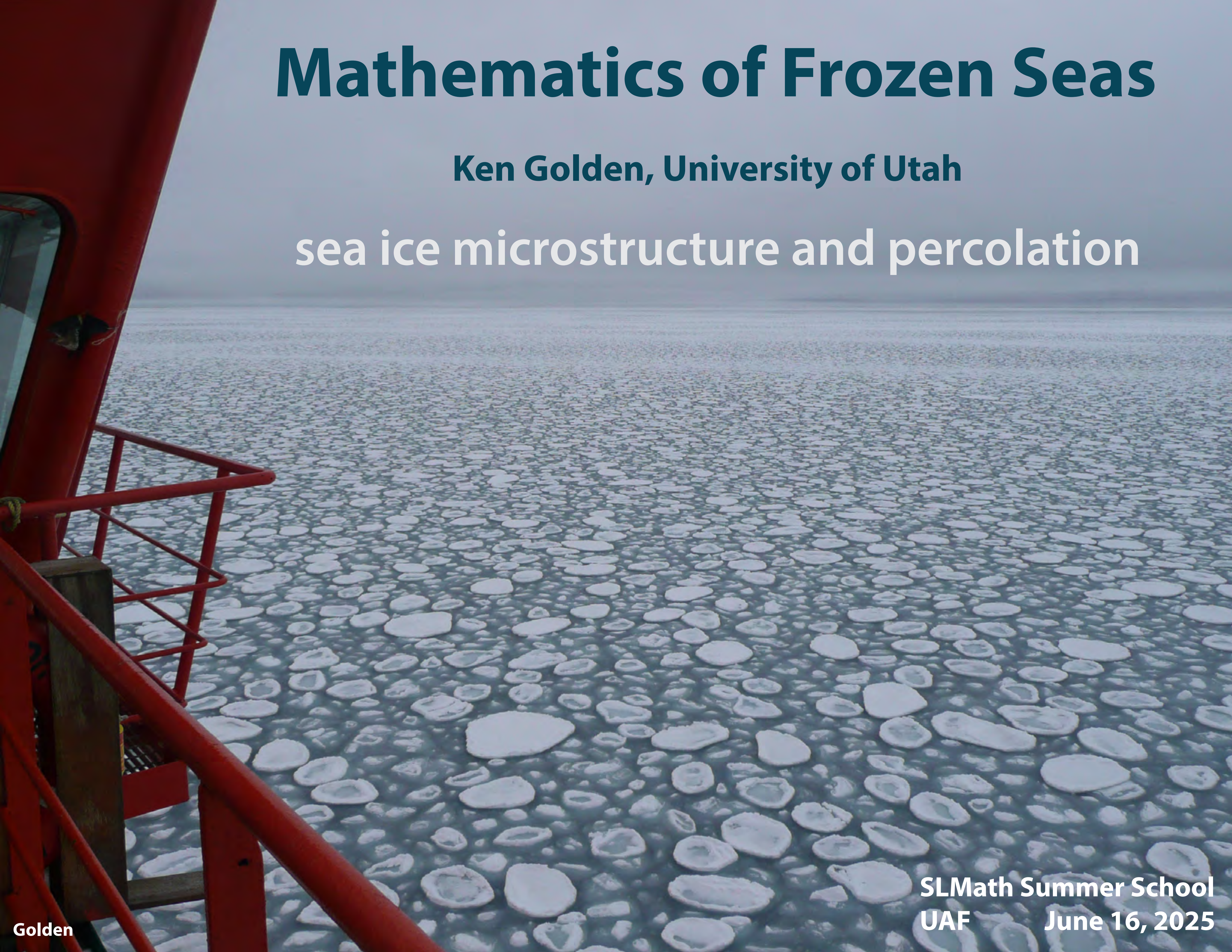
Mathematics of Frozen Seas

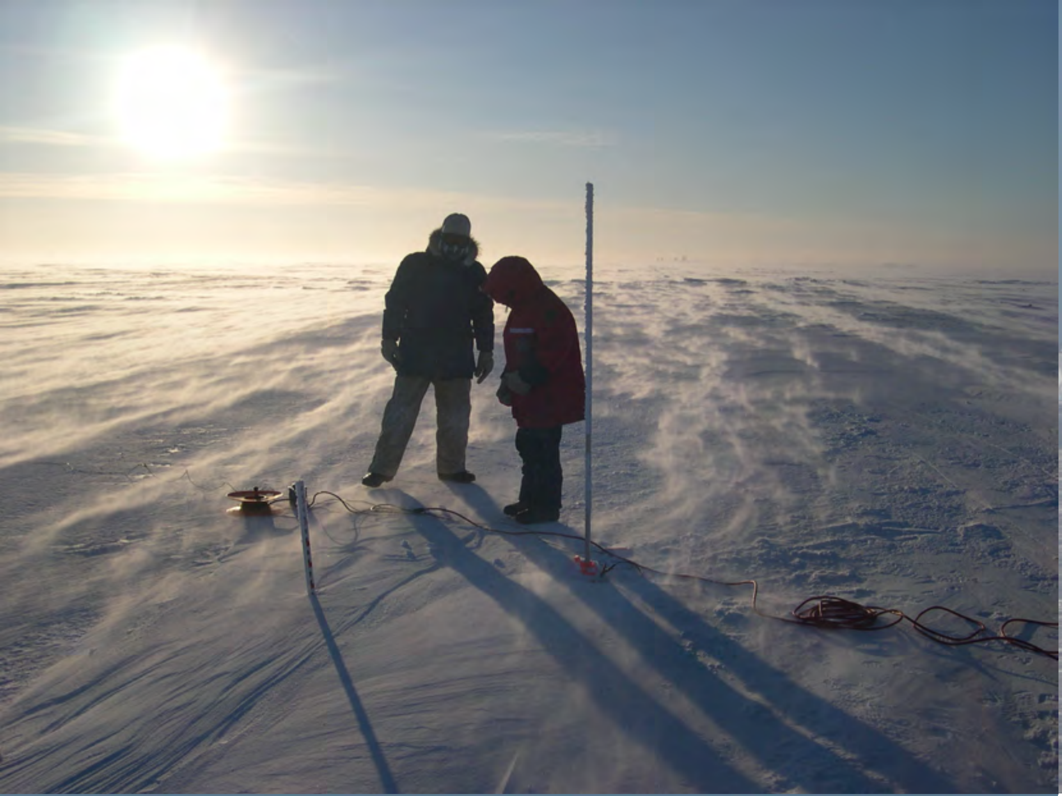
Ken Golden, University of Utah

sea ice microstructure and percolation



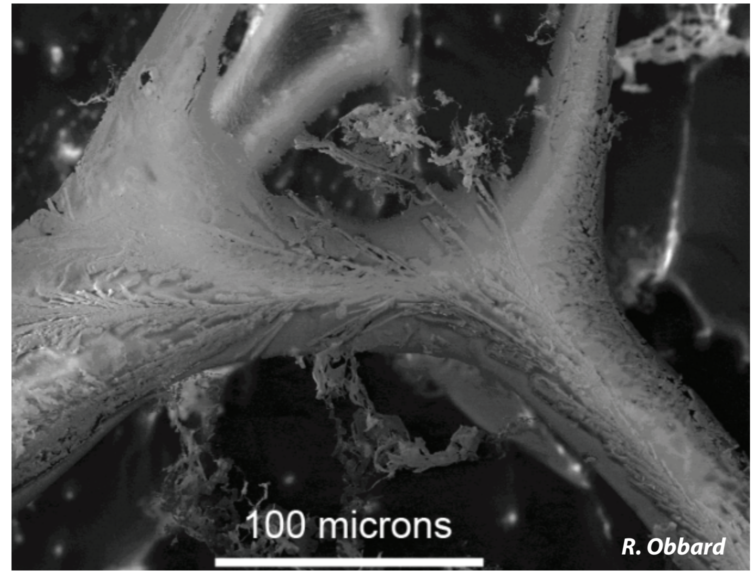
Golden

SLMath Summer School
UAF June 16, 2025



sea ice may appear to be a barren, impermeable cap ...

brine inclusions in sea ice (mm)



micro - brine channel (SEM)

brine channels (cm)

sea ice is a porous composite

pure ice with brine, air, and salt inclusions

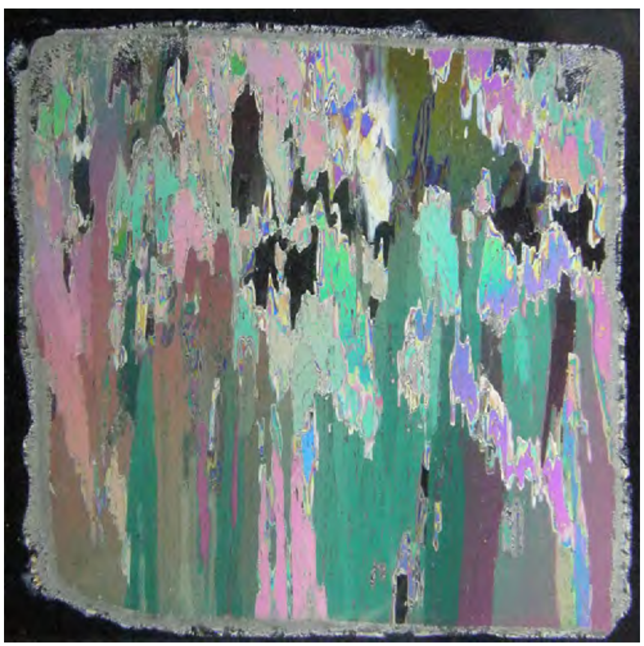
horizontal section

vertical section

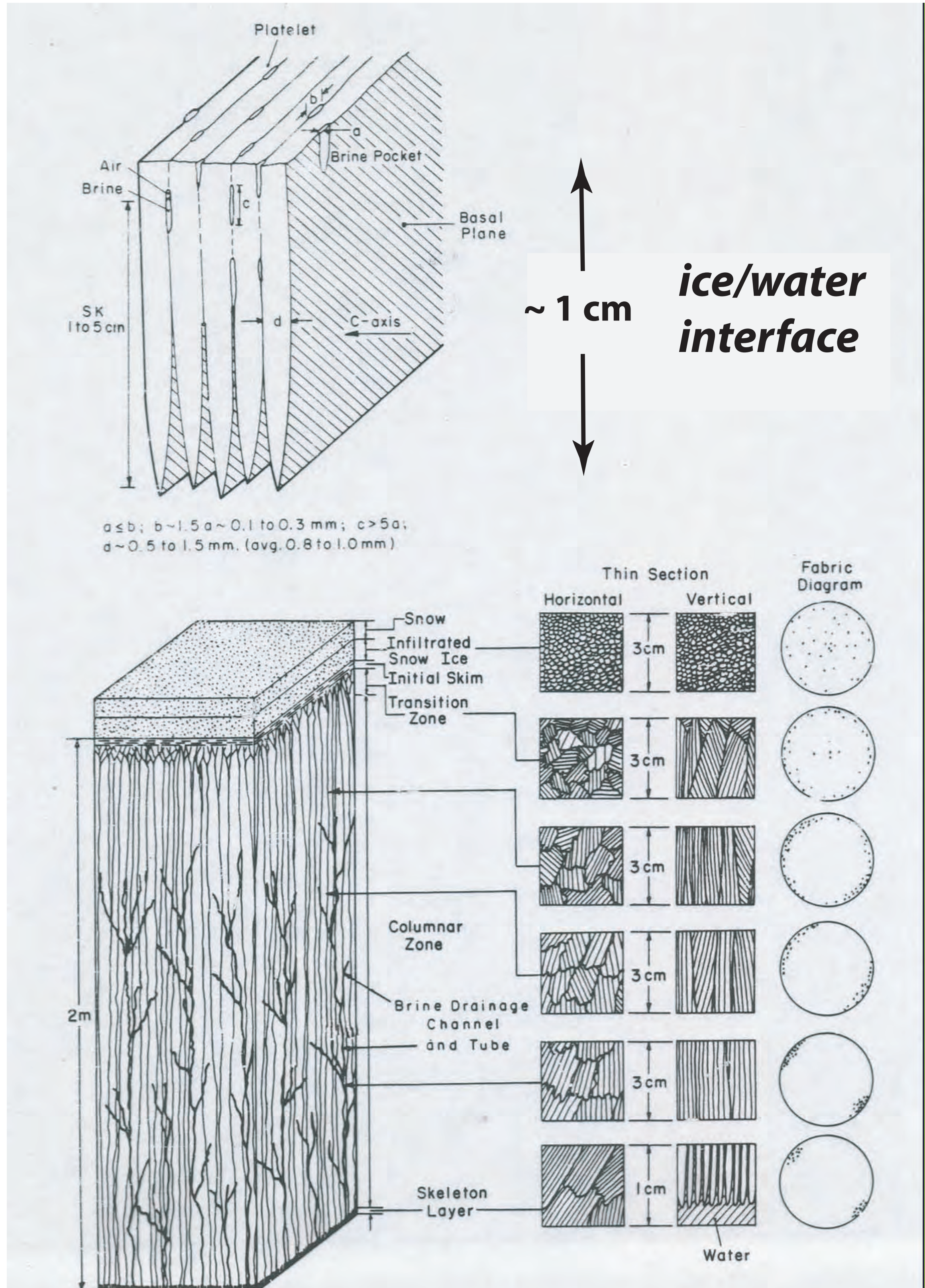
cross-sections of sea ice structure

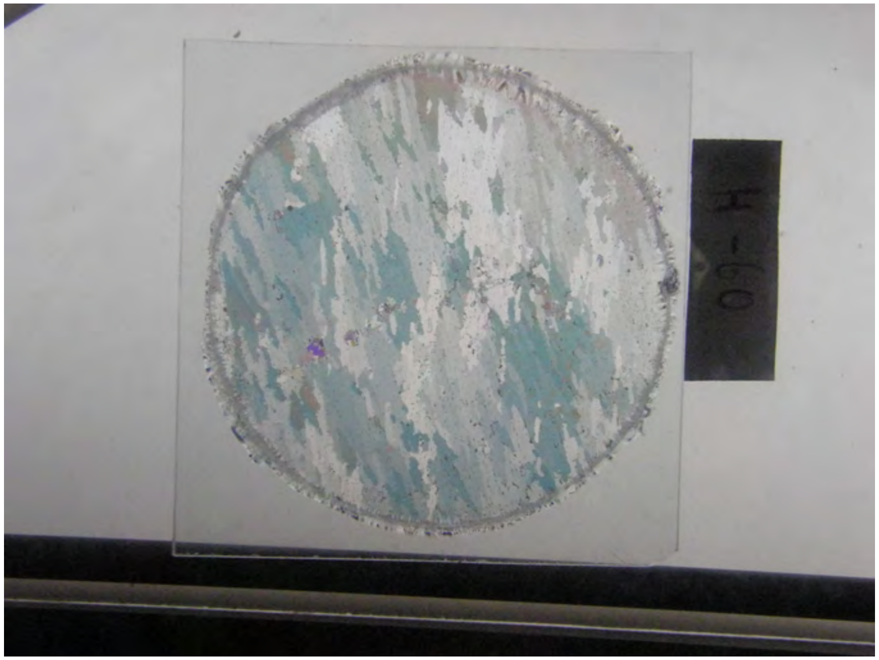
$$T_{freeze} = -1.8^{\circ}\,\mathrm{C}$$

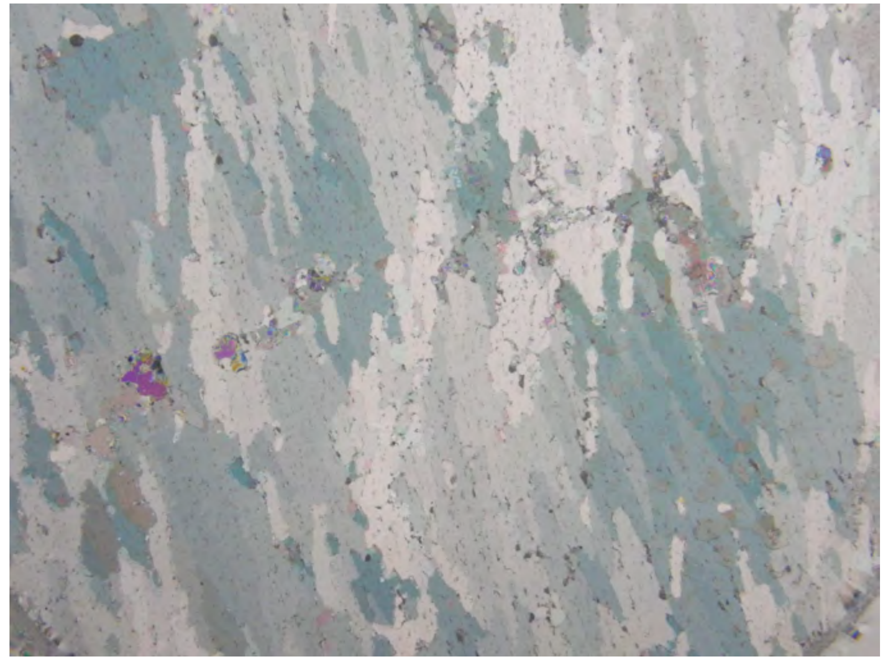
crystallographic texture

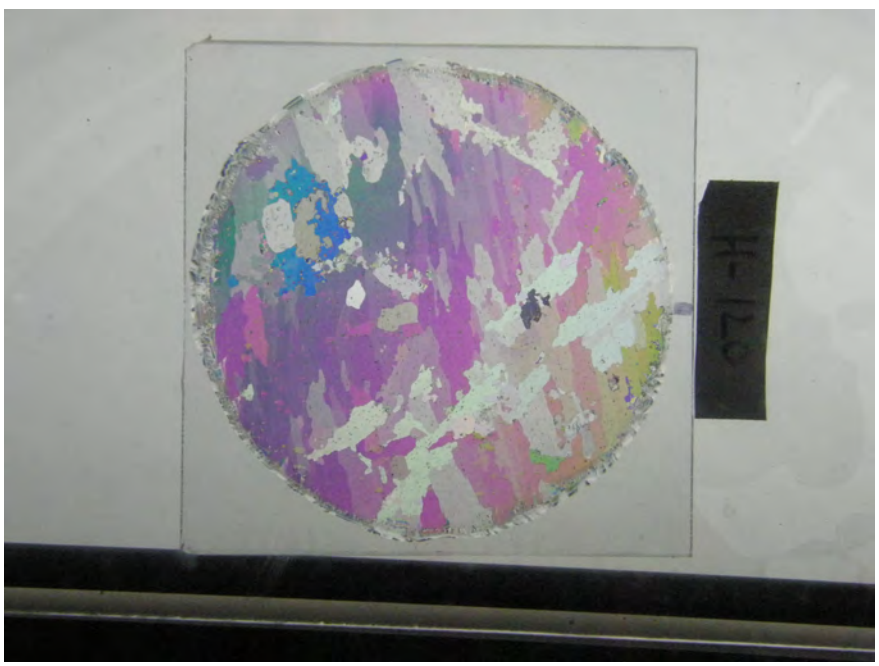


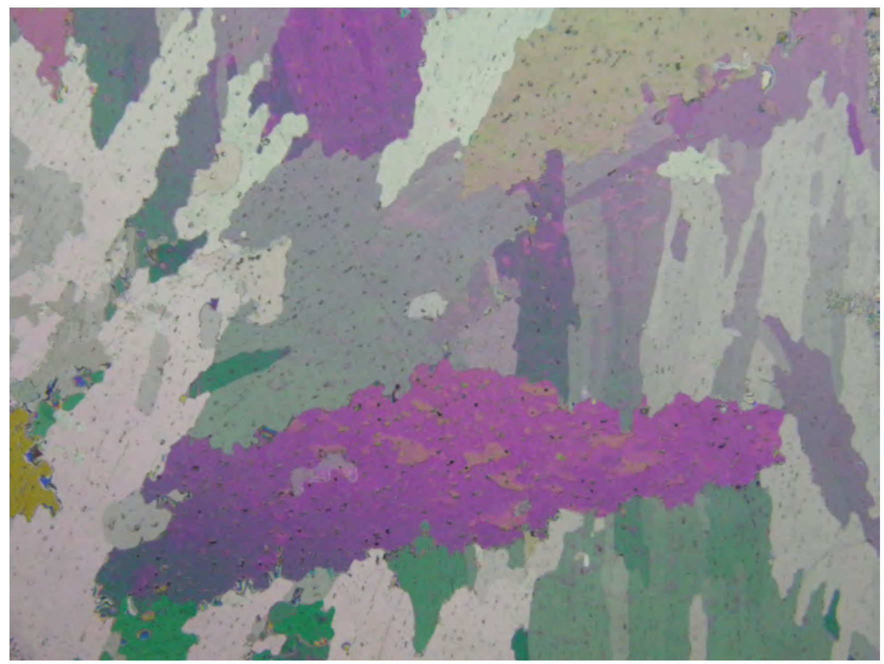
vertical thin section




H-60




H-120



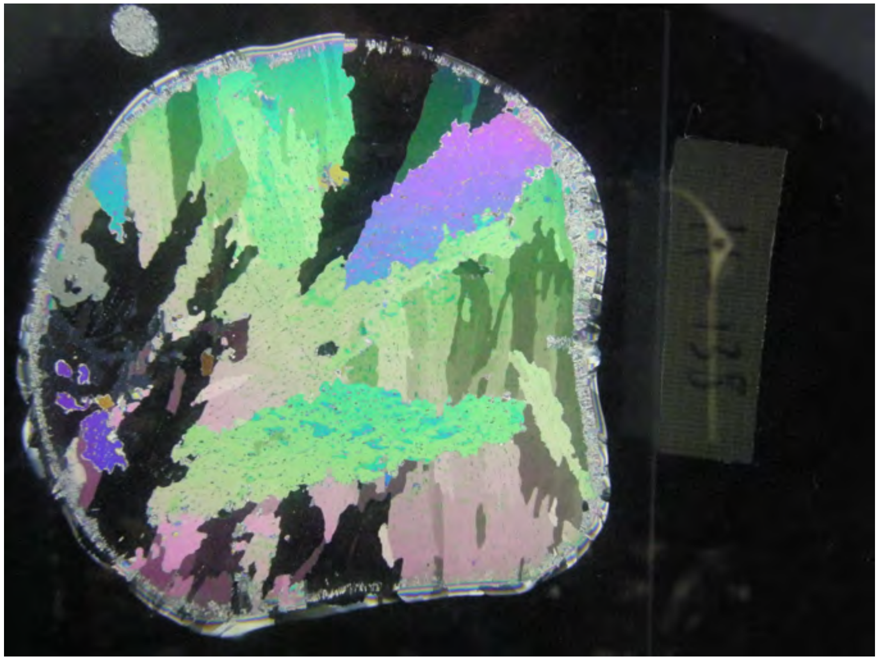
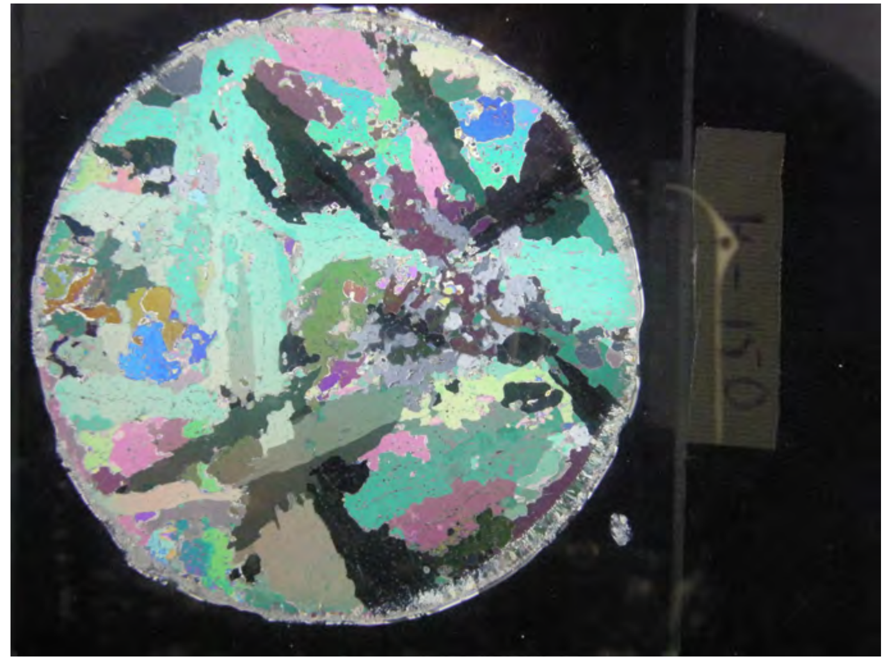
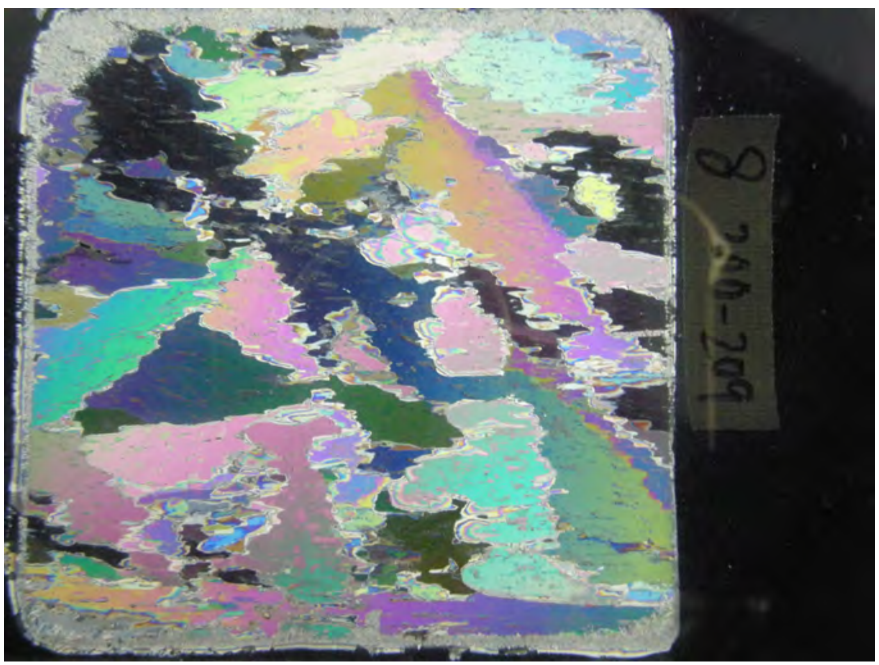
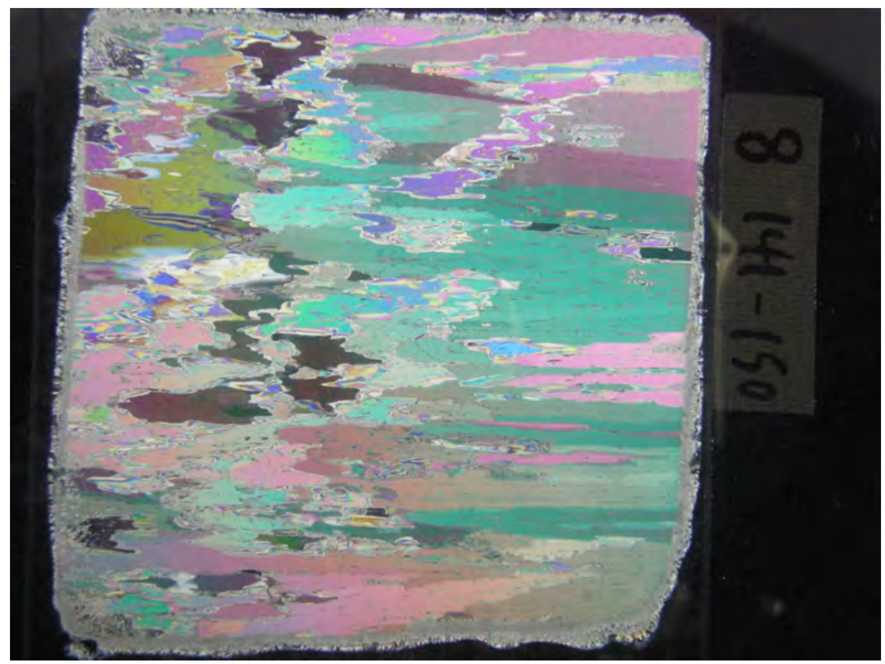





Plate 2*b*

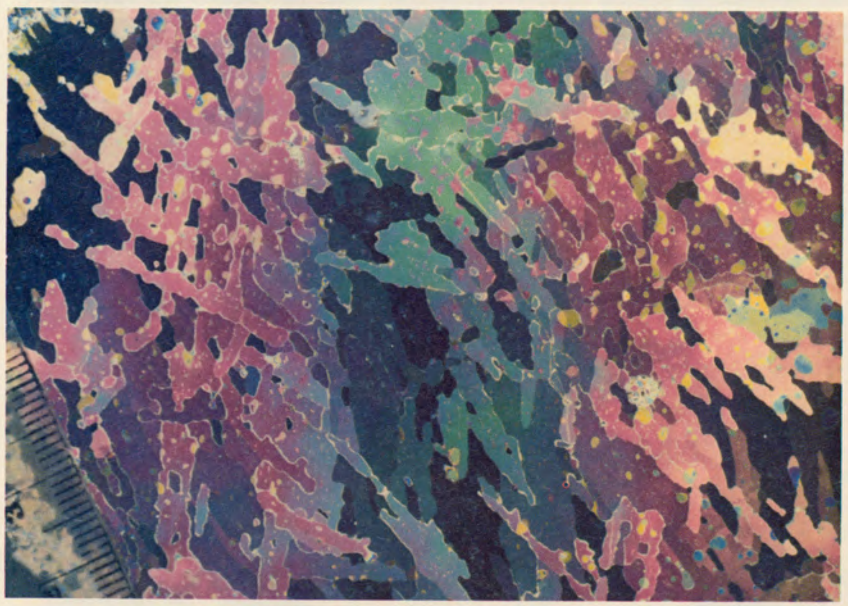


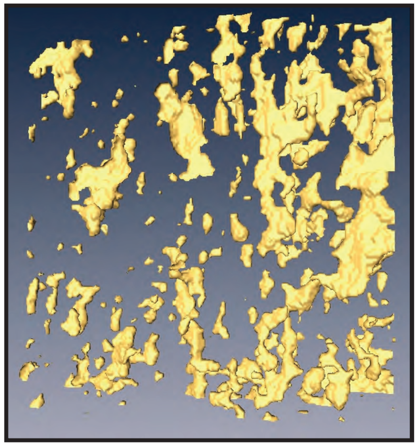
Plate 2*c*

Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

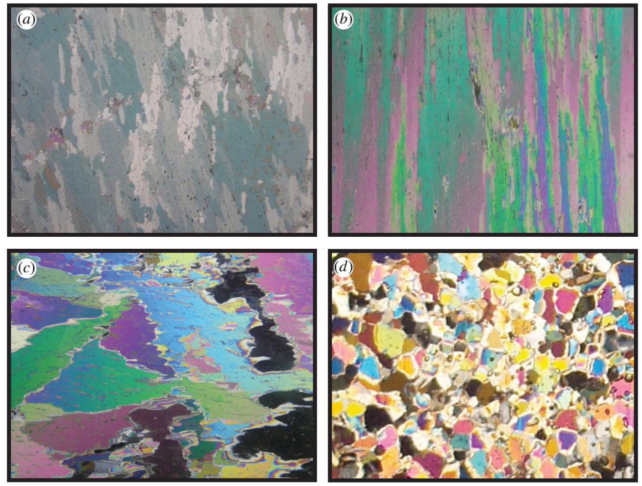
Weeks & Assur 1969



H. Eicken

Golden *et al. GRL* 2007

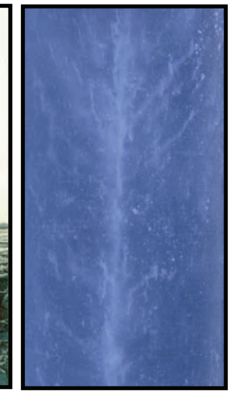
polycrystals



Gully *et al. Proc. Roy. Soc. A* 2015

brine channels

D. Cole



K. Golden

millimeters **centimeters**

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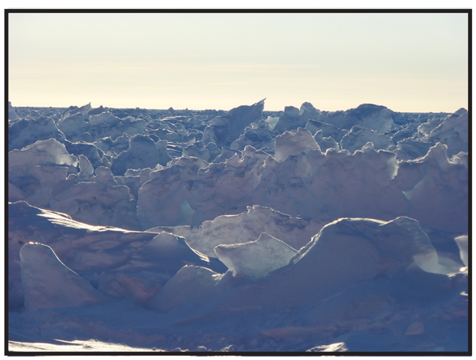
mesoscale

Arctic melt ponds

K. Frey

Antarctic pressure ridges



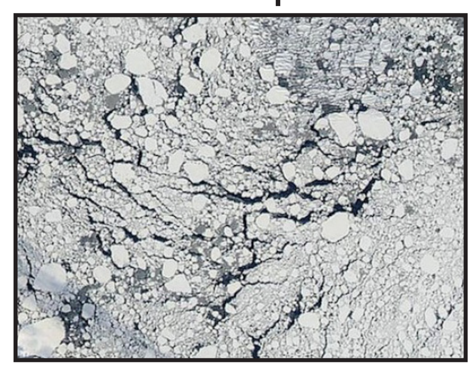
K. Golden

macroscale

sea ice floes

J. Weller

sea ice pack



NASA

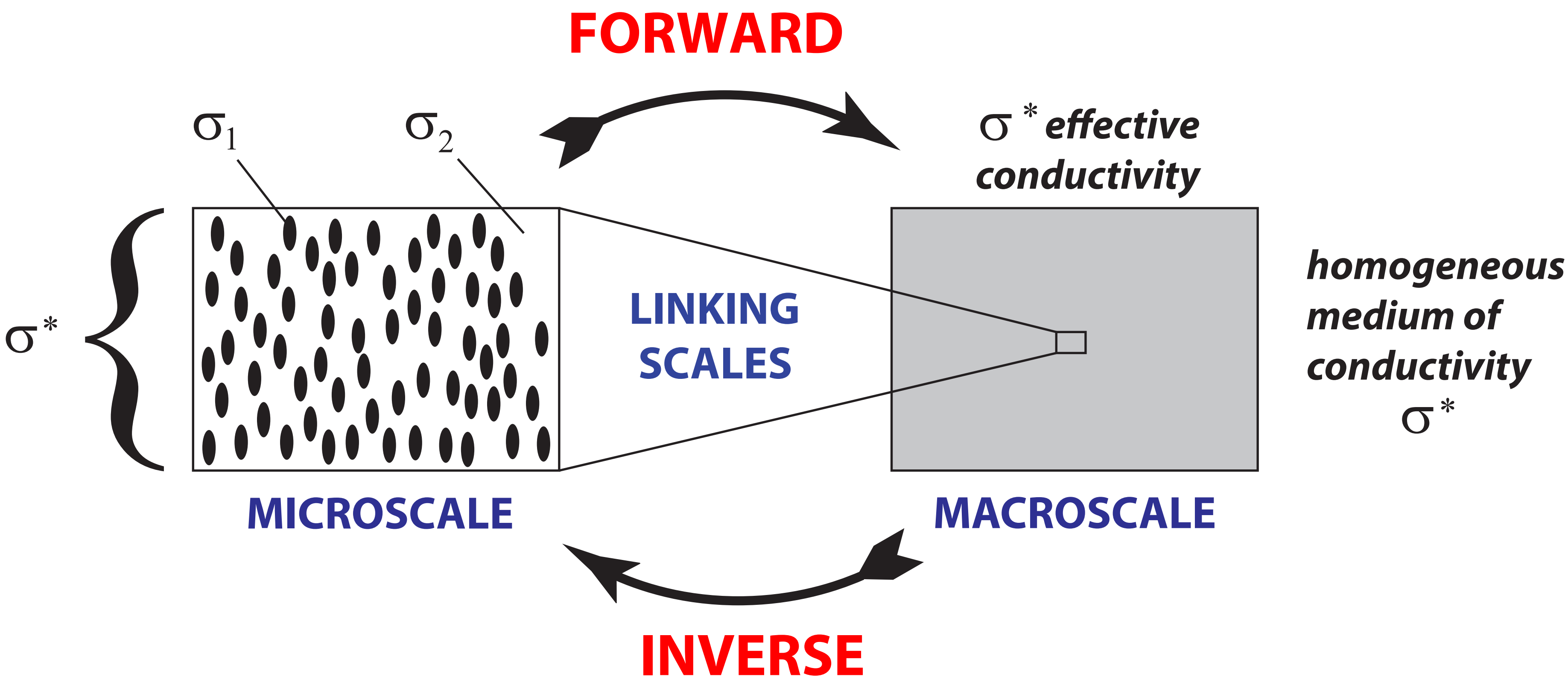
meters **kilometers**

Central theme:

How do we use “small scale” information to find effective behavior on larger scales relevant to climate and ecological models?

OBJECTIVE: advance how sea ice is represented in climate models
improve projections of fate of SEA ICE and its ECOSYSTEMS

HOMOGENIZATION for Composite Materials



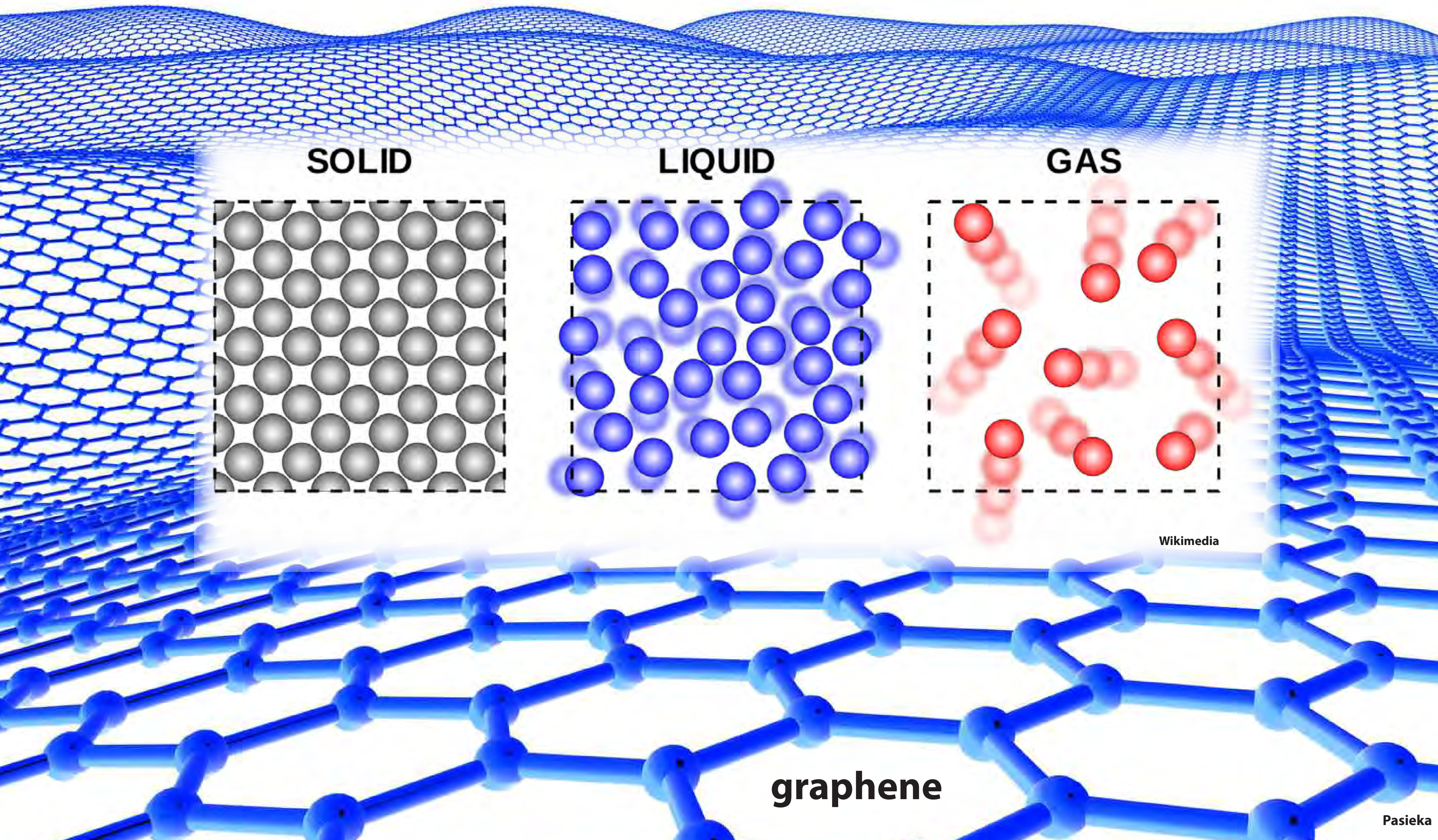
Maxwell 1873, Einstein 1906
Wiener 1912, Hashin and Shtrikman 1962

STATISTICAL PHYSICS

percolation, phase transitions
solid state, semiconductors

How do microscopic laws determine macroscopic behavior?

Banwell, Burton, Cenedese, Golden, Astrom, Physics of the Cryosphere, *Nature Reviews Physics* 2023

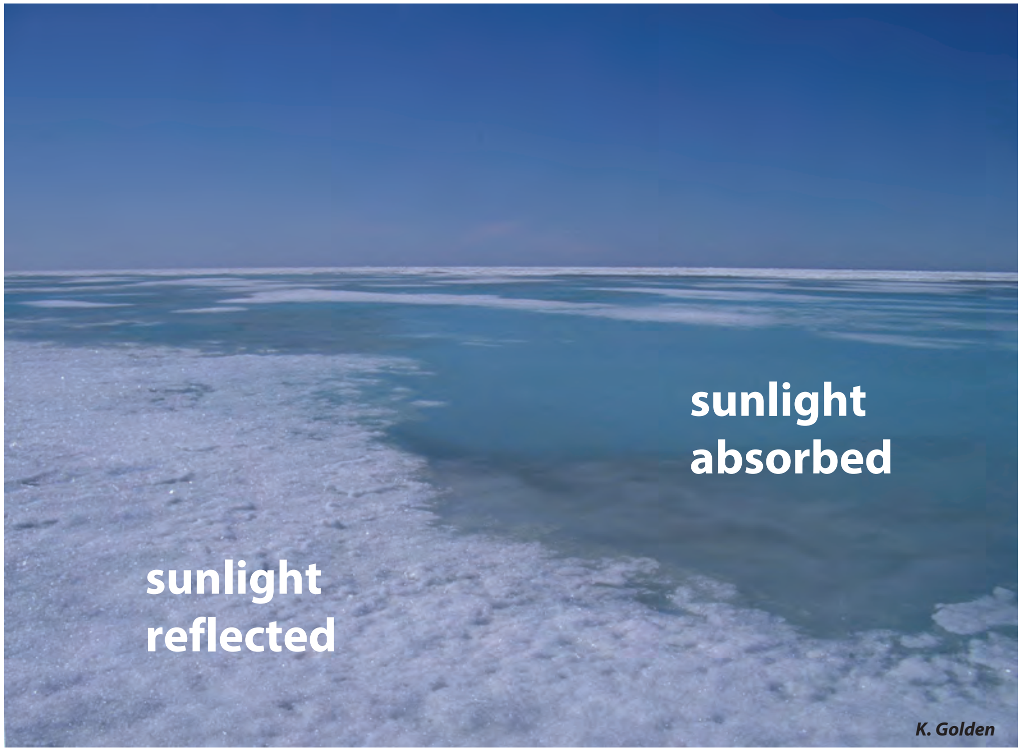


sea ice microphysics

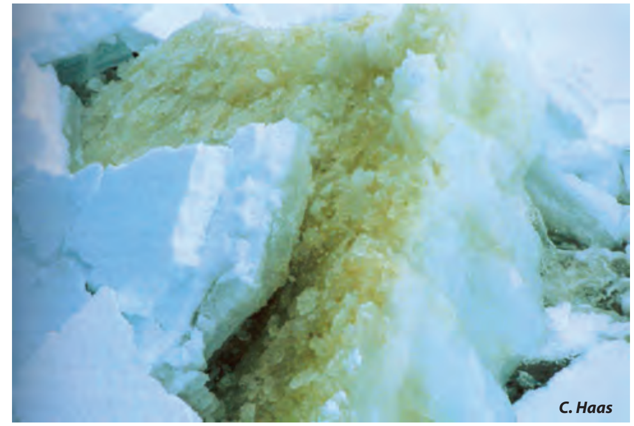
fluid transport

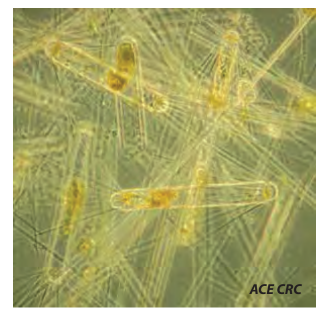
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

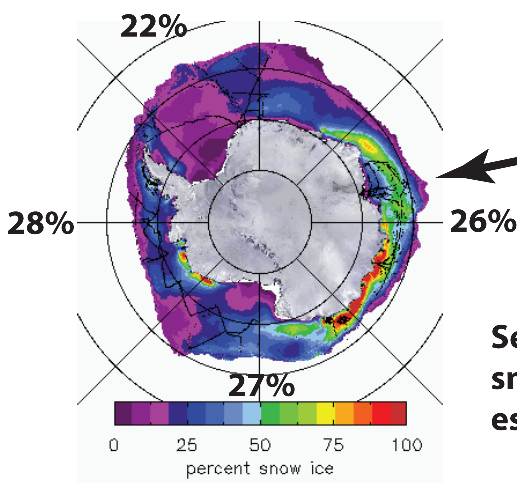
evolution of Arctic melt ponds and sea ice albedo



nutrient flux for algal communities







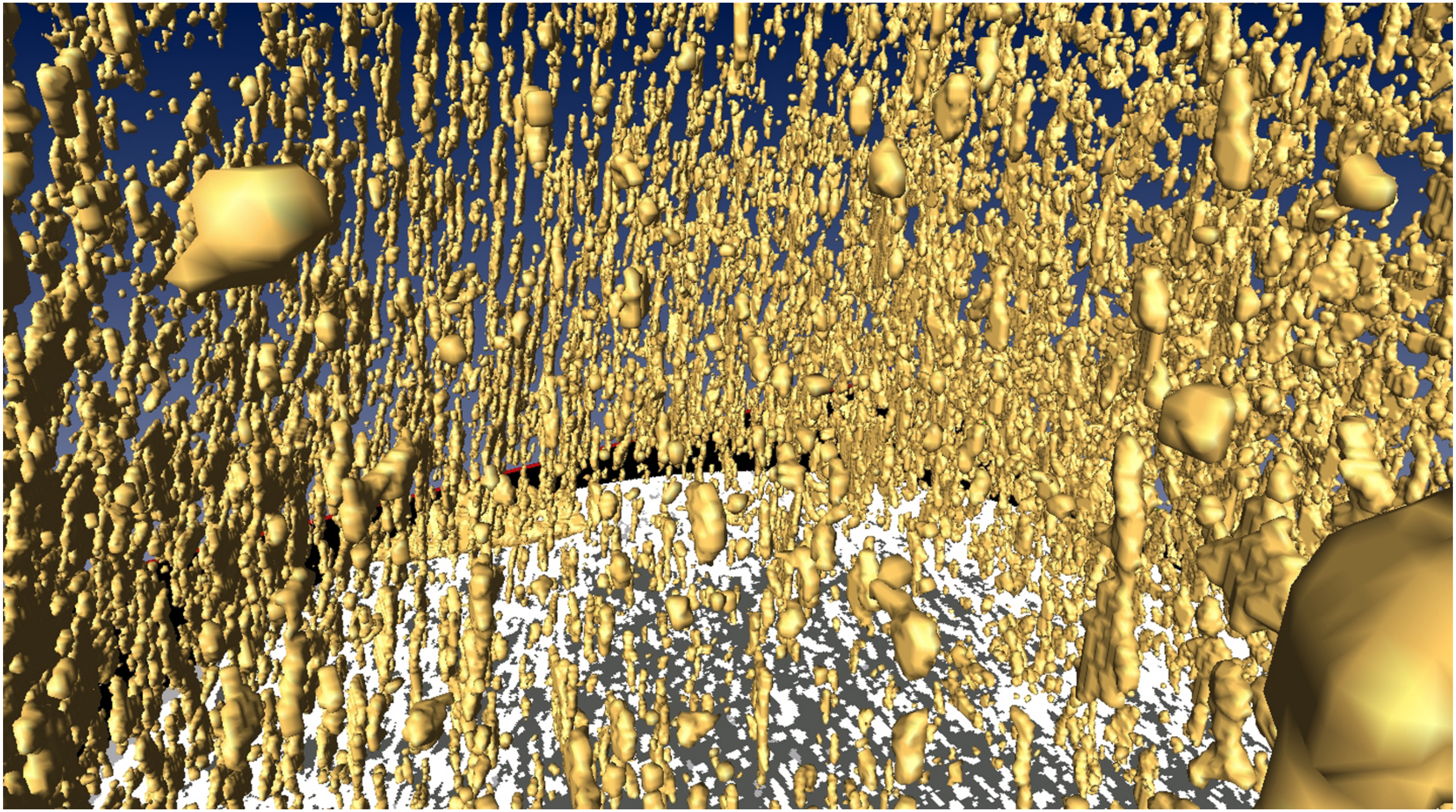
T. Maksym and T. Markus, 2008

September snow-ice estimates

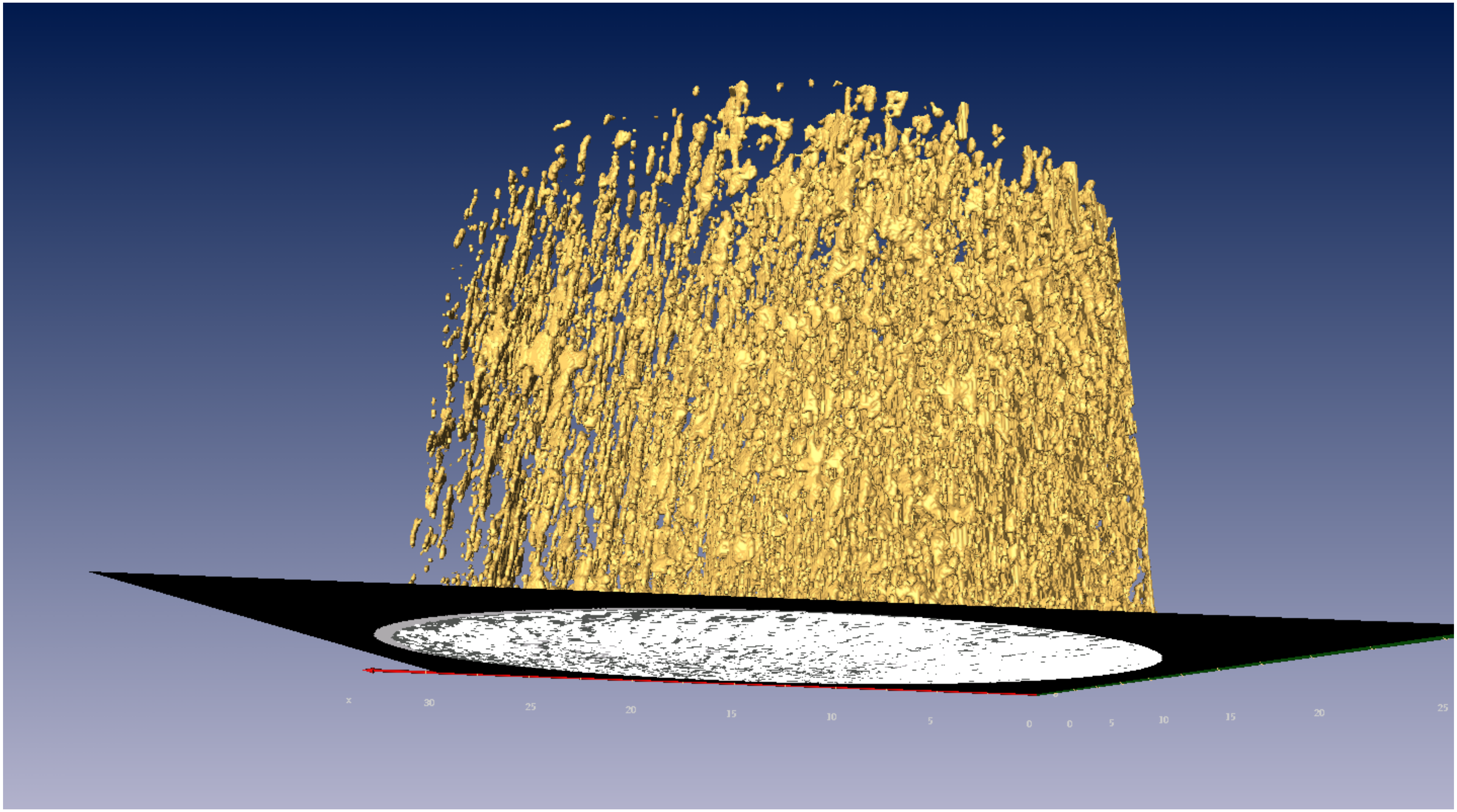
Antarctic surface flooding and snow-ice formation

- evolution of salinity profiles

- ocean-ice-air exchanges of heat, CO_2

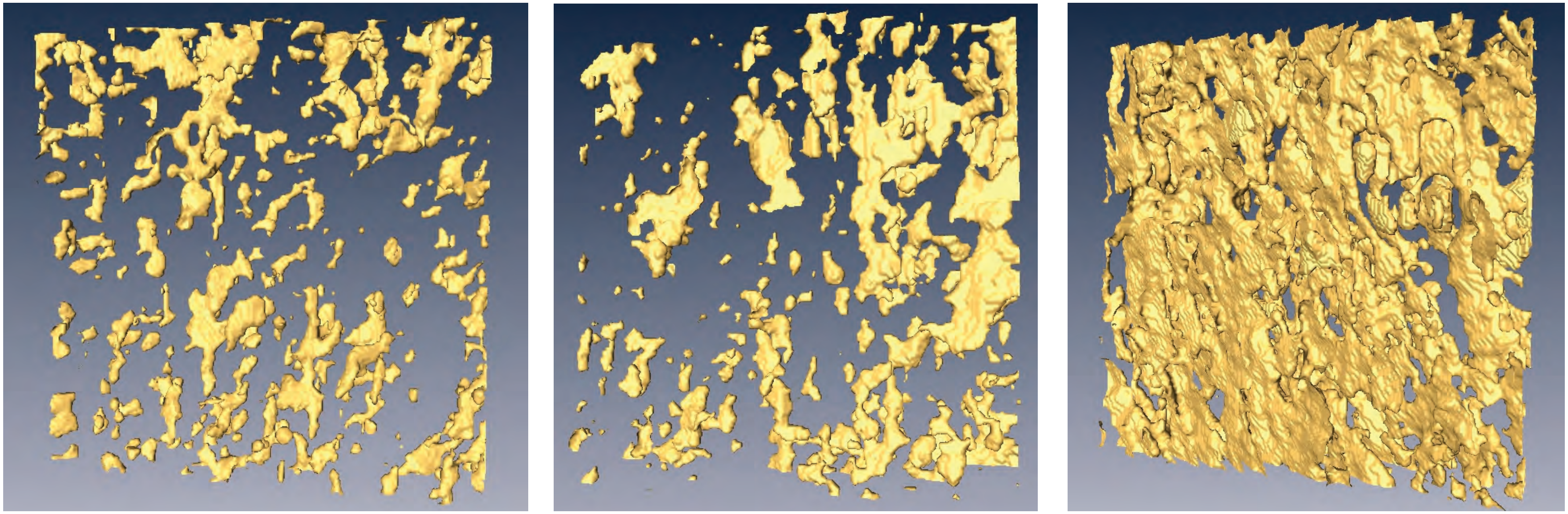


$\phi = 3.3\ \%$ $T = -18$

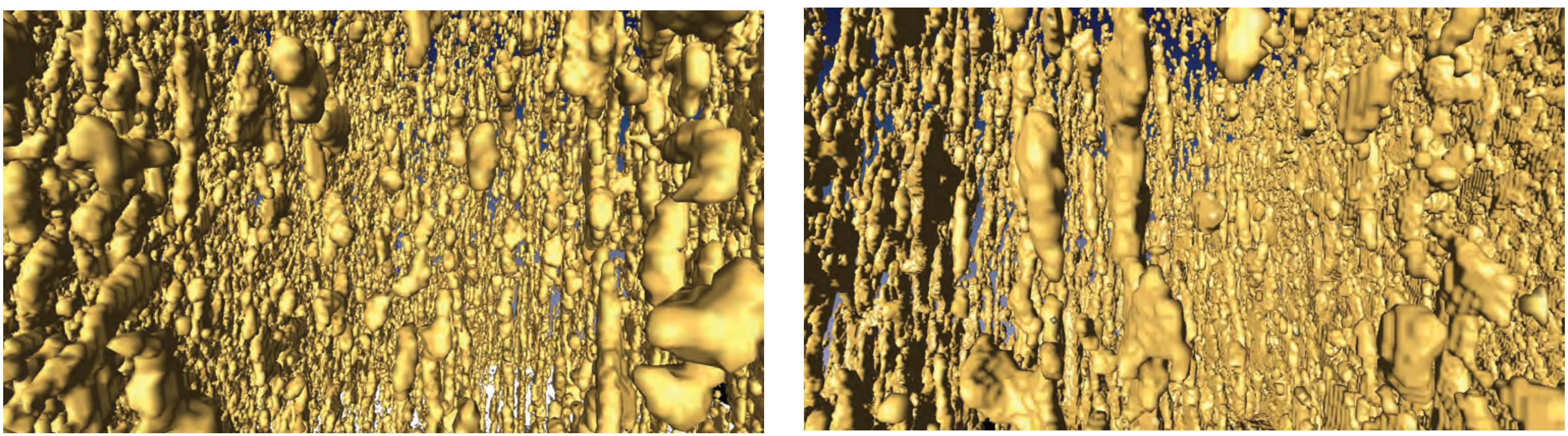


$\phi = 7.8\ \%$ $T = -6$

brine volume fraction and ***connectivity*** increase with temperature



T = -15 °C, $\phi = 0.033$ T = -6 °C, $\phi = 0.075$ T = -3 °C, $\phi = 0.143$



T = -8 °C, $\phi = 0.057$ T = -4 °C, $\phi = 0.113$

X-ray tomography for brine in sea ice

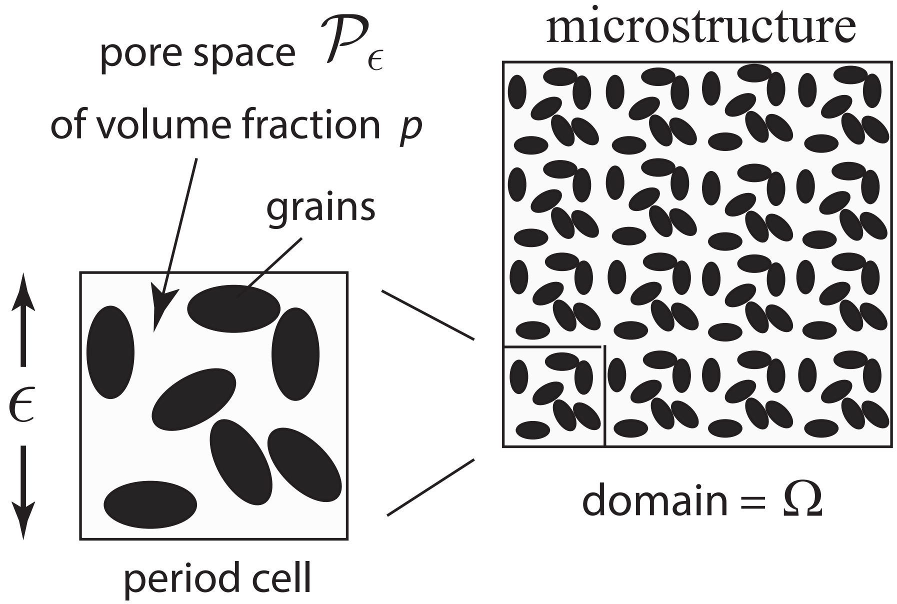
Golden *et al.*, *Geophysical Research Letters*, 2007

fluid permeability of a porous medium

HOMOGENIZE

how much fluid gets through the sample per unit time?

Stokes equations for fluid velocity $\mathbf{v}^\epsilon$, pressure p^ϵ, force $\mathbf{f}$:



$$\nabla p^\epsilon - \epsilon^2 \eta \Delta \mathbf{v}^\epsilon = \mathbf{f}, \qquad x \in \mathcal{P}_\epsilon$$

$$\nabla \cdot \mathbf{v}^\epsilon = 0, \qquad x \in \mathcal{P}_\epsilon$$

$$\mathbf{v}^\epsilon = 0, \qquad x \in \partial \mathcal{P}_\epsilon$$

$$\eta = \text{fluid viscosity}$$

via two-scale expansion

MACROSCOPIC EQUATIONS $\mathbf{v}^\epsilon \longrightarrow \mathbf{v}$, $p^\epsilon \longrightarrow p$ as $\epsilon \longrightarrow 0$

Darcy's law (f = 0)

$$\mathbf{v} = -\frac{1}{\eta}\,\mathbf{k}\,\nabla p, \qquad x \in \Omega$$

$$\nabla \cdot \mathbf{v} = 0, \qquad x \in \Omega$$

$\mathbf{k}(x)$ = **effective fluid permeability tensor**

[Keller '80, Tartar '80, Sanchez-Palencia '80, J. L. Lions '81, Allaire '89, '91,'97]

fluid permeability of a porous medium

how much water gets through the sample per unit time?

Darcy's Law

for slow viscous flow in a porous medium

averaged fluid velocity

pressure gradient

$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\,\nabla p$$

viscosity

$\mathbf{k}$ = fluid permeability tensor

HOMOGENIZATION

mathematics for analyzing effective behavior of heterogeneous systems

Critical behavior of fluid transport in sea ice

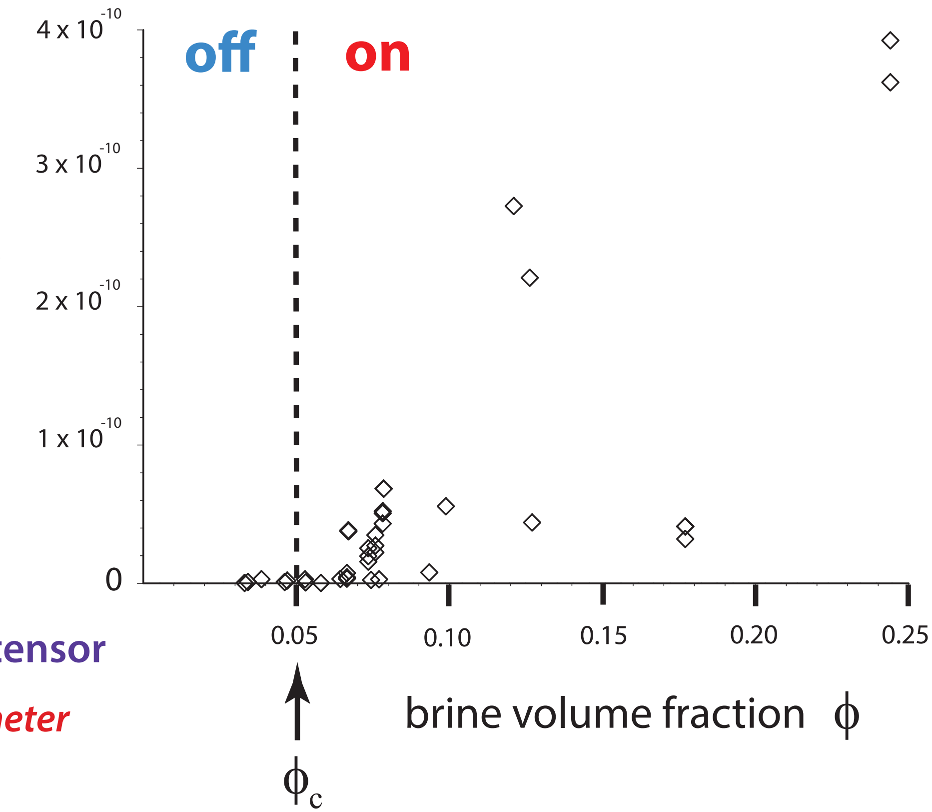
impermeable permeable

Arctic field data

vertical fluid permeability k (m^2)

Darcy's Law

$$\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$$



$\mathbf{k}$ = fluid permeability tensor

homogenized parameter

"on - off" switch for fluid flow

PERCOLATION THRESHOLD $\phi_c \approx 5\%$ $\longleftrightarrow$ $T_c \approx -5^\circ C, \ S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

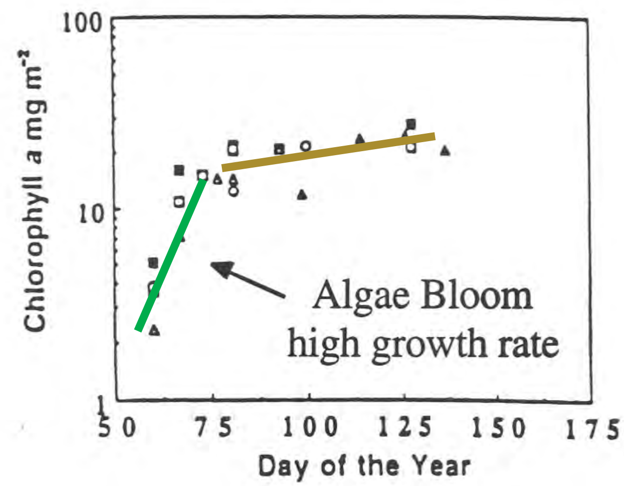
sea ice algal communities

D. Thomas 2004

nutrient replenishment controlled by ice permeability

biological activity turns on or off according to ***rule of fives***

critical behavior of microbial activity



Convection-fueled algae bloom
Ice Station Weddell

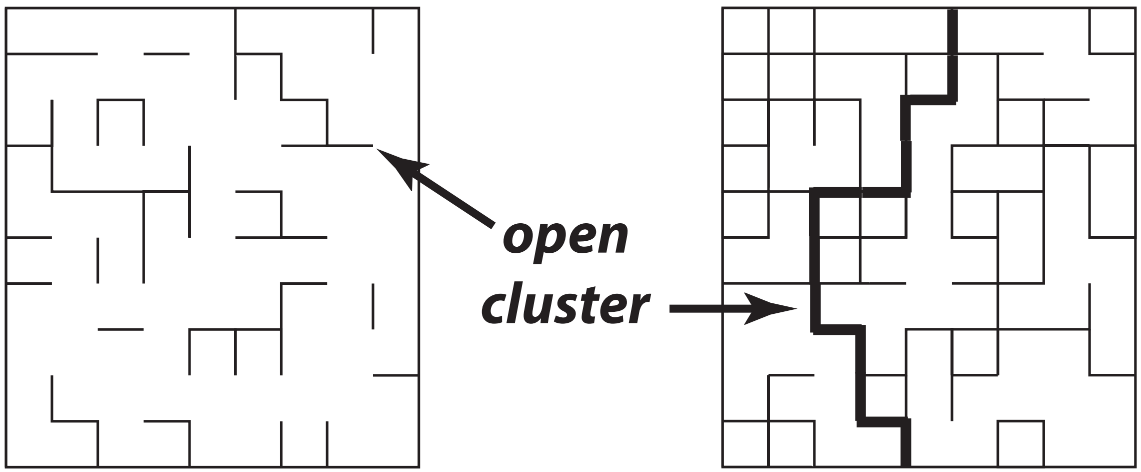
Golden, Ackley, Lytle ***Science 1998***

Fritsen, Lytle, Ackley, Sullivan ***Science 1994***

percolation theory

probabilistic theory of connectedness

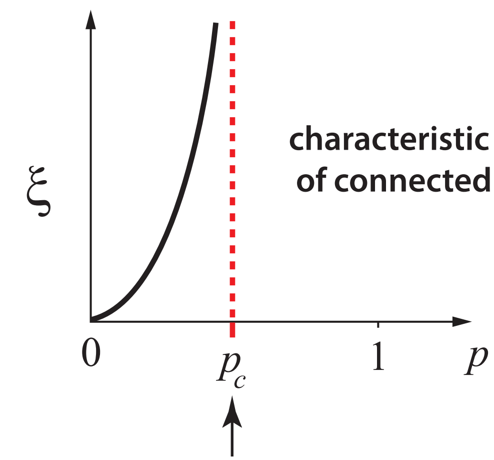
impermeable **permeable**



$p = 1/3$ $p = 2/3$

correlation length

development of long range order



percolation threshold

bond ⟶ ***open*** *with probability* p
closed *with probability 1-p*

$$\xi(p) \sim |p - p_c|^{-\nu} \qquad p \to p_c$$

ν **universal: depends only on *d***

p_c depends on type of lattice and *d*

percolation threshold

$$p_c = 1/2 \quad \text{for} \;\; d = 2$$

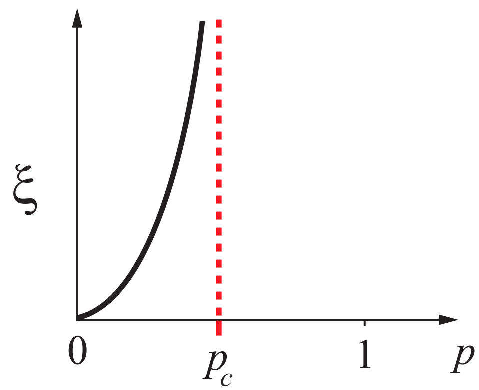
smallest *p* for which there is an infinite open cluster

order parameters in percolation theory

geometry

correlation length

characteristic scale
of connectedness

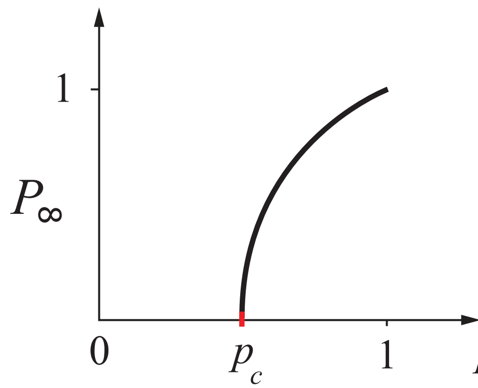


$$\xi(p) \sim |p - p_c|^{-\nu}$$

$$p \to p_c$$

infinite cluster density

probability the origin
belongs to infinte cluster

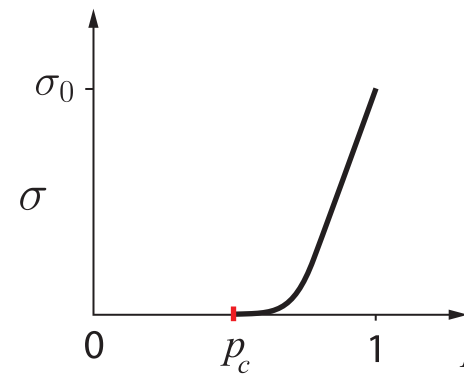


$$P_\infty(p) \sim (p - p_c)^{\beta}$$

$$p \to p_c^+$$

transport

effective conductivity or fluid permeability



$$\sigma(p) \sim \sigma_0 (p - p_c)^t$$

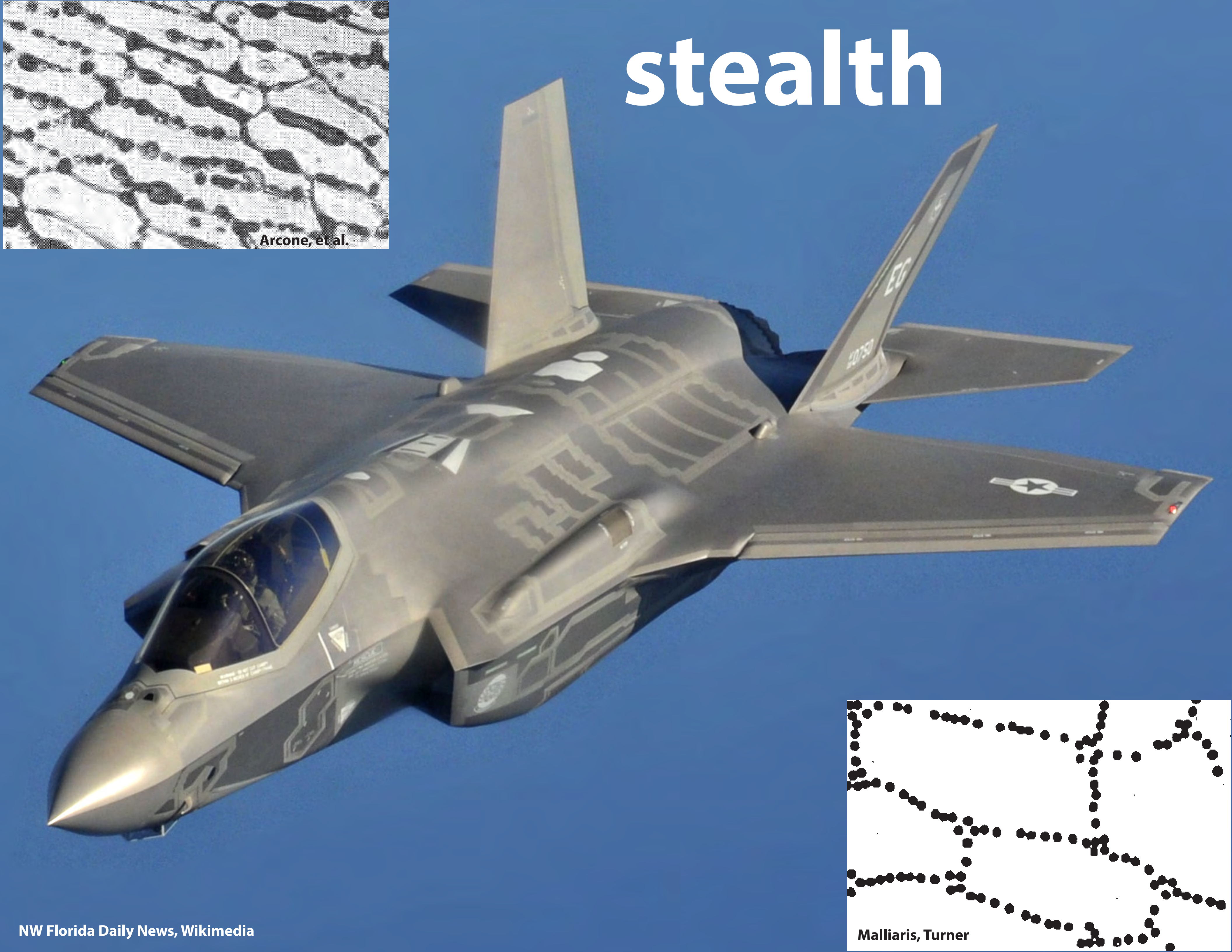
$$p \to p_c^+$$

UNIVERSAL critical exponents for lattices -- depend only on dimension

$1 \leq t \leq 2$ (for idealized model), Golden, *Phys. Rev. Lett.* 1990 ; *Comm. Math. Phys.* 1992

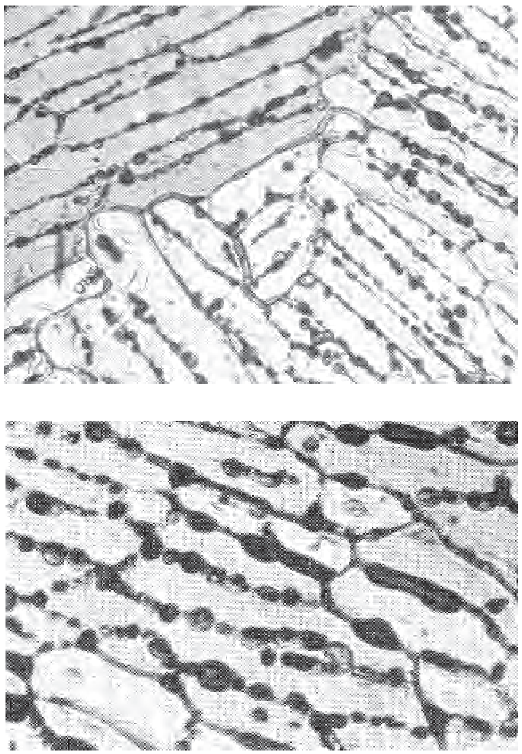
non-universal behavior in continuum

stealth

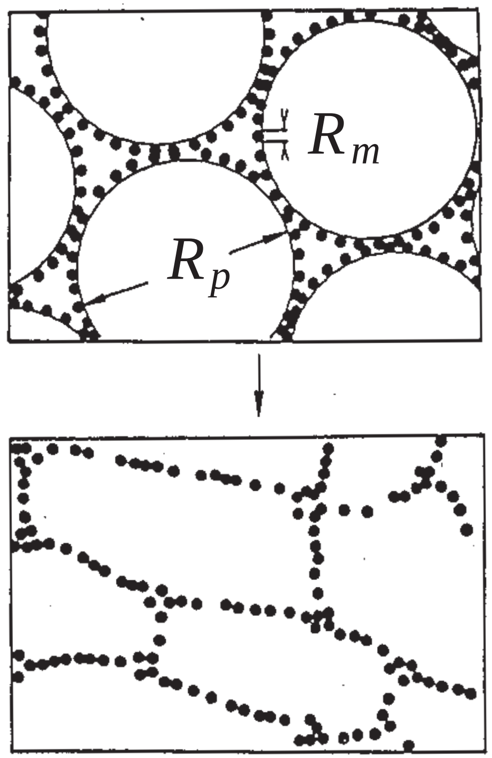

Arcone, et al.
NW Florida Daily News, Wikimedia
Malliaris, Turner

Continuum percolation model for ***stealthy*** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

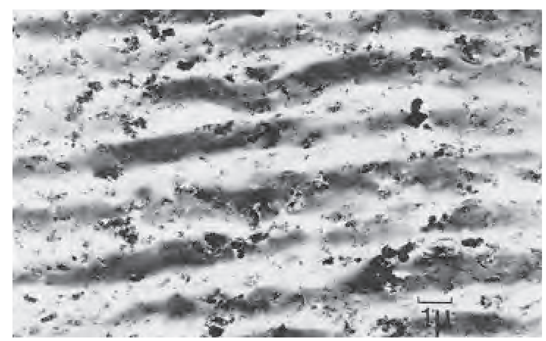
$\phi_c \approx 5\ \%$ Golden, Ackley, Lytle, *Science*, 1998



sea ice



compressed powder



radar absorbing composite

sea ice is radar absorbing

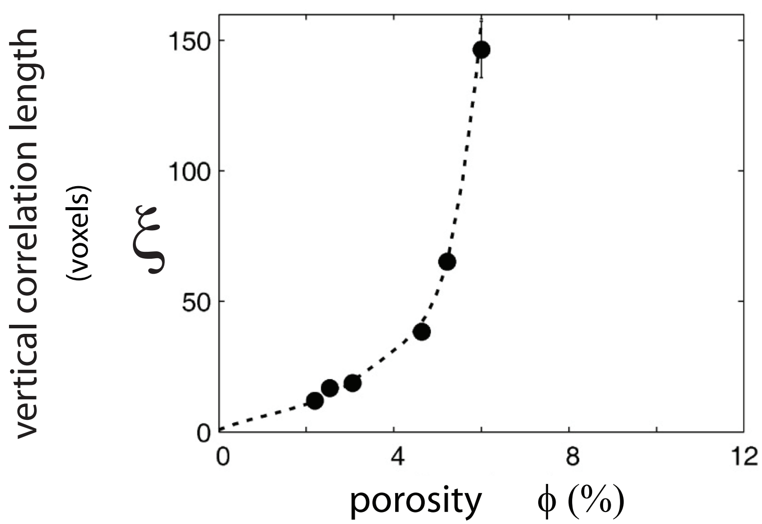
order parameters in brine percolation

geometry

correlation length

(characteristic scale of connectedness)

divergence of correlation length for single crystal CT data



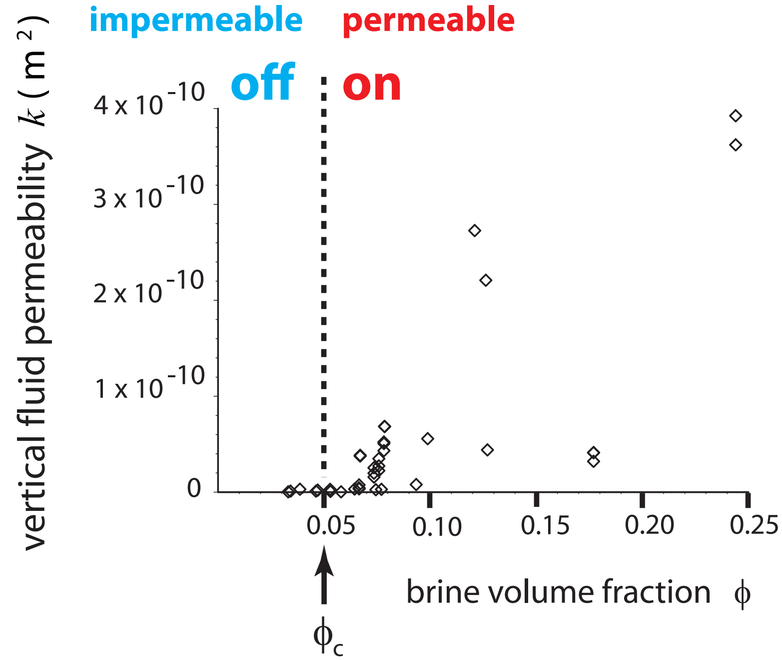
$$\xi(p) \sim |\phi - \phi_c|^{-\nu}$$

Pringle, Miner, Eicken, Golden, JGR (Oceans) 2009

transport

sea ice permeability

Arctic field data



$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical exponent t

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

exponent is UNIVERSAL lattice value $t \approx 2.0$

Golden, Ackley, Lytle *Science* 1998

Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophys. Res. Lett.* 2007

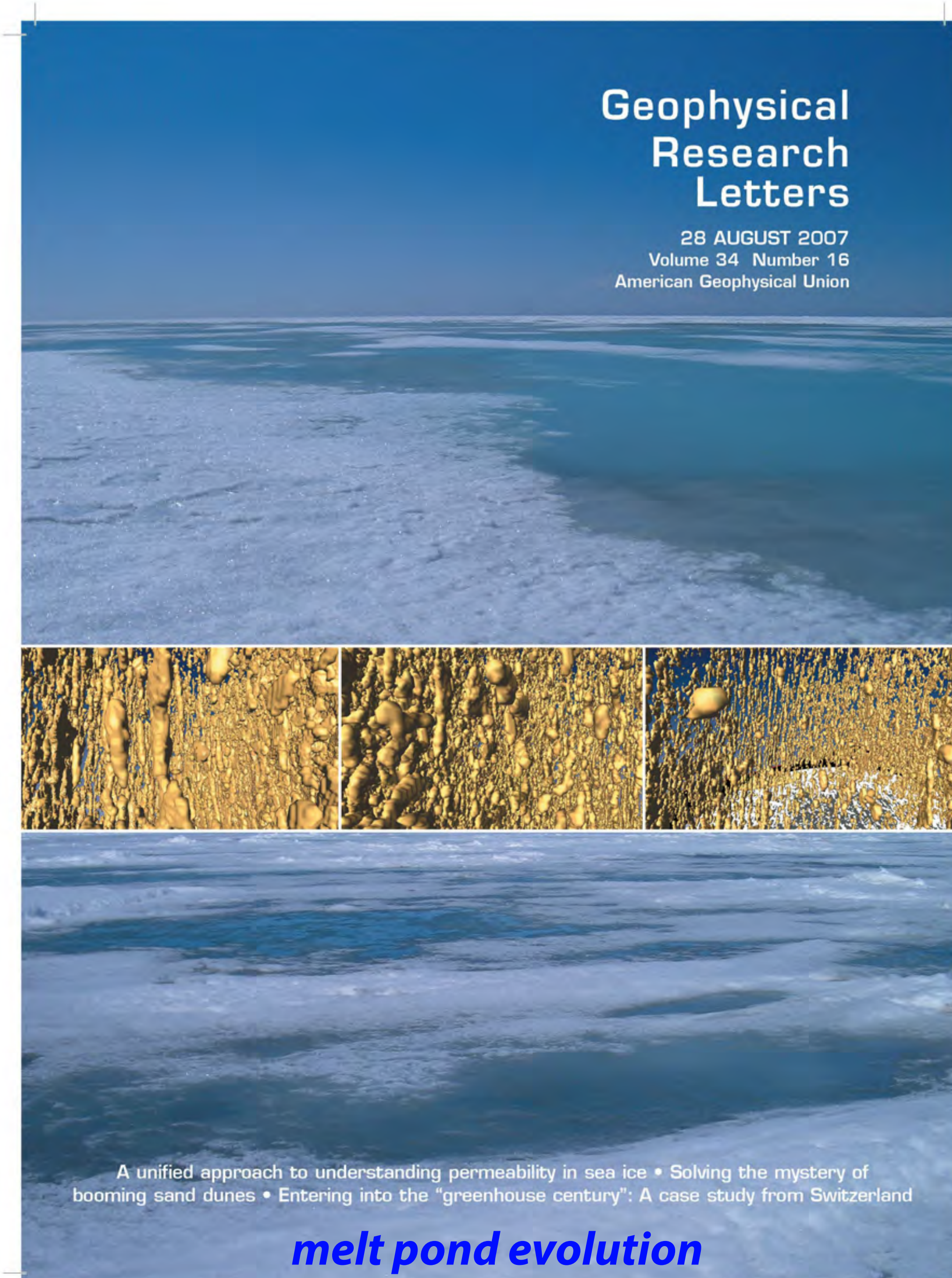
Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007

microscale

governs

mesoscale processes



melt pond evolution

percolation theory for fluid permeability

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical exponent t

$$k_0 = 3 \times 10^{-8} \ \mathrm{m}^2$$

from critical path analysis in hopping conduction

hierarchical model

rock physics

network model

rigorous bounds

X-ray tomography for brine inclusions

confirms rule of fives

brine percolation threshold of $\phi = 5\%$ for bulk fluid flow

Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009

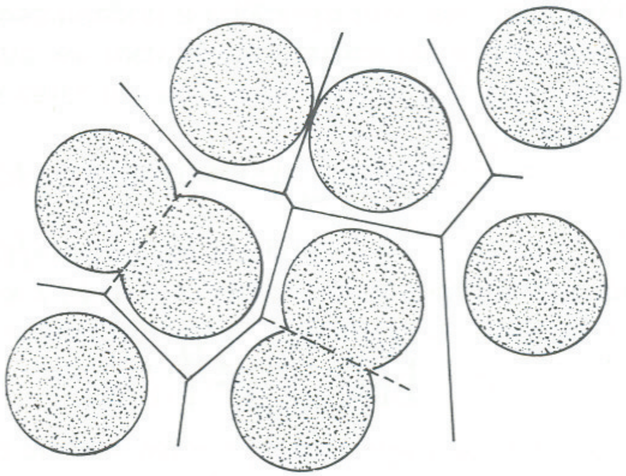
theories agree closely with field data

Non-universal behavior in the continuum:

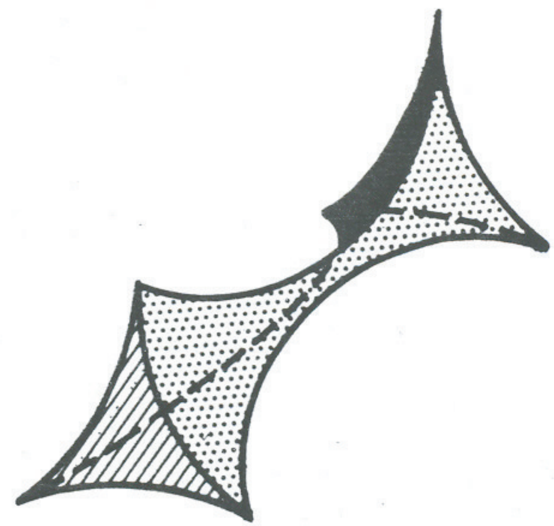
critical exponents for transport in Swiss cheese model take values different than for lattices, e.g. $t > 2$

$e \neq t$

Halperin, Feng, Sen, *Phys. Rev. Lett.* 1985



Swiss cheese model
$d = 2$



conducting neck in $d = 3$
Swiss cheese model

in general, non-universal exponents arise from a singular distribution of local conductances

In sea ice, this distribution is lognormal.

(excluding inclusions below cutoff)

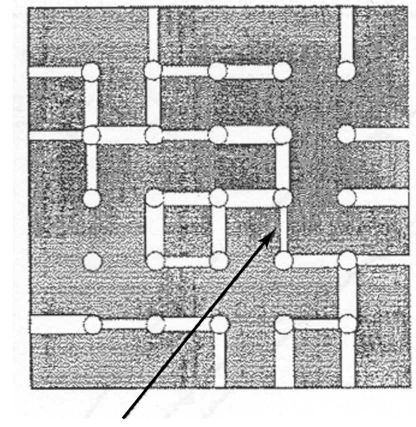
Thus, the permeability exponent for sea ice is **2**, the universal lattice value.

ESTIMATE fluid conductivity scaling factor $k_0 = r^2/8$

for media with broad range of conductances

CRITICAL PATH ANALYSIS

bottlenecks control flow



Ambegaokar, Halperin, Langer 1971: CPA in electronic hopping conduction
Friedman, Seaton 1998: CPA in fluid and electrical networks
Golden, Kozlov 1999: rigorous CPA on long-range checkerboard model

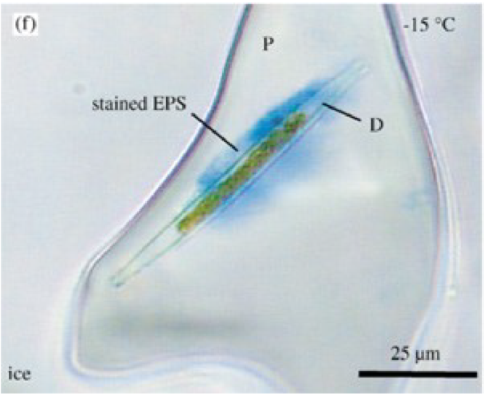
$k_0 \approx r_c^2 / 8$ critical fluid conductivity

Microstructural analyses yield $r_c \approx$ 0.5 mm

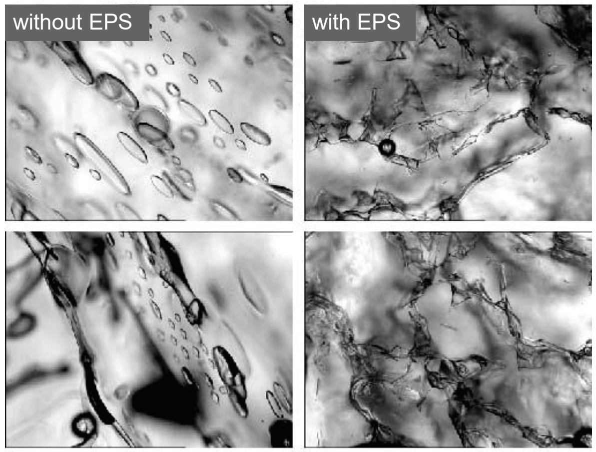
Sea ice algae secrete exopolymeric substances (EPS) affecting evolution of brine microstructure.

How does EPS affect fluid transport? **How does the biology affect the physics?**

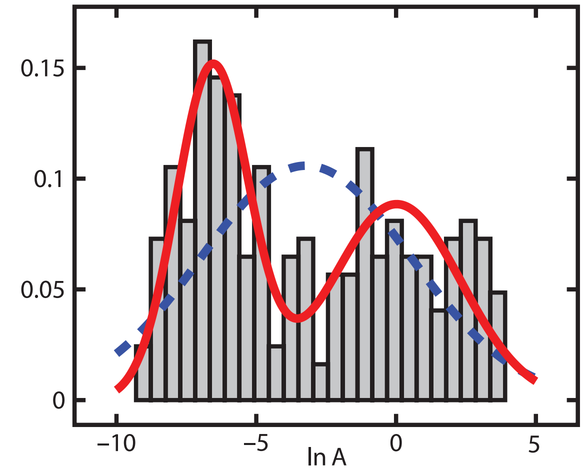
FRACTAL



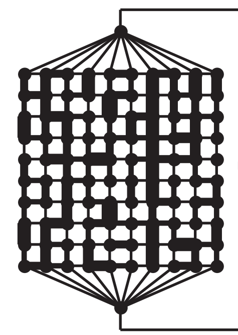
Krembs



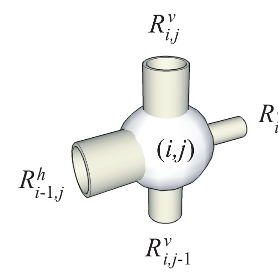
Krembs, Eicken, Deming, PNAS 2011



RANDOM PIPE MODEL



- **2D random pipe model with bimodal distribution of pipe radii**
- **Rigorous bound on permeability *k*; results predict observed drop in *k***



Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
***Multiscale Modeling and Simulation*, 2018**

Zhu, Jabini, Golden, Eicken, Morris
***Ann. Glac.* 2006**

SIAM News
June 2024

EPS - Algae Model Jajeh, Reimer, Golden

electrical transport

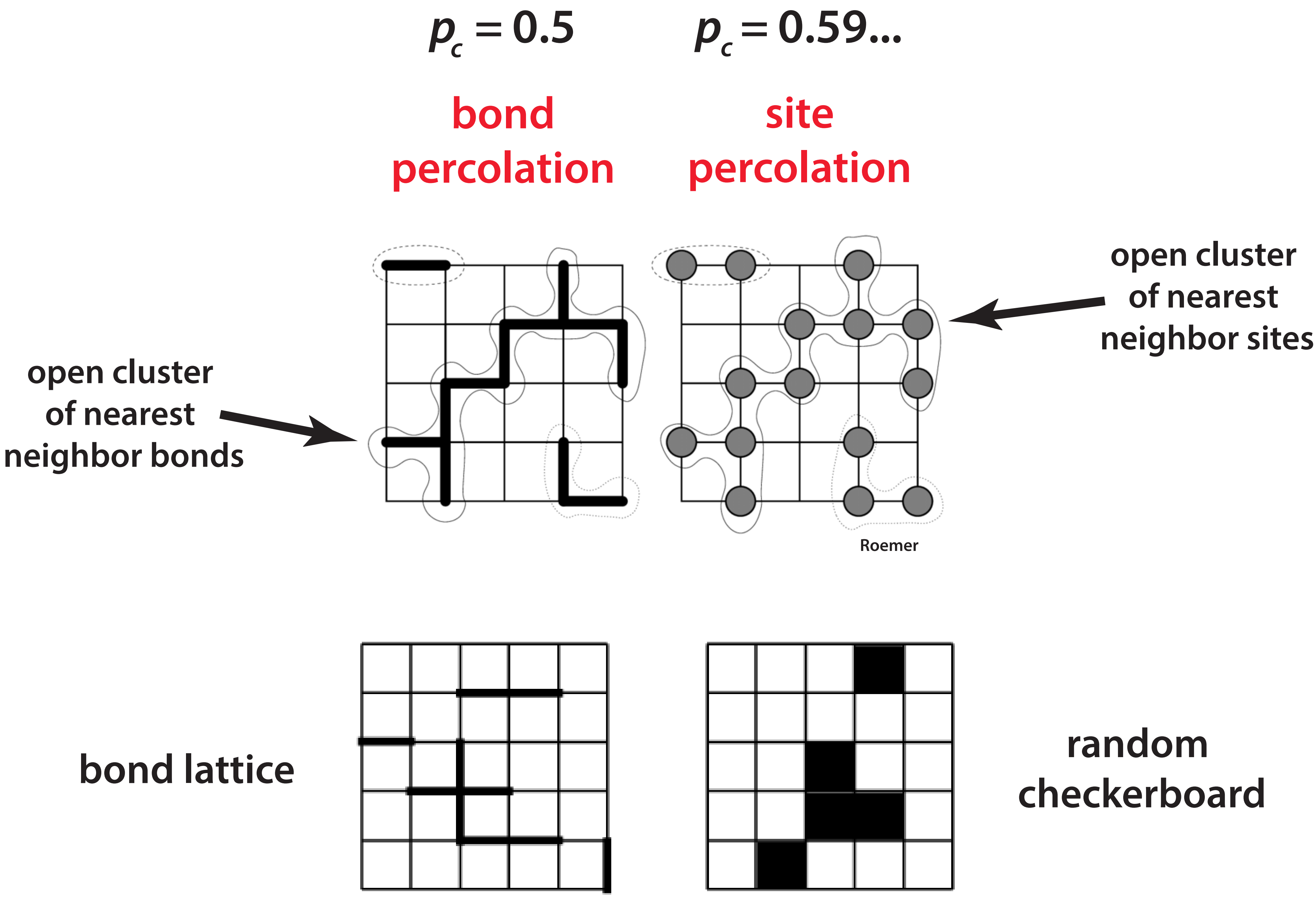
fluid transport

electrical conductance $g_e = \pi r^2 \sigma$

fluid conductance $g_f = \pi r^4 / 8\eta$

electrical conductivity $\sigma_e = \sigma$

fluid conductivity $\sigma_f = r^2 / 8\eta$



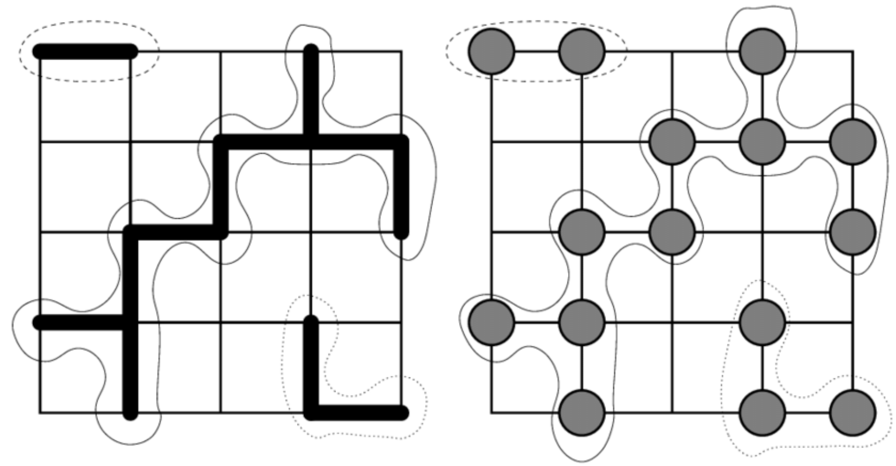
$p_c = 0.5$
$p_c = 0.59...$
bond
percolation
site
percolation
open cluster
of nearest
neighbor bonds
open cluster
of nearest
neighbor sites
Roemer
bond lattice
random
checkerboard

$p_c = 0.5$ $p_c = 0.59...$

bond percolation **site percolation**

open cluster of nearest neighbor bonds

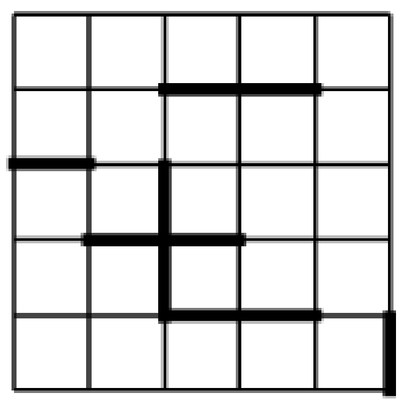
open cluster of nearest neighbor sites

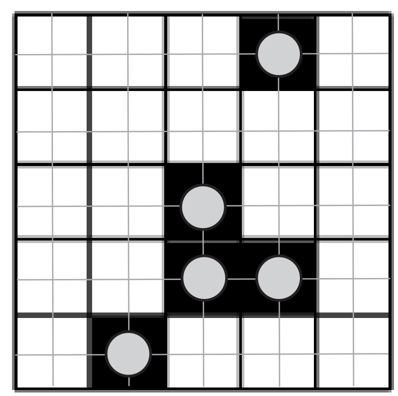


Roemer

discrete

bond lattice





continuum

random checkerboard

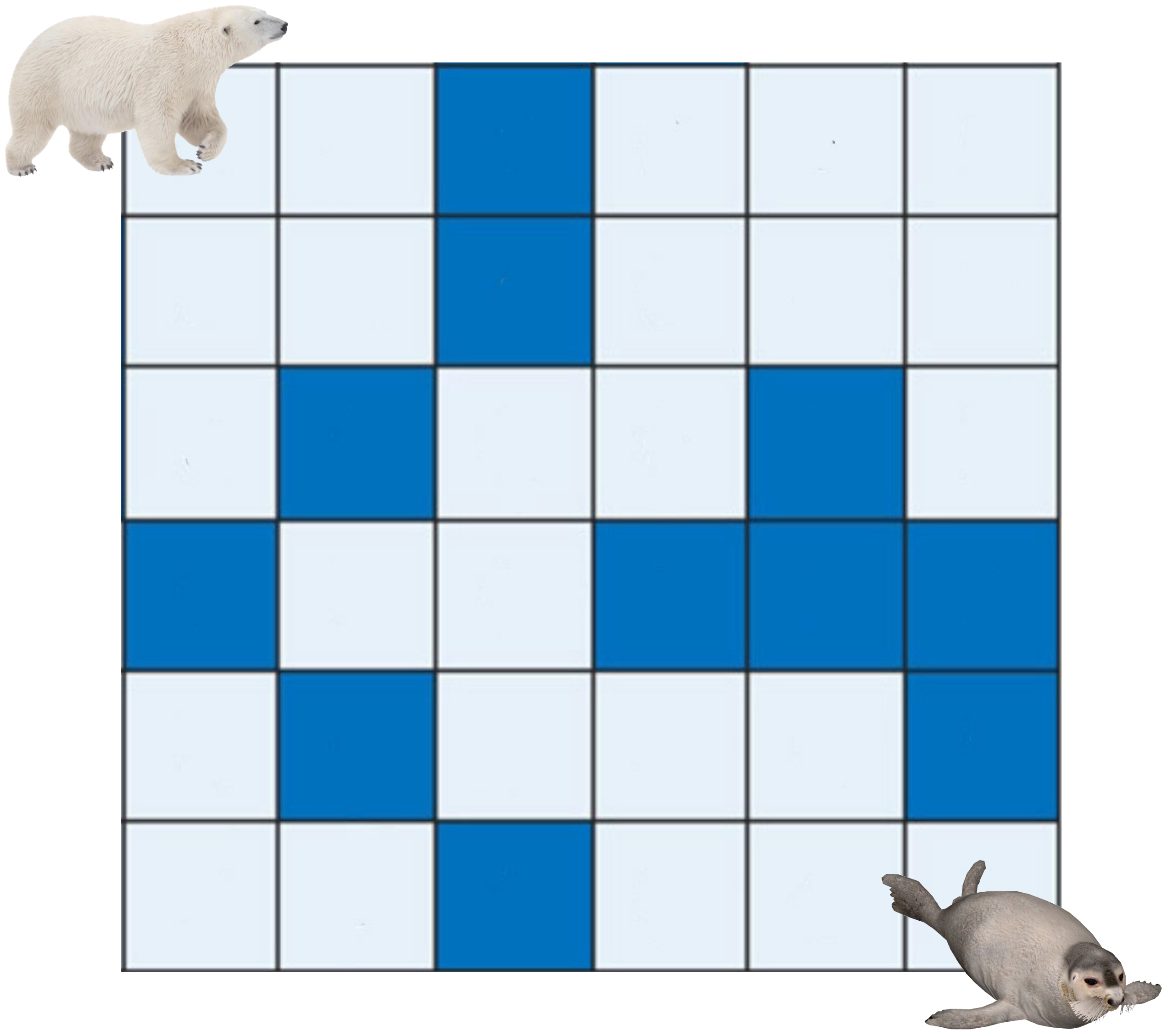
$p_c = 0.59...$

Optimal Movement of a Polar Bear in a Heterogenous Icescape

Nicole Forrester, Jody Reimer, Ken Golden 2024

Polar bears expend 5X more energy swimming than walking on sea ice.

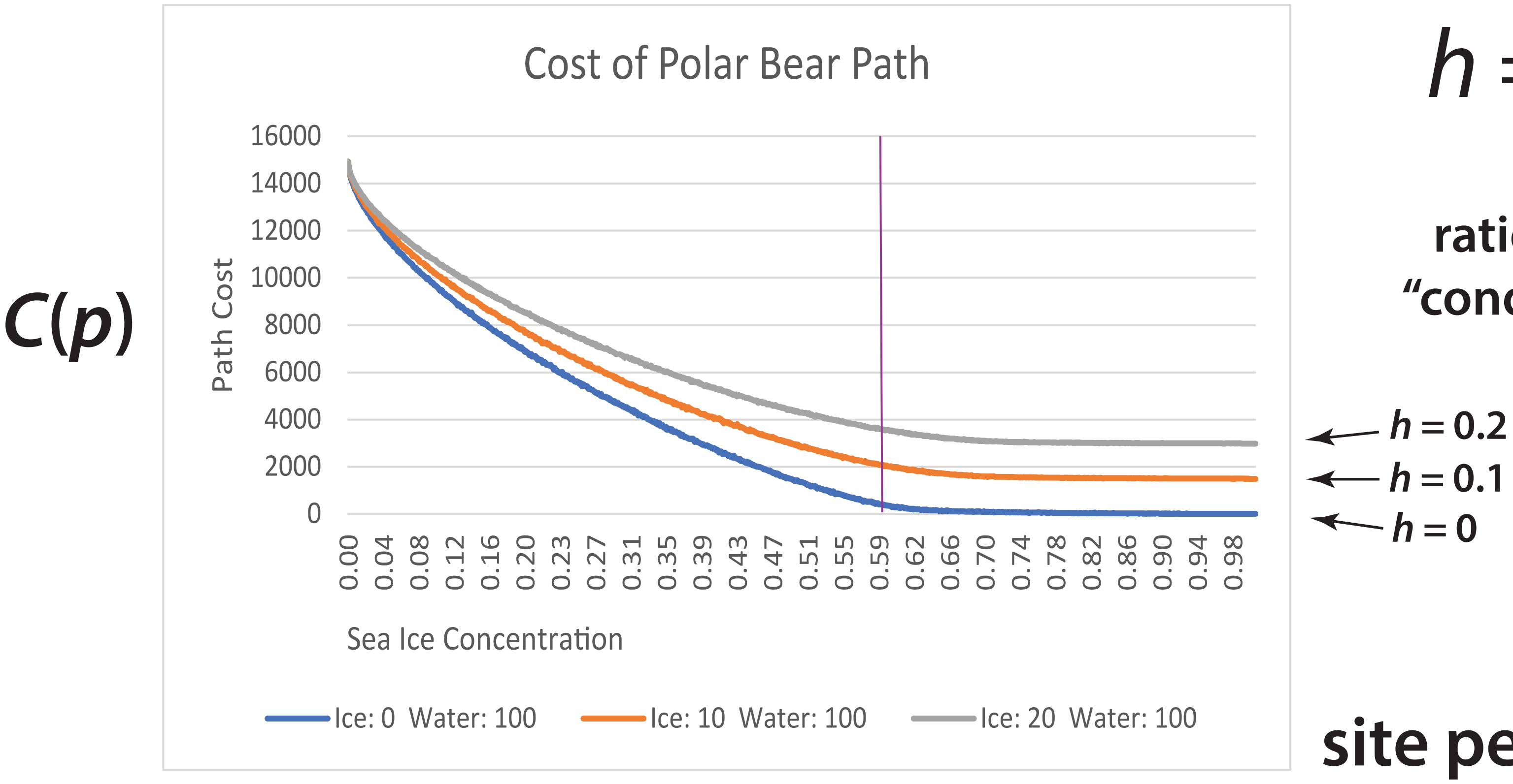
As sea ice is lost, how do polar bears optimize their movement to save energy and survive?



https://pluspng.com/img-png/polar-bear-png-polar-white-bear-png-1000.png
https://zt2downloadlibrary.fandom.com/wiki/Bearded_Seal_(Tamara_Henson)

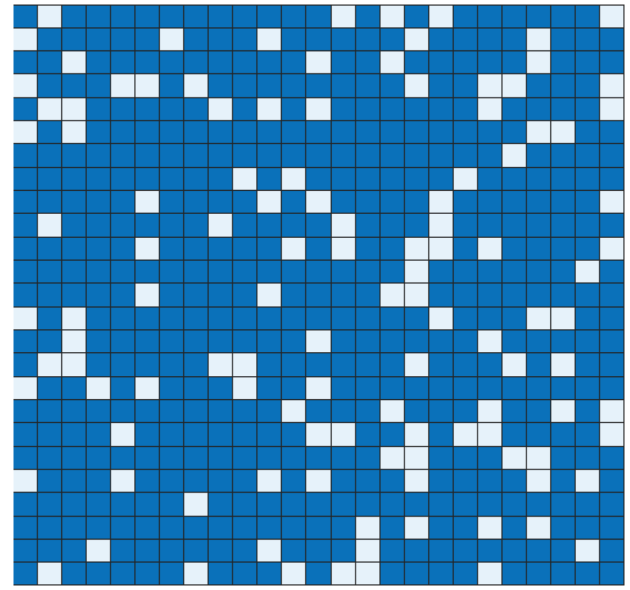
Polar Bear Percolation

To study the importance of ice connectedness, we exaggerate the data by setting the cost of walking on ice to 0 with the cost of swimming still at 5.



$$h = \frac{c_i}{c_w}$$

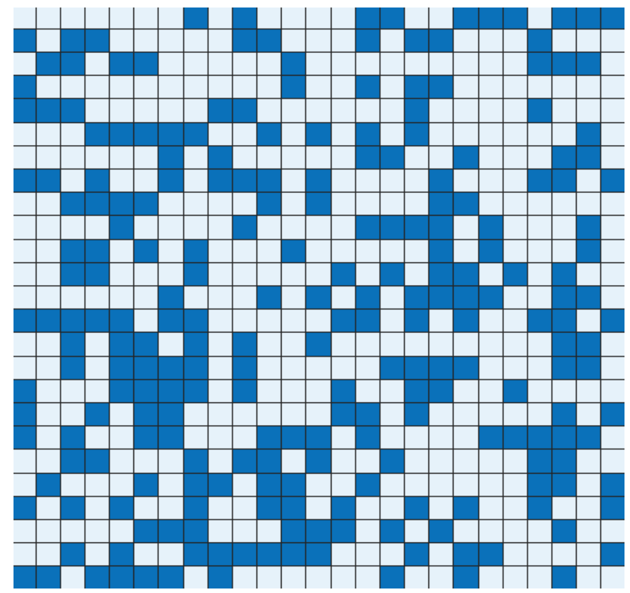
ratio of local "conductivities"



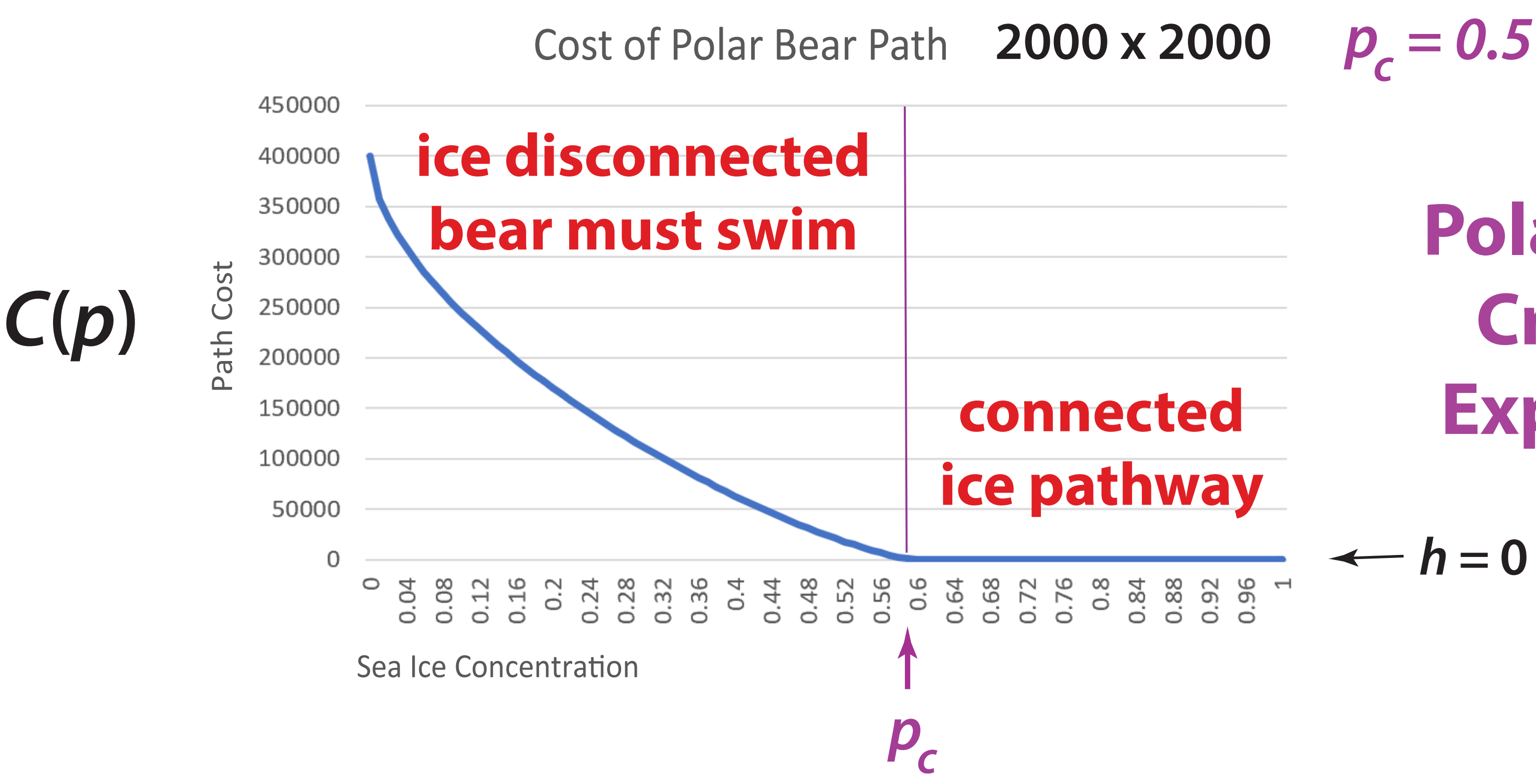
20% Ice

site percolation threshold

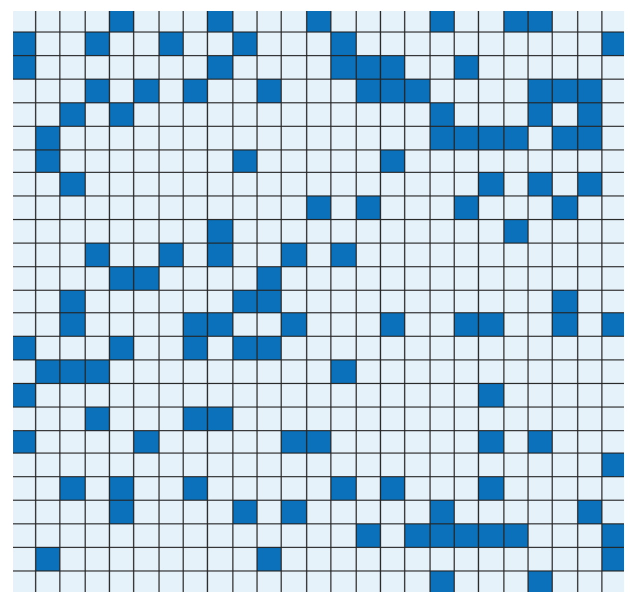
$p_c = 0.59$ for $d = 2$



60% Ice



Polar Bear Critical Exponent



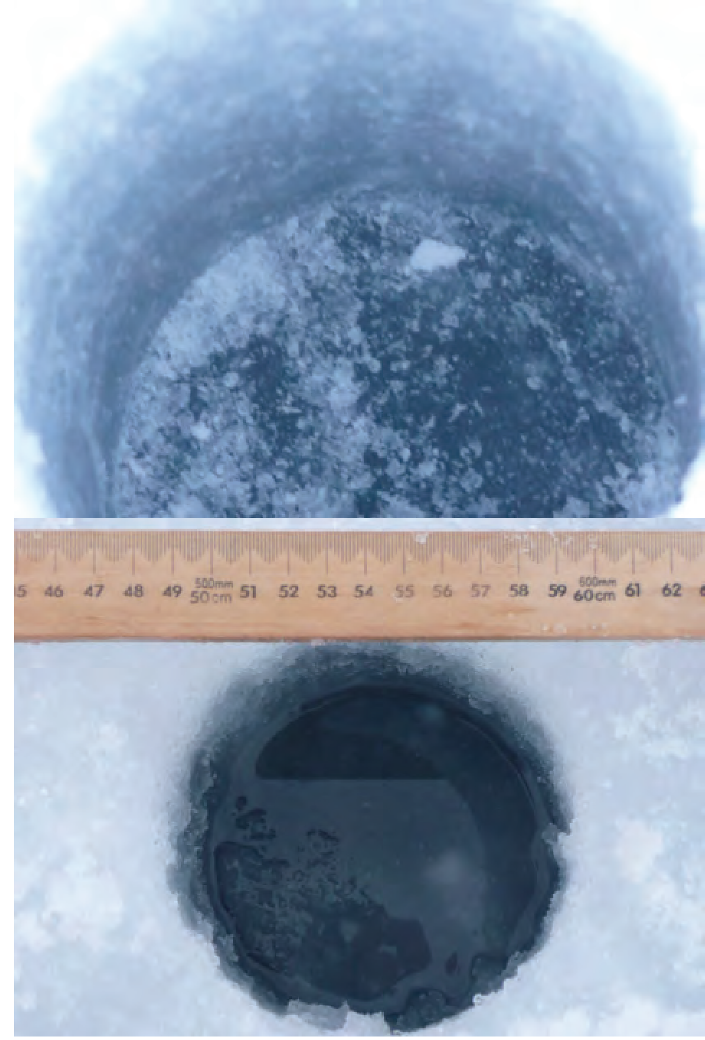
80% Ice

cores from ice sheets give climate records, e.g. CO2 data
analysis of gas in bubbles

percolation theory gives critical depth below which air bubbles are disconnected from atmosphere

Enting, Nature 1985

photo by Jan Lieser



measuring fluid permeability of Antarctic sea ice

SIPEX 2007

Arctic and Antarctic field experiments

develop electromagnetic methods of monitoring fluid transport and microstructural transitions

extensive measurements of fluid and electrical transport properties of sea ice:

2007	*Antarctic*	*SIPEX*
2010	*Antarctic*	*McMurdo Sound*
2011	*Arctic*	*Barrow AK*
2012	*Arctic*	*Barrow AK*
2012	*Antarctic*	*SIPEX II*
2013	*Arctic*	*Barrow AK*
2014	*Arctic*	*Chukchi Sea*

The Melt Pond Conundrum:

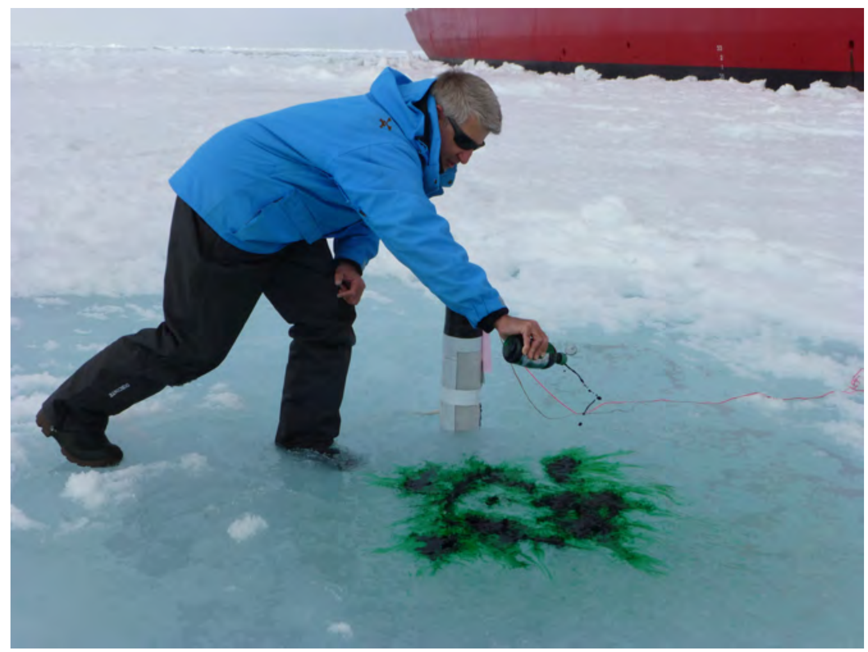
How can ponds form on top of sea ice that is highly permeable?

C. Polashenski, K. M. Golden, D. K. Perovich, E. Skyllingstad, A. Arnsten, C. Stwertka, N. Wright

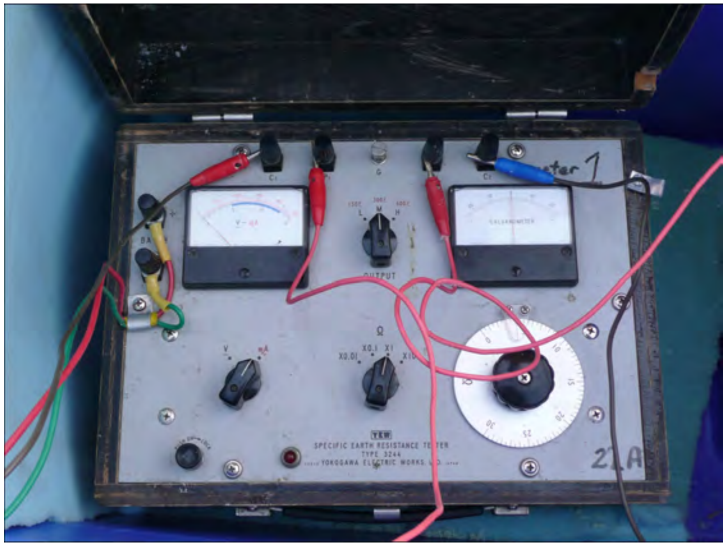
Percolation Blockage: A Process that Enables Melt Pond Formation on First Year Arctic Sea Ice

J. Geophys. Res. Oceans 2017

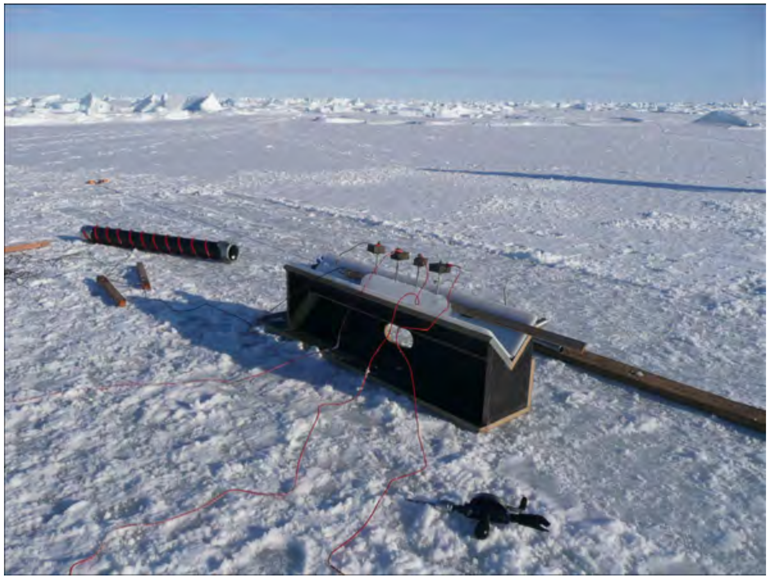
2014 Study of Under Ice Blooms in the Chuckchi Ecosystem (SUBICE) aboard USCGC Healy

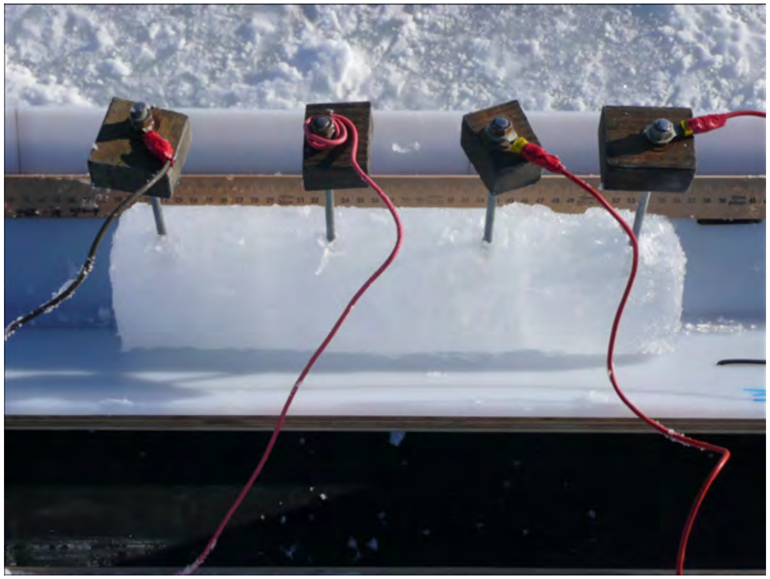


electrical measurements



Wenner array





vertical conductivity

Zhu, Golden, Gully, Sampson ***Physica B*** 2010

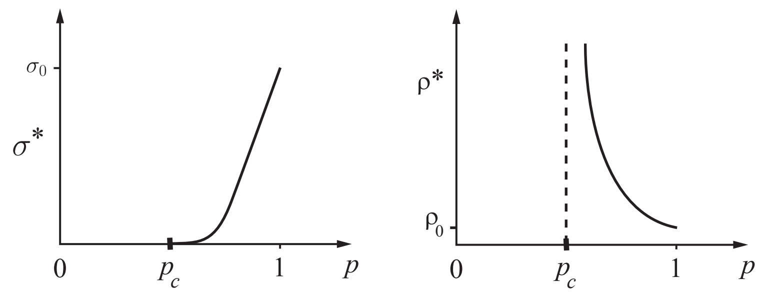
Sampson, Golden, Gully, Worby ***Deep Sea Research*** 2011

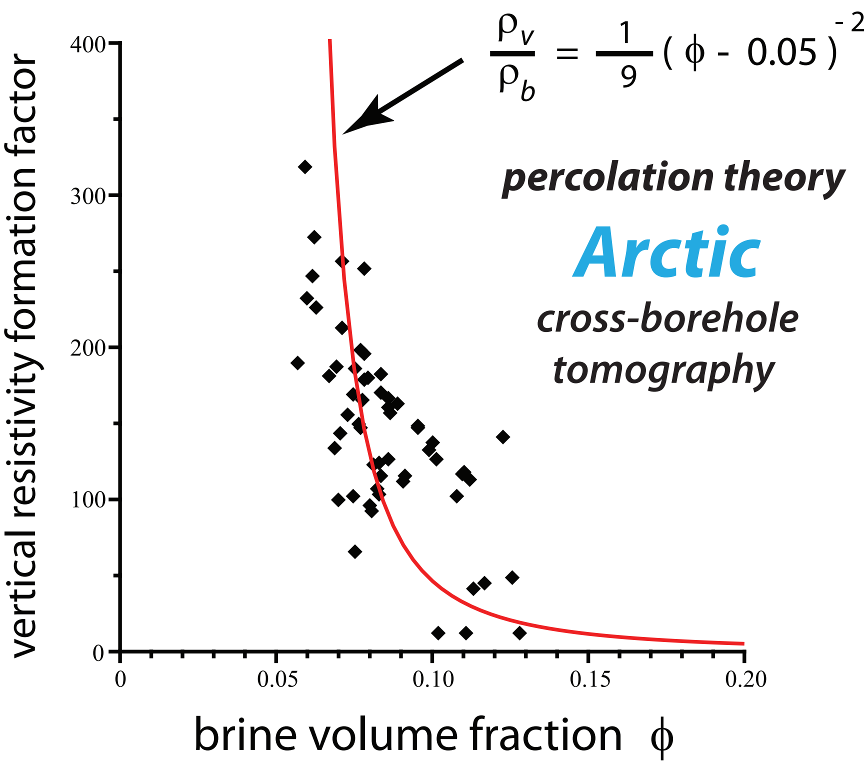
critical behavior of electrical transport in sea ice

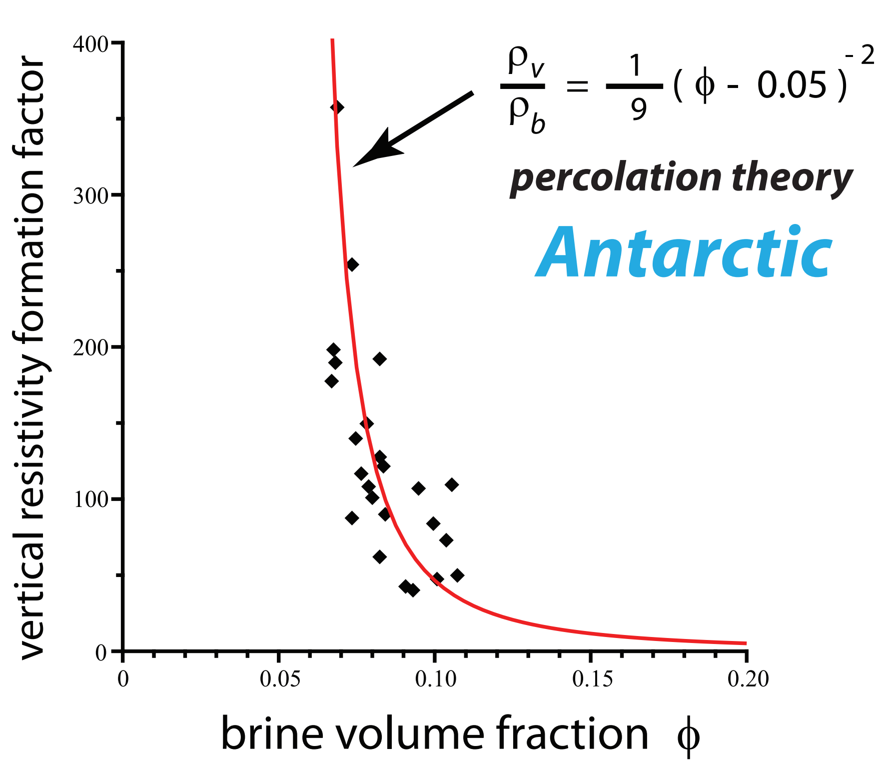
electrical signature of the on-off switch for fluid flow

same universal critical exponent as for fluid permeability

studied for over 50 years but no previous observations or theory of critical behavior







Golden, Eicken, Gully, Ingham, Jones, Lin, Reid, Sampson, Worby 2022

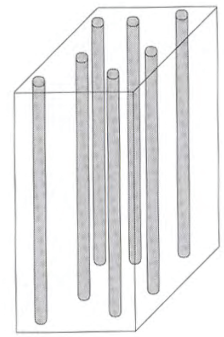
PIPE BOUNDS on vertical fluid permeability k

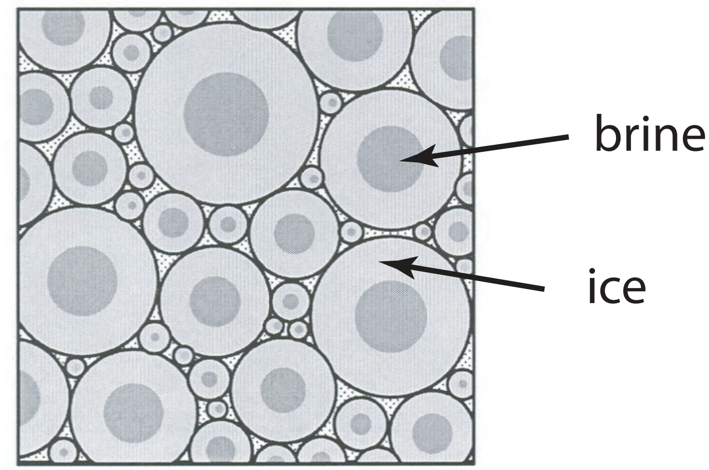
Golden, Heaton, Eicken, Lytle, Mech. Materials 2006
Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophys. Res. Lett. 2007

vertical pipes

with appropriate radii

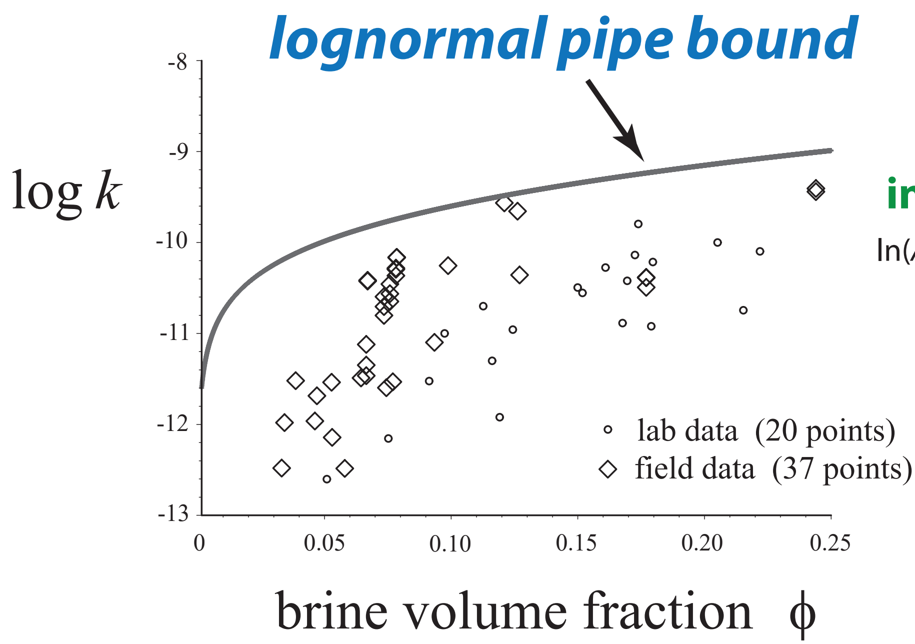
maximize k





optimal coated cylinder geometry

fluid analog of arithmetic mean upper bound for effective conductivity of composites (Wiener 1912)



Golden et al., Geophys. Res. Lett. 2007

$$k \;\leq\; \frac{\phi\langle R^4\rangle}{8\langle R^2\rangle} \;=\; \frac{\phi}{8}\langle R^2\rangle e^{\sigma^2}$$

inclusion cross sectional areas A lognormally distributed

ln(A) normally distributed, mean μ (increases with T) variance σ^2 (Gow and Perovich 96)

get bounds through variational analyis of ***trapping constant*** γ for diffusion process in pore space with absorbing BC

Torquato and Pham, *PRL* 2004

$$\mathbf{k} \;\leq\; \gamma^{-1}\mathbf{I}$$

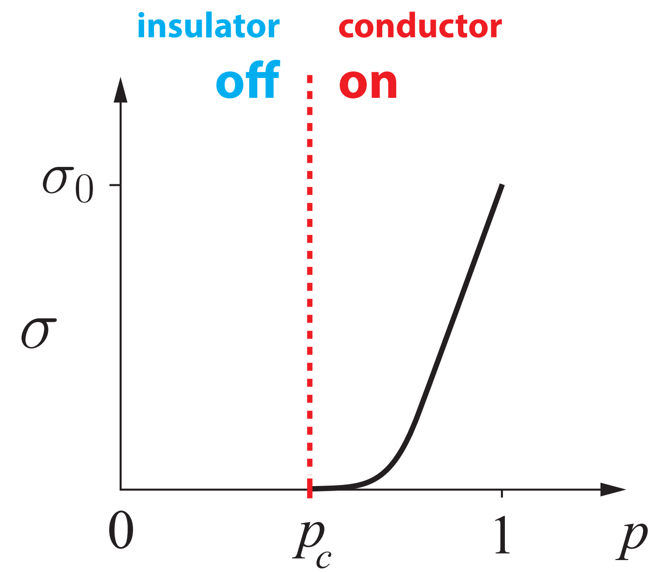
for any ergodic porous medium (Torquato 2002, 2004)

transport in percolation theory

MICRO —— lattice homogenization ——→ MACRO

local conductivity (electrical or fluid) **effective** conductivity or fluid permeability

$$\textbf{bond} \longrightarrow \begin{cases} \sigma_0 & \text{probability } p \\ 0 & \text{probability } 1 - p \end{cases}$$



percolation threshold

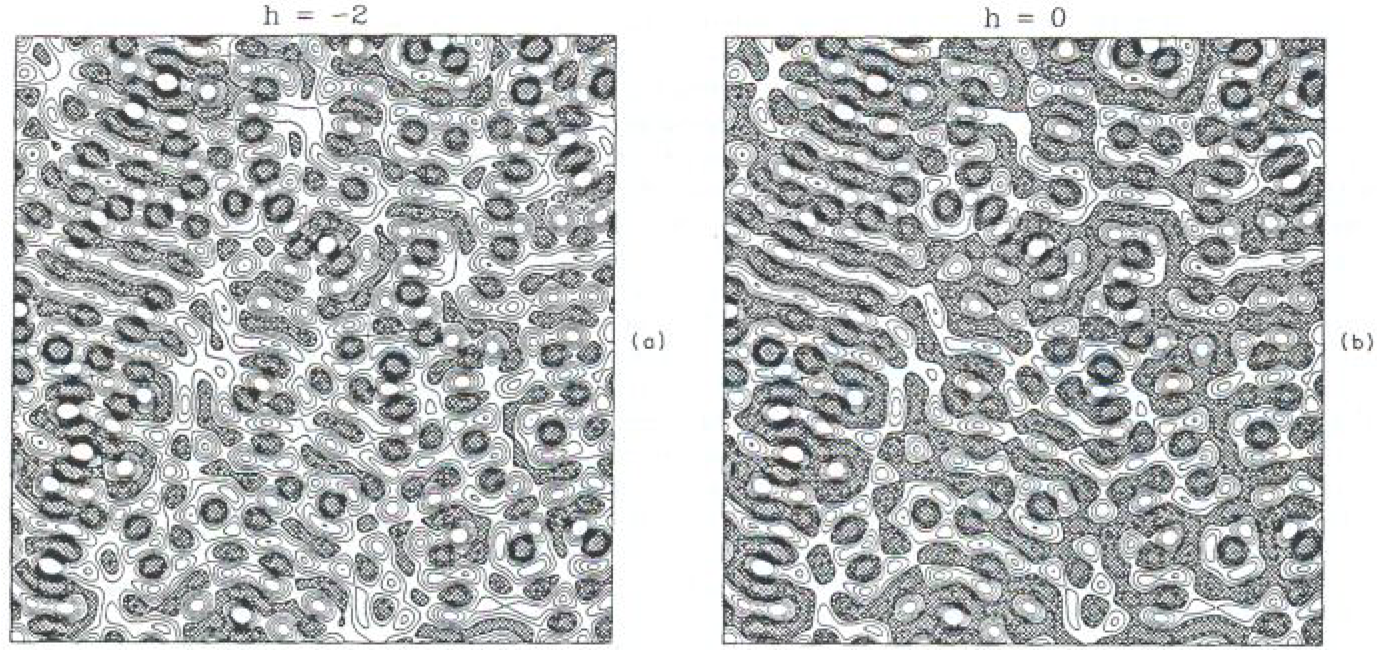
$$\sigma(p) \sim \sigma_0 (p - p_c)^t \quad p \to p_c^+$$

consider local conductivities
1 and *h* > 0
smooths, softens transition

UNIVERSAL critical exponents for lattices -- depend only on dimension

1 ≤ *t* ≤ 2 (for idealized model), Golden, *Phys. Rev. Lett.* 1990 **;** *Comm. Math. Phys.* 1992

non-universal behavior in continuum



level set percolation for a random potential