<td>2</td>
<td>23</td>
<td>$\ldots$</td>
<td>23$\ldots$N-2</td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N-1</td>
<td>N-1</td>
<td>N-2</td>
<td></td>
</tr>
<tr>
<td>N-2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Note that we can read off some maximal sets from each column above. The first column gives maximal sets of size 2, the second column gives maximal sets of size 3 and so on. Infact, if $i_1i_2 \ldots i_j$ is an entry of a column then in a big enough polygon we can get a maximal set of size $j + 1$.

Label the columns of the arrangement as $C_i, 1 \leq i \leq N - 3$. Then the size of a MIS in $C_i$ is $i + 1$ for $i < \lceil (N - 2)/2 \rceil$ and $N - (i + 1)$ for $i \geq \lceil (N - 2)/2 \rceil$ (which is infact all the elements in the column).

Now we describe the MISs we choose for our coloring. Group together elements in each column starting at the top and group size equal to the size of the MIS as given above. Then the total number of MIS or the total number of colors used will be

$$
\Sigma(A_N) = \left\lceil \frac{N - 2}{2} \right\rceil + \left\lceil \frac{N - 3}{3} \right\rceil + \ldots + \left\lceil \frac{N - 2 - k}{k} \right\rceil + \left( (N - 3) - \left\lceil \frac{N - 2}{2} \right\rceil \right)
$$

where $k < \lceil (N - 2)/2 \rceil$ and $\left( (N - 3) - \left\lceil \frac{N - 2}{2} \right\rceil \right)$ is the number of columns containing only one MIS.

Remark : The above sum is also equal to

$$
\left\lceil \frac{N - 2}{2} \right\rceil + \left\lceil \frac{N - 3}{3} \right\rceil + \ldots + \left\lceil \frac{N - 2 - k}{k} \right\rceil + \ldots + \left\lceil \frac{3}{N - 3} \right\rceil + \left\lceil \frac{2}{N - 2} \right\rceil
$$

because the terms after $k = \lceil (N-2)/2 \rceil$ are equal to 1 and there are $\left( (N - 3) - \left\lceil \frac{N - 2}{2} \right\rceil \right)$ of them.

### 1.3 Proof of Theorem 5

We know that

$$
\left\lfloor x \right\rfloor < x + 1
$$

Using the formula for $\Sigma(A_N)$ in the remark, we have

$$
\left\lceil \frac{N - 2}{2} \right\rceil + \left\lceil \frac{N - 3}{3} \right\rceil + \ldots + \left\lceil \frac{3}{N - 3} \right\rceil + \left\lceil \frac{2}{N - 2} \right\rceil < \frac{N - 2}{2} + \frac{N - 3}{3} + \ldots + \frac{2}{N - 2} + (N - 3)
$$

$$
< (N - 2) \left( \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N - 2} + 1 \right)
$$

$$
< (N - 2)(\ln(N - 2) + 1)
$$

$$
< N(\ln N + 1)
$$

Thus

$$
\chi(A_N) \leq \Sigma(A_N) < N(\ln N + 1).
$$
2 Coloring of $\mathcal{C}(S_{0,n})$

Let $\mathcal{C}(S_{0,n})$ be the curve graph of $S_{0,n}$, a sphere with $n$ punctures, $n > 4$.

**Problem 6.** *What is the chromatic number of $\mathcal{C}(S_{0,n})$?*

We will find an upper bound for the chromatic number.

2.1 Main Result

**Theorem 7.** *The chromatic number of $\mathcal{C}(S_{0,n})$, $\chi(\mathcal{C}(S_{0,n}))$ is bounded from above by $\frac{n(n-3)}{2}$.***

To prove the above theorem, we give a coloring of $\mathcal{C}(S_{0,n})$. This coloring will be done by finding independent sets (IS) and choosing distinct color for each IS. Thus the number of colors required will be the total number of mutually disjoint independent sets we find. Note that even for $n = 5, 6$ this coloring is not minimal. We will show that the number of colors used by this coloring is bounded by $\frac{n(n-3)}{2}$.

2.2 Coloring

We will think of a sphere with $n$ punctures as the plane $\mathbb{R}^2$ with $(n-1)$ punctures labelled $1, 2, \ldots, (n-1)$ and $n$th puncture at infinity. Every simple closed curve in the punctured plane determines a disc containing some punctures. Label the curve with the punctures in the disc. We think of these labels as words in the letters $1, 2, 3, \ldots, n-1$ arranged in increasing order from left to right.

For example, in $S_{0,5}$ we get the following labels :

<table>
<thead>
<tr>
<th>12</th>
<th>13</th>
<th>14</th>
<th>23</th>
<th>24</th>
<th>34</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>124</td>
<td>134</td>
<td>234</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Lemma 8.** *Any two simple closed curves with the same label always intersect. Hence all curves with a given label belong to the same color class.*

**Proof.** Let $\alpha$ and $\beta$ be the two different curves with same labels. Since they bound the same punctures but are different, say $\alpha$ has to leave the disc of $\beta$ which introduces the points of intersection. Therefore all the curves with same label form an independent set in $\mathcal{C}(S_{0,n})$ and hence can be given the same color. $\square$
Definition 9 (Length of a label). Let the length of a label be the number of letters in it.

Lemma 10. Let \( \alpha \) and \( \beta \) be two simple closed curves with different labels \( w_{\alpha} \) and \( w_{\beta} \) such that they share atleast one letter but letters of \( w_{\alpha} \) are not a subset of letters of \( w_{\beta} \) and vice-versa. Then the two curves intersect.

Proof. The curve \( \alpha \) divides the punctured plane into two components. Let \( D_1 \) be the disc containing punctures labelling \( \alpha \) and \( D_2 \) the component containing remaining punctures. Now \( \beta \) encloses a punctured disc with at least one puncture from both \( D_1 \) and \( D_2 \). This implies \( \beta \) has to intersect \( \alpha \). \( \square \)

Corollary 11. For every letter \( m \in 1, 2, \ldots, n - 2 \), any two curves with different labels but of same length and starting at \( m \) intersect each other.

For example, in \( S_{0,5} \), for \( m = 1 \), \( \{ (12), (13), (14) \} \) is an independent set and \( \{ (123), (124), (134) \} \) is another independent set. For \( m = 2 \) the independent sets are \( \{ (23), (24) \} \) and \( \{ (234) \} \) and for \( m = 3 \), the independent set is \( \{ (34) \} \). Then the total numbers of colors required is total number of independent sets.

\[
\begin{align*}
\{12, 13, 14\} & \quad \{23, 24\} & \quad \{34\} \\
\{123, 124, 134\} & \quad \{234\} &
\end{align*}
\]

So we get \( 2 + 2 + 1 = 5 \) colors.

For example, in \( S_{0,6} \) we get

\[
\begin{align*}
\{12, 13, 14, 15\} & \quad \{23, 24, 25\} & \quad \{34, 35\} & \quad \{45\} \\
\{123, 124, 125, 134, 135, 145\} & \quad \{234, 235, 245\} & \quad \{345\} &
\end{align*}
\]

\[
\begin{align*}
\{2345\}
\end{align*}
\]
This gives us $3 + 3 + 2 + 1 = 9$ colors.

In general counting the number of independent sets we get: For words starting at letter 1, we can have length at most $n-2$ and length 1 is not allowed (length one labels would correspond to inessential curves). Thus the number of color classes starting at 1 are $(n - 3)$. For words starting at letter $m$ for $1 < m < n - 1$, we can have length at most $(n-1)-(m-1)$ and length 1 is not allowed. Thus the number of color classes starting at $m$ are $(n - m - 1)$. Thus total number of color classes is

$$(n - 3) + (n - 3) + (n - 4) + \ldots + 2 + 1 = \frac{n(n - 3)}{2}.$$

This implies

$$\chi(C(S_{0,n})) \leq \frac{n(n - 3)}{2}.$$

3 Coloring of $\text{Sep}(S_{1,n})$

Let $S_{1,n}$ be a genus 1 surface with $n$ punctures, $n > 3$. Label the punctures with letters $1, 2, \ldots, n$. Let $\text{Sep}(S_{1,n})$ be a graph whose vertices are separating simple closed curves and two vertices are joined by an edge if the corresponding curves are disjoint.

**Theorem 12.**

$$\chi(\text{Sep}(S_{1,n})) \leq \frac{n(n - 3)}{2} + 1$$

**Proof.** Each separating curve divides the surface into two components, exactly one of which is a disc containing some punctures. Label such a curve by a word in the puncture letters, increasing from left to right. This is the same as in the case of sphere with $n$ punctures. So we can color the separating curves as in the case of $S_{0,n}$. We will have one extra class of curves, labelled by the word 123...$n$ which was inessential in $S_{0,n}$ but not in $S_{1,n}$. Therefore the total number of colors needed is $\frac{n(n - 3)}{2} + 1$. \qed