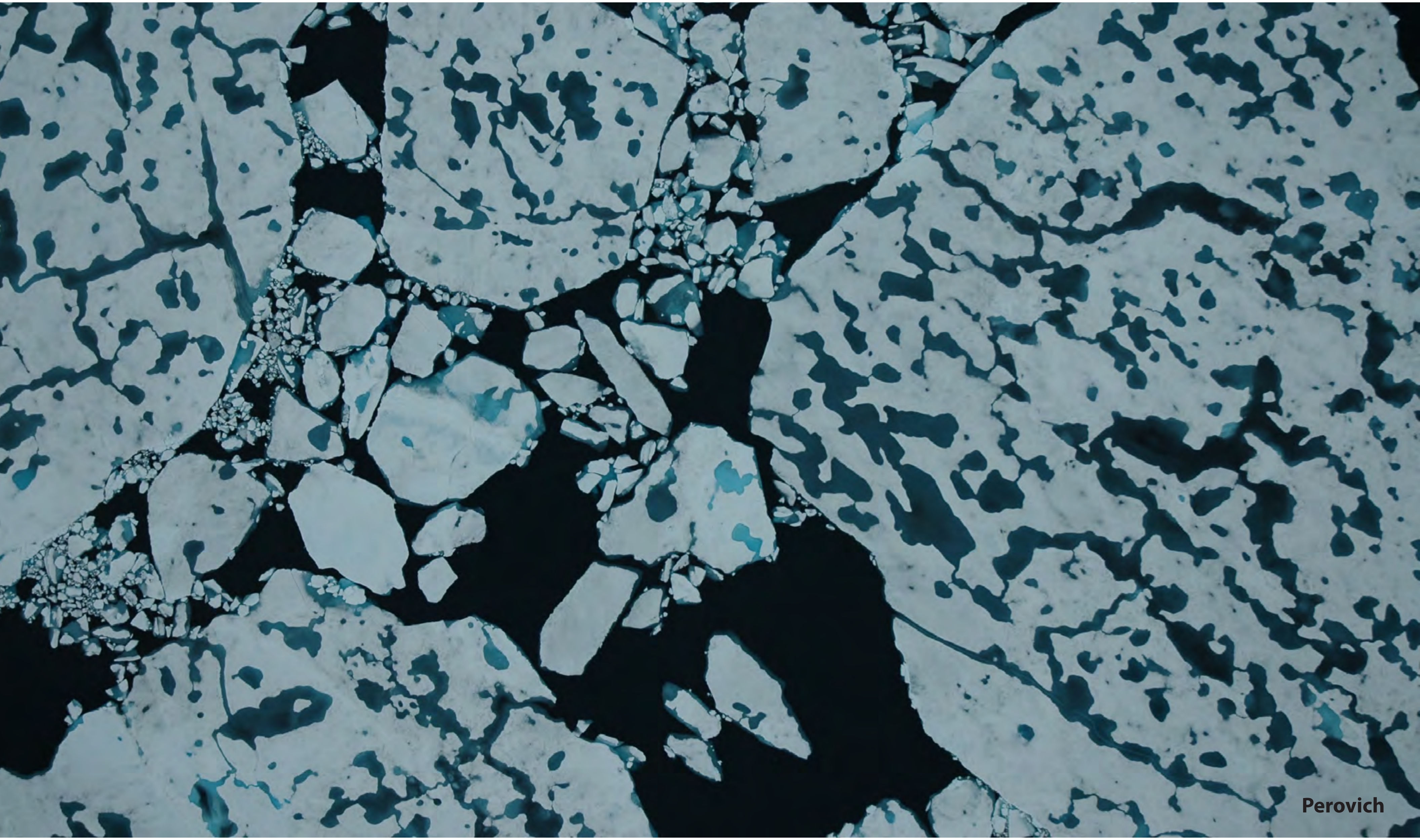
# Multiscale homogenization for sea ice and other composite materials

**Kenneth M. Golden** **University of Utah**

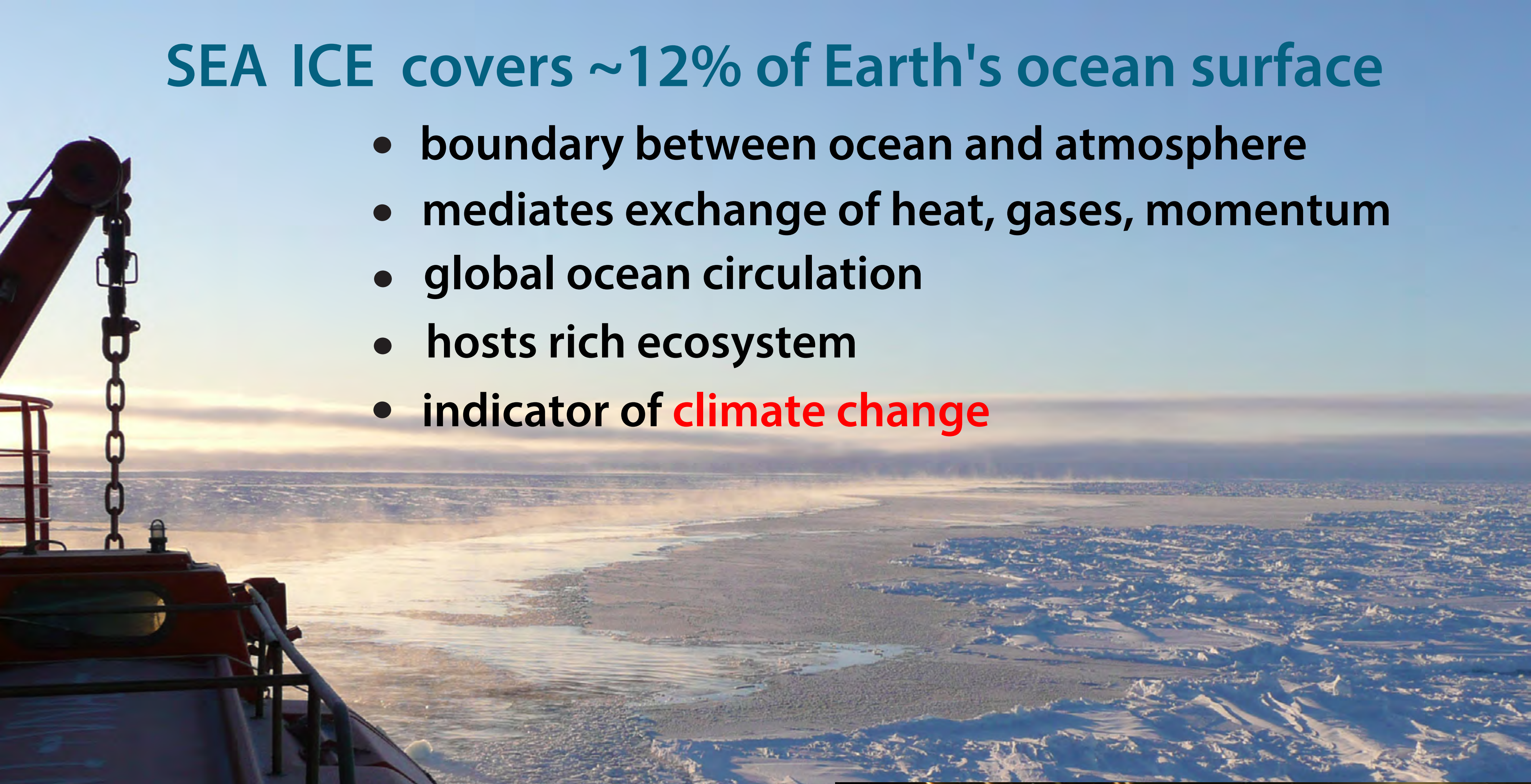


Perovich

ICIAM 2019, Valencia

# SEA ICE covers ~12% of Earth's ocean surface

- boundary between ocean and atmosphere
- mediates exchange of heat, gases, momentum
- global ocean circulation
- hosts rich ecosystem
- indicator of climate change



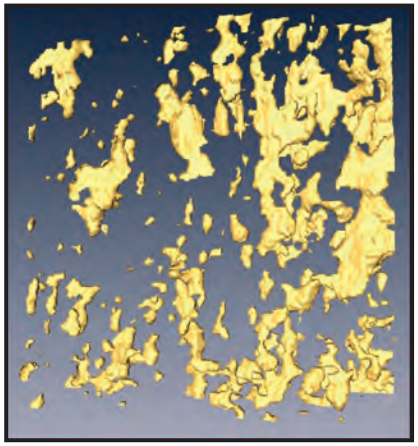
polar ice caps critical
to climate in reflecting
sunlight during summer

# Sea Ice is a Multiscale Composite Material

## *sea ice microstructure*

brine inclusions

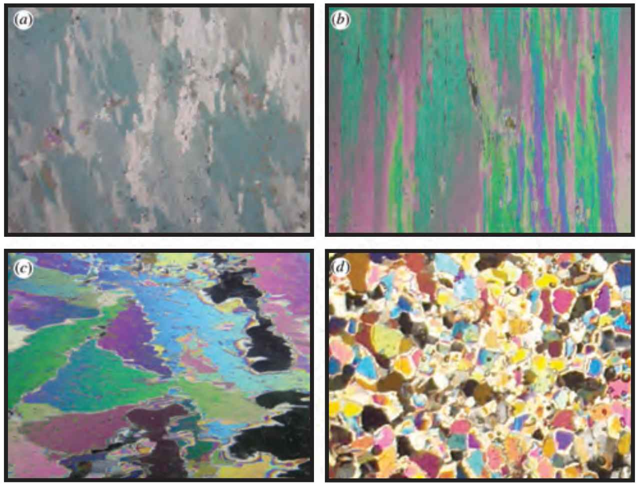
Weeks & Assur 1969



**H. Eicken**

Golden *et al. GRL* 2007

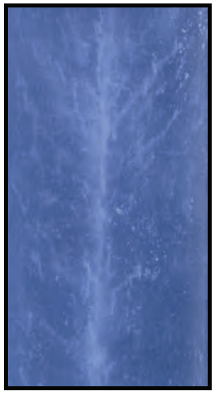
polycrystals



Gully *et al. Proc. Roy. Soc. A* 2015

brine channels

**D. Cole**



**K. Golden**

**millimeters**

**centimeters**

## *sea ice mesostructure*

Arctic melt ponds

**K. Frey**

Antarctic pressure ridges

**K. Golden**

## *sea ice macrostructure*

sea ice floes

**J. Weller**

sea ice pack

**NASA**

**meters**

**kilometers**

# *What is this talk about?* HOMOGENIZATION

**What is the role of microstructure in determining effective properties?**

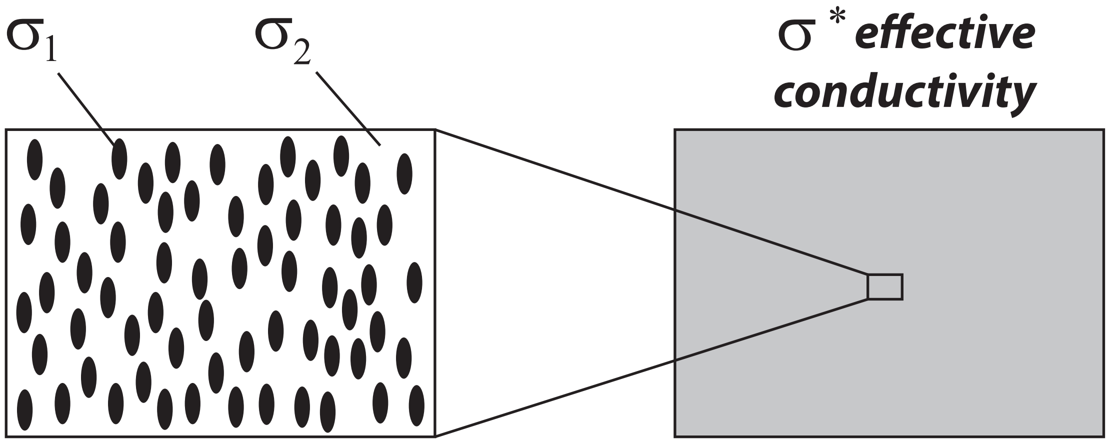
***Using methods of statistical physics and homogenization to*** **LINK SCALES** ***in the sea ice system ... rigorously compute effective behavior and improve climate models.***

1. ***Sea ice microphysics and fluid transport***
2. ***Analytic Continuation Method, integral representations***
3. ***Extension of ACM to advection diffusion, waves in sea ice***
4. ***Fractal geometry of melt pond evolution***

***Solving problems in physics of sea ice drives advances in theory of composite materials.***

**cross - pollination**

# HOMOGENIZATION - Linking Scales in Composites



***inhomogeneous medium***

***homogeneous medium***

**find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium**

*Maxwell 1873 : effective conductivity of a dilute suspension of spheres*
*Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid*

*Wiener 1912 : arithmetic and harmonic mean bounds on effective conductivity*
*Hashin and Shtrikman 1962 : variational bounds on effective conductivity*

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

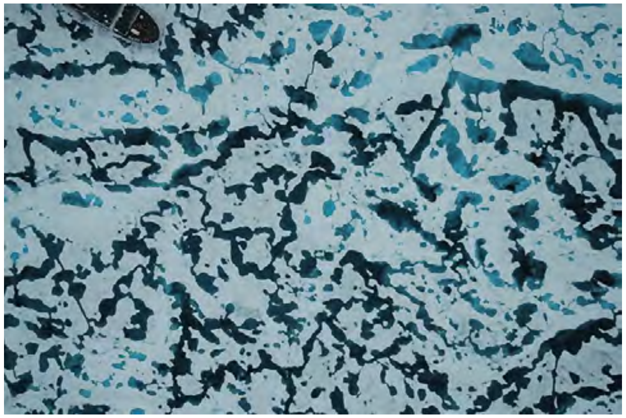
# How do scales interact in the sea ice system?

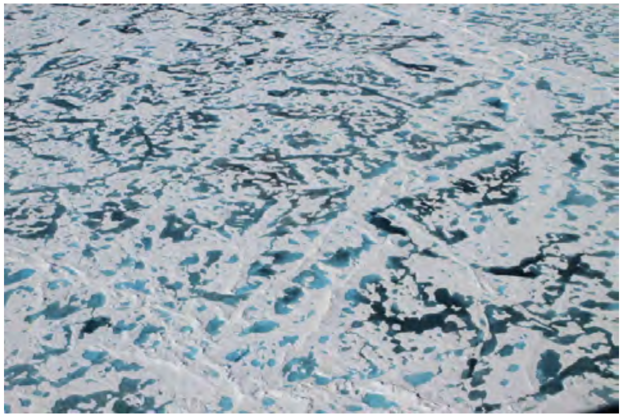
NASA

**basin scale -**
**grid scale**
**albedo**

## Linking Scales

**km**
**scale**
**melt**
**ponds**





Perovich

**km**
**scale**
**melt**
**ponds**

## Linking Scales

**mm**
**scale**
**brine**
**inclusions**

**meter**
**scale**
**snow**
**topography**

# sea ice microphysics

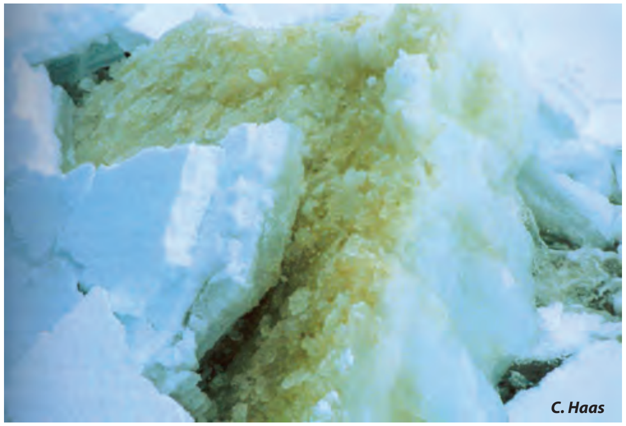
## fluid transport

# fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

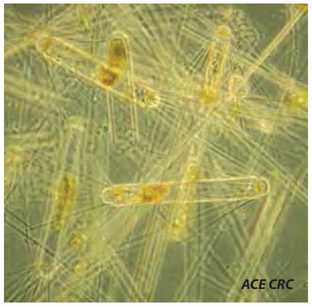
*evolution of Arctic melt ponds and sea ice albedo*

*K. Golden*

*nutrient flux for algal communities*

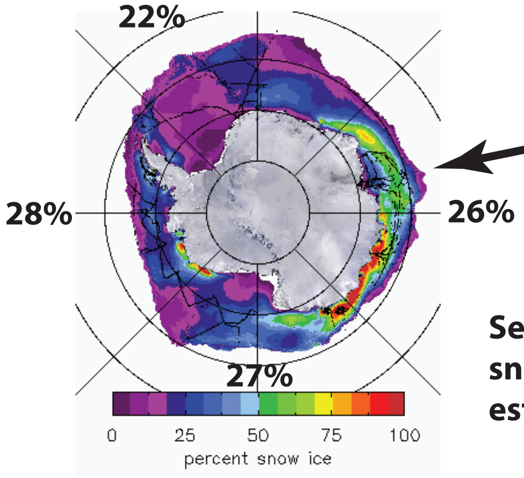


*C. Haas*



*ACE CRC*

*C. Krembs*



T. Maksym and T. Markus, 2008

**September snow-ice estimates**

*Antarctic surface flooding and snow-ice formation*

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, $CO_2$*

# fluid permeability of a porous medium

how much water gets through the sample per unit time?

## *Darcy's Law*

for slow viscous flow in a porous medium

averaged fluid velocity

pressure gradient

$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\,\nabla p$$

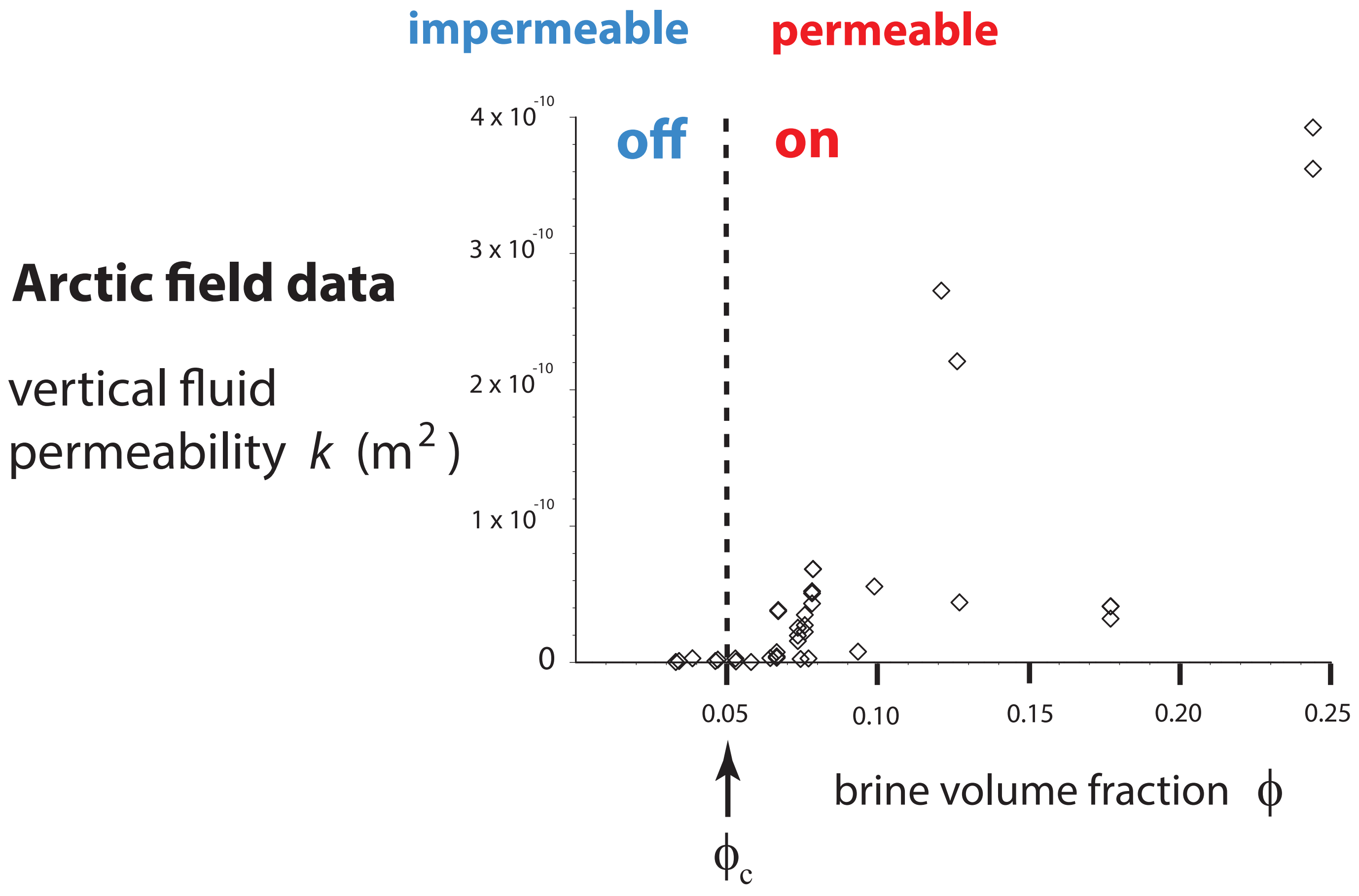
viscosity

$\mathbf{k}$ = fluid permeability tensor

## *HOMOGENIZATION*

*mathematics for analyzing effective behavior of heterogeneous systems*

# Critical behavior of fluid transport in sea ice



**Arctic field data**

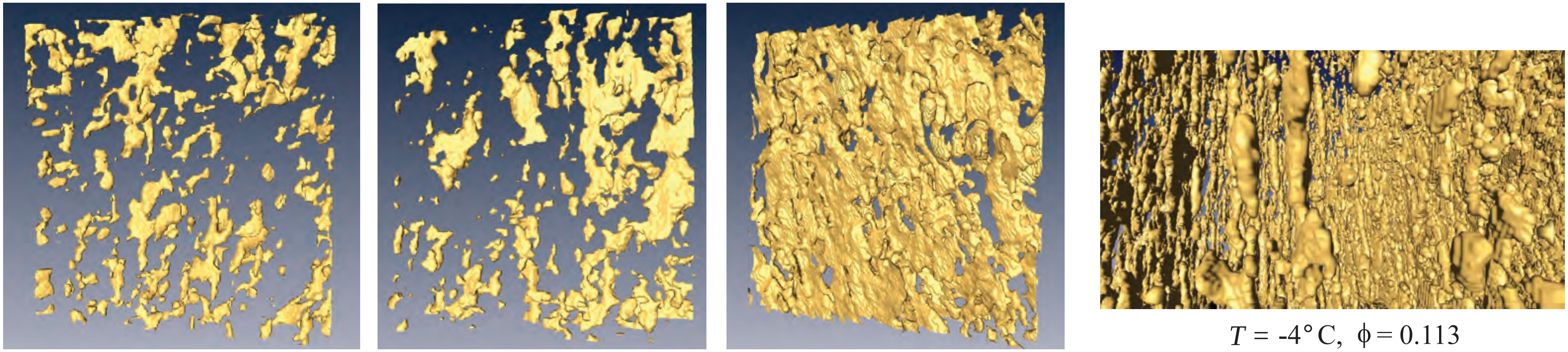
***"on - off" switch for fluid flow***

critical brine volume fraction $\phi_c \approx 5\%$ ⟷ $T_c \approx -5^\circ C$, $S \approx 5$ ppt

**RULE OF FIVES**

***Golden, Ackley, Lytle Science 1998***
***Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007***
***Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009***

# brine volume fraction and ***connectivity*** increase with temperature



$T$ = -15 °C, $\phi = 0.033$

$T$ = -6 °C, $\phi = 0.075$

$T$ = -3 °C, $\phi = 0.143$

$T$ = -4 °C, $\phi = 0.113$

**X-ray tomography for brine phase in sea ice**

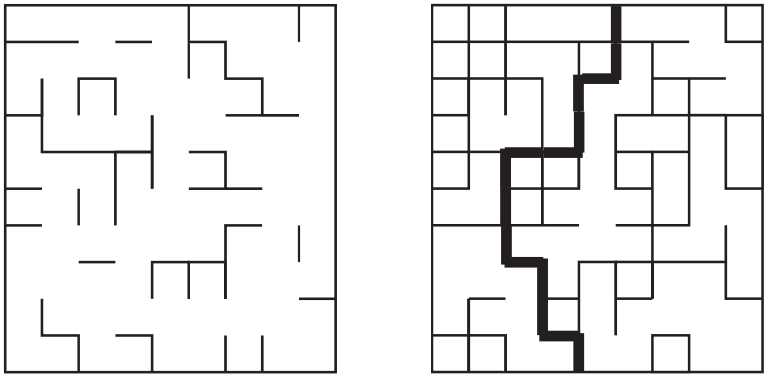
Golden, Eicken, *et al.*, *Geophysical Research Letters* 2007

## PERCOLATION THRESHOLD $\phi_c \approx 5\ \%$

Golden, Ackley, Lytle, *Science* 1998

impermeable

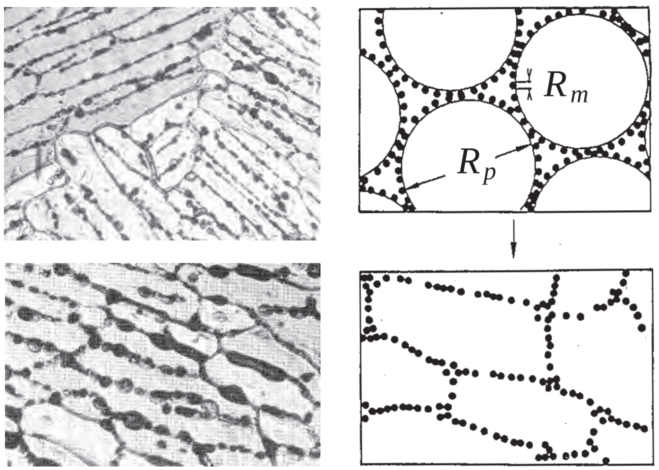
permeable



$p = 1/3$

$p = 2/3$

**lattice percolation**



Kusy, Turner
*Nature* 1971

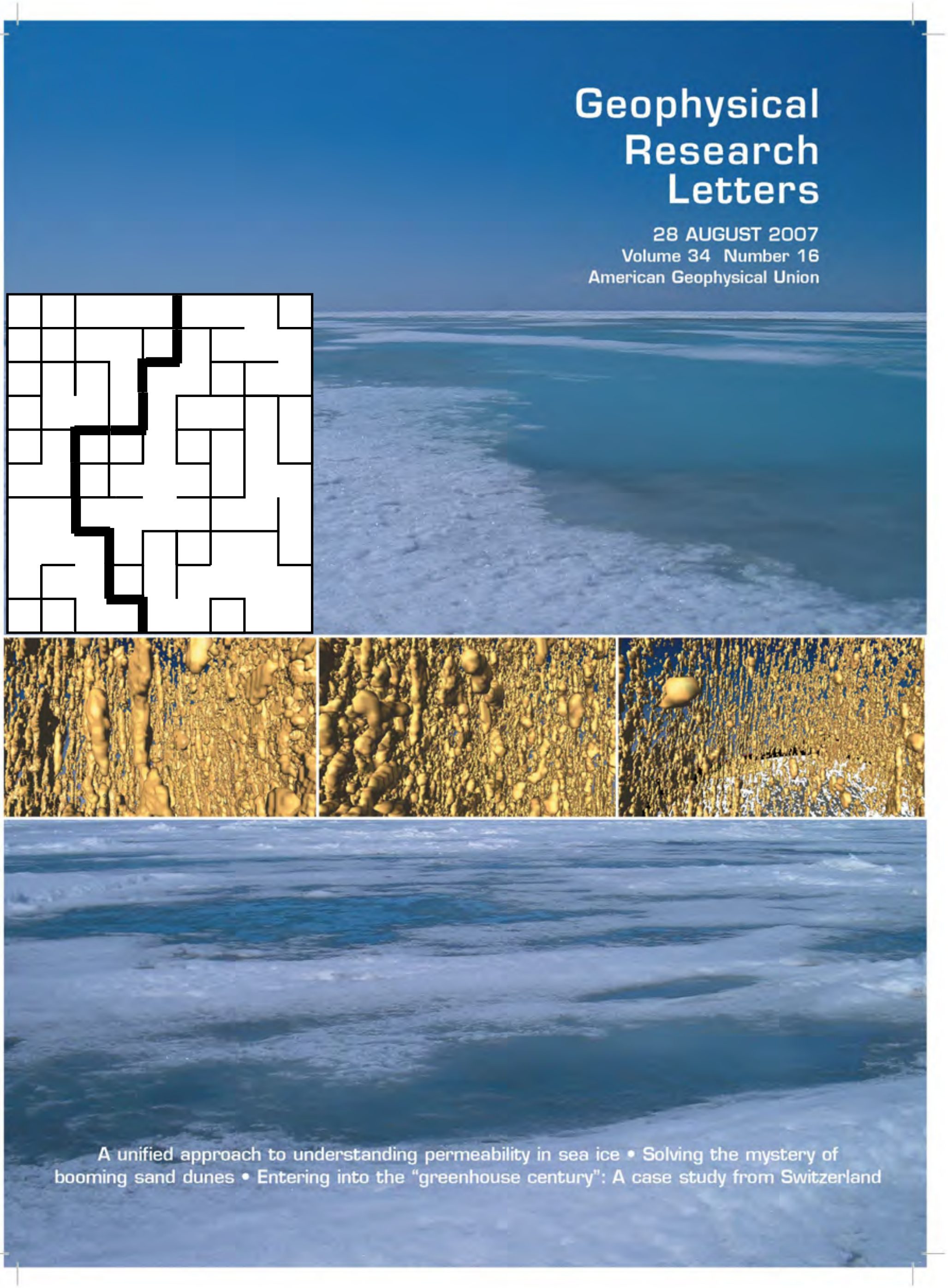
sea ice

compressed powder

**continuum percolation**

# *Thermal evolution of permeability and microstructure in sea ice*

***Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007***



Geophysical Research Letters

28 AUGUST 2007
Volume 34 Number 16
American Geophysical Union

A unified approach to understanding permeability in sea ice • Solving the mystery of booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

**micro-scale**

**controls**

**macro-scale**

**processes**

***percolation theory***

$$k(\phi) = k_0(\phi - 0.05)^2 \longleftarrow \text{critical exponent } t$$

$$k_0 = 3 \times 10^{-8} \ \mathrm{m}^2$$

***hierarchical model***
***network model***
***rigorous bounds***

**agree closely**
**with field data**

**X-ray tomography for brine inclusions**

***unprecedented look***
***at thermal evolution***
***of brine phase and***
***its connectivity***

## ***confirms rule of fives***

***Pringle, Miner, Eicken, Golden***
***J. Geophys. Res. 2009***

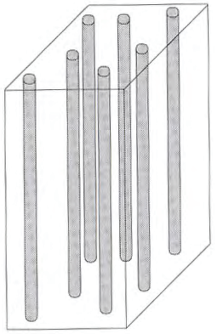
# PIPE BOUNDS on vertical fluid permeability $k$

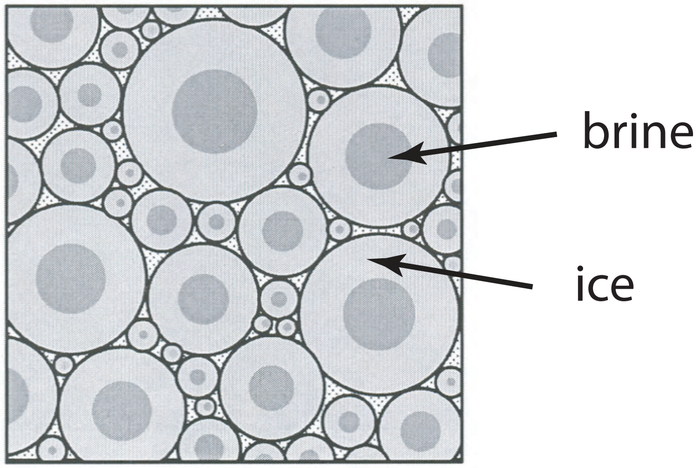
***Golden, Heaton, Eicken, Lytle, Mech. Materials 2006***
***Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophys. Res. Lett. 2007***

vertical pipes

with appropriate radii

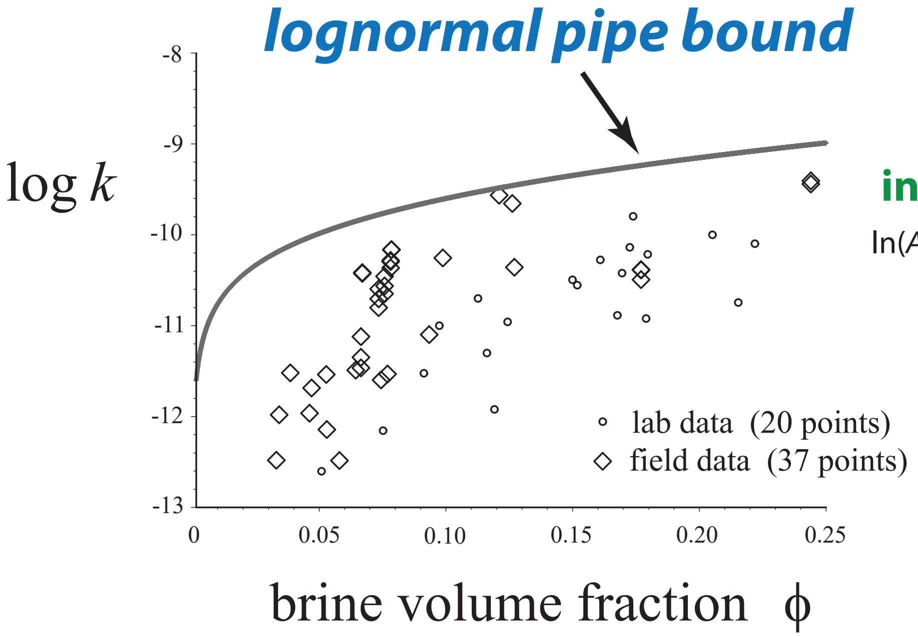
maximize $k$





**optimal coated cylinder geometry**

***fluid analog of arithmetic mean upper bound for effective conductivity of composites (Wiener 1912)***



***Golden et al., Geophys. Res. Lett. 2007***

$$k \;\leq\; \frac{\phi\langle R^4\rangle}{8\langle R^2\rangle} \;=\; \frac{\phi}{8}\langle R^2\rangle e^{\boldsymbol{\sigma}^2}$$

**inclusion cross sectional areas A lognormally distributed**

ln($A$) normally distributed, mean $\mu$ (increases with T) variance $\sigma^2$ (Gow and Perovich 96)

get bounds through variational analyis of ***trapping constant*** $\gamma$ for diffusion process in pore space with absorbing BC

**Torquato and Pham, *PRL* 2004**

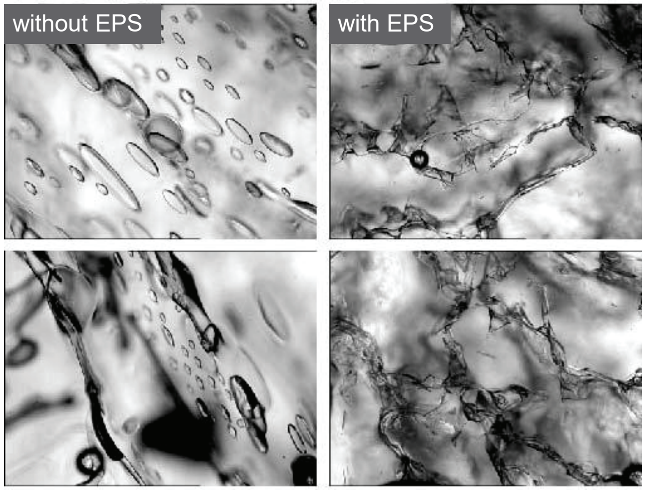
$$\mathbf{k} \;\leq\; \gamma^{-1}\mathbf{I}$$

for any ergodic porous medium (Torquato 2002, 2004)

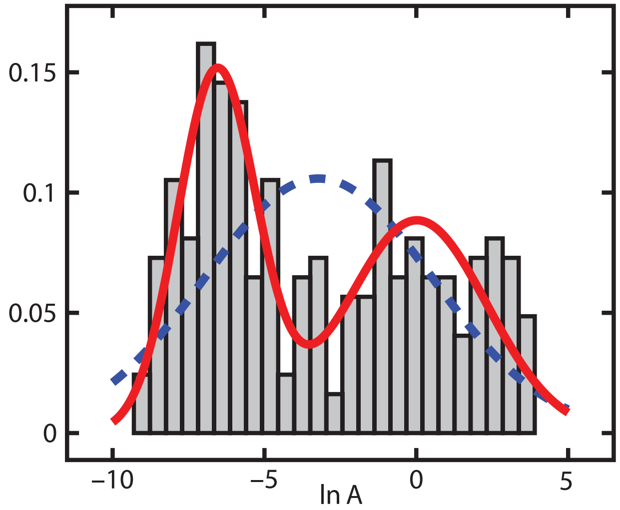
**BACTERIAL FORAGING**

# Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

## How does EPS affect fluid transport?



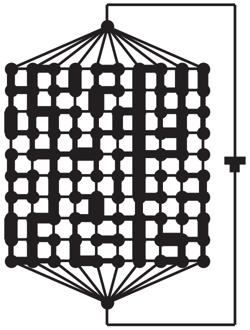
**Krembs, Eicken, Deming, PNAS 2011**

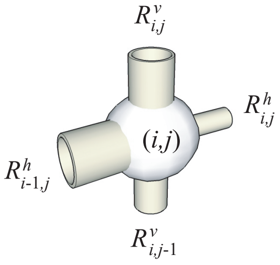


## *RANDOM PIPE MODEL*

- **Bimodal** lognormal distribution for brine inclusions
- Develop random pipe network model with bimodal distribution; Use numerical methods that can handle larger variances in sizes.
- Results predict observed drop in fluid permeability *k*.
- **Rigorous bound on *k* for bimodal distribution of pore sizes**

**Steffen, Epshteyn, Zhu, Bowler, Deming, Golden**
***Multiscale Modeling and Simulation*, 2018**





**Zhu, Jabini, Golden, Eicken, Morris**
***Ann. Glac.* 2006**

## How does the biology affect the physics?

ISSN 0002-9920

# Notices

of the American Mathematical Society

May 2009

Volume 56, Number 5

Climate Change and the Mathematics of Transport in Sea Ice

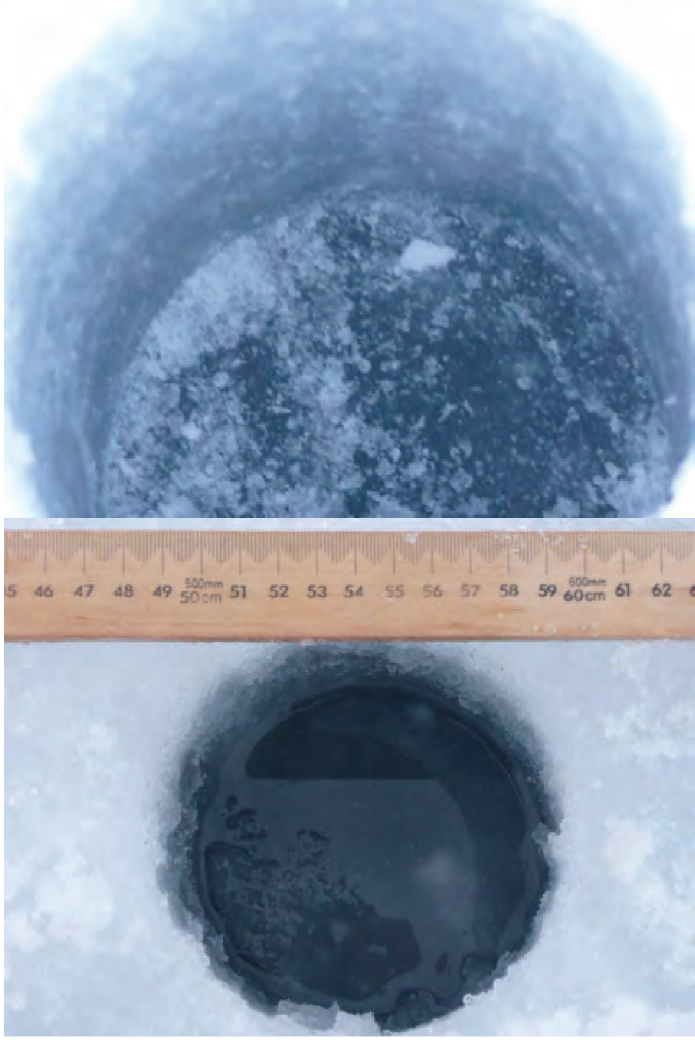
**page 562**

Mathematics and the Internet: A Source of Enormous Confusion and Great Potential

**page 586**

*Real analysis in polar coordinates (see page 613)*

*photo by Jan Lieser*



***measuring fluid permeability of Antarctic sea ice***

***SIPEX 2007***

# Remote sensing of sea ice

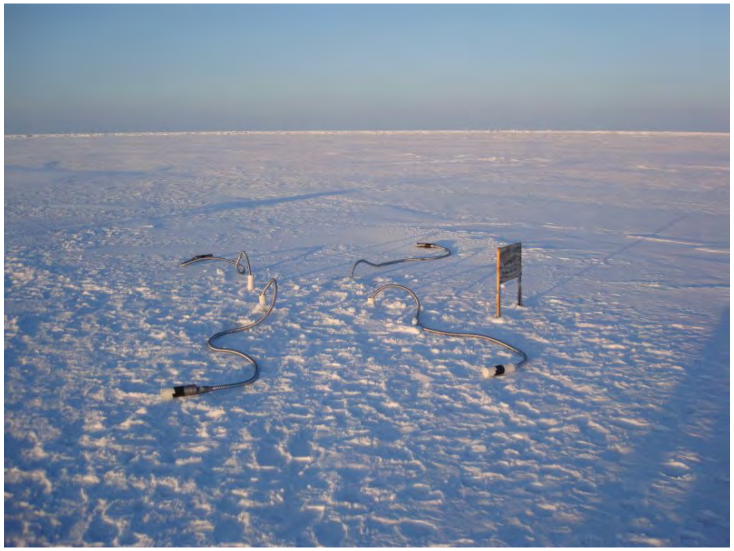
***sea ice thickness***
***ice concentration***

## *INVERSE PROBLEM*

Recover sea ice properties from electromagnetic (EM) data

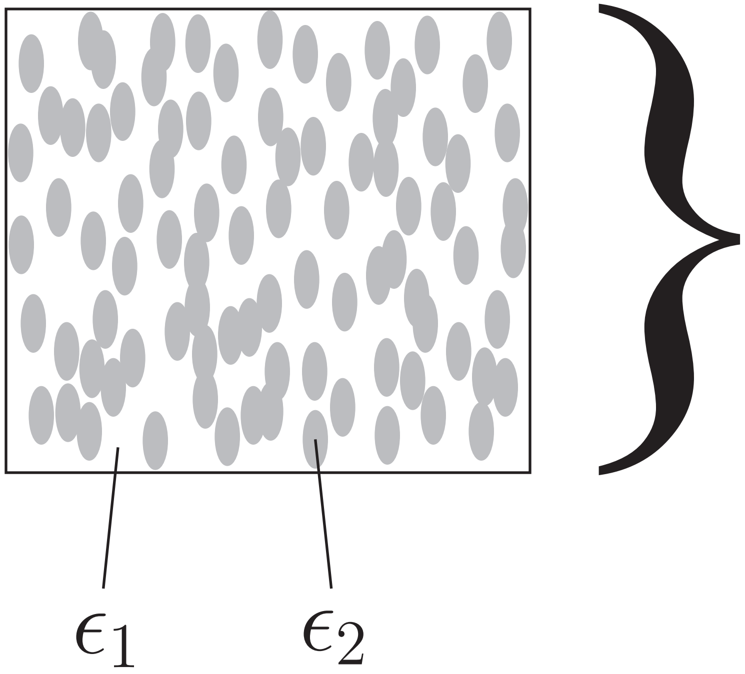
$\varepsilon^*$

effective complex permittivity
(dielectric constant, conductivity)



***brine volume fraction***
***brine inclusion connectivity***

# Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$\epsilon^*$$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

$p_1$ , $p_2$ = volume fractions of the components

$$\epsilon^* = \epsilon^* \left( \frac{\epsilon_1}{\epsilon_2} \text{ , composite geometry} \right)$$

**What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?**

# *Analytic Continuation Method for Homogenization*

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

## Stieltjes integral representation for homogenized parameter

*separates geometry from parameters*

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z} \qquad s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

material parameters

$\mu$

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = $p_1$
- higher moments depend on $n$-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla \cdot$$

$\chi =$ characteristic function of the brine phase

$$E = s\,(s + \Gamma\chi)^{-1} e_k$$

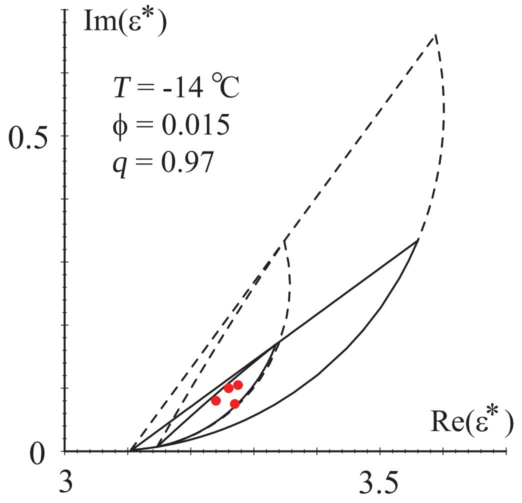
## $\Gamma\chi$ : microscale $\longrightarrow$ macroscale

## $\Gamma\chi$ *links scales*

***Golden and Papanicolaou, Comm. Math. Phys. 1983***

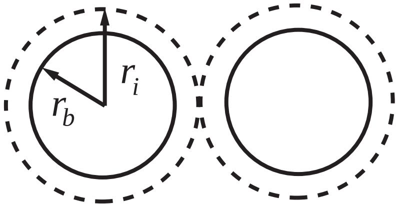
# forward and inverse bounds on the complex permittivity of sea ice

## forward bounds



4.75 GHz data

***matrix particle***



$$q = r_b / r_i$$

$$0 < q < 1$$

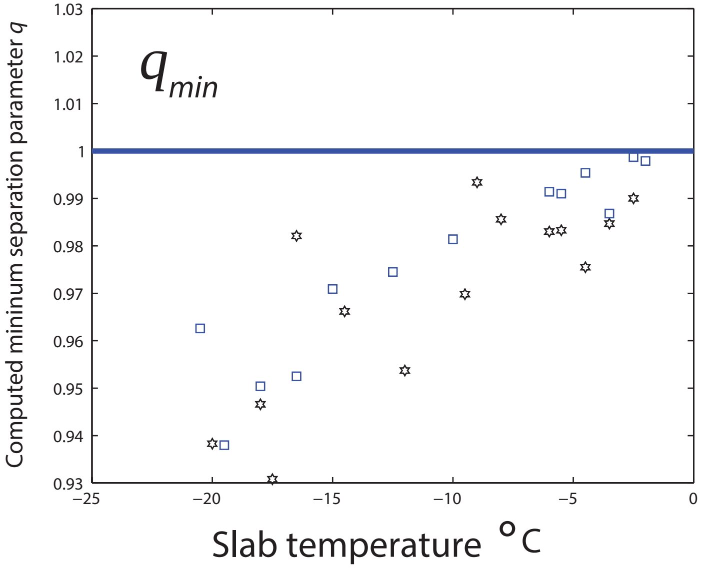
***Golden 1995, 1997***

***Bruno 1991***

### inverse bounds and recovery of brine porosity

***Gully, Backstrom, Eicken, Golden***
***Physica B, 2007***

## inverse bounds



### inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε*

**rigorous inverse bound**
**on spectral gap**

*construct algebraic curves which bound*
*admissible region in (p,q)-space*

***Orum, Cherkaev, Golden***
***Proc. Roy. Soc. A, 2012***

# direct calculation of spectral measures

**Murphy, Hohenegger, Cherkaev, Golden, *Comm. Math. Sci.* 2015**

- depends only on the composite geometry
- discretization of microstructural image gives binary network
- fundamental operator becomes a random matrix
- spectral measure computed from eigenvalues and eigenvectors

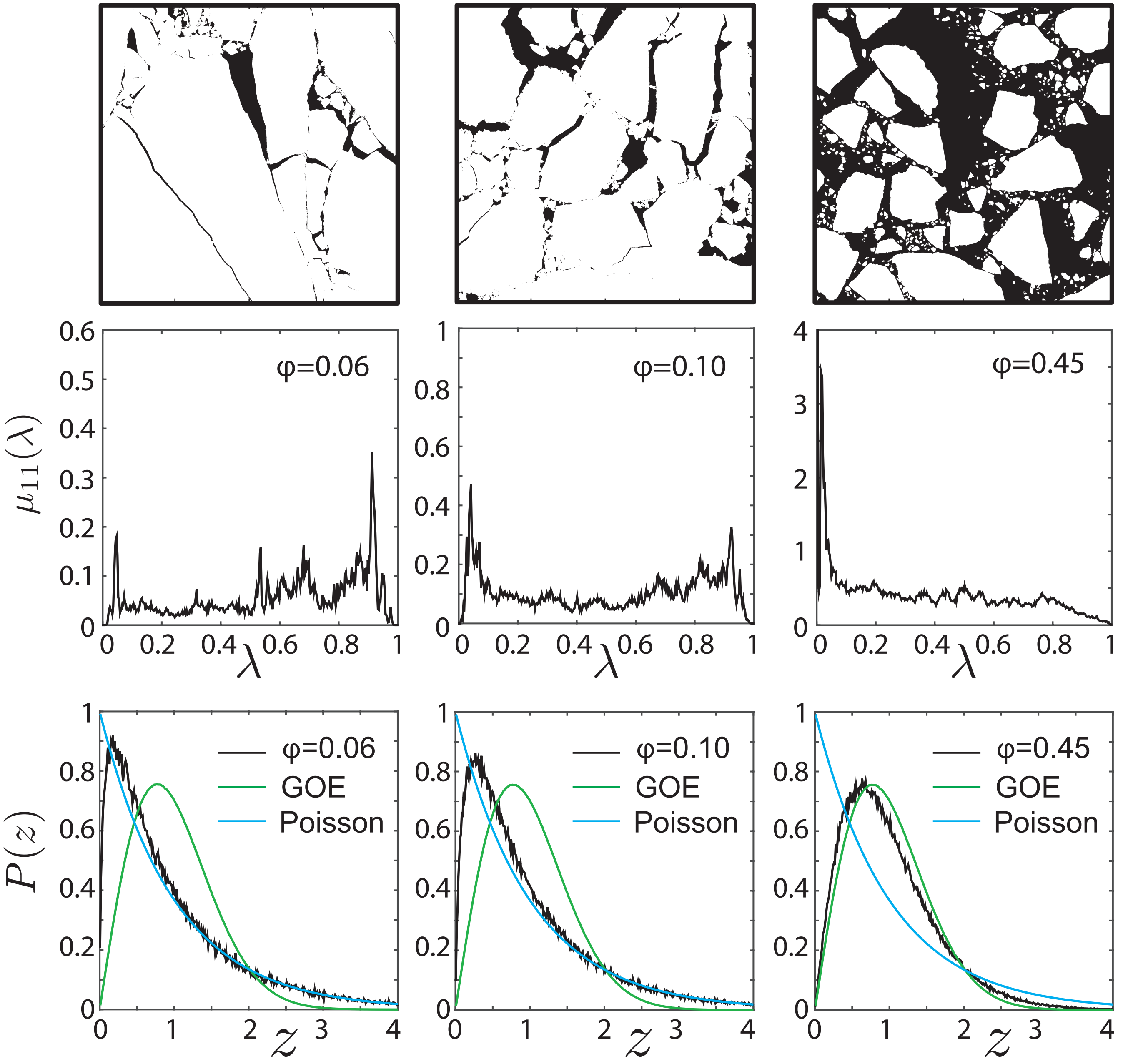
**once we have the spectral measure $\mu$ it can be used in Stieltjes integrals for other transport coefficients:**

***electrical and thermal conductivity, complex permittivity, magnetic permeability, diffusion, fluid flow properties***

earlier studies of spectral measures

Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

# Spectral computations for sea ice floe configurations



**spectral measures**

**eigenvalue spacing distributions**

***UNIVERSAL Wigner-Dyson distribution***

**uncorrelated** → **level repulsion**

***ANDERSON TRANSITION***

**Murphy, Cherkaev, Golden**
***Phys. Rev. Lett. 2017***

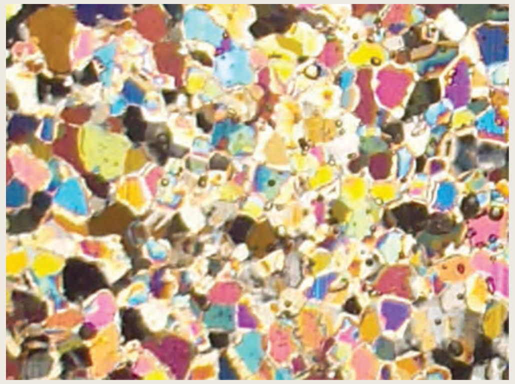
# Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**

  ***orientation statistics***
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



**Proc. Roy. Soc. A 8 Feb 2015**

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS OF SCIENTIFIC PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

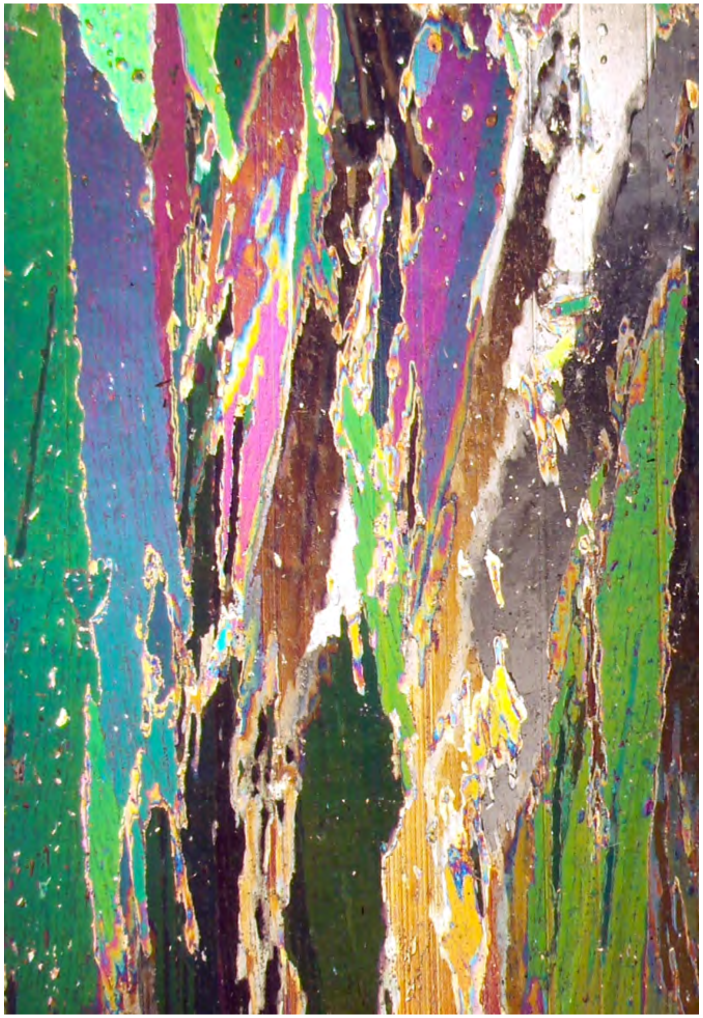
THE ROYAL SOCIETY PUBLISHING

21/01/15 1:47 PM

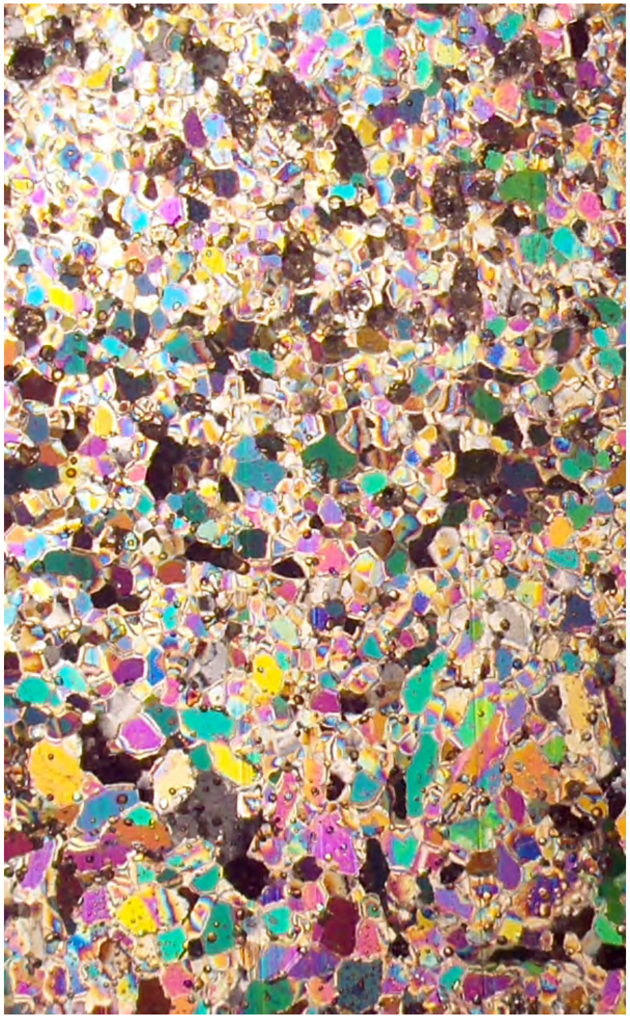
# higher threshold for fluid flow in Antarctic granular sea ice

columnar

**5%**



granular



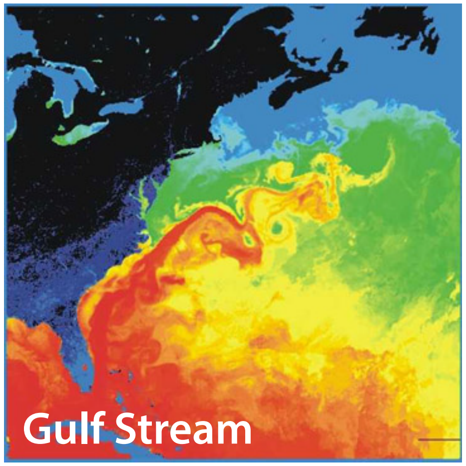
**10%**

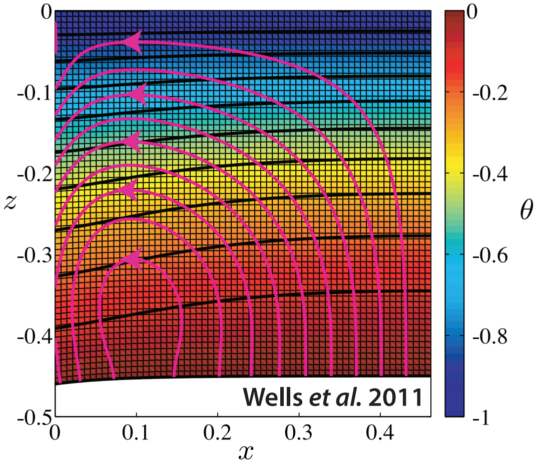
***Golden, Sampson, Gully, Lubbers, Tison 2019***

# *advection enhanced diffusion*

## effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere





**advection diffusion equation with a velocity field** $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

***homogenize***

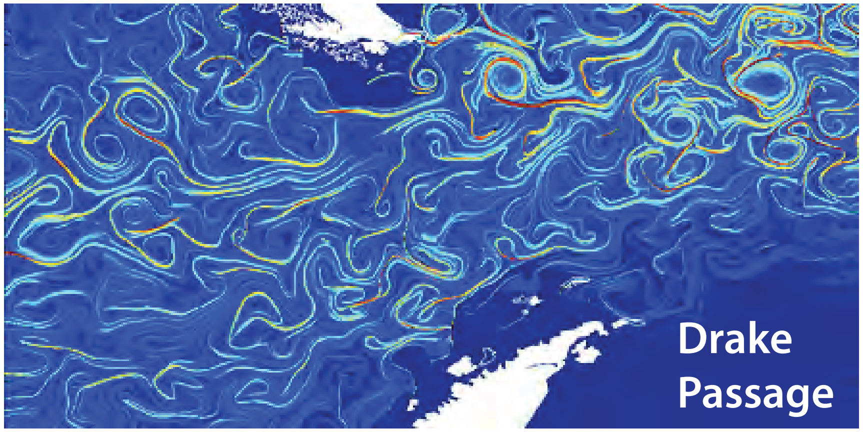
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

$\kappa^*$ ***effective diffusivity***

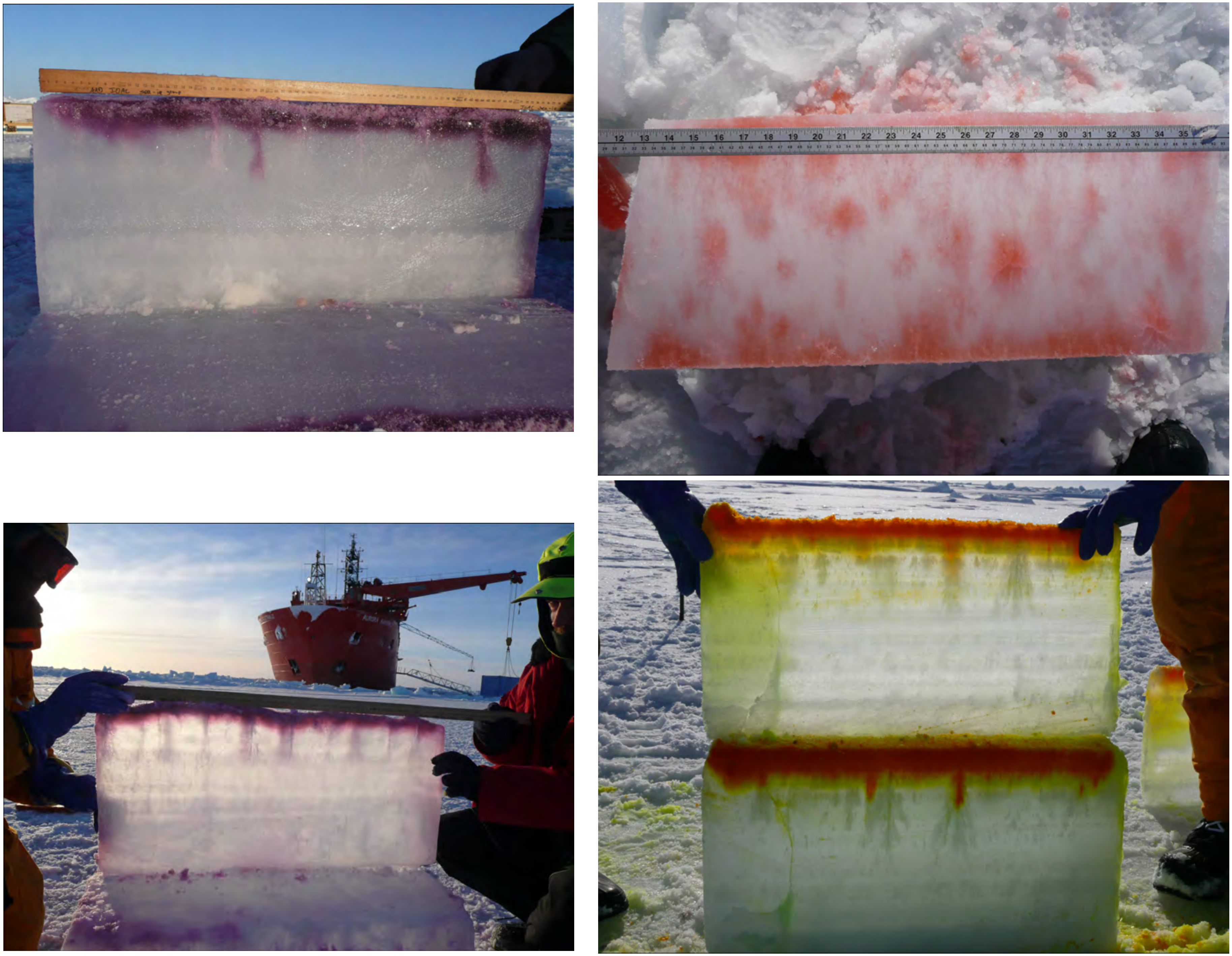
***Stieltjes integral for*** $\kappa^*$ ***with spectral measure***

***Avellaneda and Majda, PRL 89, CMP 91***

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019



# tracers flowing through inverted sea ice blocks



# Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019

$$\kappa^* = \kappa \left( 1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2} \right), \quad F(\kappa) \ = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}$$

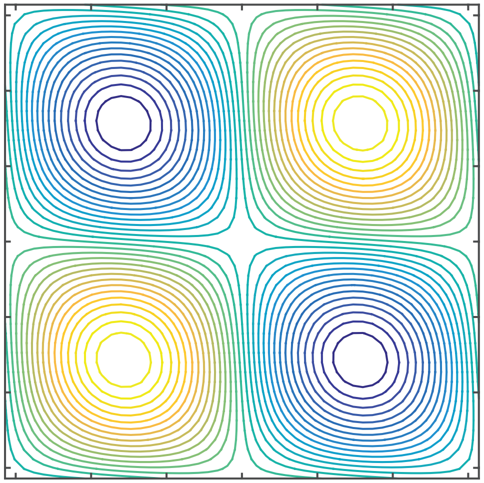
- $\mu$ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H \Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$ , $\Delta$ is the Laplace operator
- $i\Gamma H \Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the $\kappa$-plane

separation of material properties and flow field

spectral measure calculations

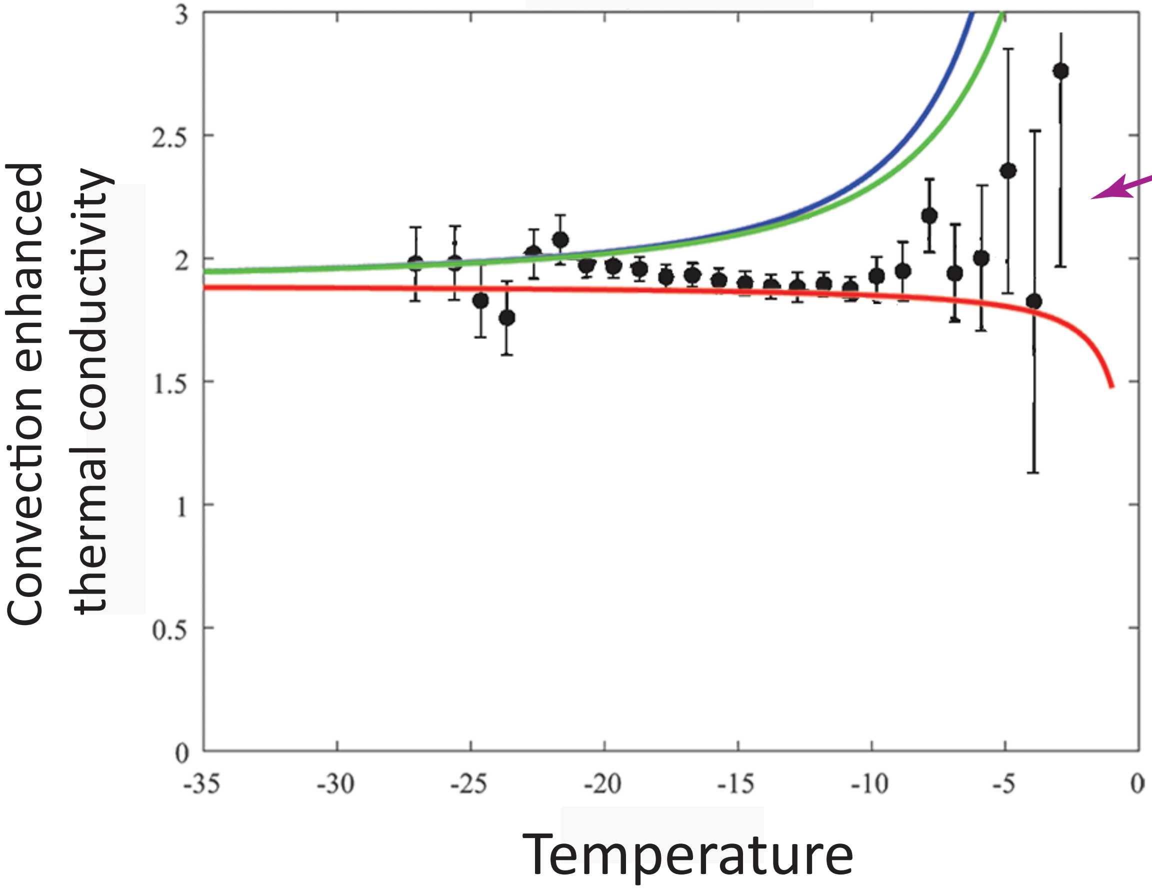
# Rigorous bounds on convection enhanced thermal conductivity of sea ice

**Kraitzman, Hardenbrook, Murphy, Zhu, Cherkaev, Strong, Golden 2019**



cat's eye flow model for
brine convection cells

similar bounds
for shear flows



**data Trodahl et al. 2001**

rigorous Padé bounds from Stieltjes integral +
analytical calculations of moments of measure

**rigorous bounds assuming information on flow field INSIDE inclusions**

Kraitzman, Cherkaev, Golden
*SIAM J. Appl. Math* (in revision), 2019

# Floe Scale Model of Anomalous Diffusion in Sea Ice Dynamics

**Huy Dinh, Elena Cherkaev, Court Strong, Ken Golden 2019**

$$\left\langle \left| \mathbf{x}(t) - \mathbf{x}(0) - \langle \mathbf{x}(t) - \mathbf{x}(0) \rangle \right|^2 \right\rangle \sim t^{\alpha}$$

$\alpha =$ Hurst exponent, a measure of anomalous diffusion

Statistic of bouy position data. Detects ice pack crowding and advective forcing.

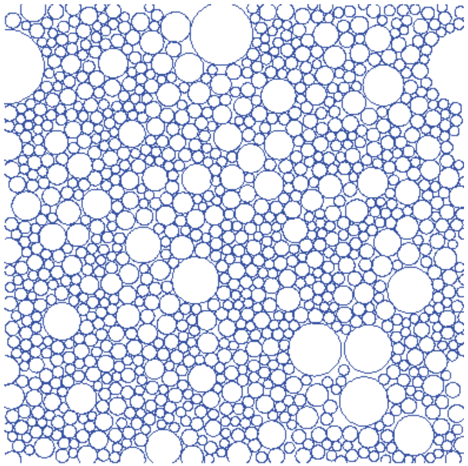
**J.V. Lukovich, J.K. Hutchings, D.G. Barber** ***Annals of Glaciology*** **2015**

$\alpha = 1$ Sparse packing, random advective forcing field.

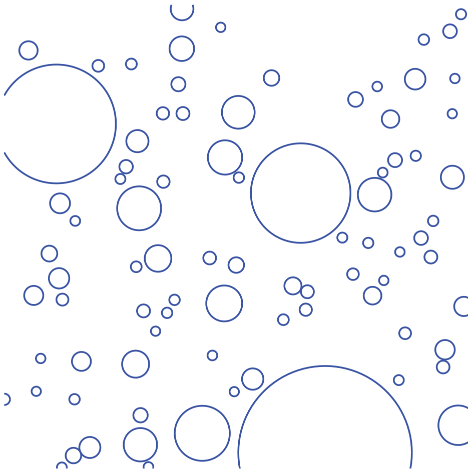
$\alpha < 1$ Dense packing, crowding dominates advection.

$\alpha = 5/4$ Sparse packing, shear dominates advection.

$\alpha = 5/3$ Sparse packing, vorticity dominates advection.



$k$ = 2.9 Concentration = 0.8



$k$ = 1.7 Concentration = 0.1

## Model Approximations

Power Law Size Distribution: $N(D) \sim D^{-k}$

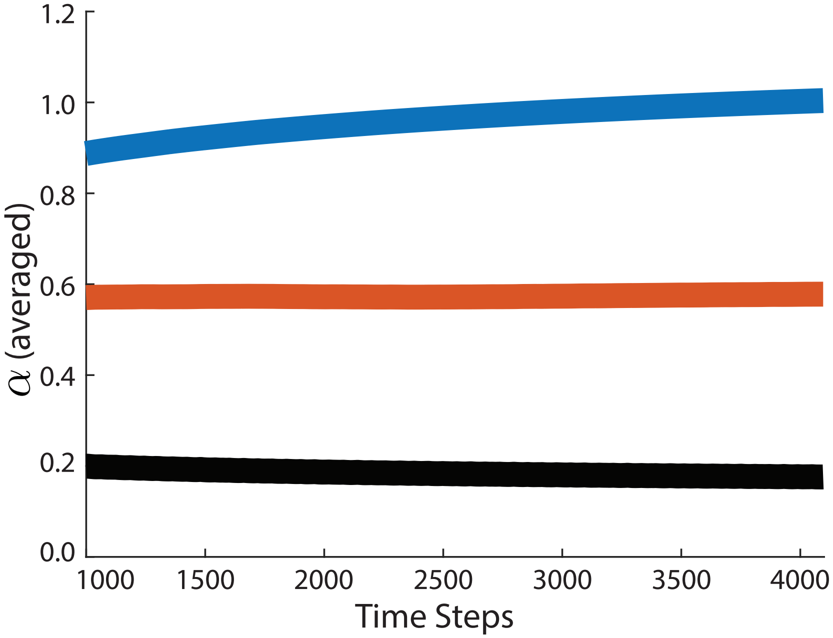
T. Toyota, S. Takatsuji, M. Nakayama *Geophysical Review Letters* 2006

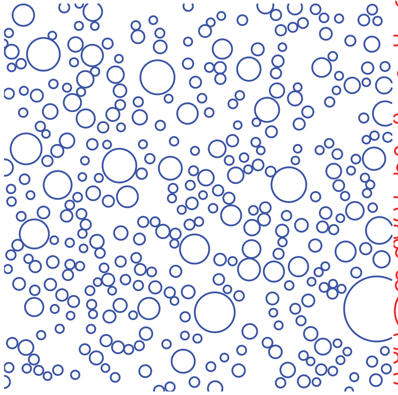
Floe-Floe Interactions: Linear Elastic Collisions

Advective Forcing: Passive, Linear Drag Law

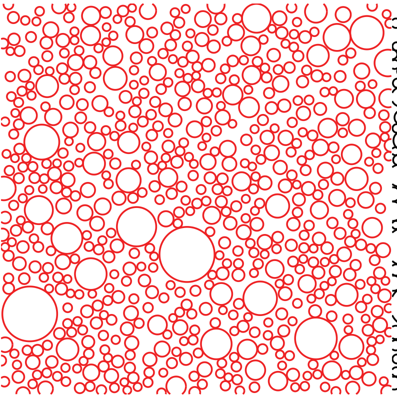
# Model Results

Crowding in random advective forcing.

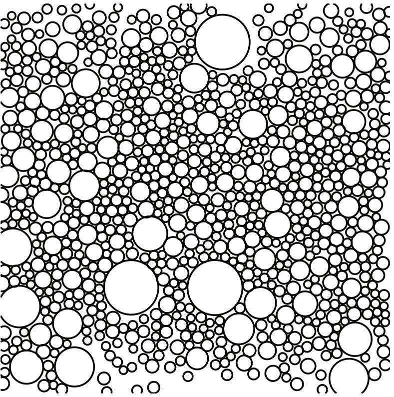




Concentration = 0.3
$k$ = 2.9

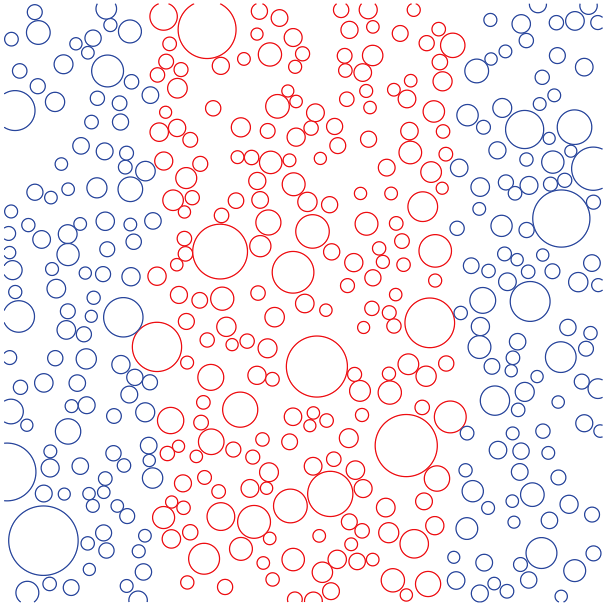


Concentration = 0.5
$k$ = 2.9



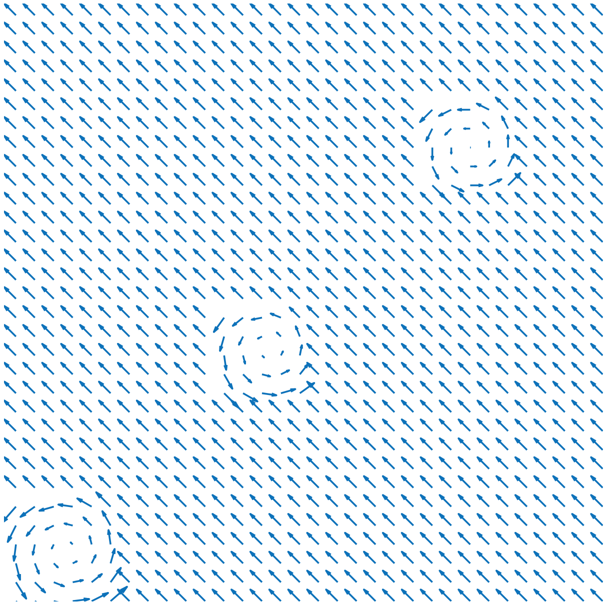
Concentration = 0.7
$k$ = 2.9

Concentration = 0.3 $k$ = 2.9



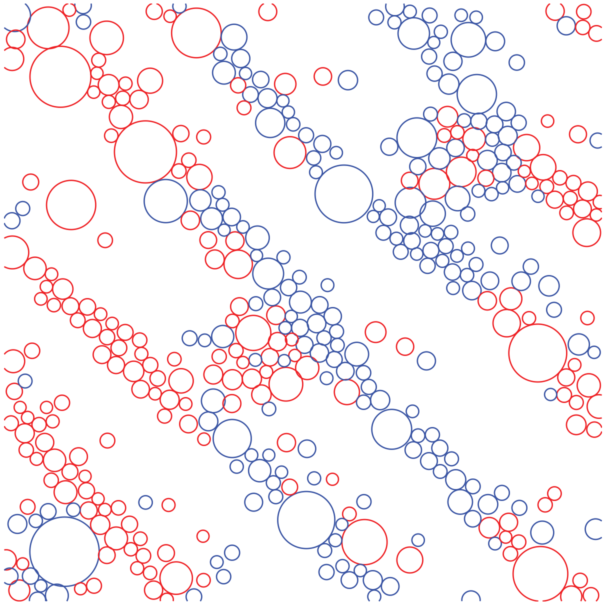
$+$

Vorticity Dominated Drift

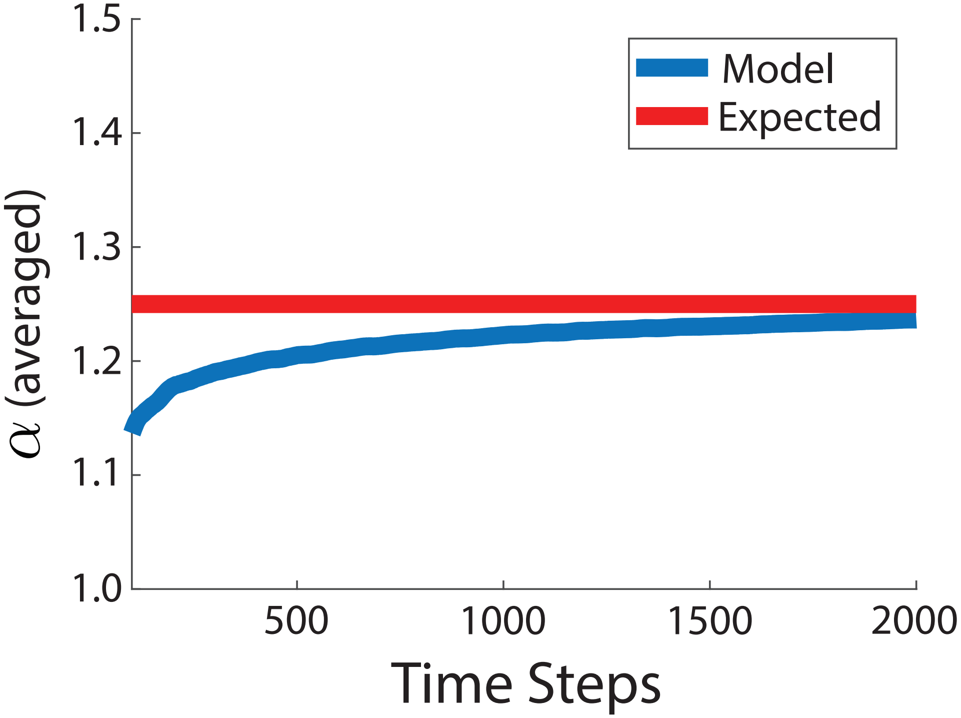


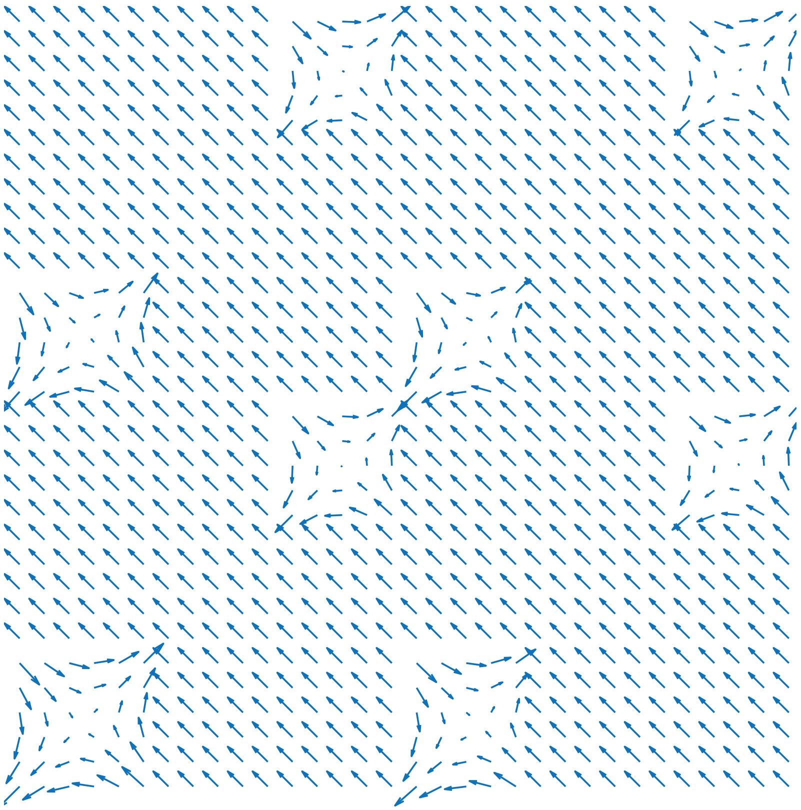
$=$

$\alpha = 5/3$



# Model Results

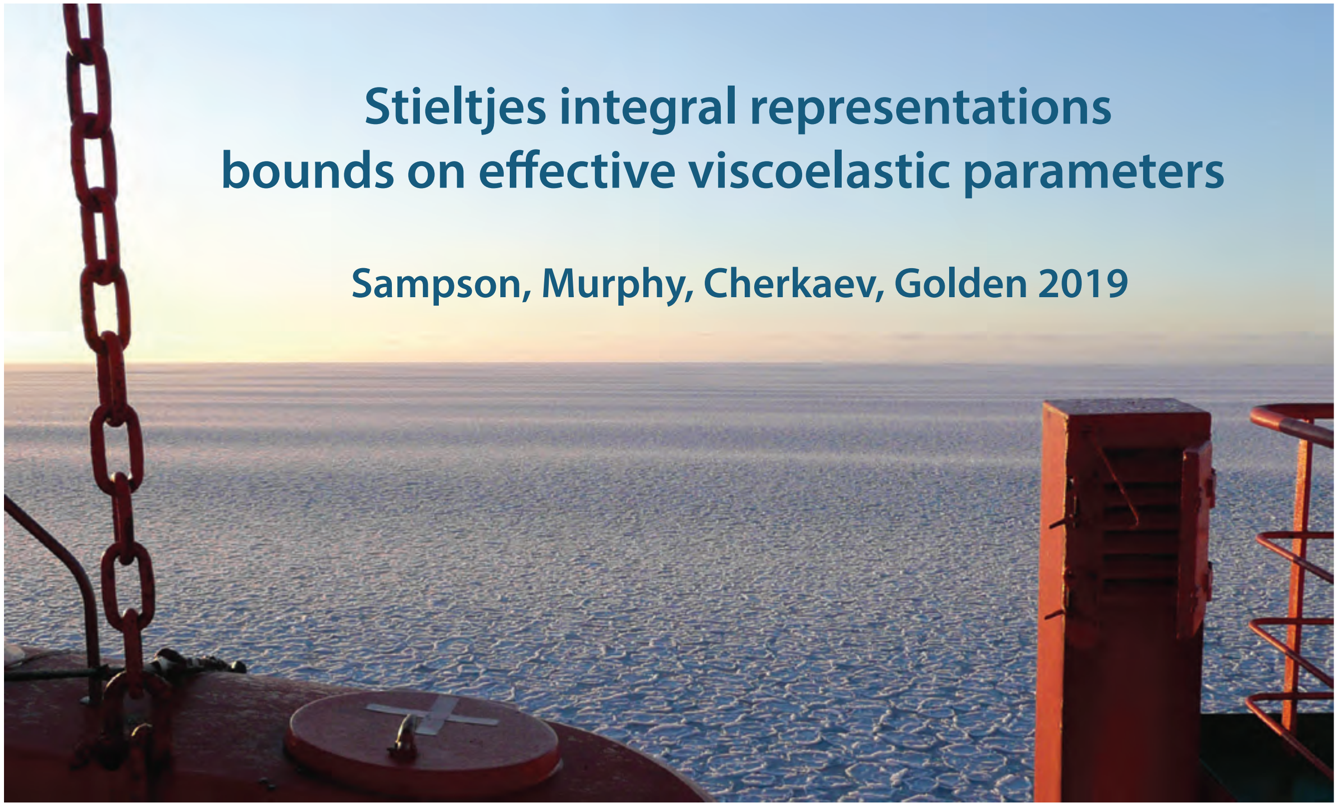

Sparse Packing, Shear Dominated Drift



Expected $\alpha = 5/4$

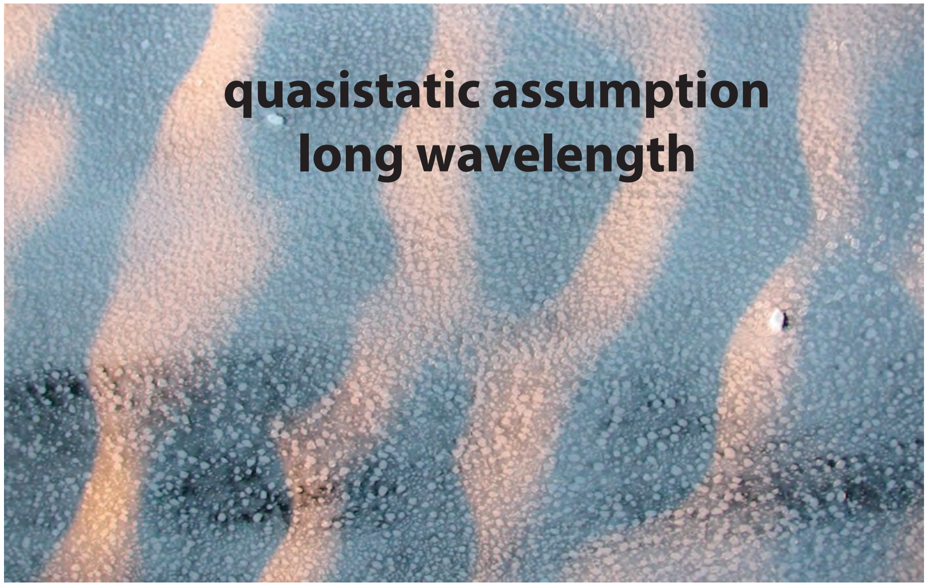
$k$ = 2.9 Concentration = 0.3

# wave propagation in the marginal ice zone



Stieltjes integral representations
bounds on effective viscoelastic parameters

Sampson, Murphy, Cherkaev, Golden 2019



**quasistatic assumption**
**long wavelength**

# *melt pond formation and albedo evolution:*

- ***major drivers in polar climate***
- ***key challenge for global climate models***

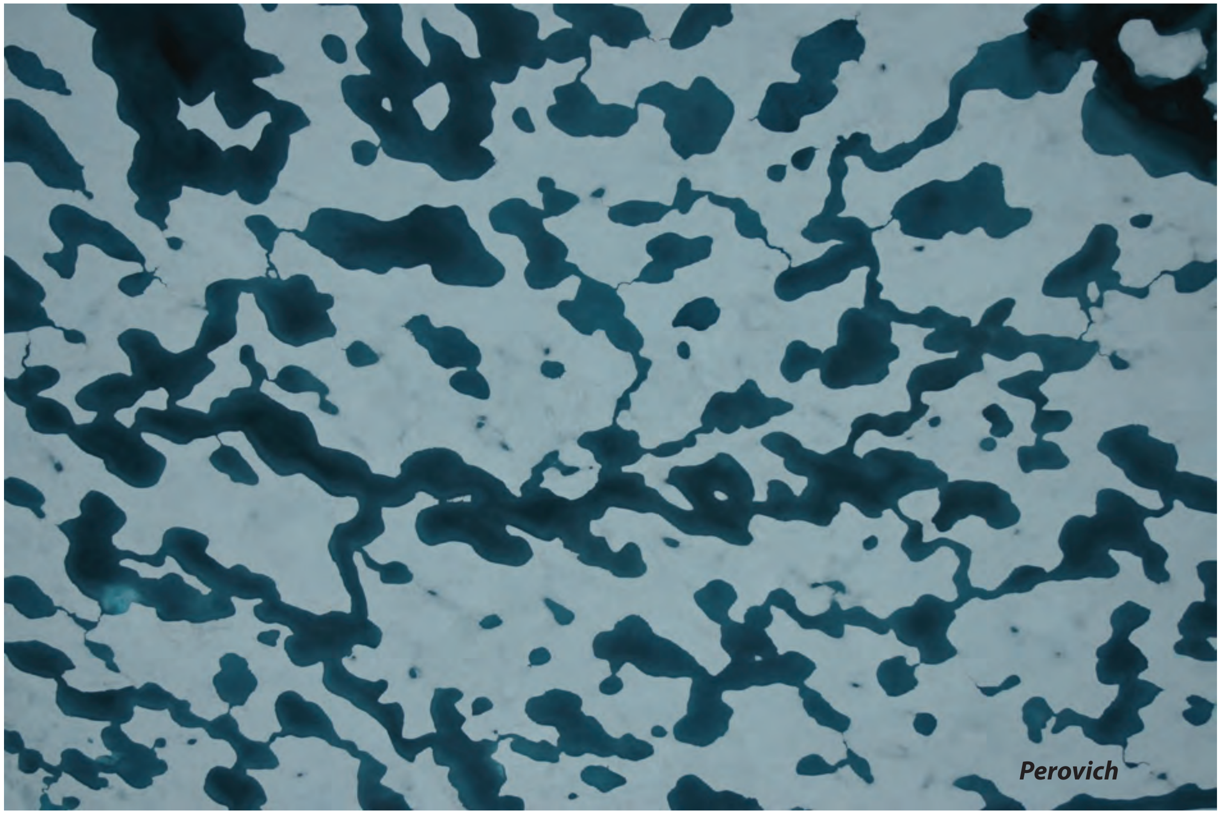
**numerical models of melt pond evolution, including topography, drainage (permeability), etc.**

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

Flocco, Feltham, Hunke 2012



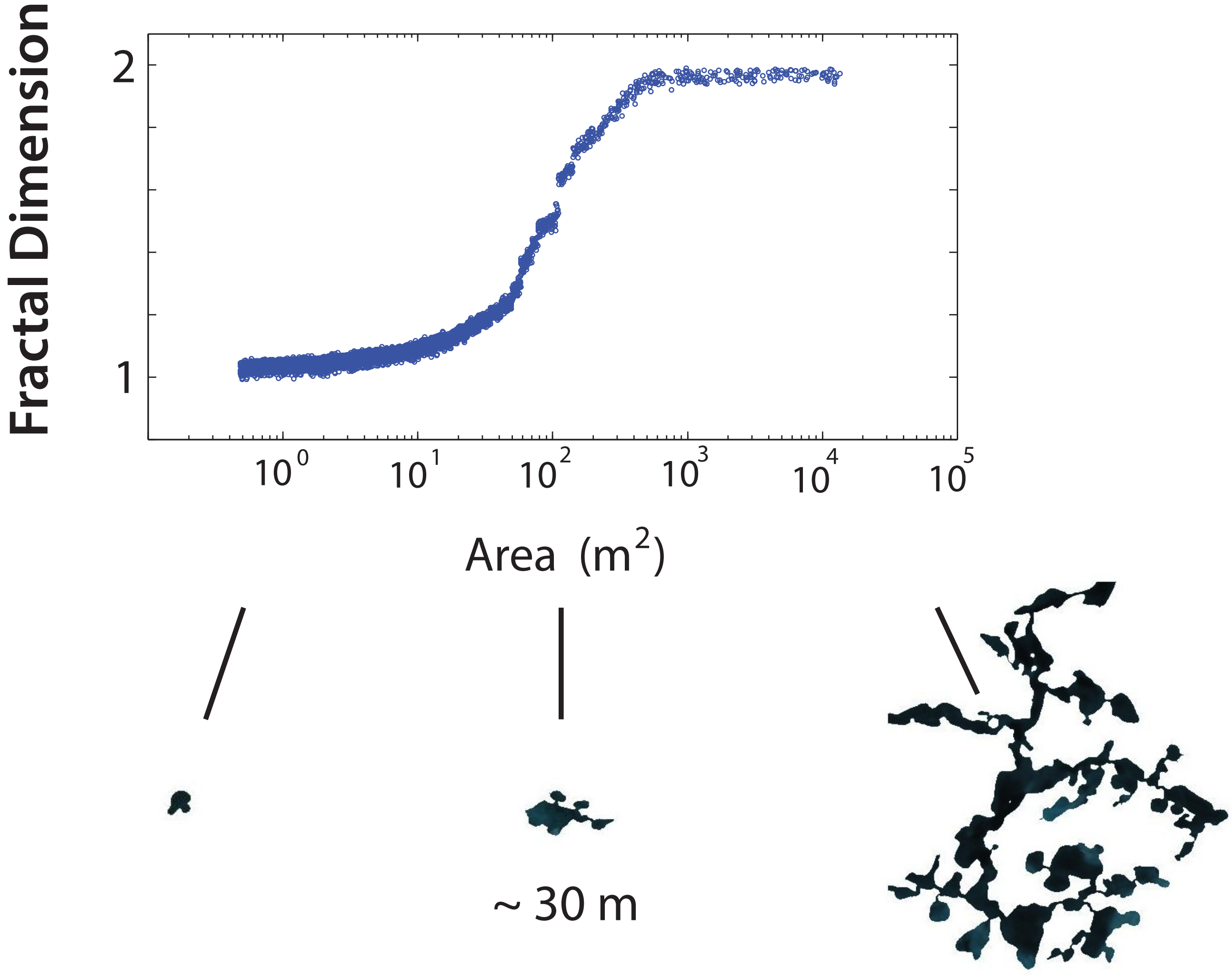
**Are there universal features of the evolution similar to phase transitions in statistical physics?**

# *Transition in the fractal geometry of Arctic melt ponds*

**Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden**

*The Cryosphere, 2012*

**complexity grows with length scale**

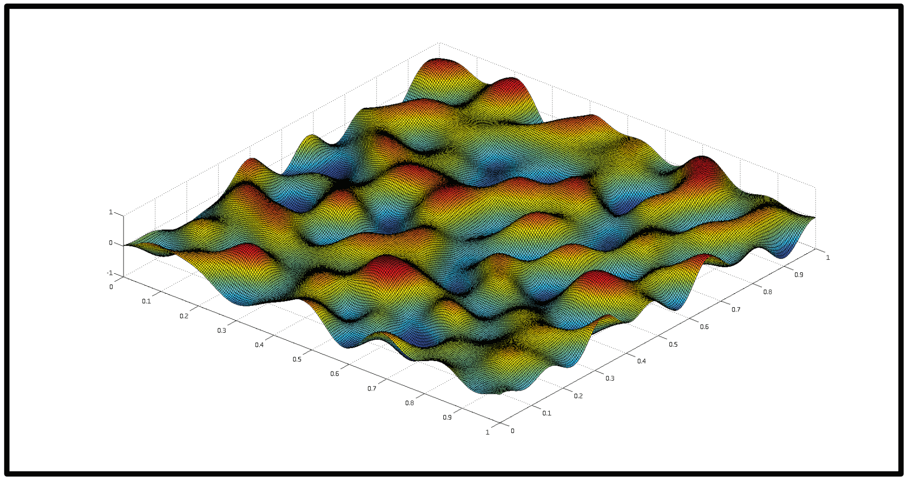


***simple pond*** ***transitional pond*** ***complex pond***

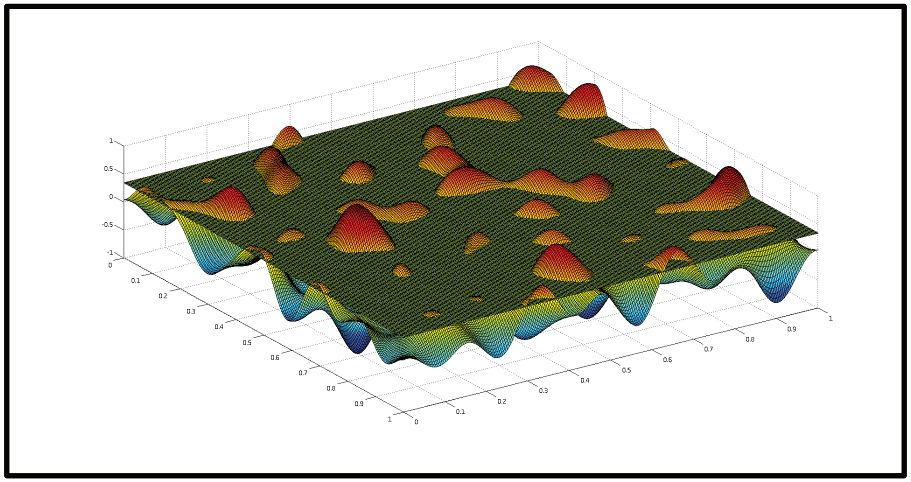
# Continuum percolation model for melt pond evolution

## *level sets of random surfaces*

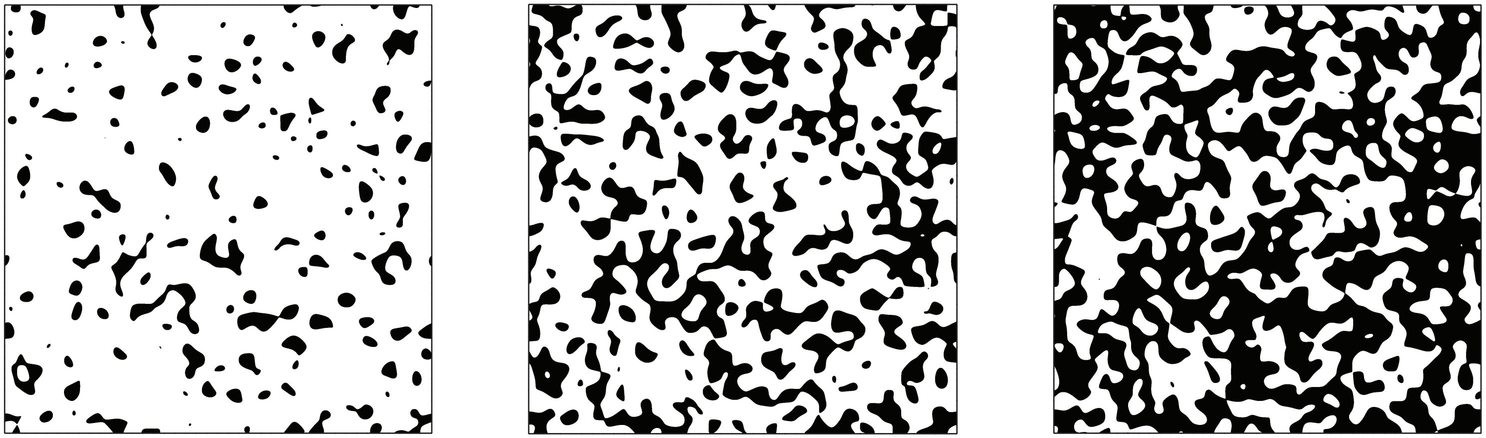
*Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018*



random Fourier series representation of surface topography

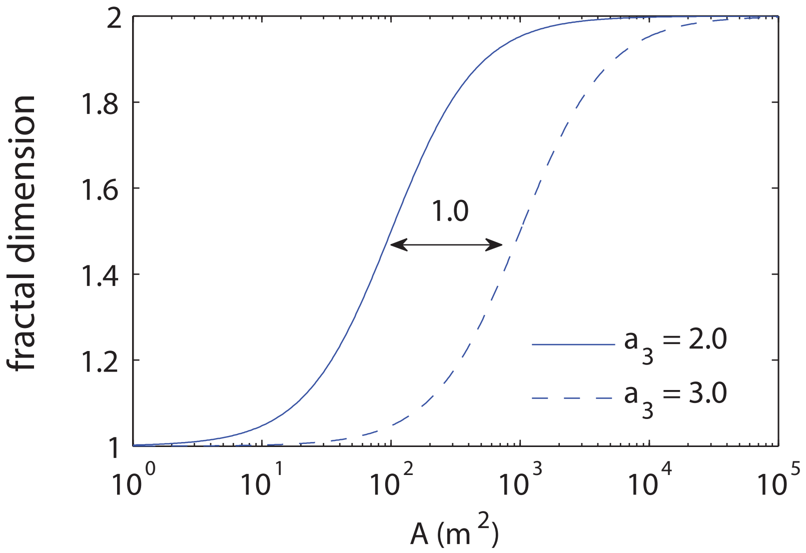


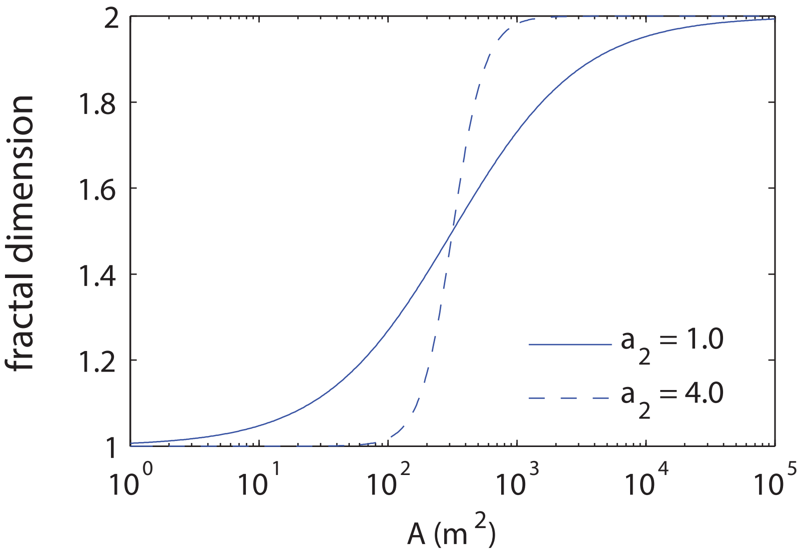
## intersections of a plane with the surface define melt ponds



*electronic transport in disordered media*  *diffusion in turbulent plasmas*  *Isichenko, Rev. Mod. Phys., 1992*

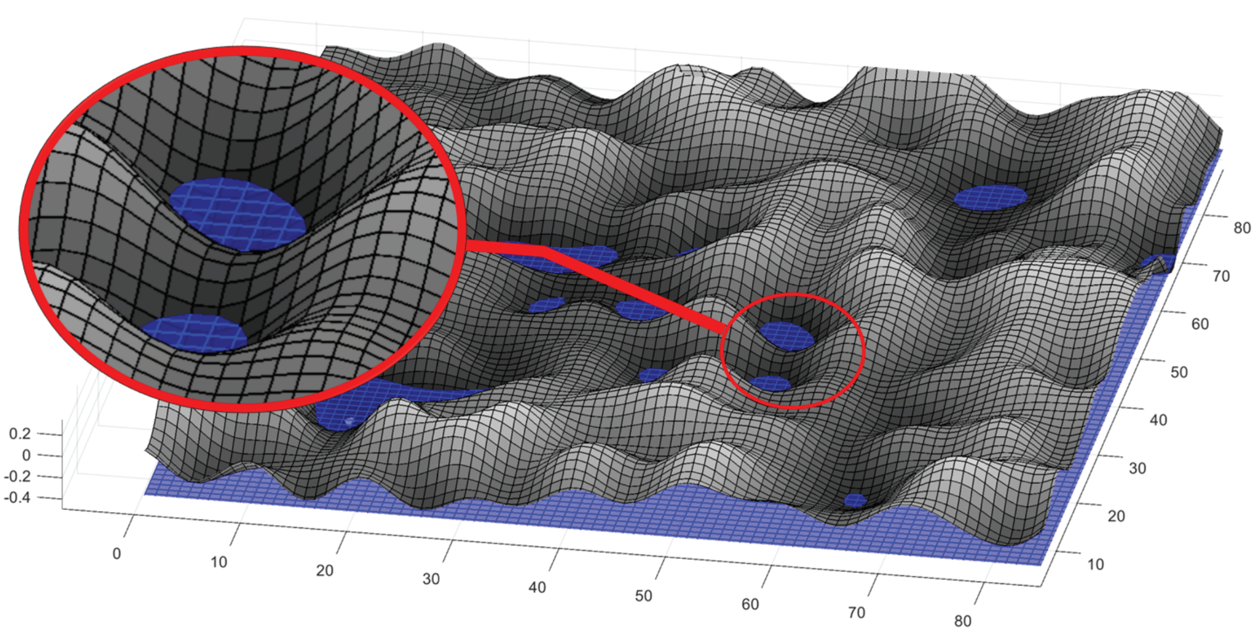
# fractal dimension curves depend on statistical parameters defining random surface

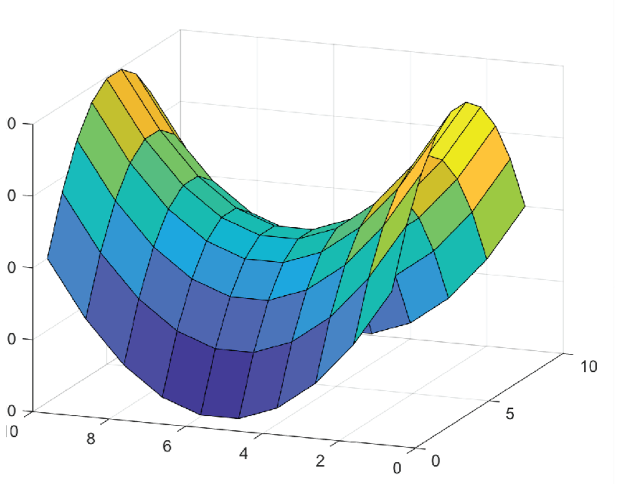




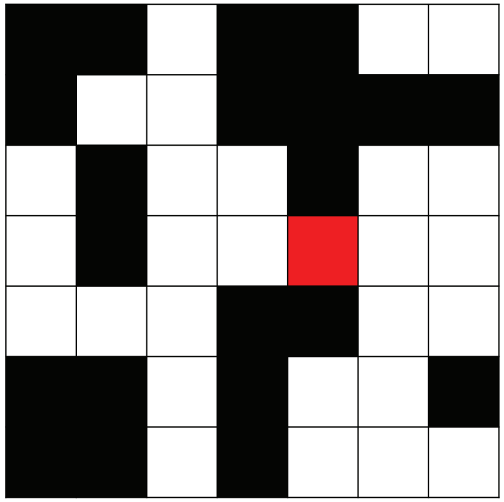
# Saddle Points: The Key to Melt Pond Evolution

## Ryleigh Moore, Jacob Jones, Dane Gollero, Court Strong, Ken Golden 2019

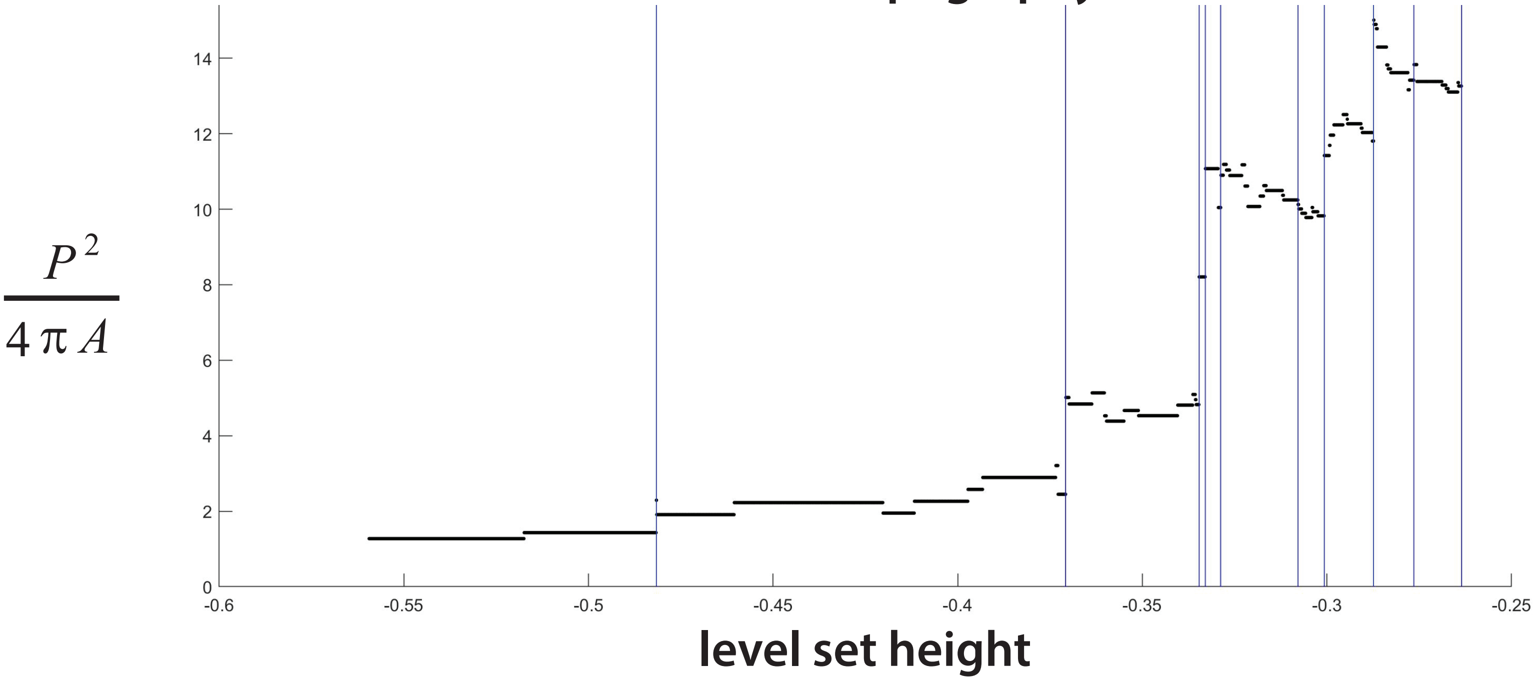




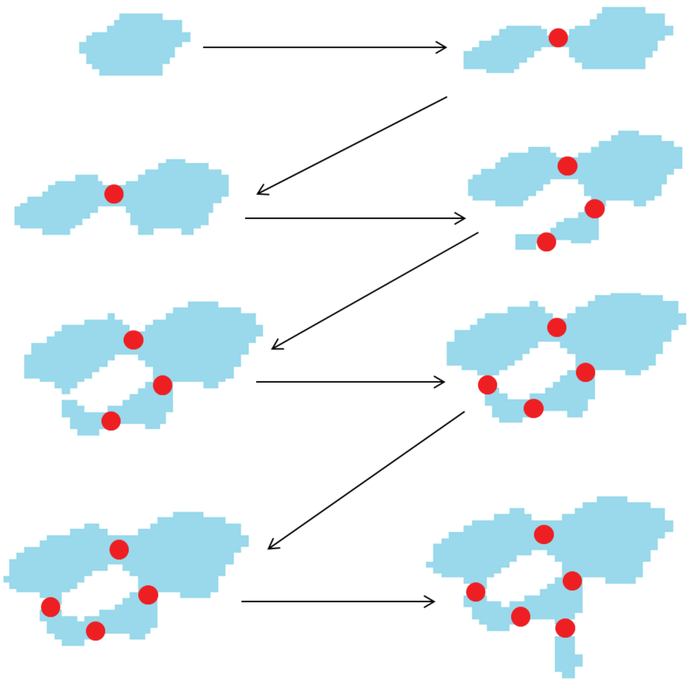
- Ponds connect through saddle points (Morse Theory).
- Red bond in lattice percolation theory ~ saddle point.



# Evolution of **Isoperimetric Quotient** with Melt Pond Growth
(from real snow topography data)



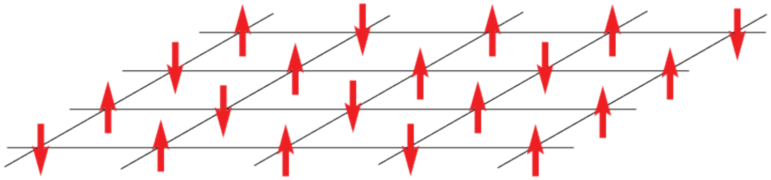
## pond coalescence and thickening



In the graph, we follow a single pond's growth. The vertical lines denote when the pond goes through a **saddle point**.

We see that large jumps in fractal dimension occur through **saddle points**.

# Ising Model for a Ferromagnet

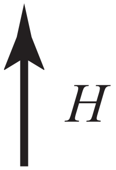


$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

**blue**

**white**

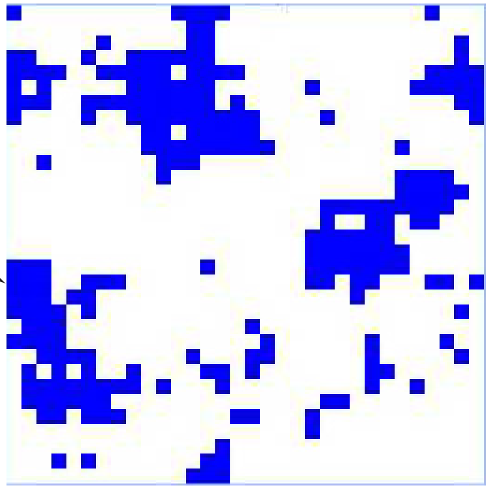
**applied magnetic field**



$$\mathcal{H} = -H \sum_{i} s_i - J \sum_{<i,j>} s_i s_j$$

**nearest neighbor Ising Hamiltonian**

ferromagnetic interaction $J \geq 0$



***islands or ponds of like spins***

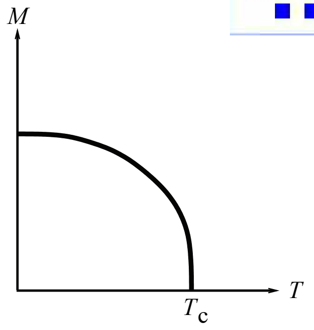
**magnetization**

$$M(T,H) = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_{j} s_j \right\rangle$$

**homogenized parameter like effective conductivity**

***Stieltjes integral representation for $M$***

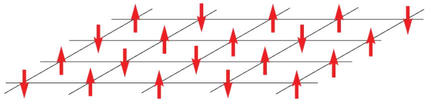
***Baker, PRL 1968***



***Curie point critical temperature***

# Ising model for ferromagnets ⟶ Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.* 2019



2–D Ising Model

$$\mathcal{H}_\omega = -J \sum_{<i,j>}^{N} s_i s_j - \sum_{i}^{N} H_i \, s_i \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

random magnetic field represents snow topography

**magnetization** $M = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$

**pond coverage** $\frac{(M+1)}{2}$

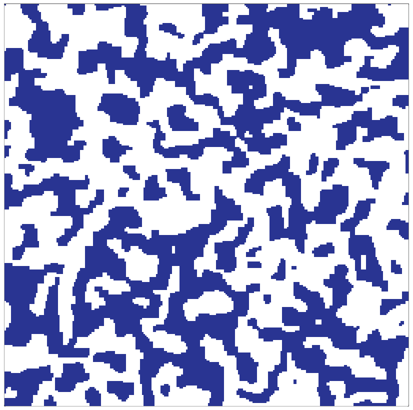
***~ albedo***

only nearest neighbor patches interact

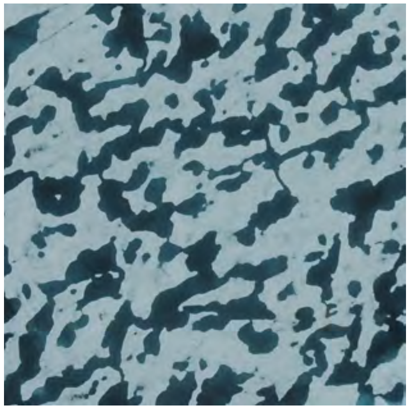
Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system “flows” toward metastable equilibria.

**Melt ponds are metastable islands of like spins.**

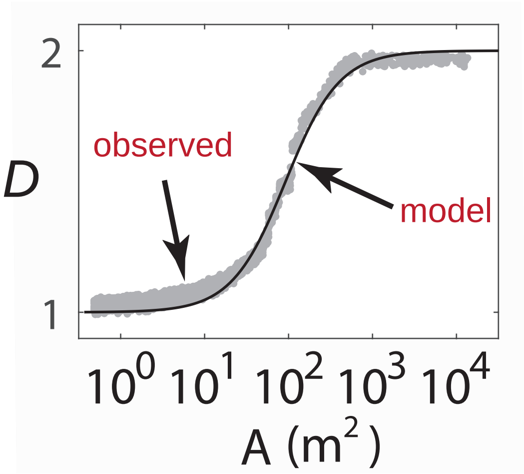
Order from Disorder



**Ising model**



**melt pond photo (Perovich)**



pond size distribution exponent

observed -1.5

**(Perovich, *et al.* 2002)**

model -1.58

**ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data**

# The distribution of solar energy under ponded first-year sea ice

Horvat, Flocco, Rees Jones, Roach, Golden, ***in revision,*** 2019

- Model for 3D light field under ponded sea ice.
- Distribution of solar energy at depth influenced by ***shape and connectivity*** of melt ponds, as well as area fraction.
- Aggregate properties of the sub-ice light field, such as a significant enhancement of available solar energy under the ice, are controlled by parameter closely related to pond fractal geometry.
- Model and analysis explain how melt pond geometry ***homogenizes*** under-ice light field, affecting habitability.

**Pond geometry affects the ecology of the Arctic Ocean.**

# The Melt Pond Conundrum:

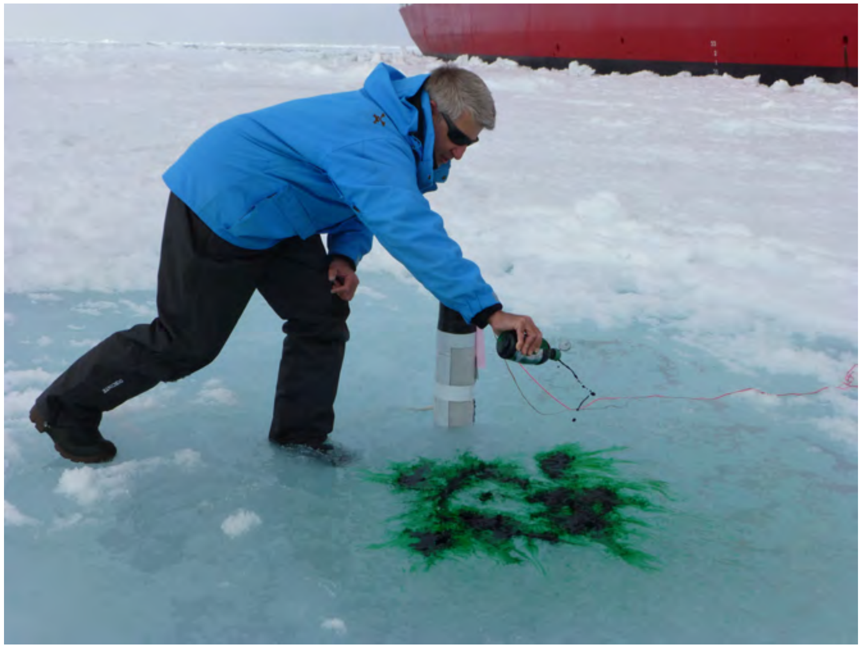
## *How can ponds form on top of sea ice that is highly permeable?*

C. Polashenski, K. M. Golden, D. K. Perovich, E. Skyllingstad, A. Arnsten, C. Stwertka, N. Wright

**Percolation Blockage: A Process that Enables Melt Pond Formation on First Year Arctic Sea Ice**

*J. Geophys. Res. Oceans* 2017

*2014 Study of Under Ice Blooms in the Chuckchi Ecosystem (SUBICE)*
*aboard USCGC Healy*



# Conclusions

1. Sea ice is a fascinating multiscale composite with structure similar to many other natural and man-made materials.

2. Mathematical methods developed for sea ice advance the theory of composites in general.

2. Homogenization and statistical physics help ***link scales in sea ice and composites***; provide rigorous methods for finding effective behavior; advance sea ice representations in climate models.

3. Fluid flow through sea ice mediates melt pond evolution and many processes important to climate change and polar ecosystems.

5. Field experiments are essential to developing relevant mathematics.

6. Our research will help to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

# THANK YOU

## Office of Naval Research

Applied and Computational Analysis Program

Arctic and Global Prediction Program

## National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

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Australian Government
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Australian Antarctic Division

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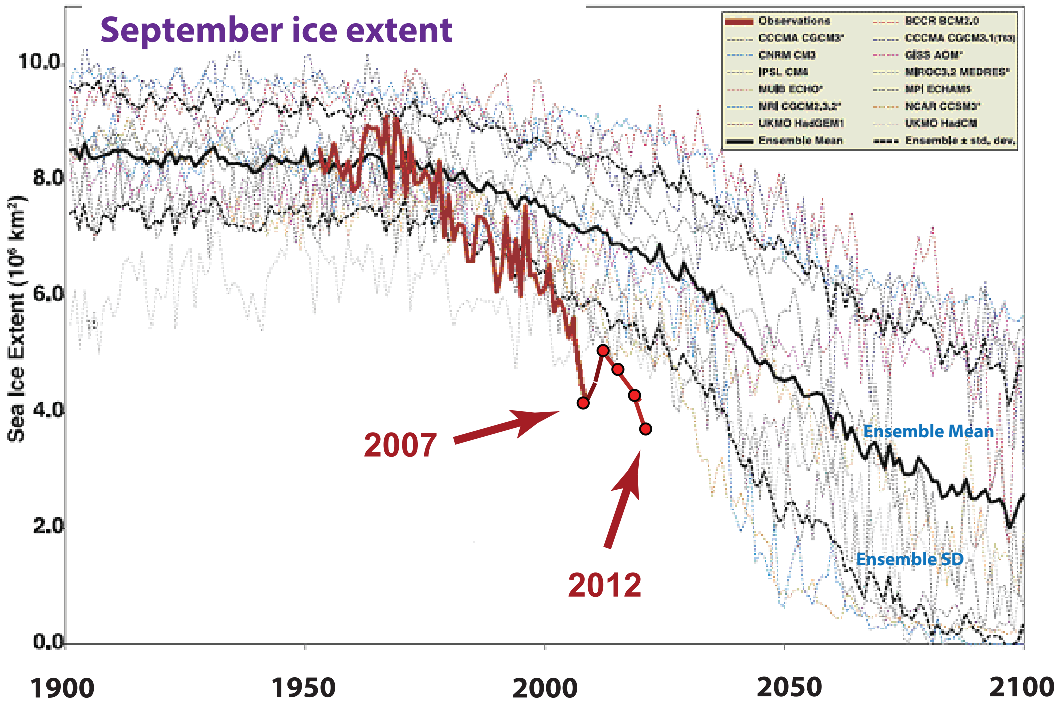
UAF
UNIVERSITY OF ALASKA FAIRBANKS

THE UNIVERSITY OF UTAH

***Buchanan Bay, Antarctica Mertz Glacier Polynya Experiment July 1999***

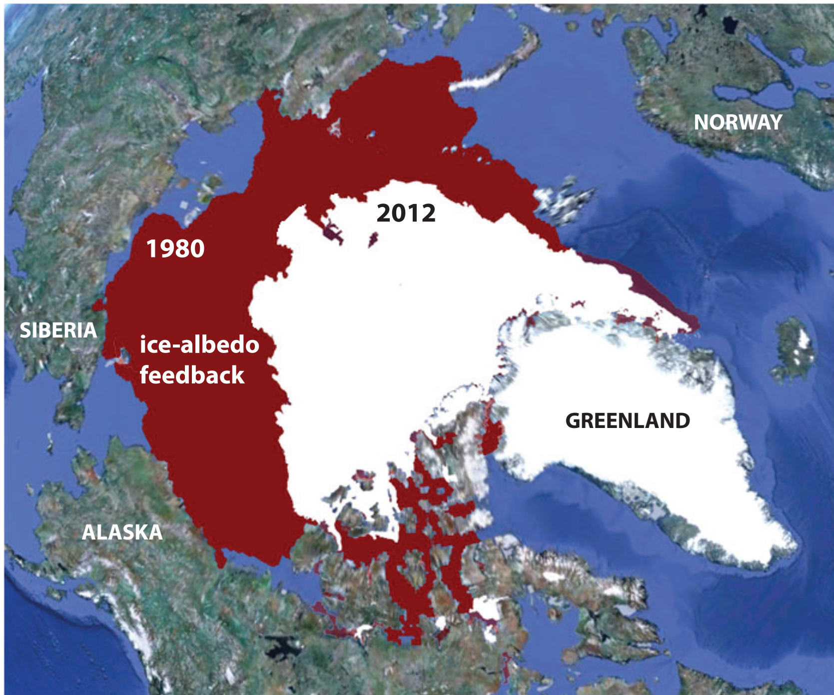
# *Arctic sea ice decline: faster than predicted by climate models*



*Stroeve et al., GRL, 2007*
*Stroeve et al., GRL, 2012*

## Change in Arctic Sea Ice Extent

**September 1980 -- 7.8 million square kilometers**
**September 2012 -- 3.4 million square kilometers**



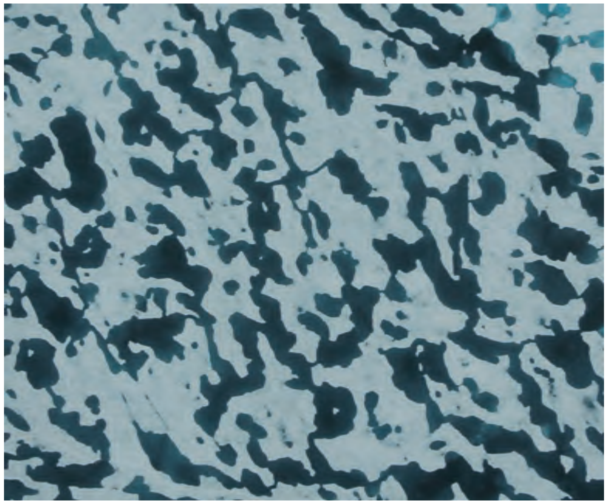
# challenge

**represent sea ice more realistically in climate models**

*account for key processes*

***such as melt pond evolution***

***How do patterns of dark and light evolve?***



Impact of melt ponds on Arctic sea ice simulations from 1990 to 2007

Flocco, Schroeder, Feltham, Hunke, JGR Oceans 2012

**For simulations with ponds September ice volume is nearly 40% lower.**

**... and other sub-grid scale structures and processes**

***linkage of scales***