Advection Diffusion in the Polar Sea-Ice Cover

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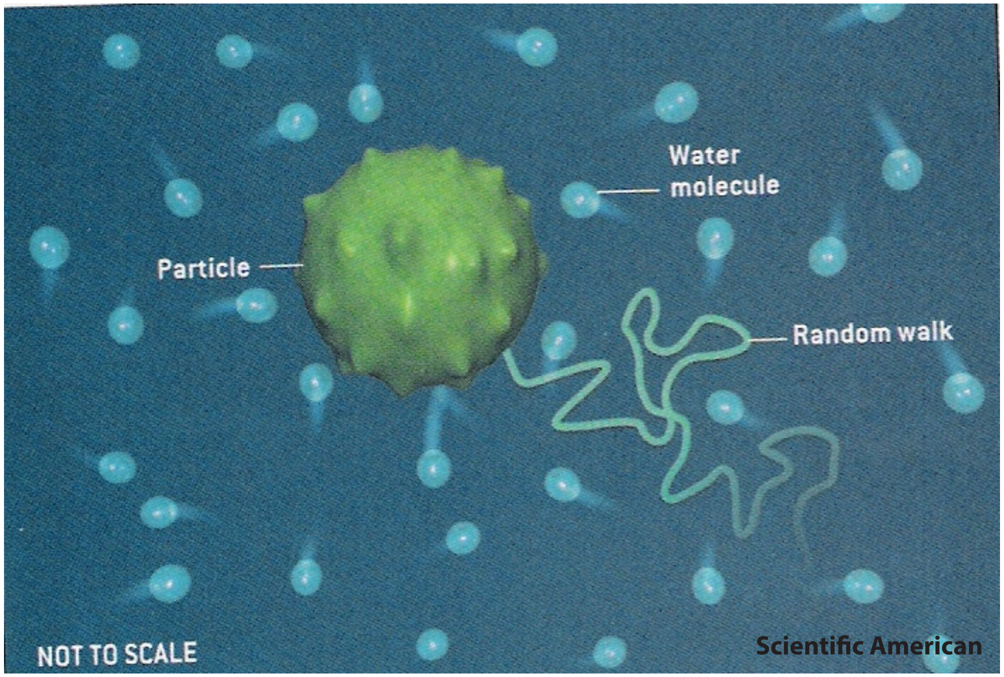
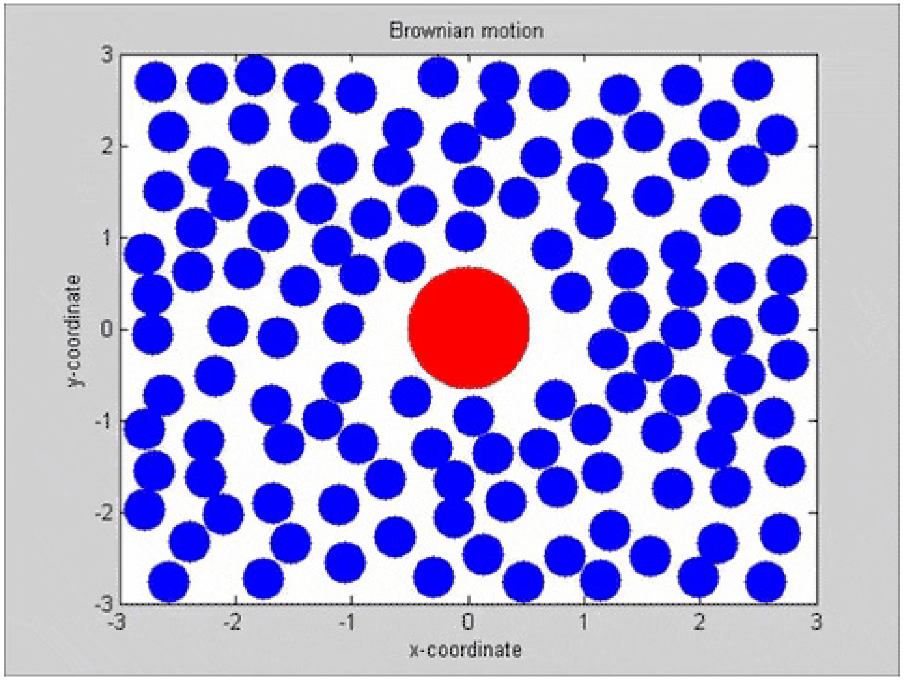
IGS, Winnipeg   23 August 2019

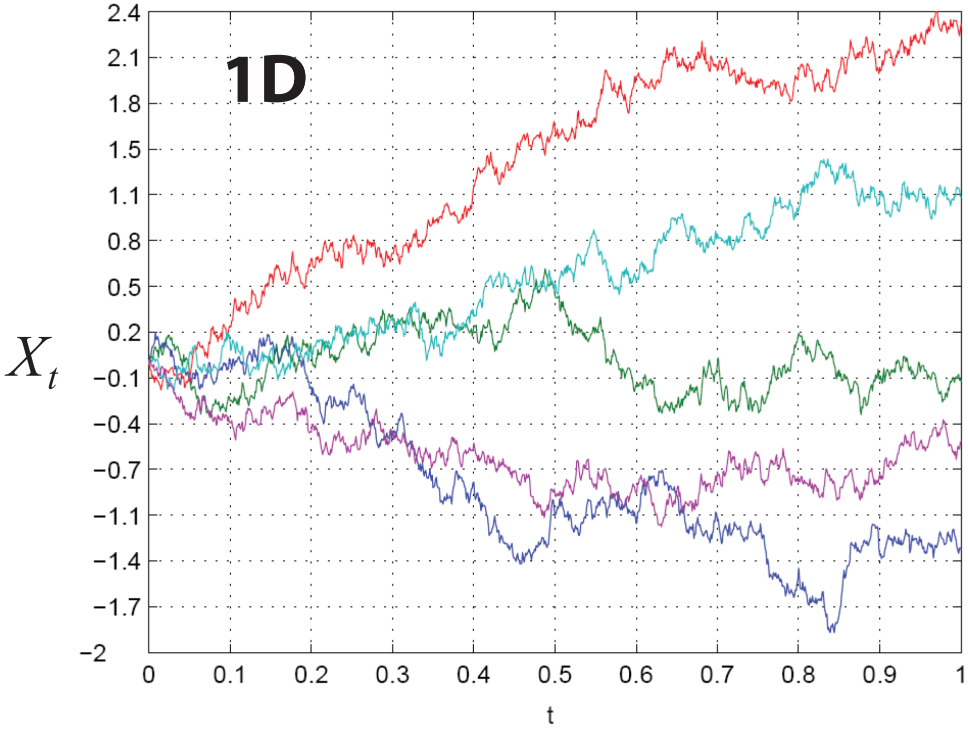
Brownian motion and diffusion

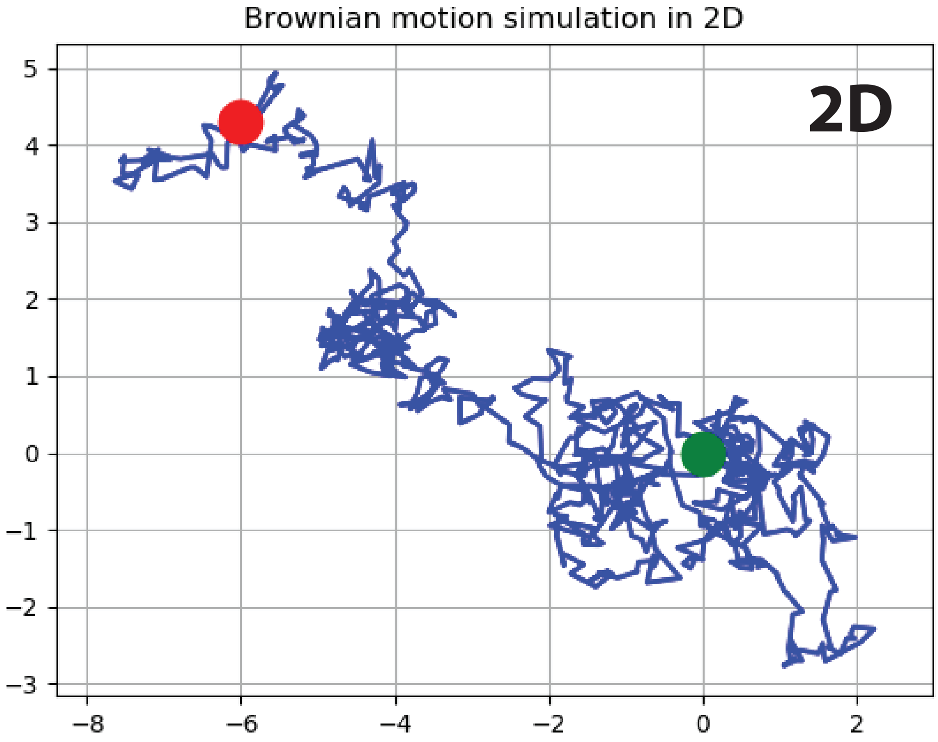


Scientific American



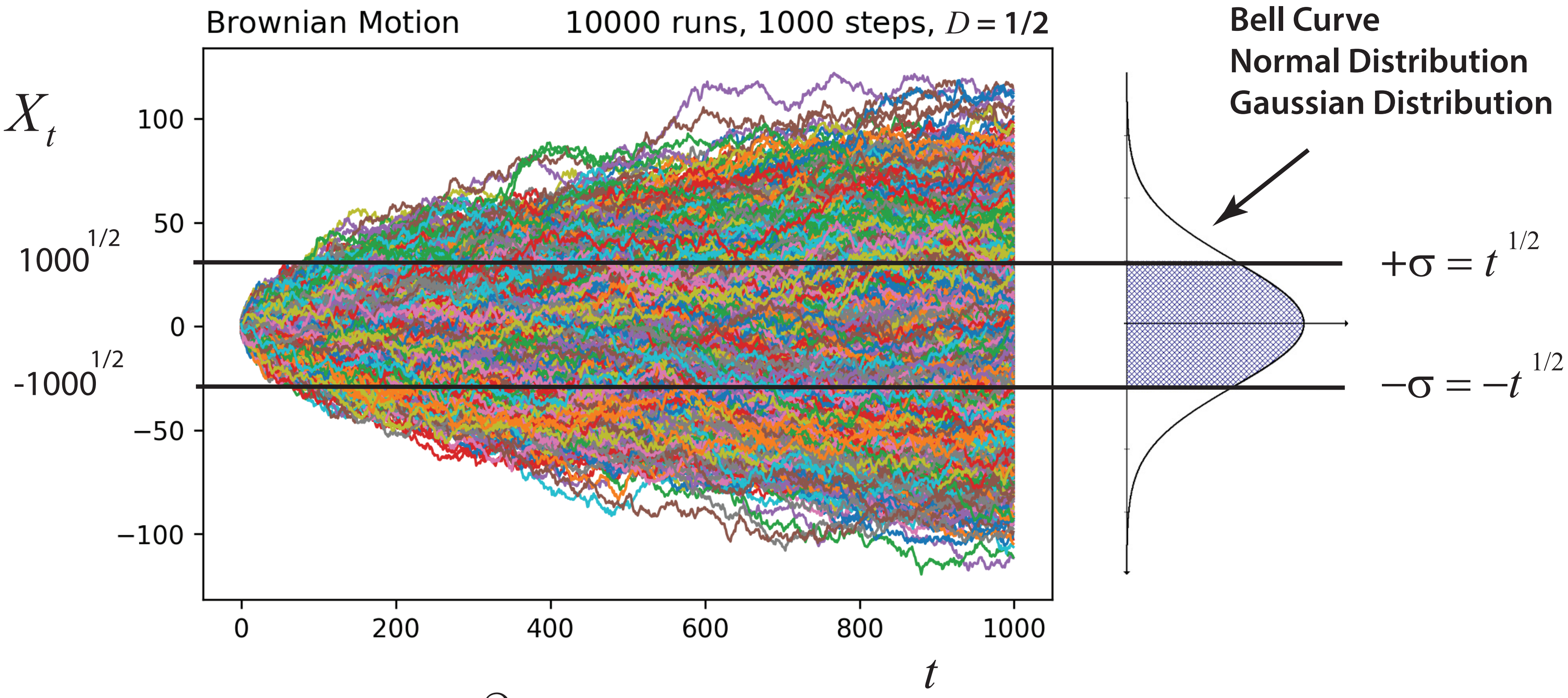
Einstein: $\langle X^2 \rangle \sim (\ ?\ )\, t$





FSU

Brownian motion and the diffusion equation



Bell Curve
Normal Distribution
Gaussian Distribution

diffusion equation

$$\frac{\partial u}{\partial t} = D\,\nabla^2 u$$

$u(x,0) = \delta_0(x)$

$$u(x,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-x^2/4Dt}$$

variance $\sigma^2 = 2Dt$

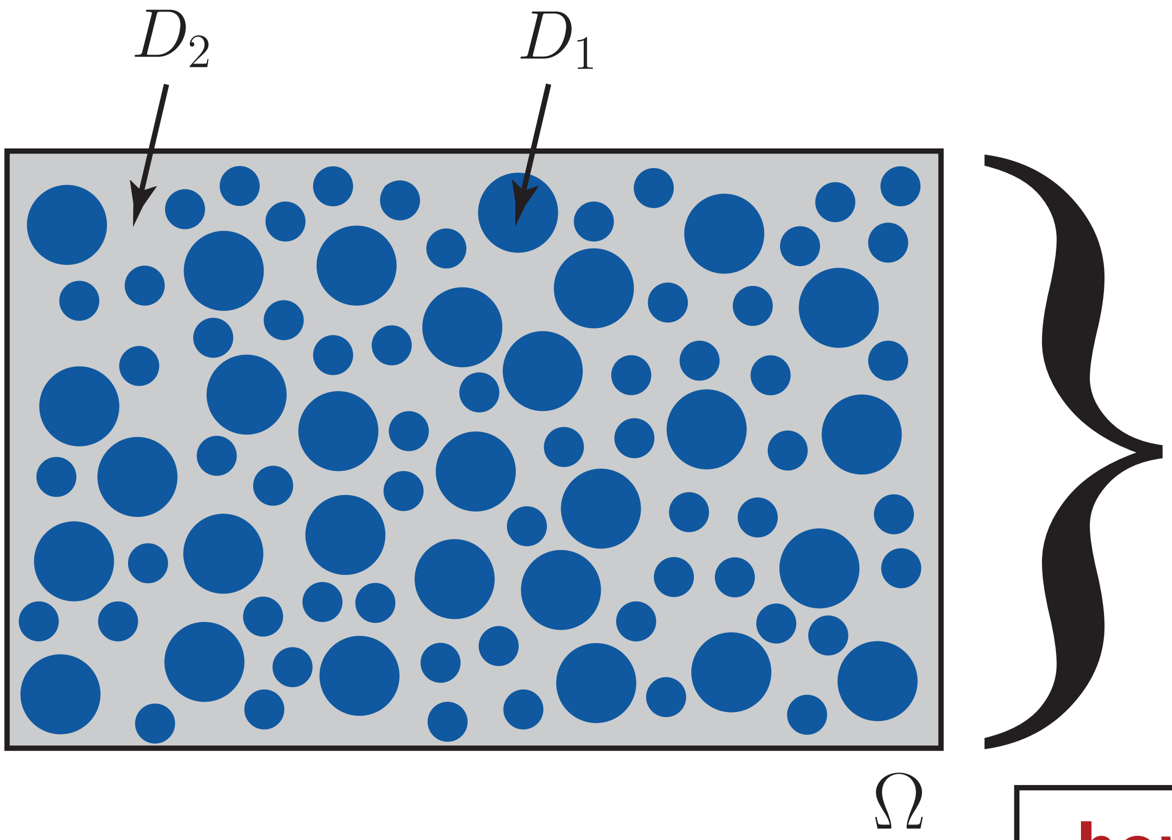
$D = \frac{1}{2}$

mean squared displacement

$$\boxed{\langle X_t^2 \rangle = 2Dt = t^1}$$

$$\boxed{\langle |X_t| \rangle = t^{1/2}}$$

Homogenization for diffusion in two phase media



D^* **effective diffusivity**

local diffusivity $D(x) = \begin{cases} D_1 & x \in \Omega_1 \\ D_2 & x \in \Omega_2 \end{cases}$

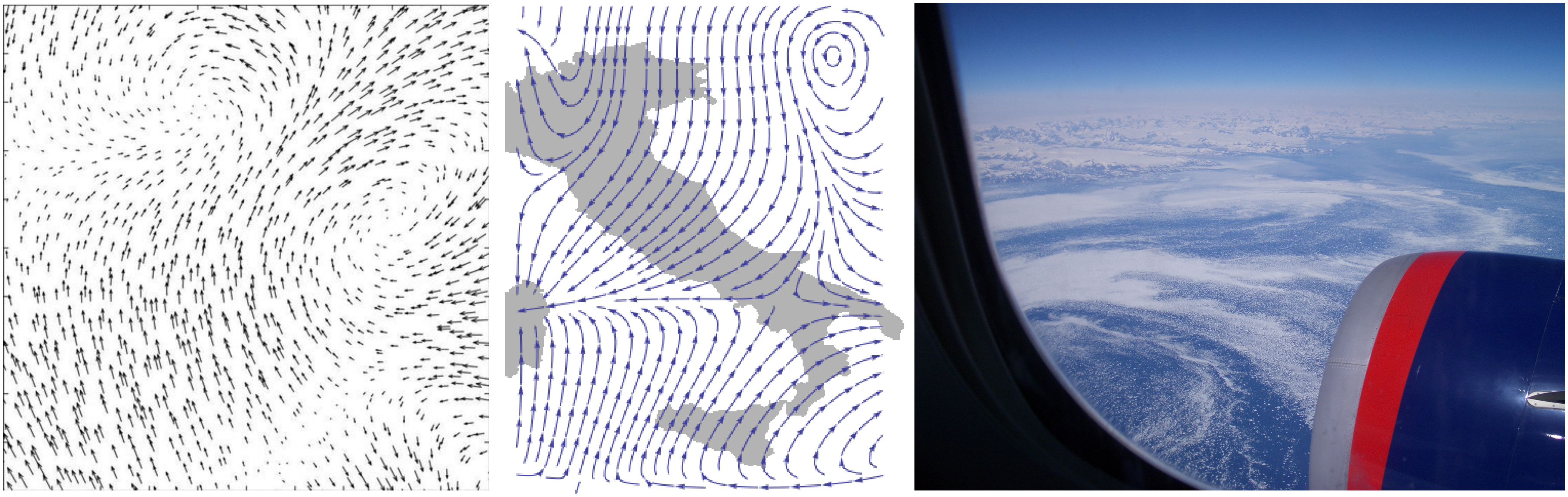
$$\frac{\partial u}{\partial t} = \nabla \cdot (D(x) \nabla u)$$

homogenized parameter captures effective behavior

$$\langle X_t^2 \rangle \sim 2D^* t$$

$$t \longrightarrow \infty$$

Homogenization for advection diffusion



$$\frac{\partial u}{\partial t} = D\,\nabla^2 u - \mathrm{v}\cdot\nabla u \qquad \nabla\cdot\mathrm{v} = 0$$

$$\Big\downarrow$$

homogenize

$$\langle X_t^2\rangle \;\sim\; 2D^*\,t$$

$$\frac{\partial\,\overline{u}}{\partial t} = D^*\,\nabla^2\,\overline{u}$$

$$t \longrightarrow \infty$$

What is this talk about?

Effective behavior of advection diffusion processes
in the sea ice system - HOMOGENIZATION.

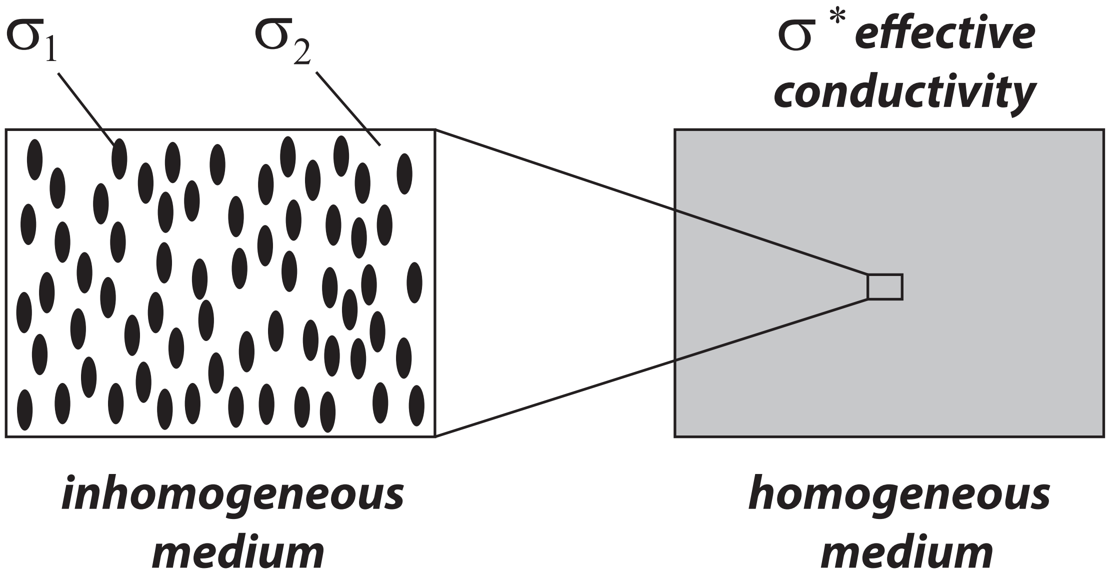
diffusion of ice floes under advective forcing
convection enhanced thermal transport in sea ice

1. Introduce rigorous homogenization framework via E&M

2. Integral representation for effective diffusivity; bounds

3. Numerical model for anomalous diffusion of ice floes

HOMOGENIZATION - Linking Scales in Composites



find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

Wiener 1912 : arithmetic and harmonic mean bounds on effective conductivity
Hashin and Shtrikman 1962 : variational bounds on effective conductivity

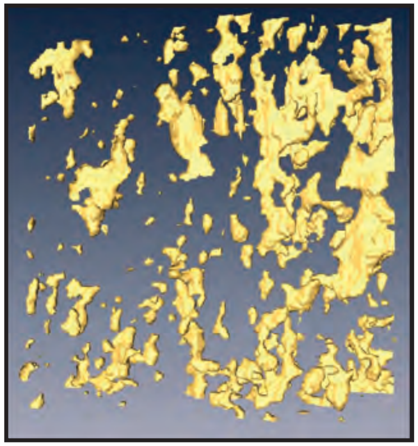
widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

Sea Ice is a Multiscale Composite Material

sea ice microstructure

brine inclusions

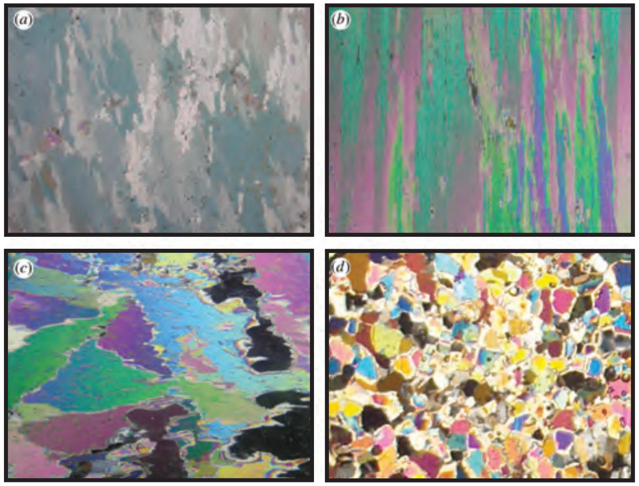
Weeks & Assur 1969



H. Eicken

Golden *et al. GRL* 2007

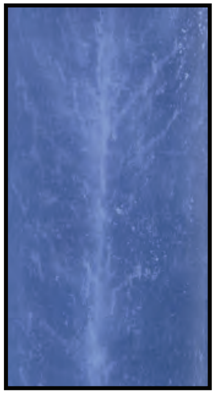
polycrystals



Gully *et al. Proc. Roy. Soc. A* 2015

brine channels

D. Cole



K. Golden

millimeters

centimeters

---

sea ice mesostructure

sea ice macrostructure

Arctic melt ponds

K. Frey

Antarctic pressure ridges

K. Golden

sea ice floes

J. Weller

sea ice pack

NASA

meters

kilometers

Remote sensing of sea ice

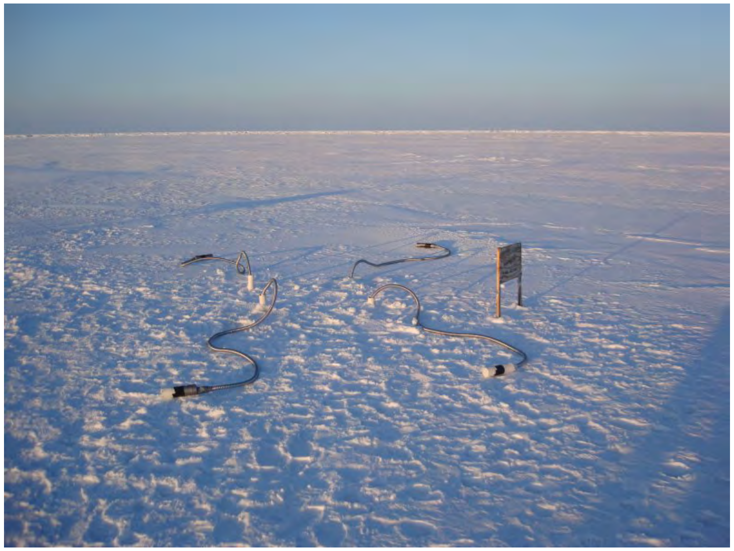
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

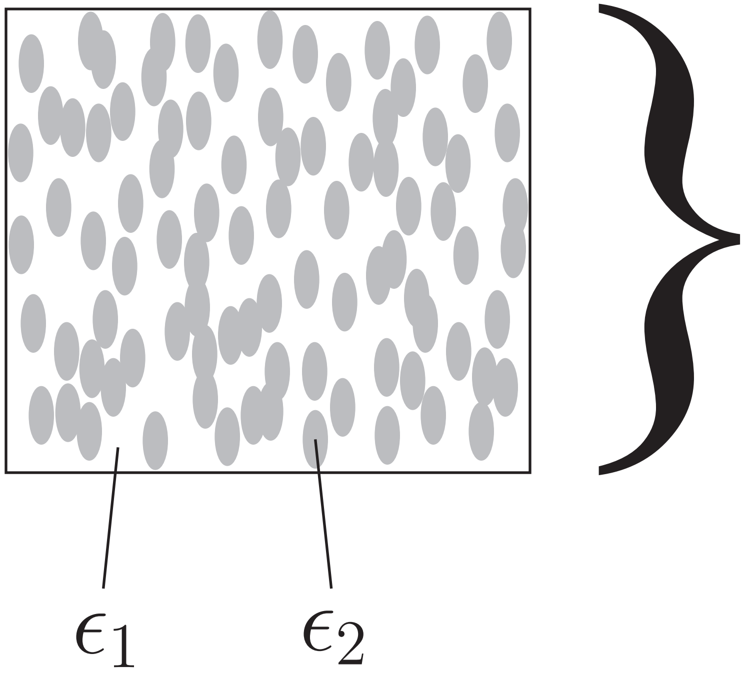
ε^*

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$\epsilon^*$$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \text{ , composite geometry } \right)$$

What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?

Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

separates geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

material parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

$$E = s\,(s + \Gamma\chi)^{-1} e_k$$

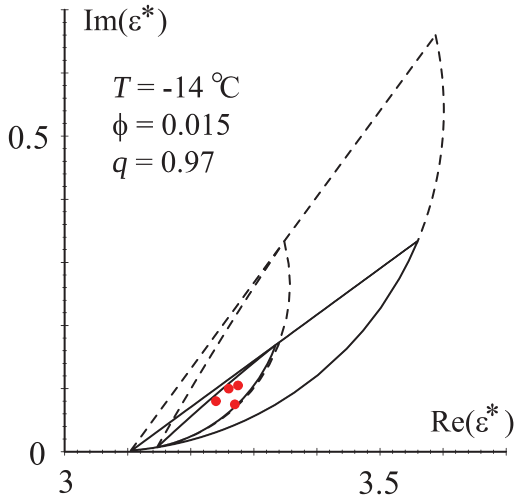
$\Gamma\chi$: microscale $\longrightarrow$ macroscale

$\Gamma\chi$ *links scales*

Golden and Papanicolaou, Comm. Math. Phys. 1983

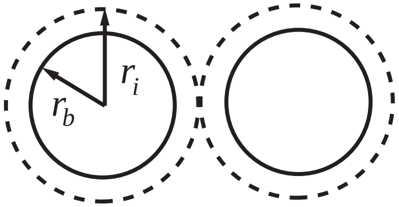
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle



$$q = r_b / r_i$$

$$0 < q < 1$$

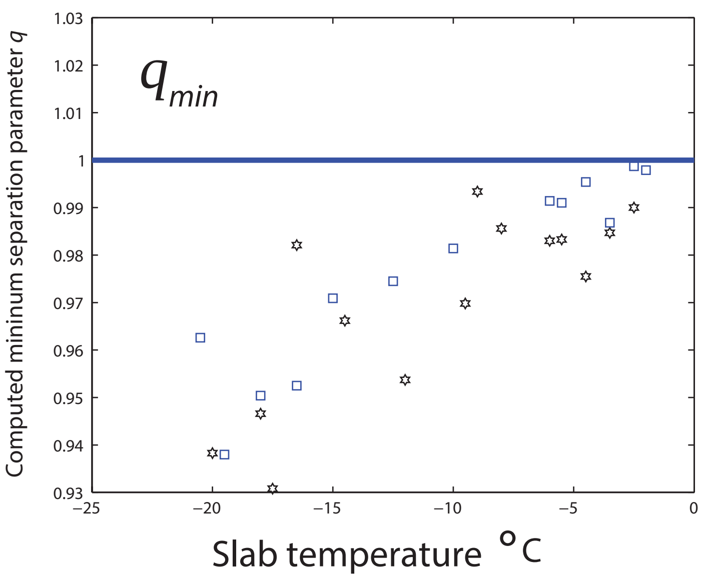
Golden 1995, 1997

Bruno 1991

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inverse bounds



inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

rigorous inverse bound
on spectral gap

construct algebraic curves which bound
admissible region in (p,q)-space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

direct calculation of spectral measures

Murphy, Hohenegger, Cherkaev, Golden, *Comm. Math. Sci.* 2015

- depends only on the composite geometry
- discretization of microstructural image gives binary network
- fundamental operator becomes a random matrix
- spectral measure computed from eigenvalues and eigenvectors

once we have the spectral measure μ it can be used in Stieltjes integrals for other transport coefficients:

electrical and thermal conductivity, complex permittivity, magnetic permeability, diffusion, fluid flow properties

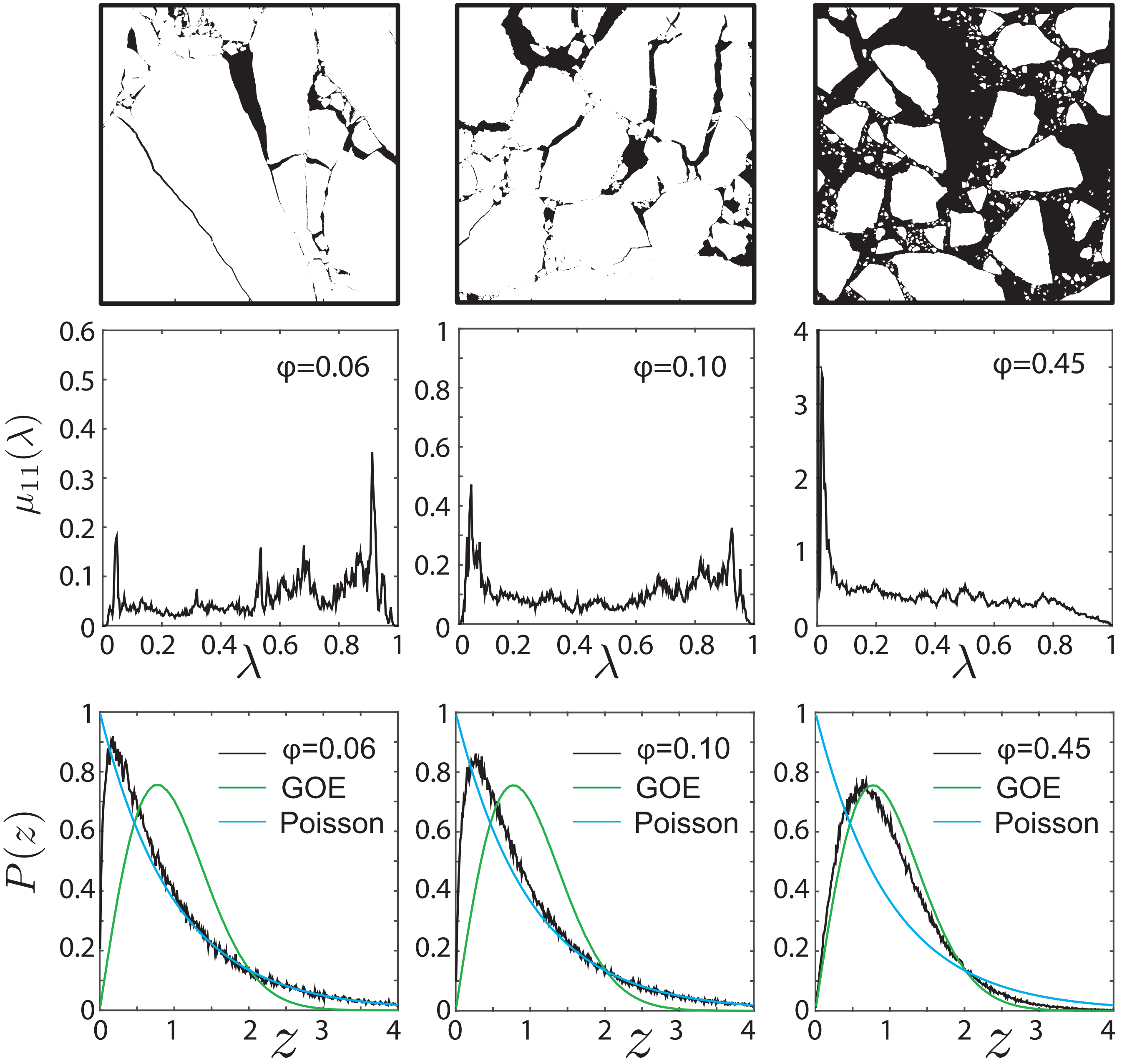
earlier studies of spectral measures

Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Spectral computations for sea ice floe configurations

spectral measures

eigenvalue spacing distributions



UNIVERSAL Wigner-Dyson distribution

uncorrelated → **level repulsion**

ANDERSON TRANSITION

Murphy, Cherkaev, Golden
Phys. Rev. Lett. 2017

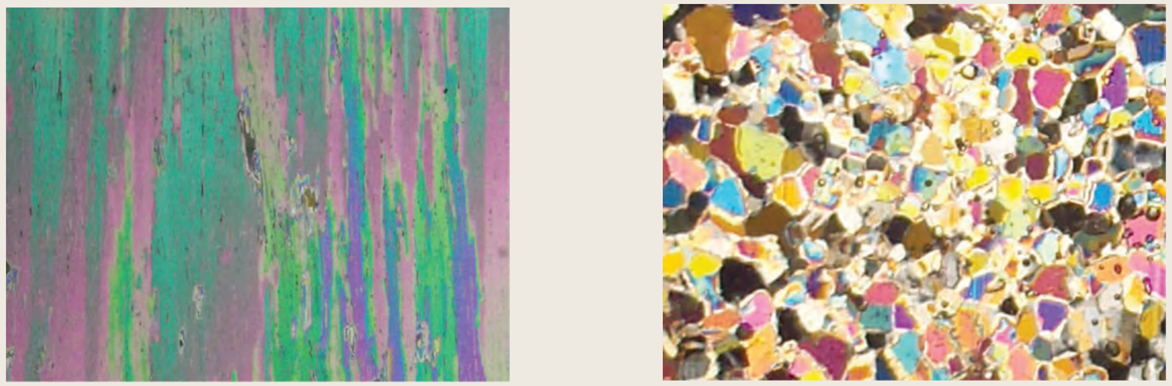
Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**

  orientation statistics
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A   8 Feb 2015

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS
OF SCIENTIFIC
PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

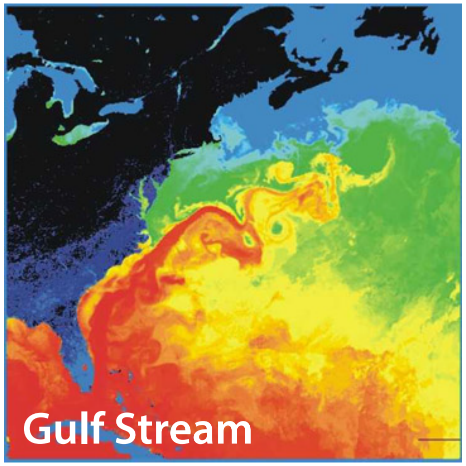
THE
ROYAL
SOCIETY
PUBLISHING

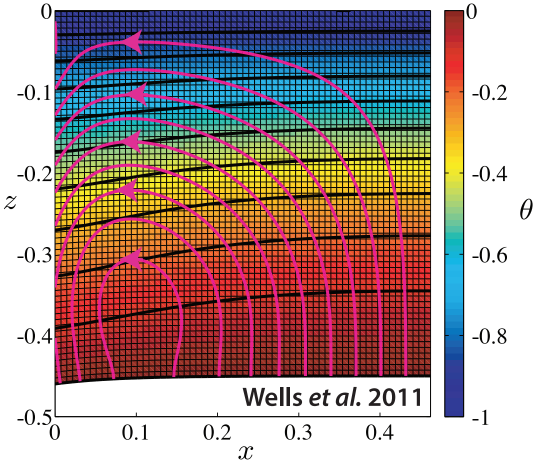
21/01/15 1:47 PM

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere





advection diffusion equation with a velocity field v

$$\frac{\partial u}{\partial t} = \kappa \, \nabla^2 u - \mathrm{v} \cdot \nabla u$$

$$\nabla \cdot \mathrm{v} = 0$$

↓ ***homogenize***

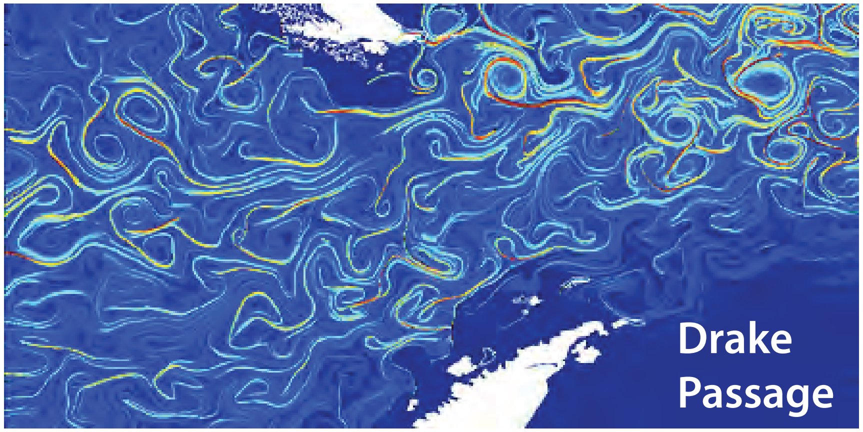
$$\frac{\partial \, \overline{u}}{\partial t} = \kappa^* \, \nabla^2 \, \overline{u}$$

κ^* ***effective diffusivity***

Stieltjes integral for κ^* ***with spectral measure***

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019



Stieltjes integral for κ^* with spectral measure

composites

Golden and Papanicolaou, CMP 1983

$$\frac{\epsilon^*}{\epsilon_2} = 1 - \int_0^1 \frac{d\mu(\lambda)}{s - \lambda}$$

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

advection diffusion

Avellaneda and Majda, PRL 89, CMP 91

$$\frac{\kappa^*}{\kappa} = 1 - \int_0^\infty \frac{d\rho(z)}{t - z}$$

$t = -1/\xi^2$, ξ = Péclet number

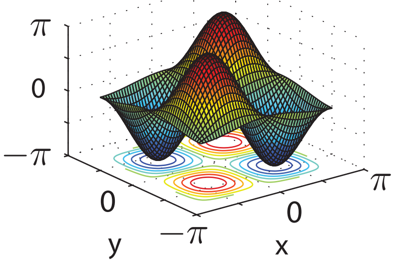
- computations of spectral measures and effective diffusivity for model flows; new representations, moment calculations

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019

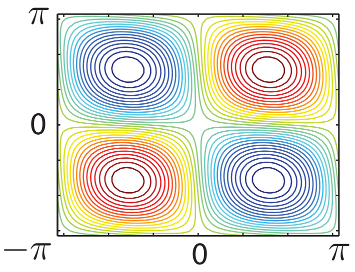
- rigorous bounds and computations for convection enhanced thermal conductivity of sea ice

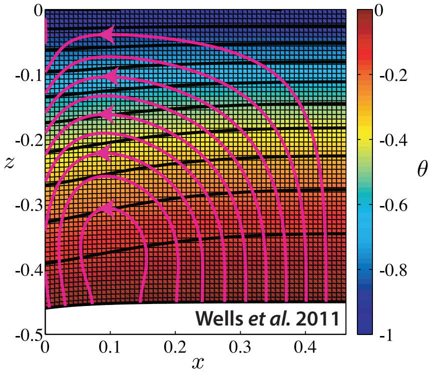
Kraitzman, Hardenbrook, Murphy, Zhu, Cherkaev, Strong, Golden 2019

stream function



streamlines





Wells *et al.* 2011

Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019

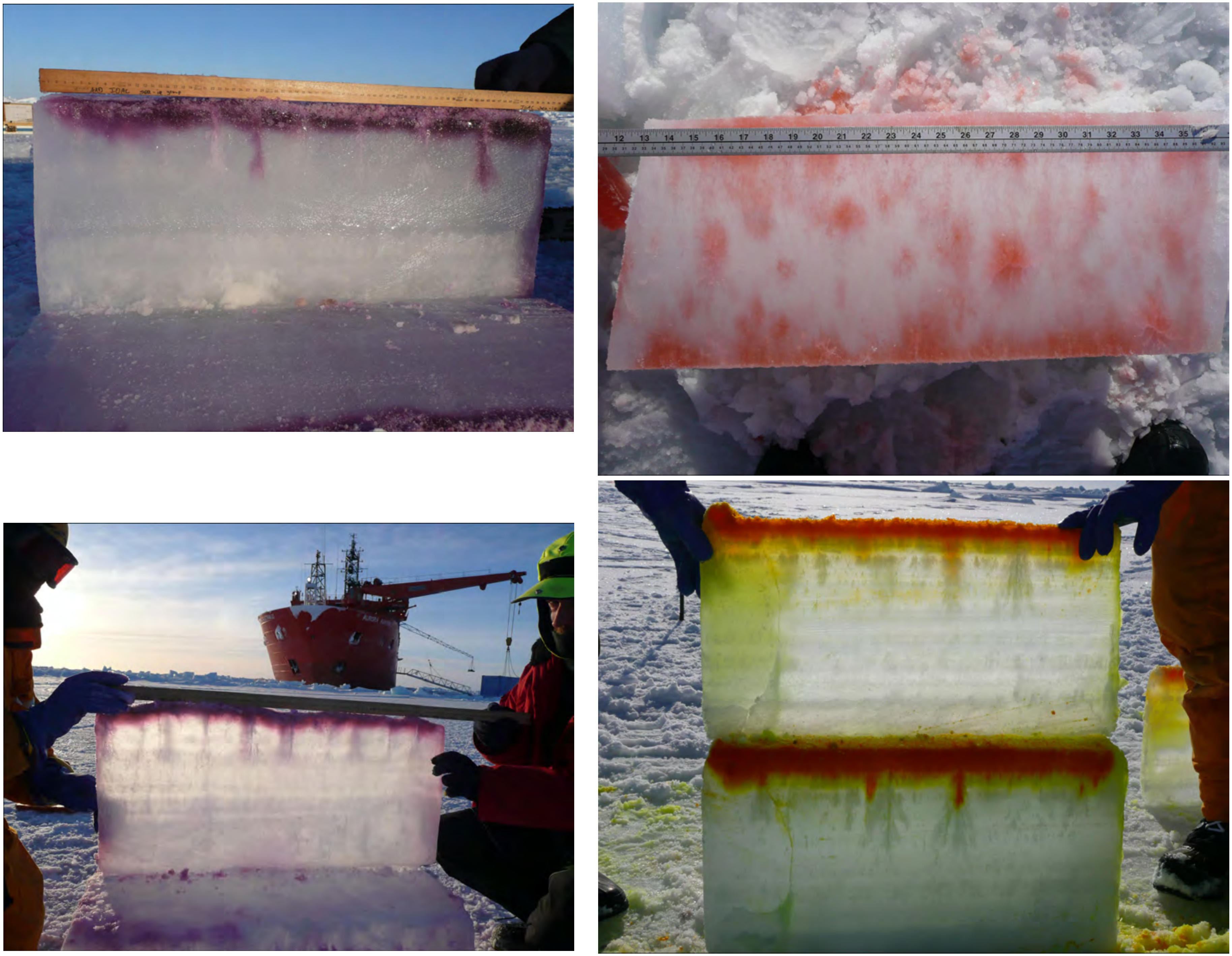
$$\kappa^* = \kappa\left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}\right), \quad F(\kappa) = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}$$

- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H\Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$, Δ is the Laplace operator
- $i\Gamma H\Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ-plane

separation of material properties and flow field

spectral measure calculations

tracers flowing through inverted sea ice blocks



fluid permeability of a porous medium

how much water gets
through the sample
per unit time?

Darcy's Law

for slow viscous flow in a porous medium

averaged
fluid velocity

pressure
gradient

$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\,\nabla p$$

viscosity

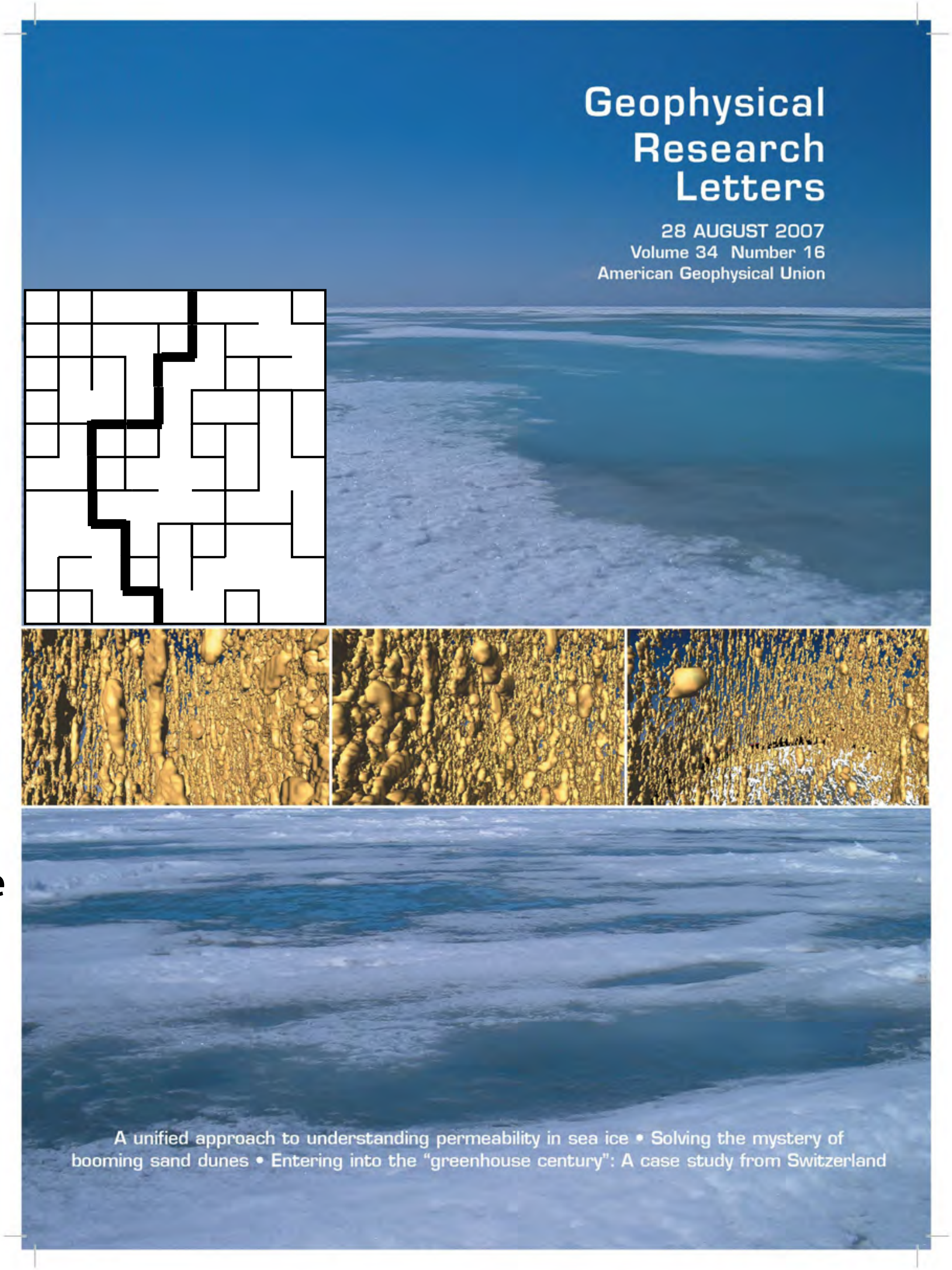
$\mathbf{k}$ = fluid permeability tensor

HOMOGENIZATION

mathematics for analyzing effective behavior of heterogeneous systems

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007



Geophysical
Research
Letters

28 AUGUST 2007
Volume 34 Number 16
American Geophysical Union

A unified approach to understanding permeability in sea ice • Solving the mystery of booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

micro-scale

controls

macro-scale

processes

percolation theory

$$k(\phi) = k_0 (\phi - 0.05)^2 \longleftarrow \text{critical exponent } t$$

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

hierarchical model
network model
rigorous bounds

agree closely
with field data

X-ray tomography for brine inclusions

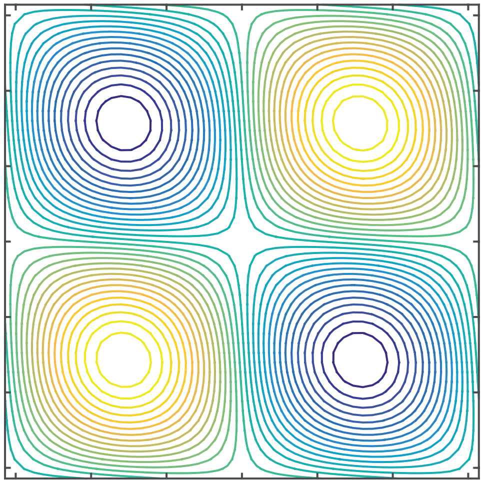
unprecedented look
at thermal evolution
of brine phase and
its connectivity

confirms rule of fives

Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009

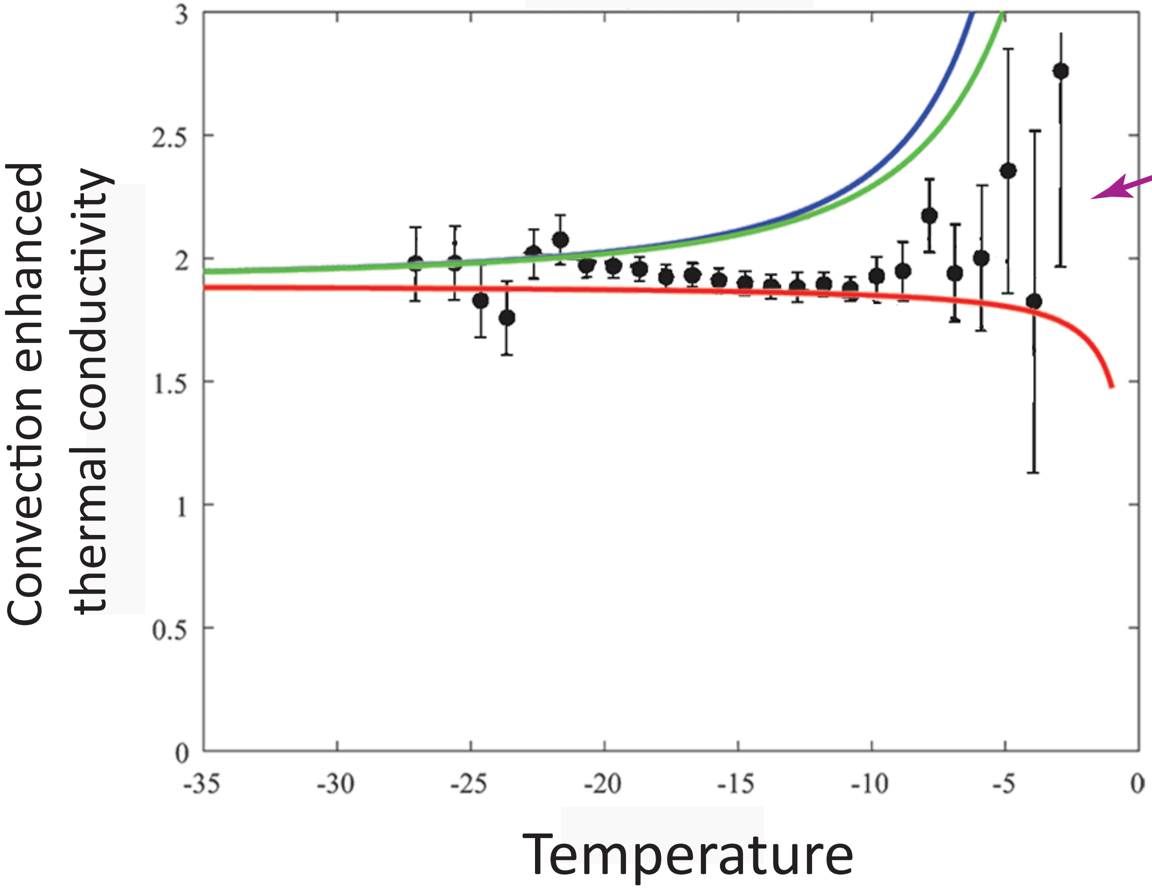
Rigorous bounds on convection enhanced thermal conductivity of sea ice

Kraitzman, Hardenbrook, Murphy, Zhu, Cherkaev, Strong, Golden 2019



cat's eye flow model for
brine convection cells

similar bounds
for shear flows



**data
Trodahl
et al. 2001**

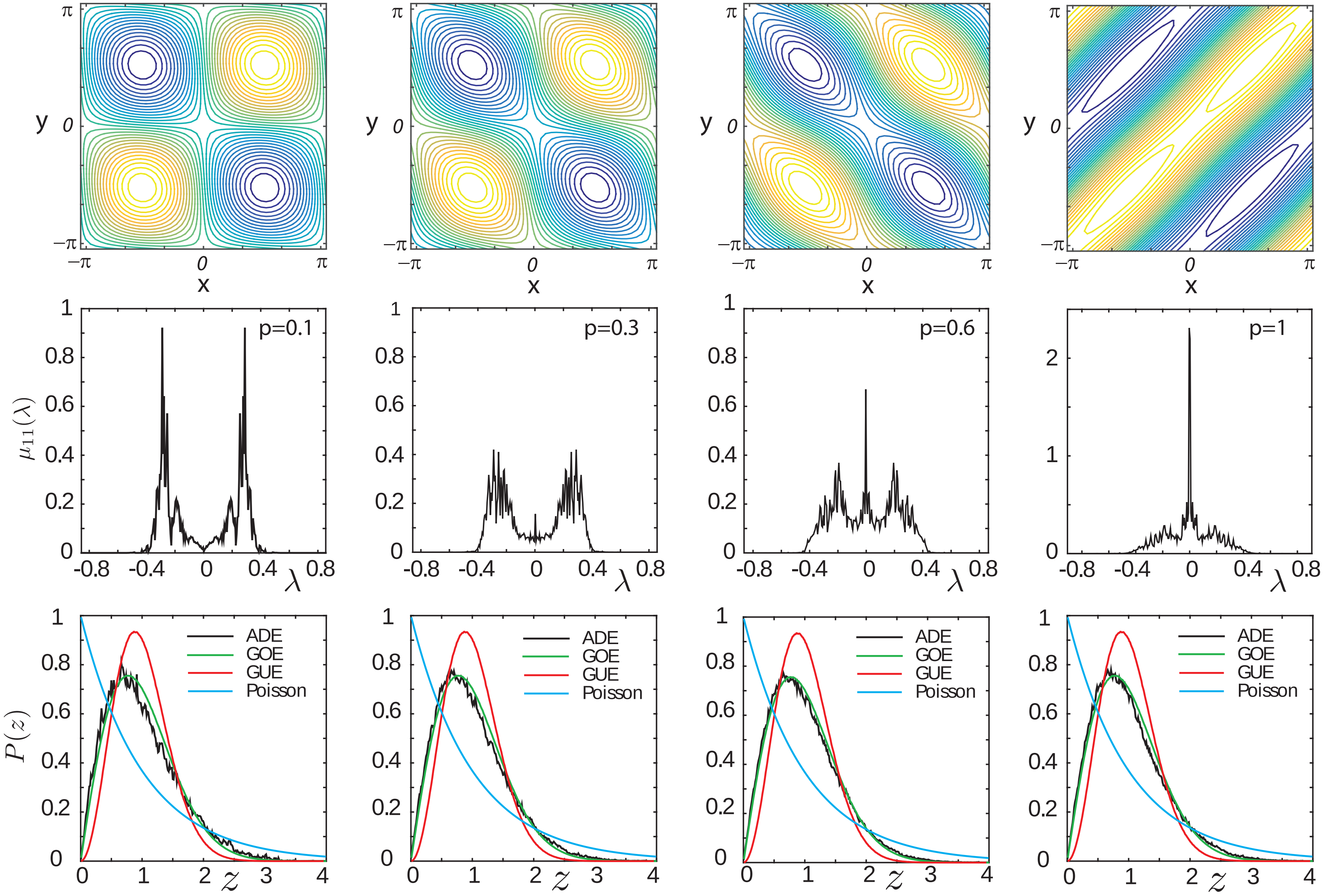
rigorous Padé bounds from Stieltjes integral +
analytical calculations of moments of measure

**rigorous bounds assuming information
on flow field INSIDE inclusions**

Kraitzman, Cherkaev, Golden
SIAM J. Appl. Math (in revision), 2019

Spectral measures and eigenvalue spacings for cat's eye flow

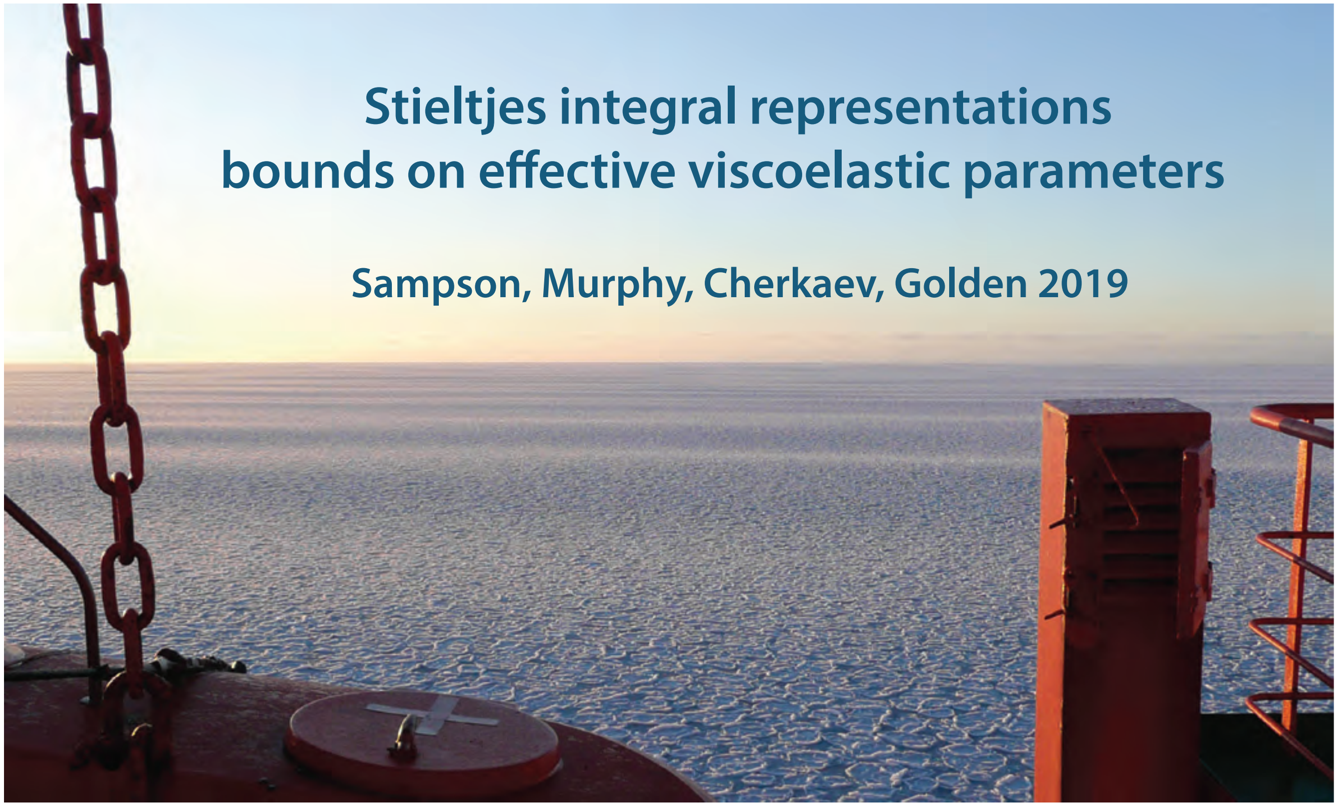
$H(x,y) = \sin(x)\sin(y) + A\cos(x)\cos(y), \quad A \sim U(-p,p)$



Murphy, Cherkaev, Xin, Golden, 2017

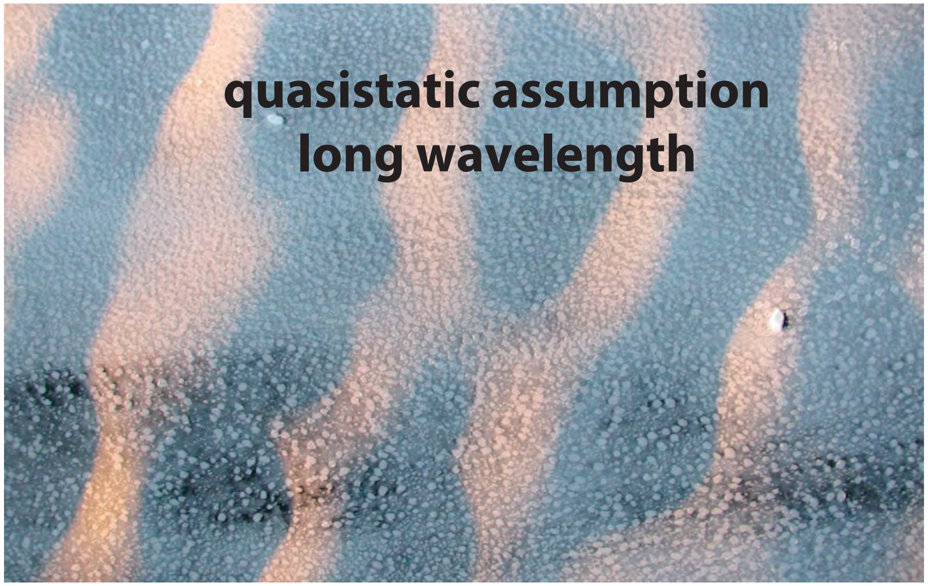
wave propagation in the marginal ice zone

ocean-
atmosphere
interaction



Stieltjes integral representations
bounds on effective viscoelastic parameters

Sampson, Murphy, Cherkaev, Golden 2019



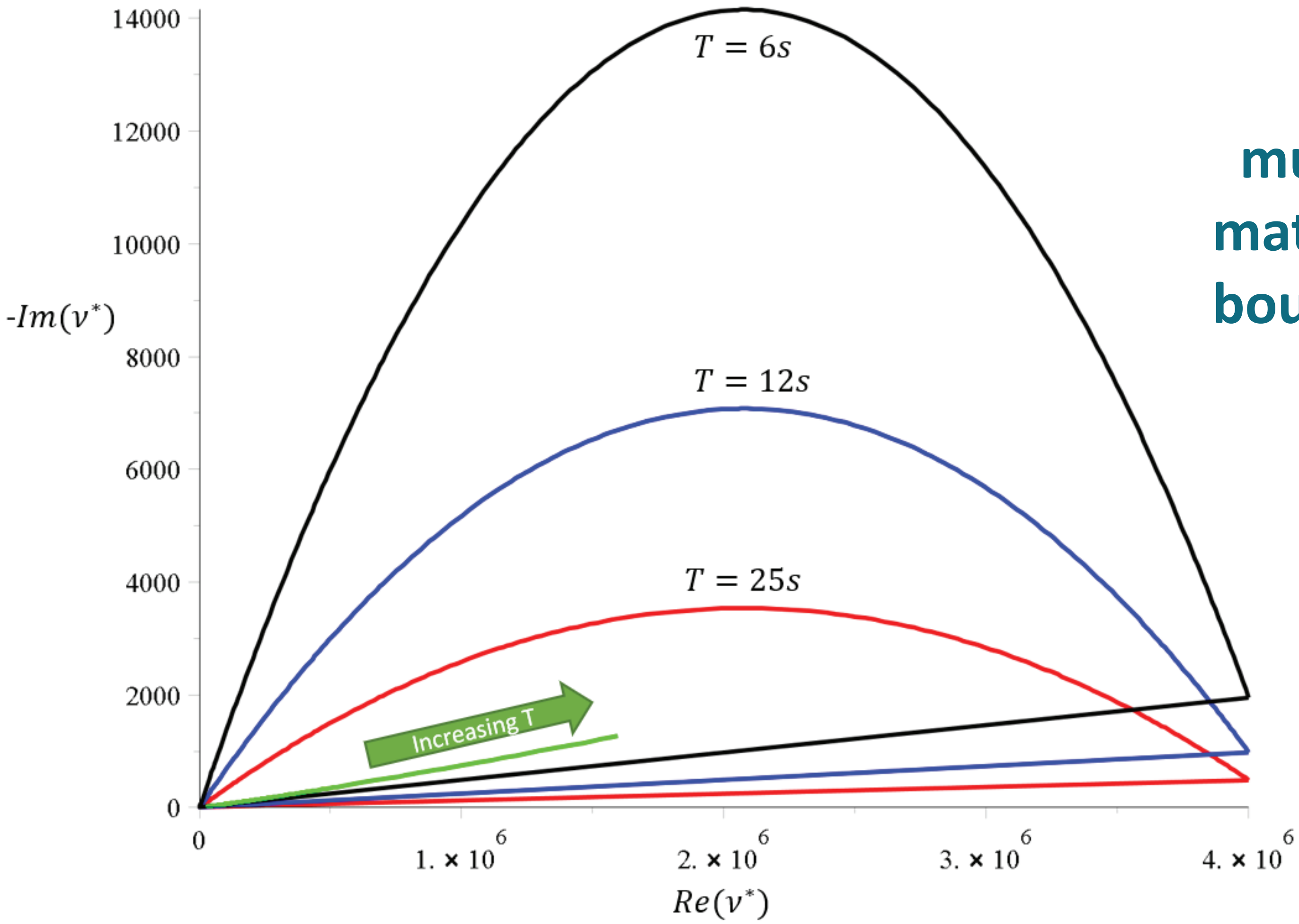
quasistatic assumption
long wavelength

bounds on the effective complex viscoelasticity

complex elementary bounds
(fixed area fraction of floes)

$\nu_1 = 10^7 + i\,4875$ **pancake ice**

$\nu_2 = 5 + i\,0.0975$ **slush / frazil**



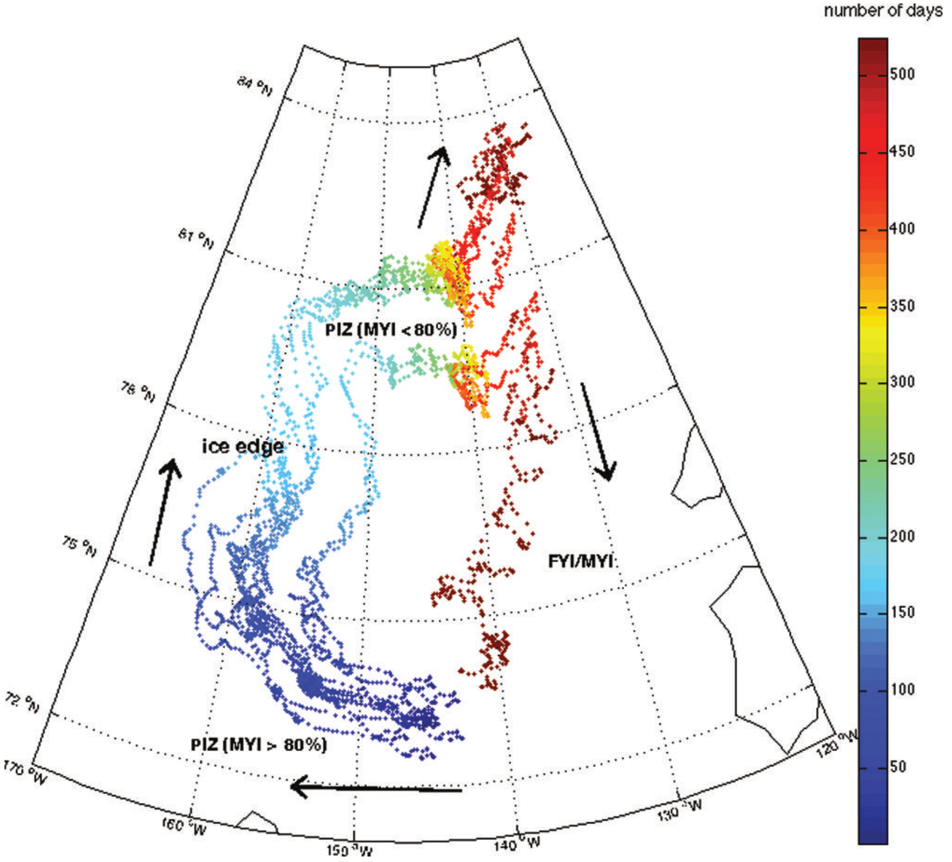
+
much tighter
matrix particle
bounds + data

Sampson, Murphy, Cherkaev, Golden 2019

Anomalous diffusion in sea ice dynamics

Ice floe diffusion in winds and currents

Jennifer Lukovich, Jennifer Hutchings,
David Barber, *Ann. Glac.* 2015



- On short time scales floes observed (buoy data) to exhibit Brownian-like behavior, but they are also being advected by winds and currents.

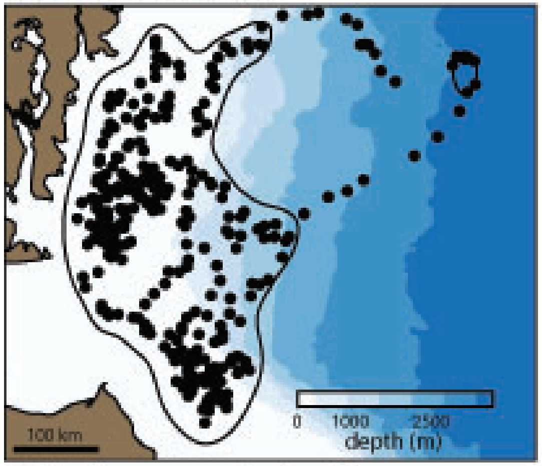
- Effective behavior is purely diffusive, sub-diffusive or super-diffusive depending on ice pack and advective conditions - Hurst exponent.

Home ranges in moving habitats: polar bears and sea ice

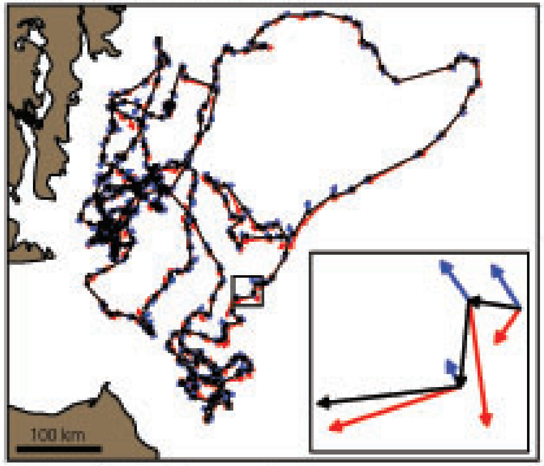
"diffusive" polar bear motion on drifting sea ice

Marie Auger-Méthé, Mark Lewis, Andrew Derocher, *Ecography*, 2016

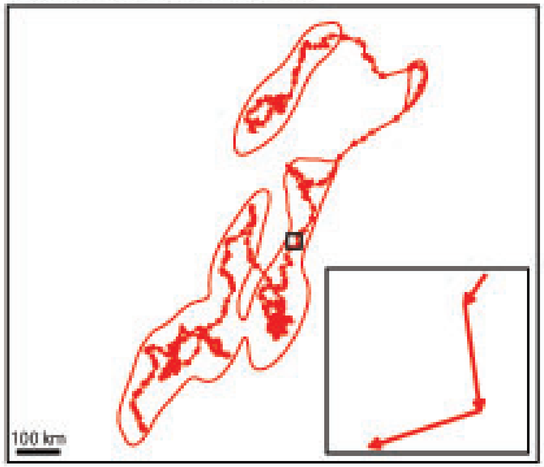
(a) geographic home range



(b) voluntary movement vs. ice drift



(c) area of habitat encountered



Floe Scale Model of Anomalous Diffusion in Sea Ice Dynamics

Huy Dinh, Elena Cherkaev, Court Strong, Ken Golden 2019

$$\left\langle \left|\mathbf{x}(t) - \mathbf{x}(0) - \langle \mathbf{x}(t) - \mathbf{x}(0) \rangle\right|^2 \right\rangle \sim t^\alpha$$

$\alpha =$ Hurst exponent, a measure of anomalous diffusion.

Measured from bouy position data. Detects ice pack crowding and advective forcing.

J.V. Lukovich, J.K. Hutchings, D.G. Barber ***Annals of Glaciology*** **2015**

diffusive	$\alpha = 1$	Sparse packing, uncorrelated advective field.
sub-diffusive	$\alpha < 1$	Dense packing, crowding dominates advection.
super-diffusive	$\alpha = 5/4$	Sparse packing, shear dominates advection.
	$\alpha = 5/3$	Sparse packing, vorticity dominates advection.

Goal: Develop numerical model to analyze regimes of transport in terms of ice pack crowding and advective conditions.

Model Approximations

Floes $\approx$ Discs

Forces on Disc $= \quad F_{drag} + F_{collision}$

A. Herman ***Physical Review E*** **2011**

Floe-Floe Interactions: Linear Elastic Collisions

$F_{collision}$ follows Hooke's Law.

Advective Forcing: Passive, Linear Drag Law

v is the advective velocity field.

F_{drag} is proportional to relative velocity.

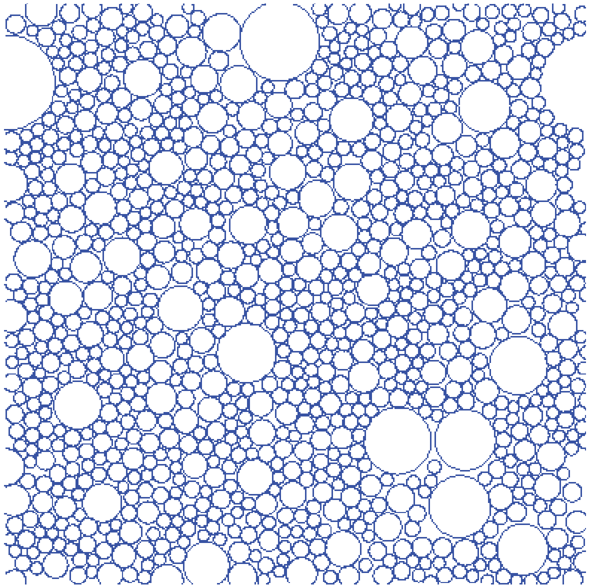
Ice Pack Characteristics

ϕ = sea ice concentration (floe area fraction)

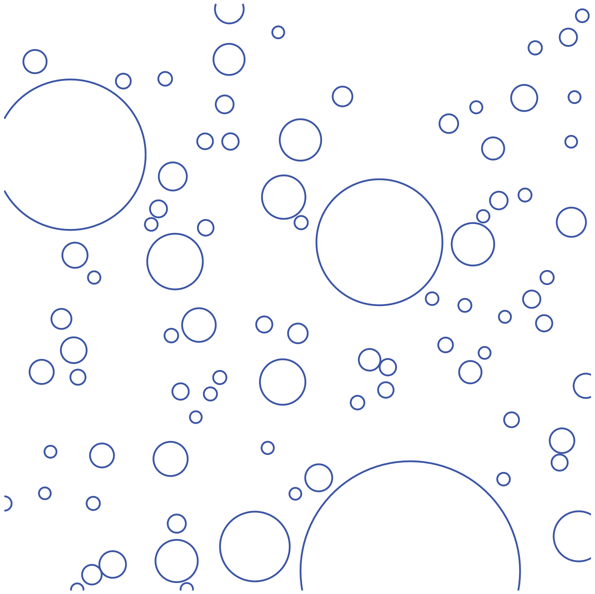
Power Law Size Distribution: $N(D) \sim D^{-k}$

T. Toyota, S. Takatsuji, M. Nakayama Geophysical Review Letters 2006

k = floe diameter exponent



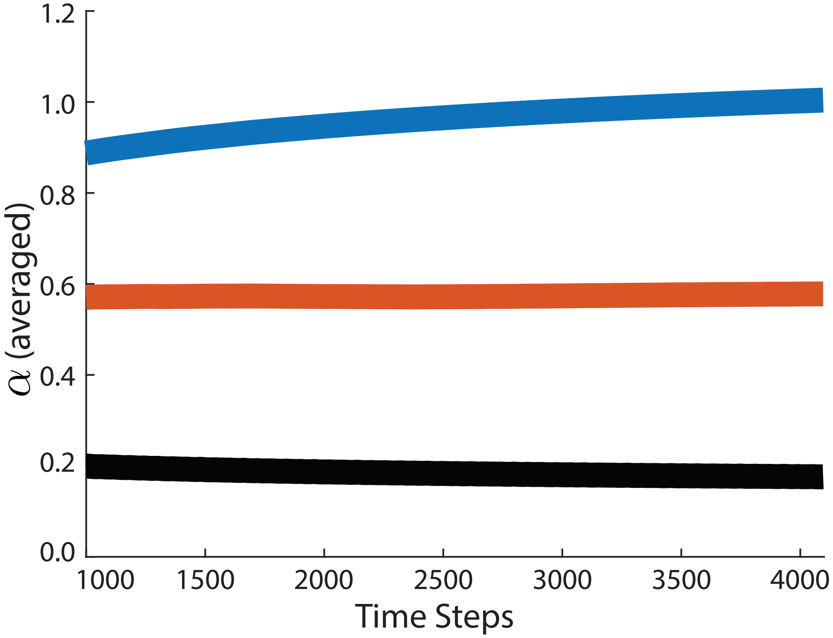
k = 2.9, ϕ = 0.8

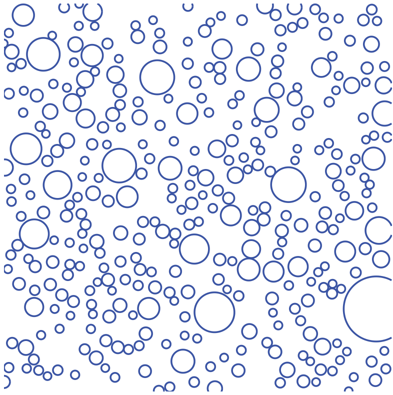


k = 1.7, ϕ = 0.1

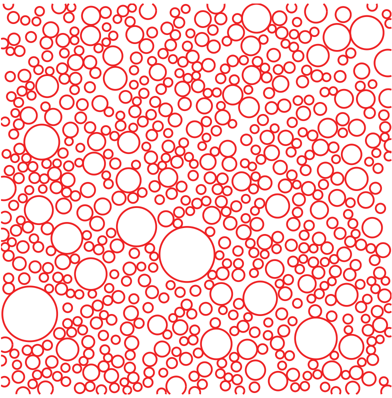
Model Results

Crowding in random advective forcing.

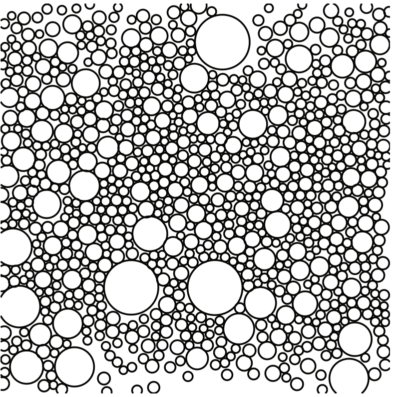




Concentration = 0.3
k = 2.9

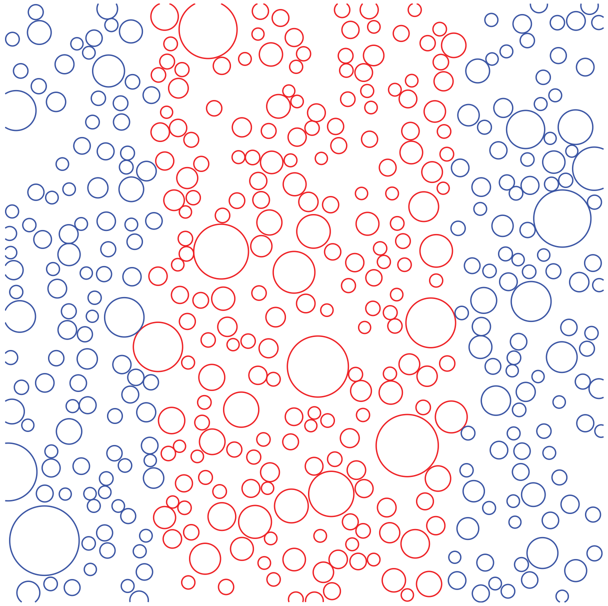


Concentration = 0.5
k = 2.9



Concentration = 0.7
k = 2.9

Concentration = 0.3 k = 2.9



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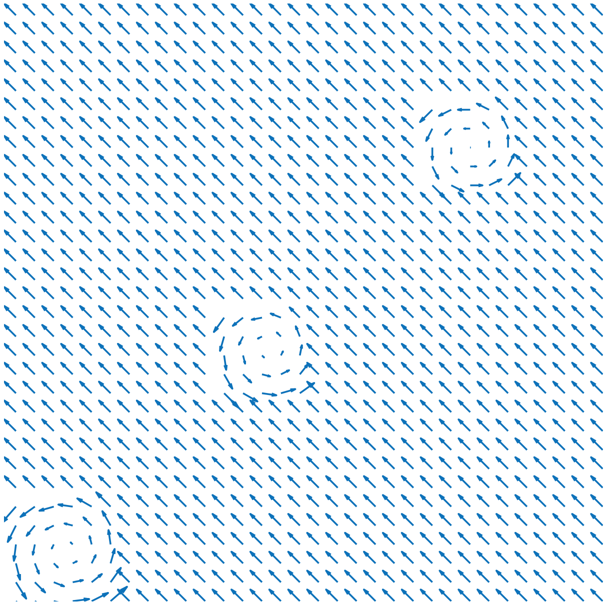
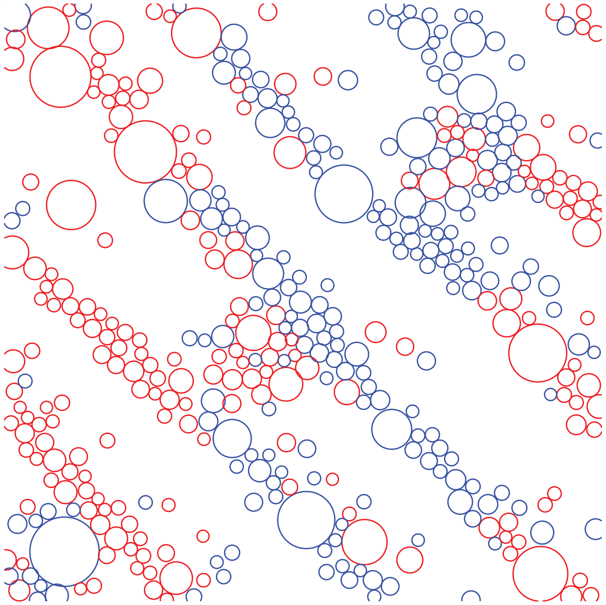
Vorticity Dominated Drift



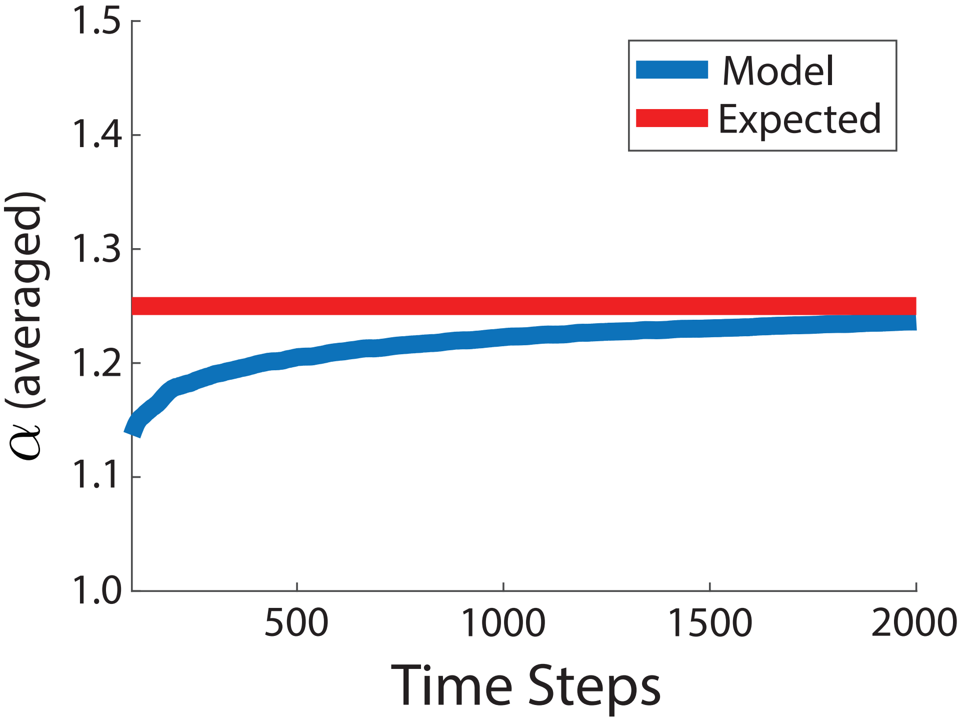
=

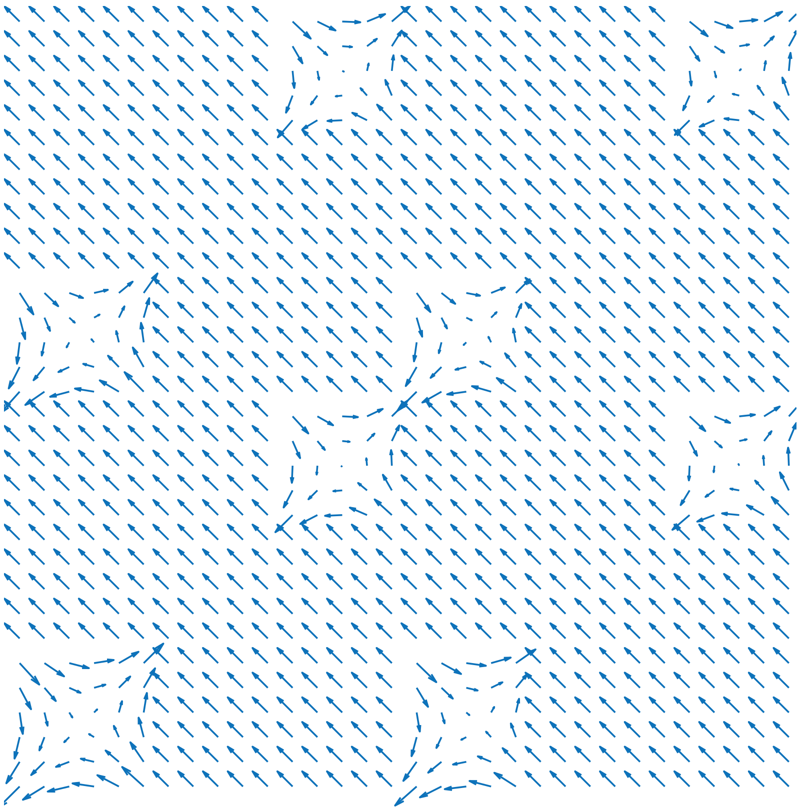
$\alpha = 5/3$



Model Results

Sparse Packing, Shear Dominated Drift





Expected $\alpha = 5/4$

k = 2.9 Concentration = 0.3

Conclusions

1. Advection diffusion processes arise naturally in the sea ice system.

2. The effective diffusivity characterizes *homogenized* behavior over long length and time scales.

3. Stieltjes integrals provide a powerful framework for rigorously calculating effective parameters.

4. This framework yields a mathematical theory for thermal transport in sea ice with convection.

5. We developed a floe scale model to study observed anomalous diffusion in sea ice dynamics.

THANK YOU

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