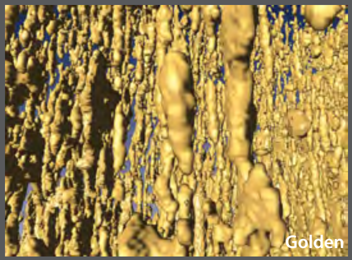
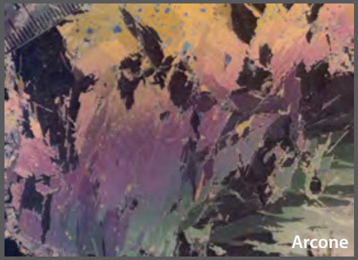
millimeters



Golden

centimeters



Arcone

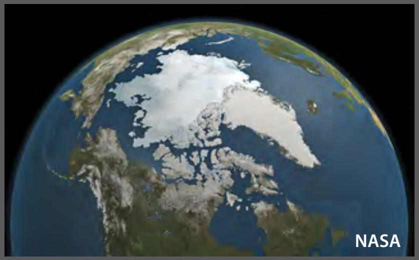
meters

Darelius

kilometers

NASA

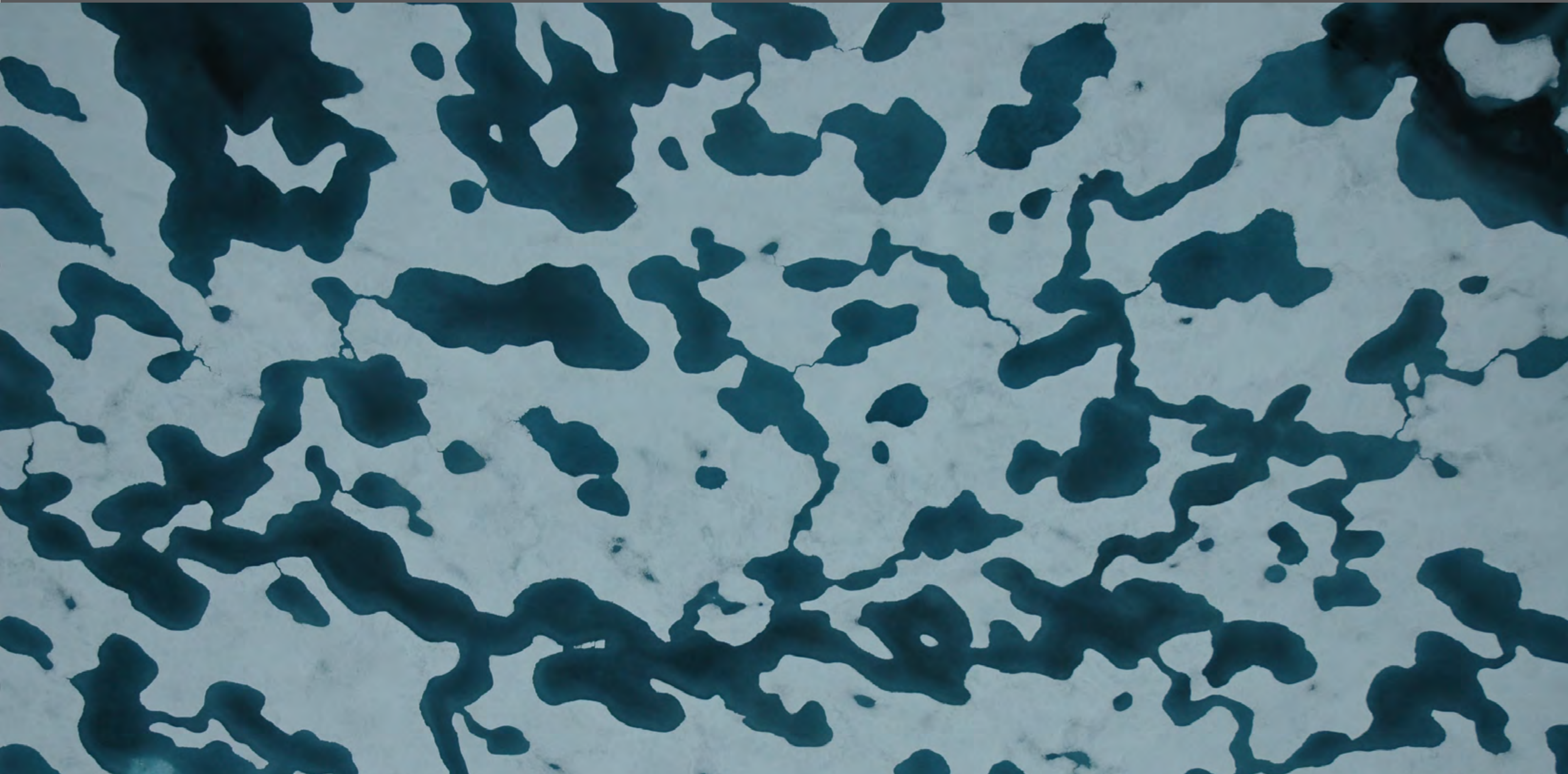
10^3 kilometers



NASA

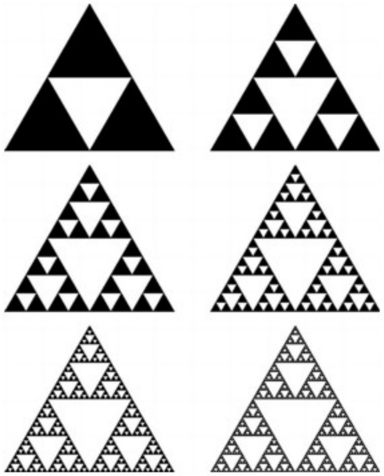
Fractal Geometry of Sea Ice Structures

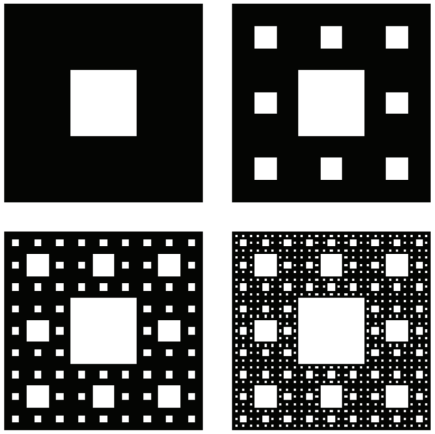
Ken Golden, University of Utah



fractals and multiscale structure

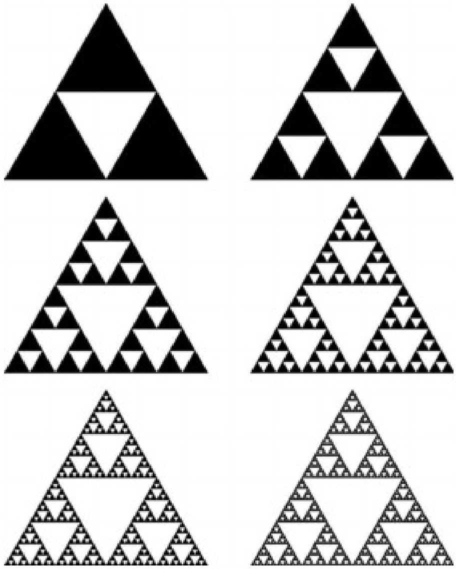
© 2004 Douglas R. Cootey www.cootey.com/fractals





fractals

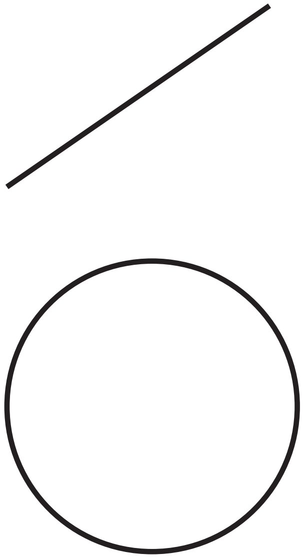
self-similar structure
non-integer dimension



$$D = \frac{\log 3}{\log 2} = 1.585...$$

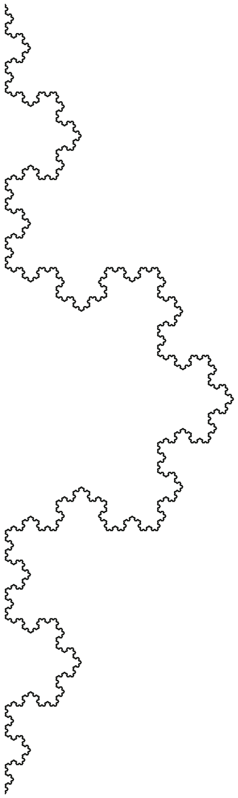
fractal curves in the plane

they wiggle so much that their dimension is >1



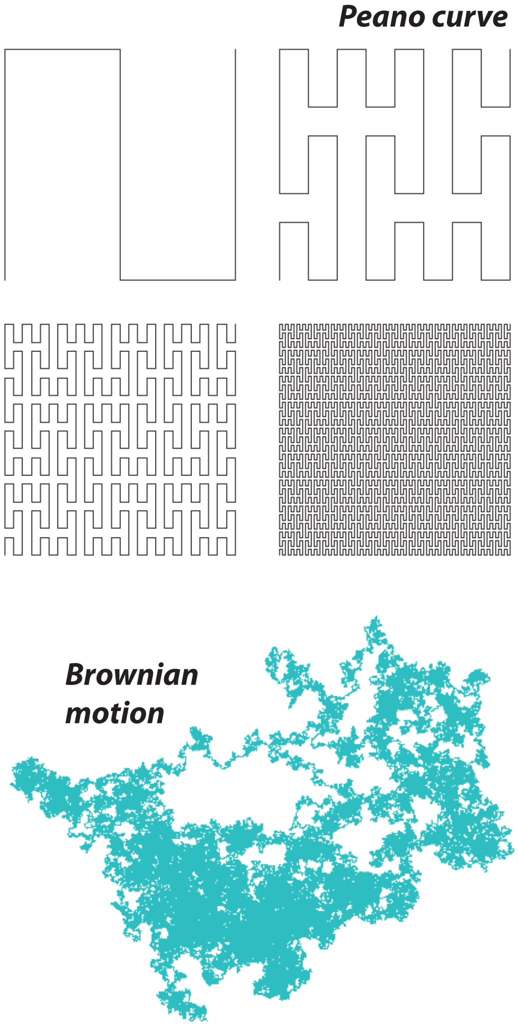
simple curves

D = 1



Koch snowflake

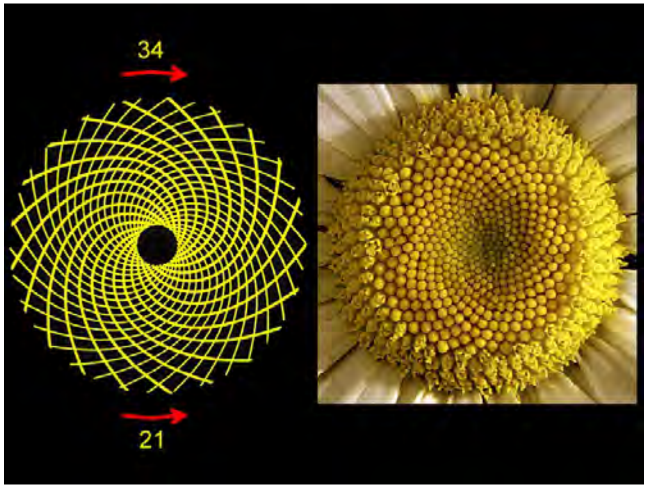
D = 1.26

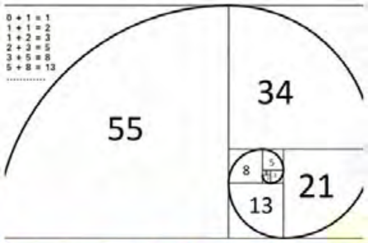


space filling curves

D = 2

Phyllotaxis, fractals and the Fibonacci sequence





Fibonacci sequence

1,1,2,3,5,8,13,21,34,55,...

34/21 ~ Golden Ratio 1.618...

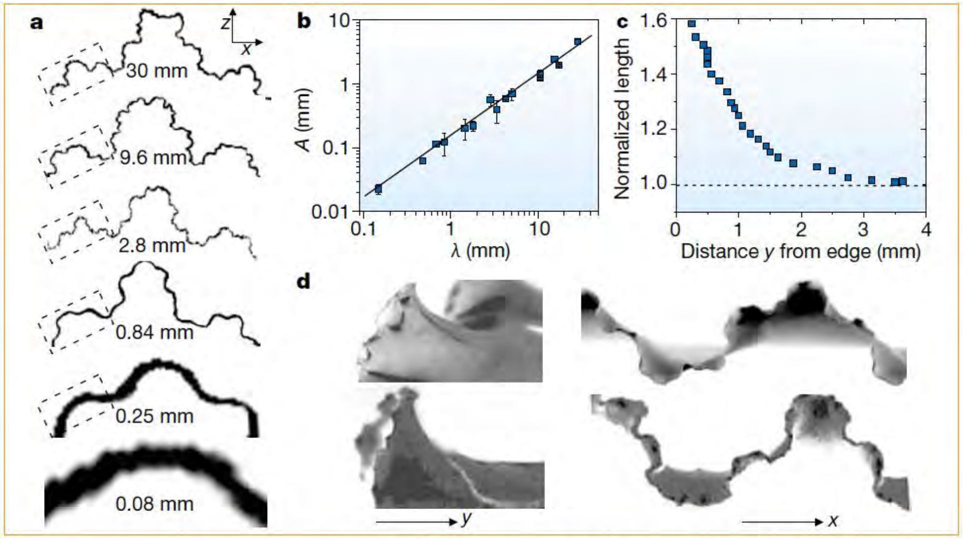


Figure 1 Buckling cascade in a deformed plastic sheet. **a,** Different magnifications of the edge of the sheet (0.012 mm thick). Successive pictures show the dotted boxed region on the left of the previous picture magnified by 3.2; the width of each image is indicated.

fractal dimension

$L = 1$
$M = 1$

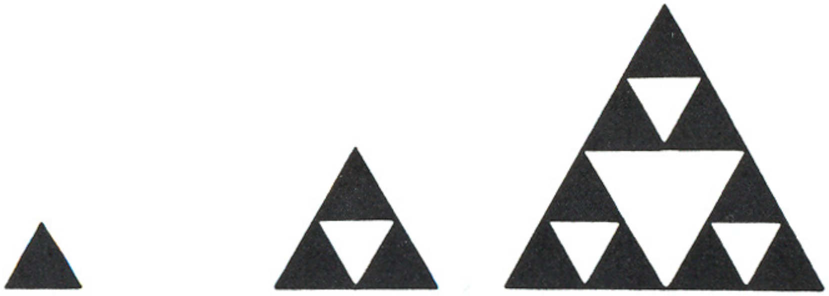
$L = 2$
$M = 4$

$L = 3$
$M = 9$

....

$$\text{Mass} \sim L^{d}$$

$$d = 2$$



$L = 1$
$M = 1$

$L = 2$
$M = 3$

$L = 4 = 2^{2}$
$M = 9 = 3^{2}$

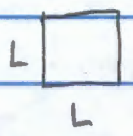
....

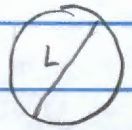
$$\text{Mass} \sim L^{d_f}$$

$$d_f = \alpha = \frac{\log 3}{\log 2} = 1.58...$$

Fractal Dimension
Sierpinski Gasket

two dim



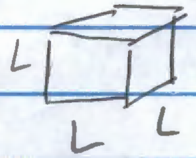


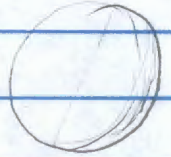
Mass = Area = L^2

Mass $= \pi\left(\frac{L}{2}\right)^2 = \left(\frac{\pi}{4}\right)L^2$

$M = 2^2 = 4$

three dim





Mass = Vol = L^3

Mass $= \frac{4}{3}\pi\left(\frac{L}{2}\right)^3 = \left(\frac{\pi}{6}\right)L^3$

$m = 2^3 = 8$

$$M = cL^d$$

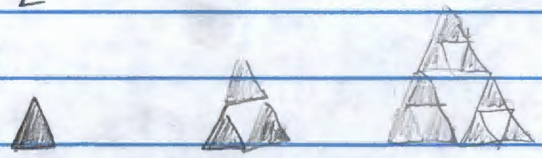
d = dimension

density $\rho = \frac{M}{L^2} = cL^{d-2}$

$d = 2$

For a fractal replace d by d_f ← fractal dimension

$$\log\rho = \underbrace{(d_f - 2)}_{\text{slope}}\log L + \text{const}$$

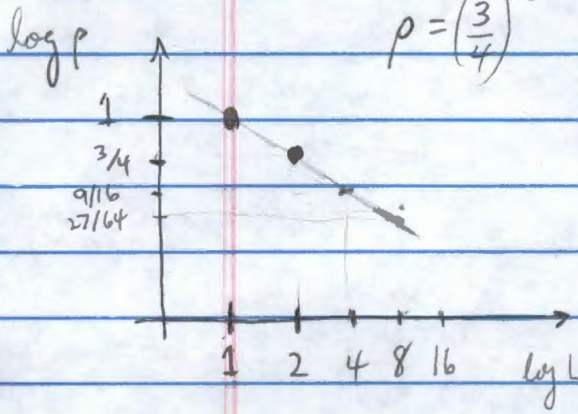


$L = 2^0$ $L = 2^1$ $L = 2^2$

$M = 3^0$ $M = 3^1$ $M = 3^2$ - - -

$\rho = \left(\frac{3}{4}\right)^0$ $\rho = \left(\frac{3}{4}\right)^1$ $\rho = \left(\frac{3}{4}\right)^2$

$\boxed{\rho \to 0 \text{ as } L \to \infty}$



$$\text{slope} = \frac{\log(3/4) - \log 1}{\log 2 - \log 1} = \frac{\log 3 - 2\log 2}{\log 2}$$

$$= \frac{\log 3}{\log 2} - 2$$

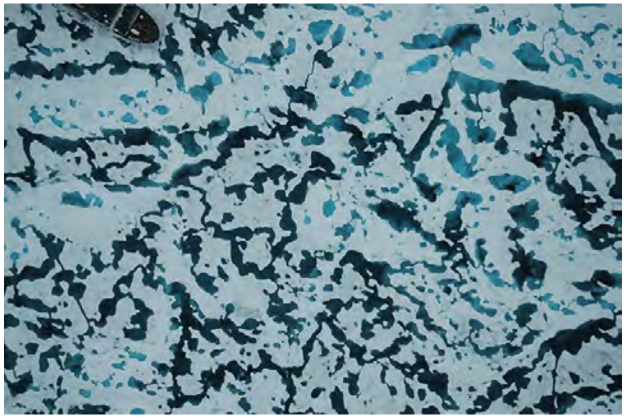
$$d_f = 1.58\ldots$$

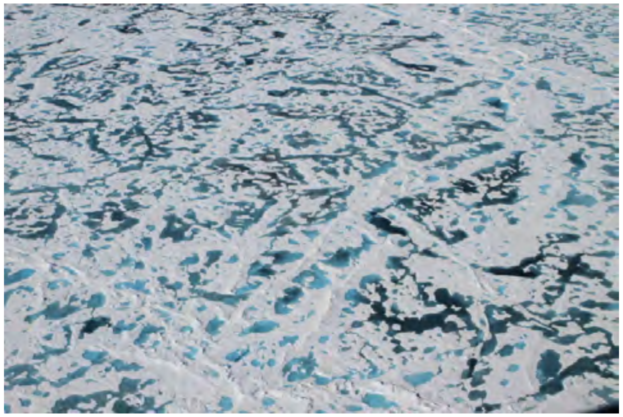

NASA

basin scale -
grid scale
albedo

Linking Scales

km
scale
melt
ponds

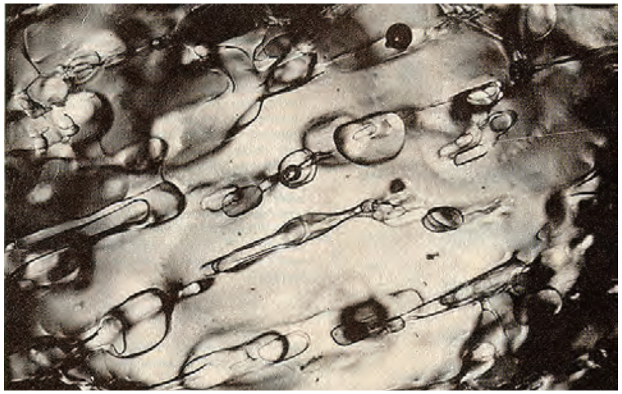

Perovich


Ramsayer / NASA

km
scale
melt
ponds

Linking Scales

mm
scale
brine
inclusions

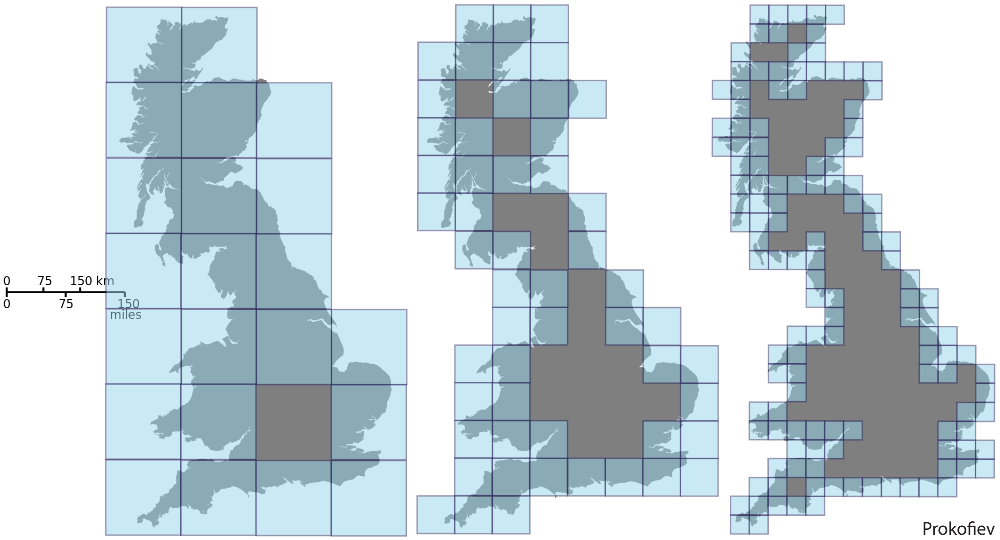

Weeks & Assur


Colon

meter
scale
snow
topography

Thermal Evolution of Brine Fractal Geometry in Sea Ice

Nash Ward, Daniel Hallman, Benjamin Murphy, Jody Reimer,
Marc Oggier, Megan O’Sadnick, Elena Cherkaev and Kenneth Golden, 2025

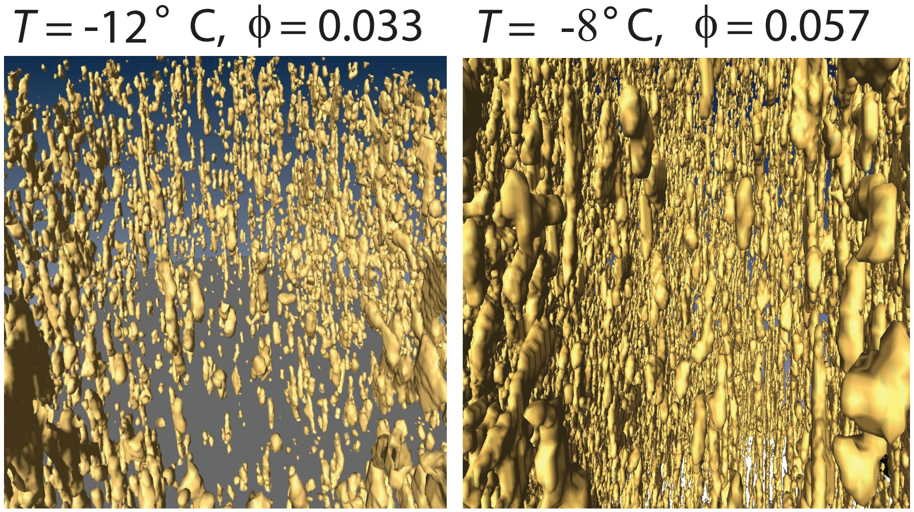


Prokofiev

fractal dimension of the coastline of Great Britain by box counting

$$N(\epsilon) \sim \epsilon^{-D}$$

brine channels and inclusions “look” like fractals (from 30 yrs ago)

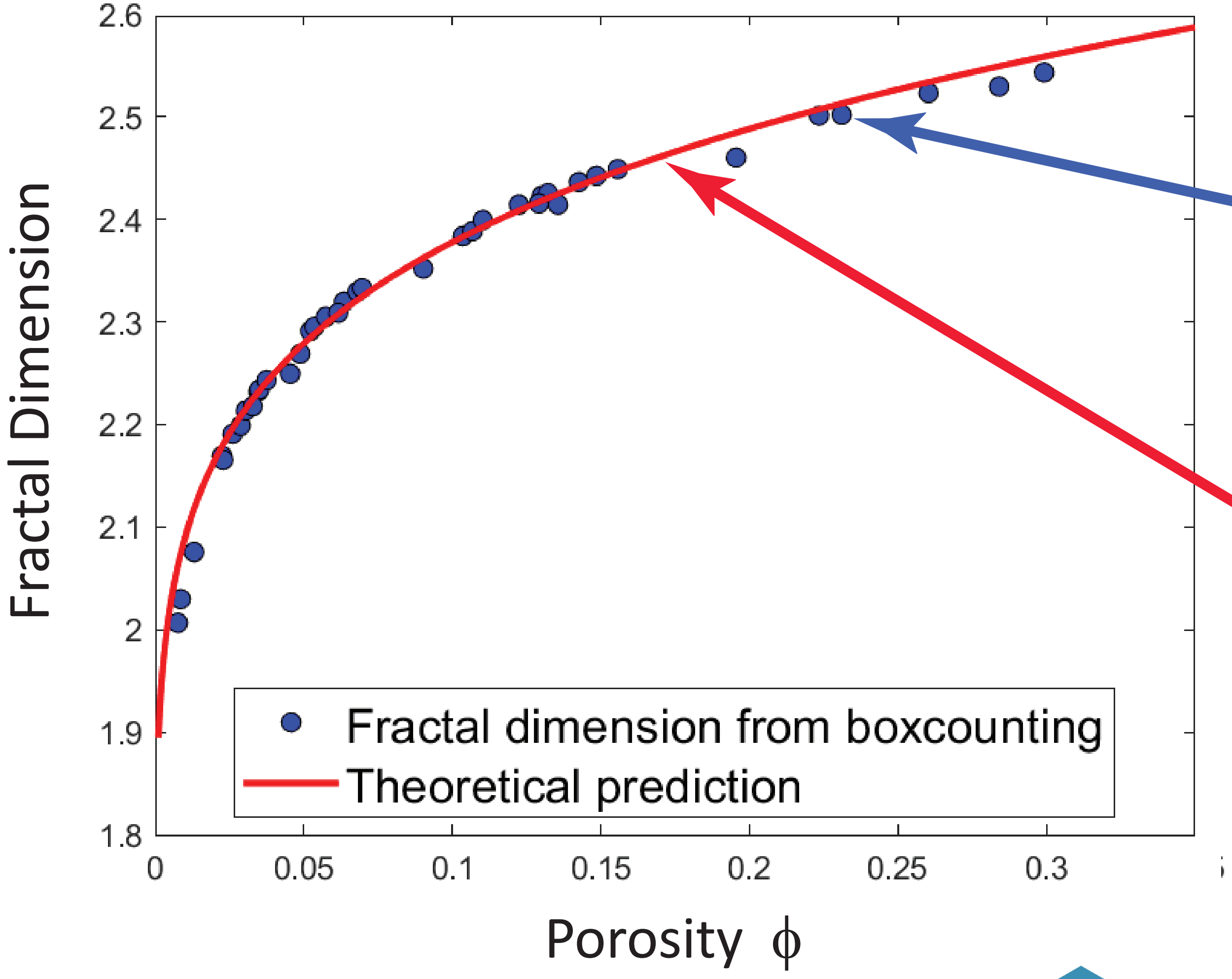


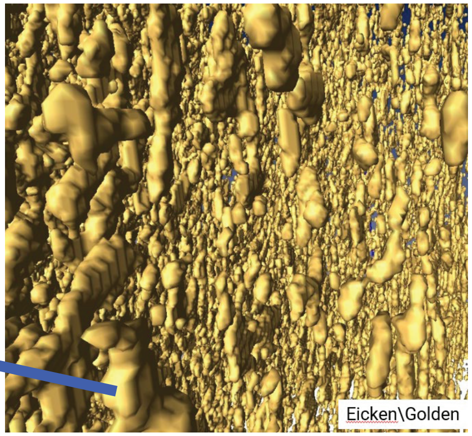
Golden, Eicken, et al. *GRL*, 2007

X-ray computed tomography of brine in sea ice

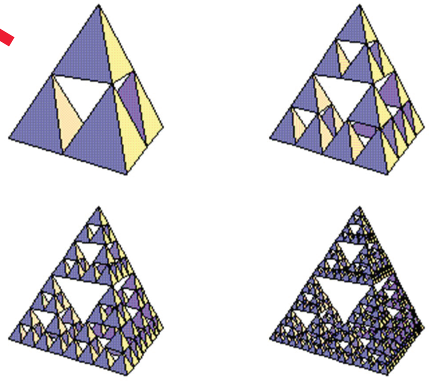
columnar and granular

The first quantitative study of the fractal dimension of brine in sea ice and its strong dependence on temperature and porosity.





Follows same curve as exactly self-similar Sierpinski tetrahedron



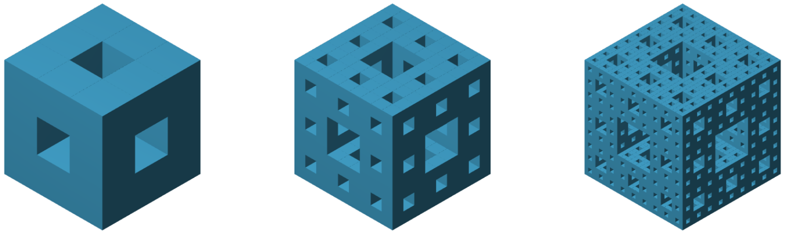
D. Eppstein

red curve

$$F_d = d_E - \frac{\ln \phi}{\ln(\lambda_{min}/\lambda_{max})}$$

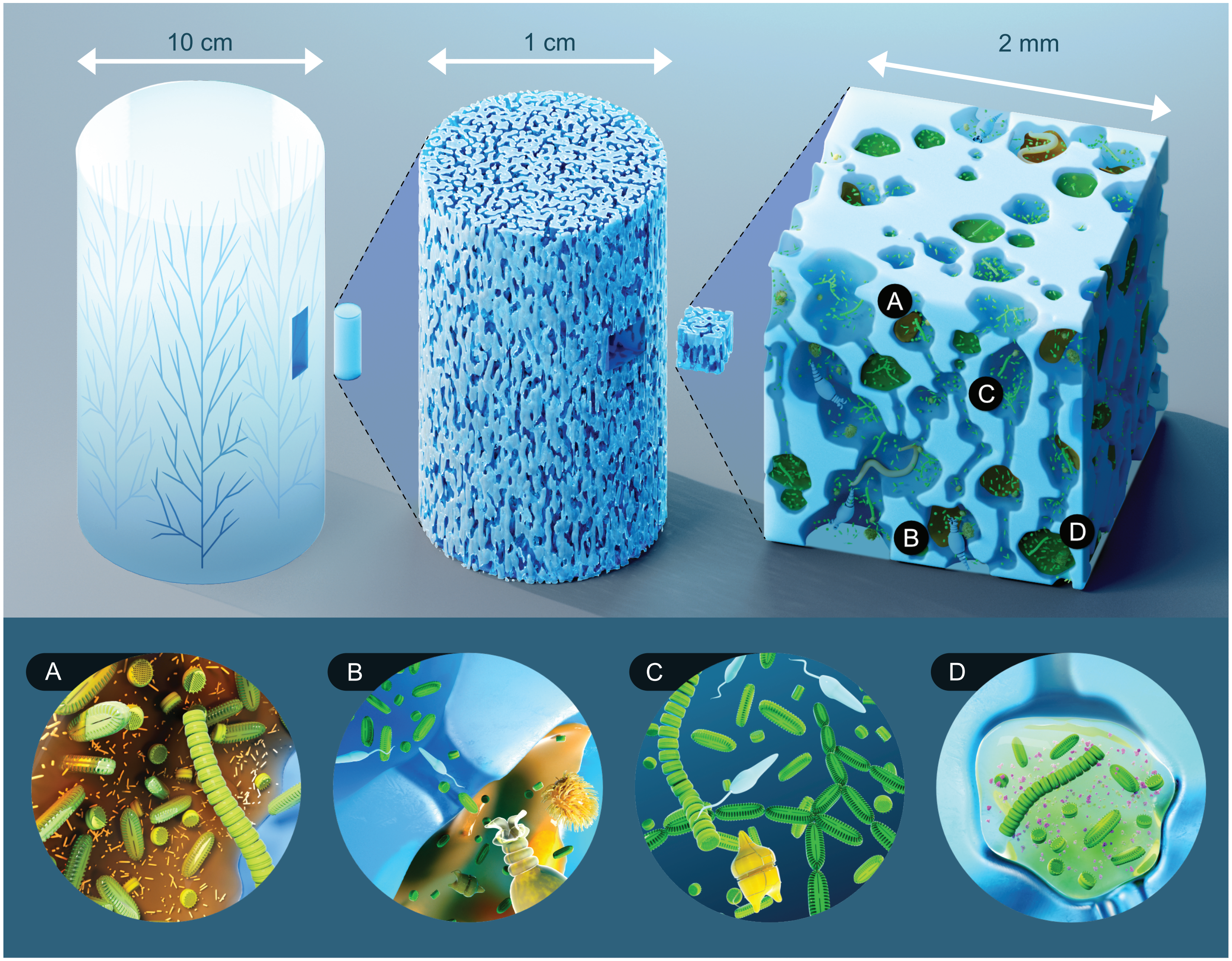
Katz and Thompson,1985; Yu and Li, 2001

discovered for sandstones
statistically self-similar porous media



Fractal geometry of brine in sea ice, Ward, *et al.* 2025

Implications of brine fractal geometry on sea ice ecology and biogeochemistry



Brine inclusions are home to ice endemic organisms, e.g., bacteria, diatoms, flagellates, rotifers, nematodes.

The habitability of sea ice for these organisms is inextricably linked to its complex brine geometry.

(A) Many sea ice organisms attach themselves to inclusion walls; inclusions with a higher fractal dimension have greater surface area for colonization.

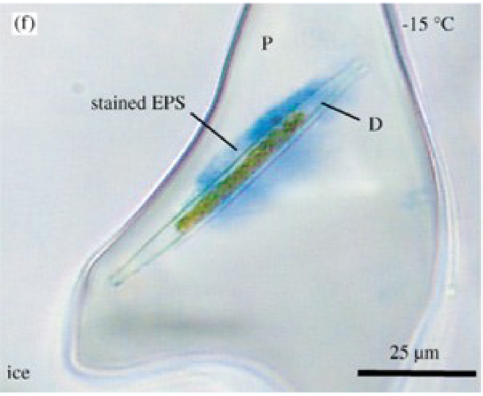
(B) Narrow channels prevent the passage of larger organisms, leading to refuges where smaller organisms can multiply without being grazed, as in (C).

(D) Ice algae secrete extracellular polymeric substances (EPS) which alter incusion geometry and may further increase the fractal dimension.

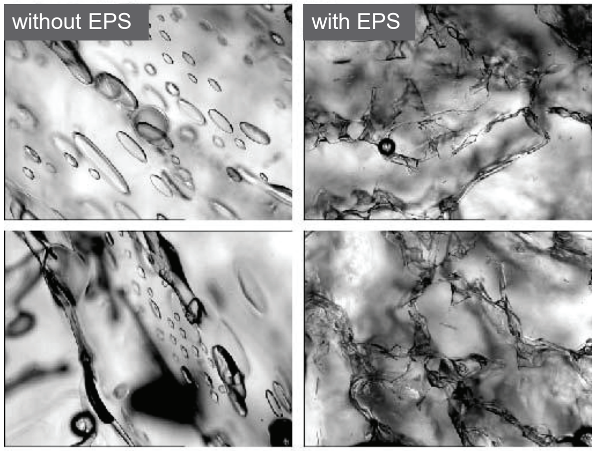
Sea ice algae secrete exopolymeric substances (EPS) affecting evolution of brine microstructure.

How does EPS affect fluid transport? **How does the biology affect the physics?**

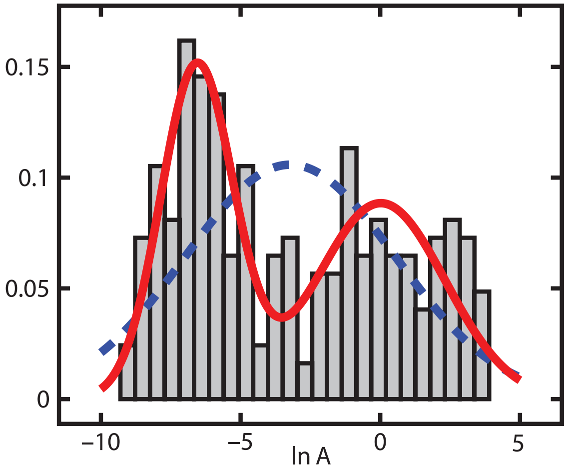
FRACTAL



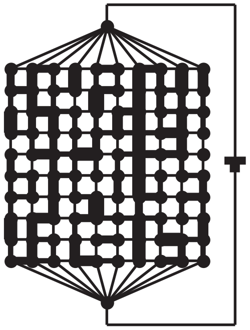
Krembs

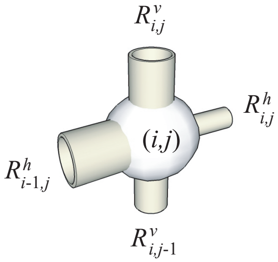


Krembs, Eicken, Deming, PNAS 2011



RANDOM PIPE MODEL





- 2D random pipe model with bimodal distribution of pipe radii

- Rigorous bound on permeability *k;* results predict observed drop in *k*

Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
***Multiscale Modeling and Simulation*, 2018**

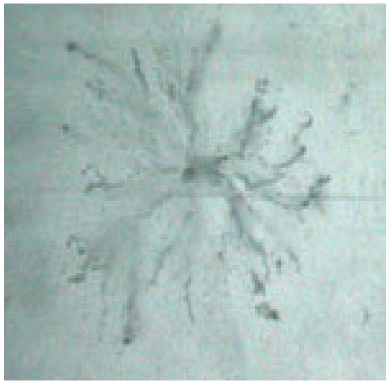
Zhu, Jabini, Golden,
Eicken, Morris
***Ann. Glac.* 2006**

SIAM News
June 2024

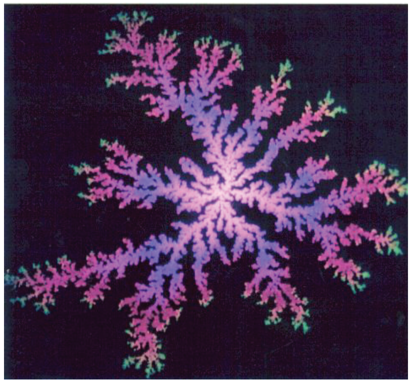
EPS - Algae Model Jajeh, Reimer, Golden

fractal microstructures

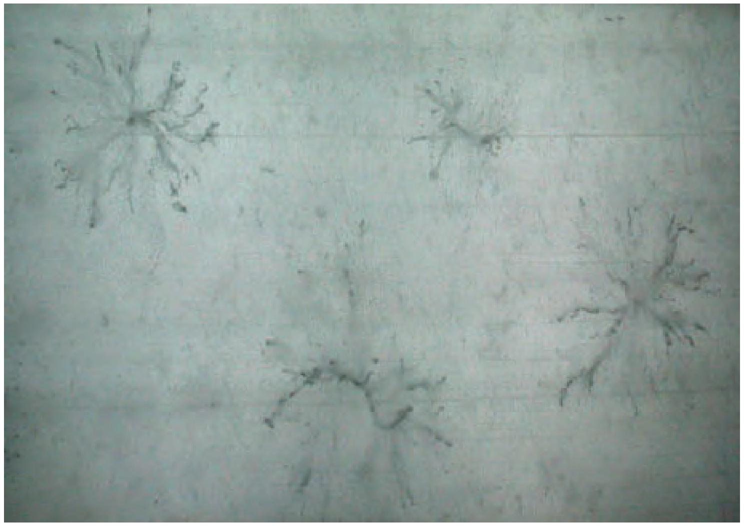
electrorheological fluid
with metal spheres



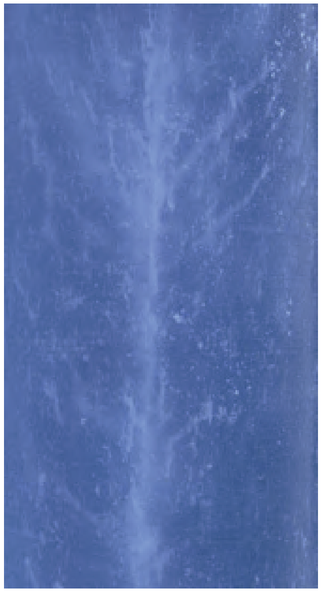
brine channel
in sea ice



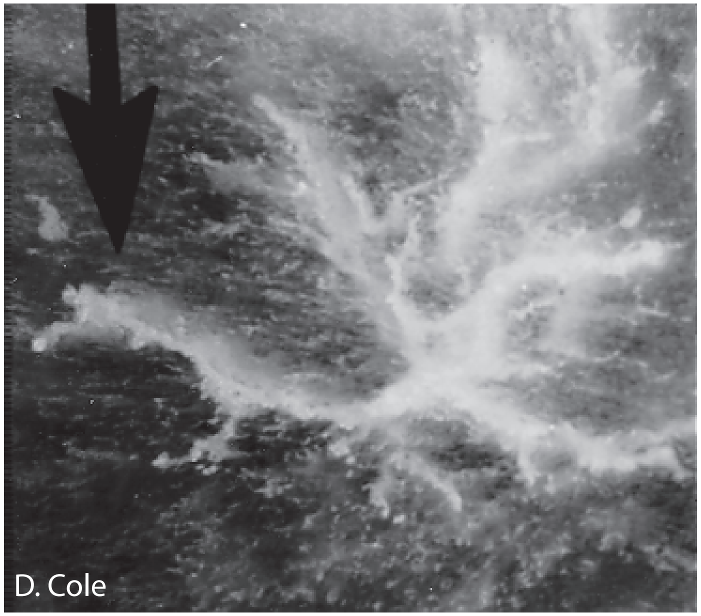
diffusion limited
aggregation



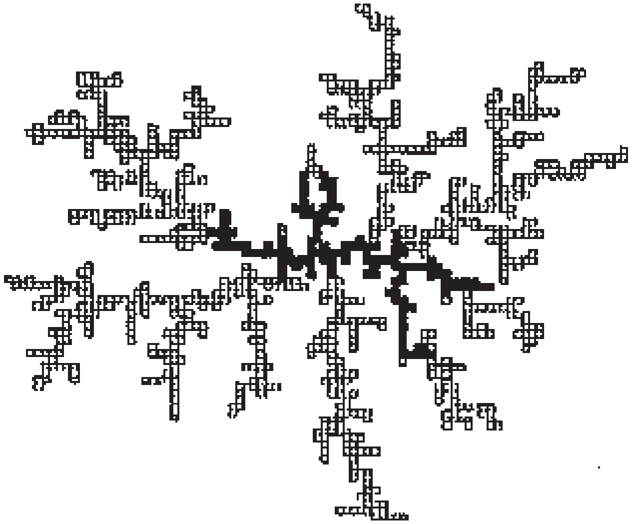
brine channels

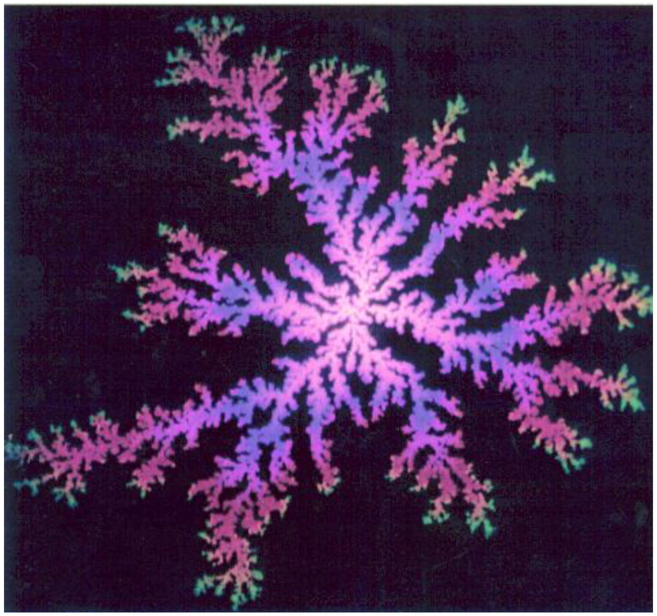


fractal structure of brine channels

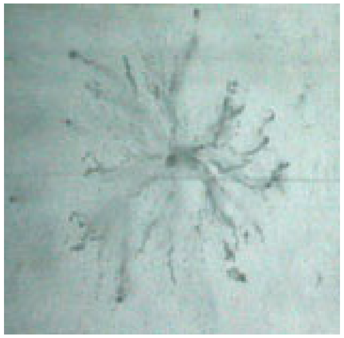


D. Cole





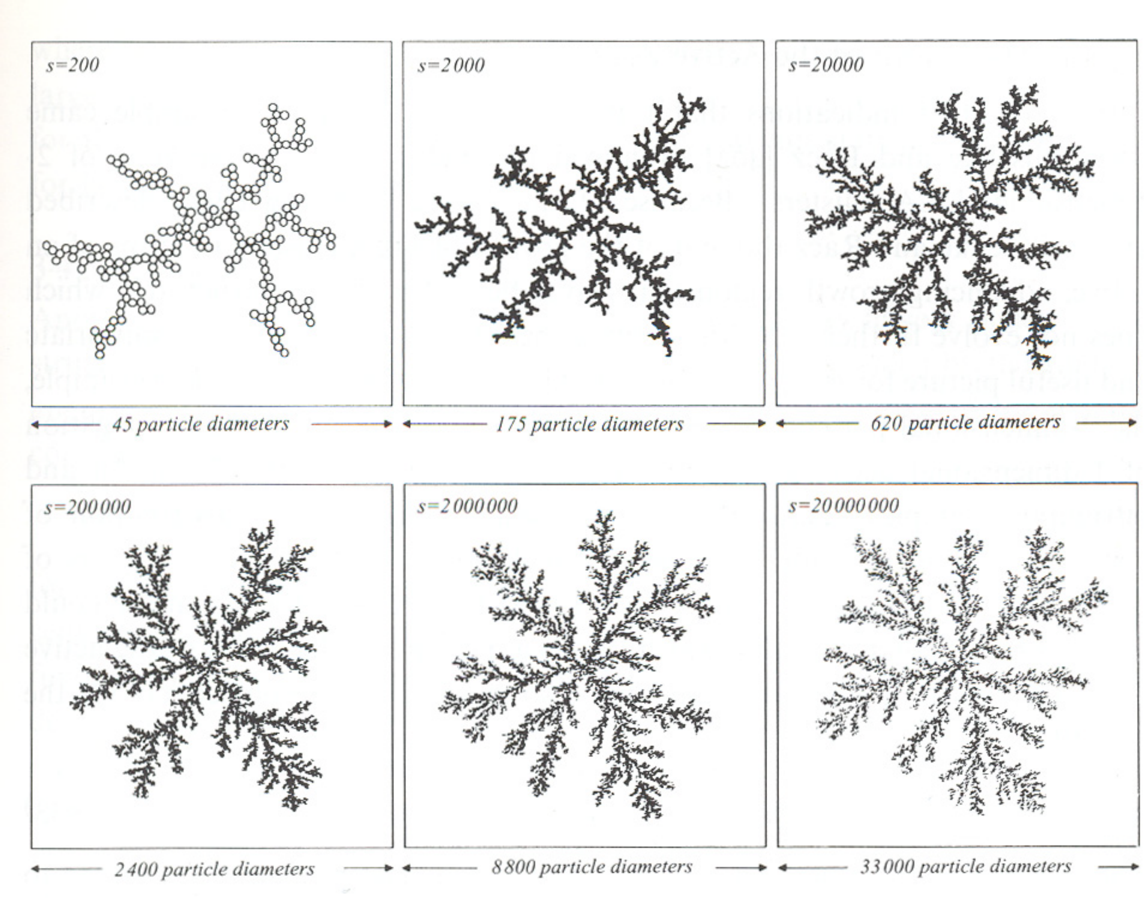
brine drainage



Diffusion Limited Aggregation (DLA) model
cluster has fractal dimension :

$d_f = 1.71$ in two dimensions

self similarity of DLA



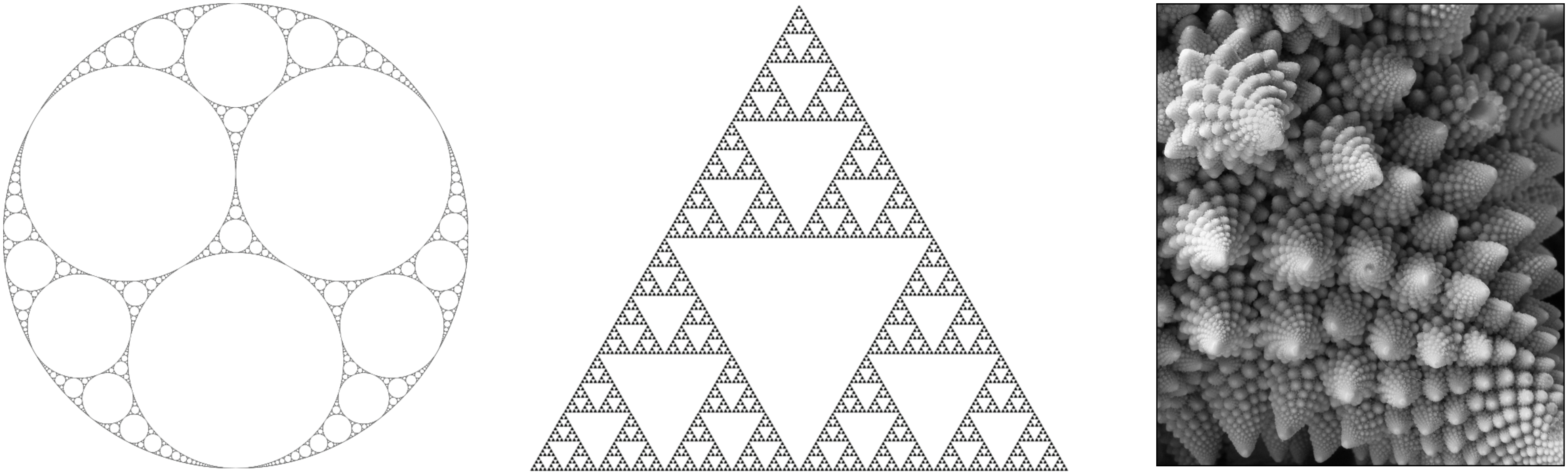
P. Meakin

the sea ice pack is a ***fractal***

dispalying self-similar structure on many scales

floe size distribution important in dynamics (fracture), thermodynamics (melting)

bigger floes easier to break, smaller floes easier to melt

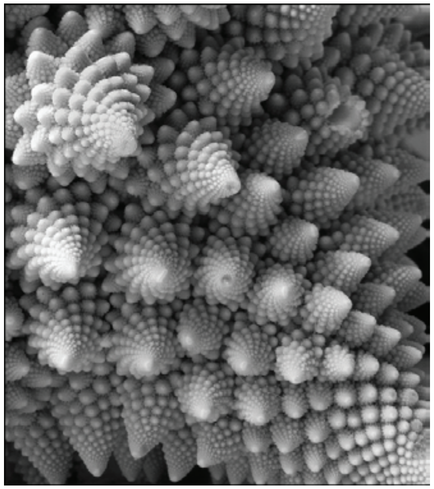


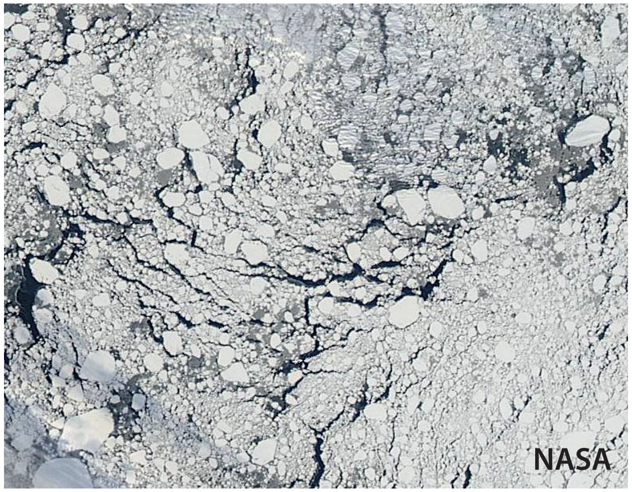
the sea ice pack is a *fractal*

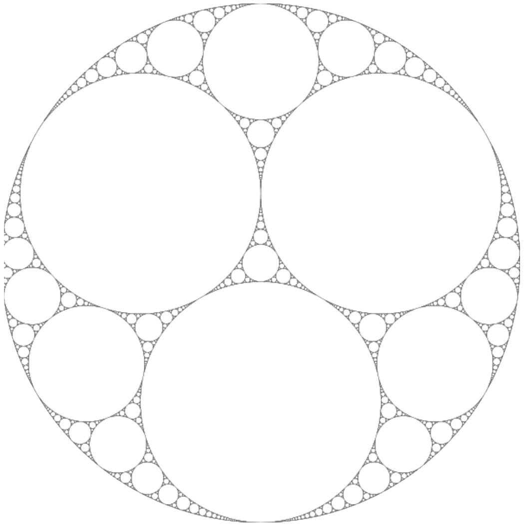
displaying self-similar structure on many scales


J. Robertson


E. Walker




NASA



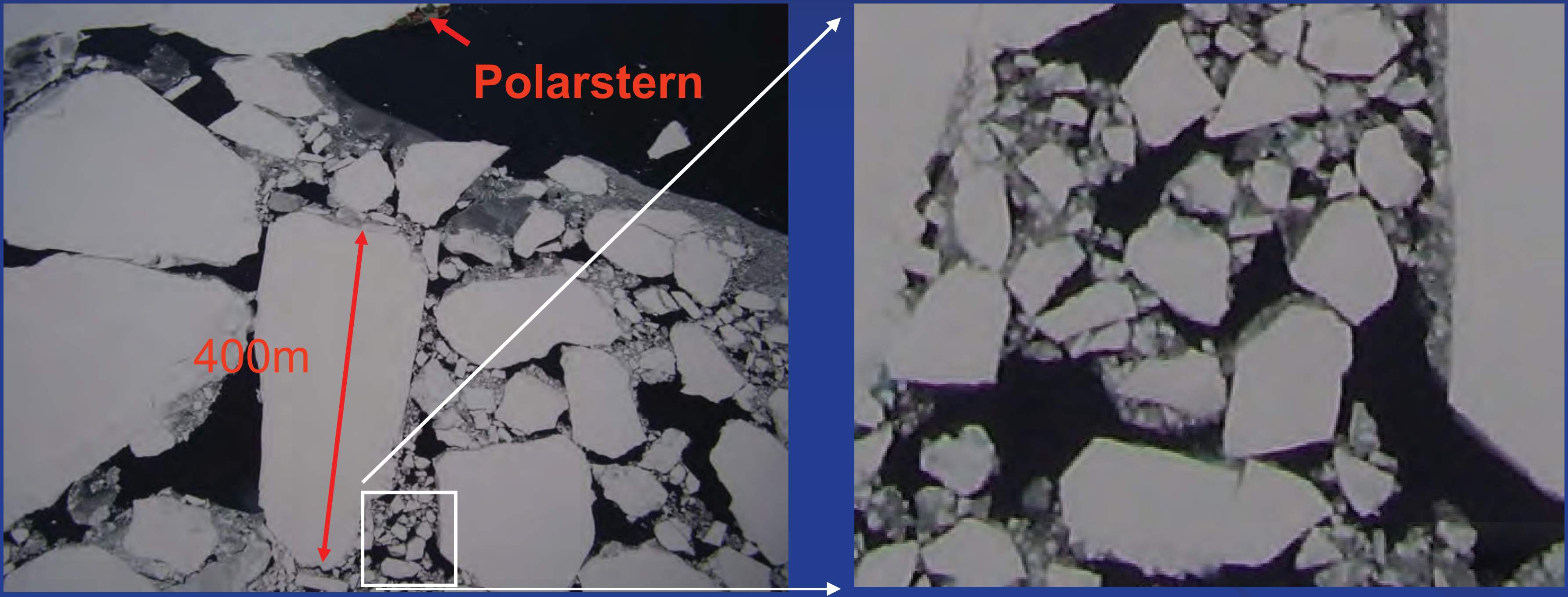
floe size distribution, area-perimeter relations, etc. important in dynamics (fracture), thermodynamics (melting)

Toyota, et al. Geophys. Res. Lett. 2006
Rothrock and Thorndike, J. Geophys. Res. 1984

The sea ice pack has fractal structure.

Self-similarity of sea ice floes

Weddell Sea, Antarctica

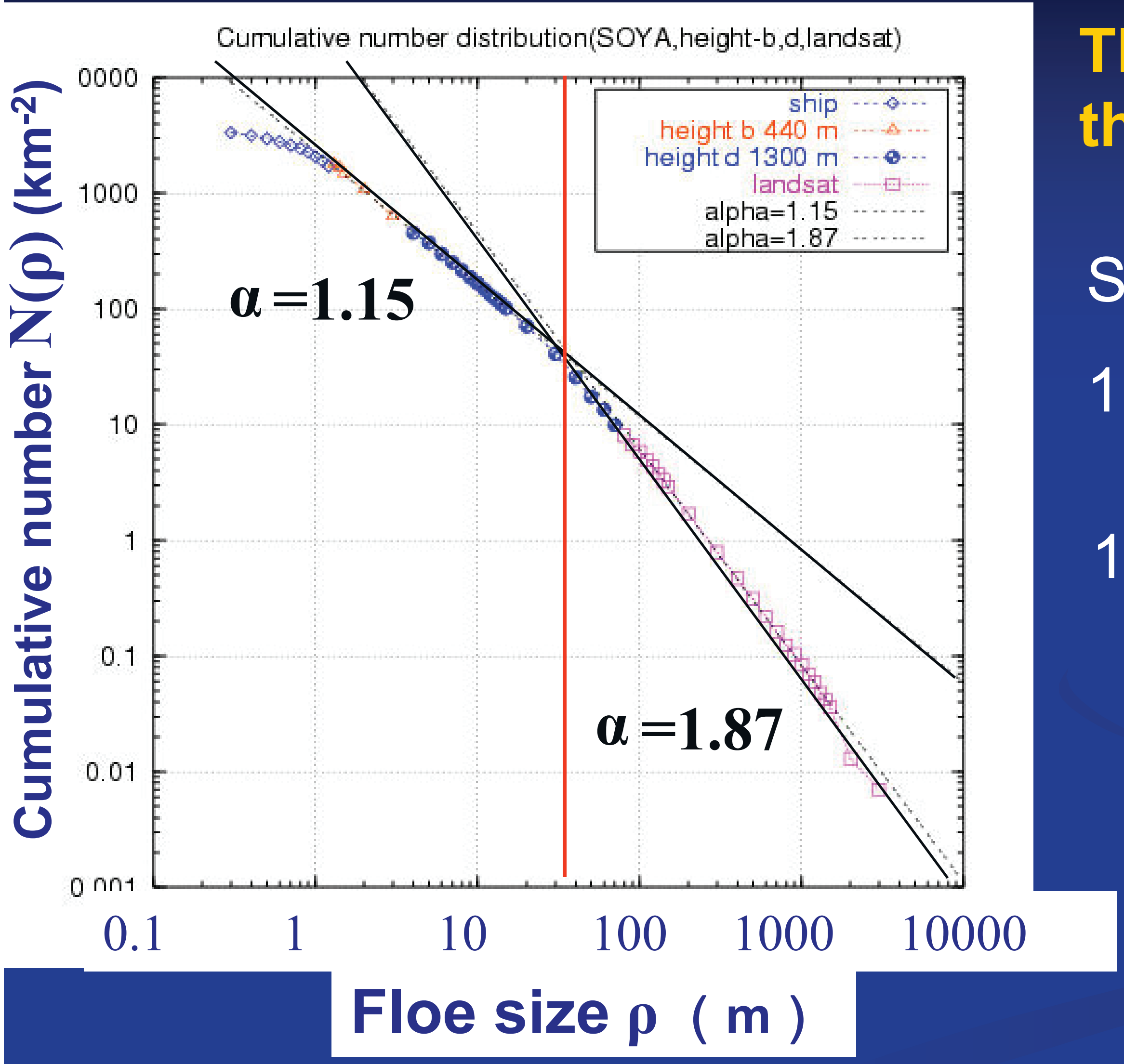


fractal dimensions of Okhotsk Sea ice pack
smaller scales D~1.2, larger scales D~1.9

fractal dim. *vs.* floe size exponent
Adam Dorsky, Nash Ward, Ken Golden 2025

Toyota, *et al. Geophys. Res. Lett.* 2006
Rothrock and Thorndike, *J. Geophys. Res. 1984*

Results from Okhotsk Sea ice



There are two regimes in the ice floe distribution.

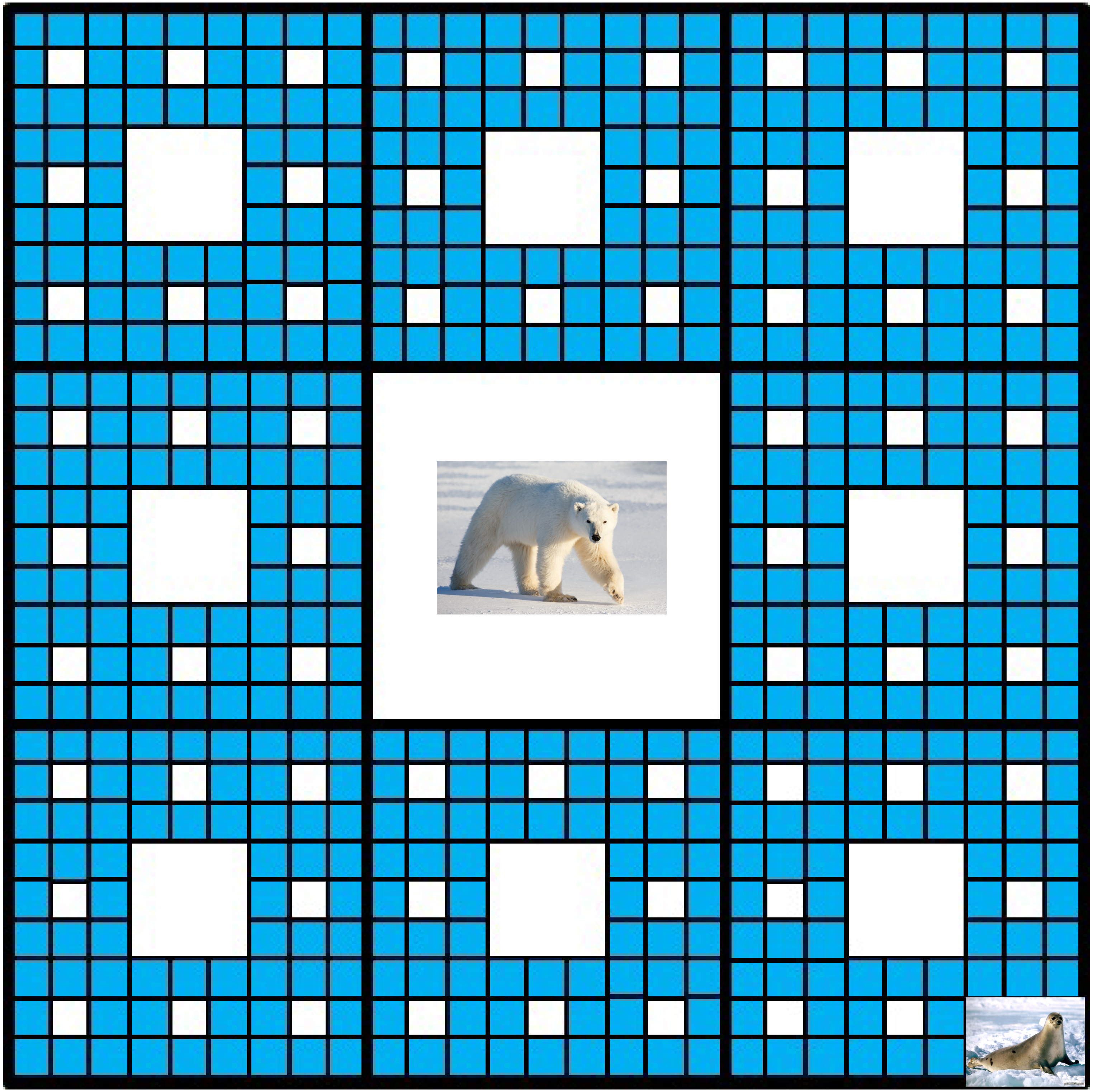
Size

1 ~ 20 m : α = 1.15 ± 0.02

100 ~ 1500 m :

α = 1.87 ± 0.02

(Toyota, Takatsuji et al., 2006)



polar bear foraging in a fractal icescape

Nicole Forrester
Jody Reimer
Ken Golden

It costs the polar bear 5 times the energy to swim through water than to walk on sea ice.

What pathway to a seal minimizes energy spent?

melt pond formation and albedo evolution:

- *major drivers in polar climate*
- *key challenge for global climate models*

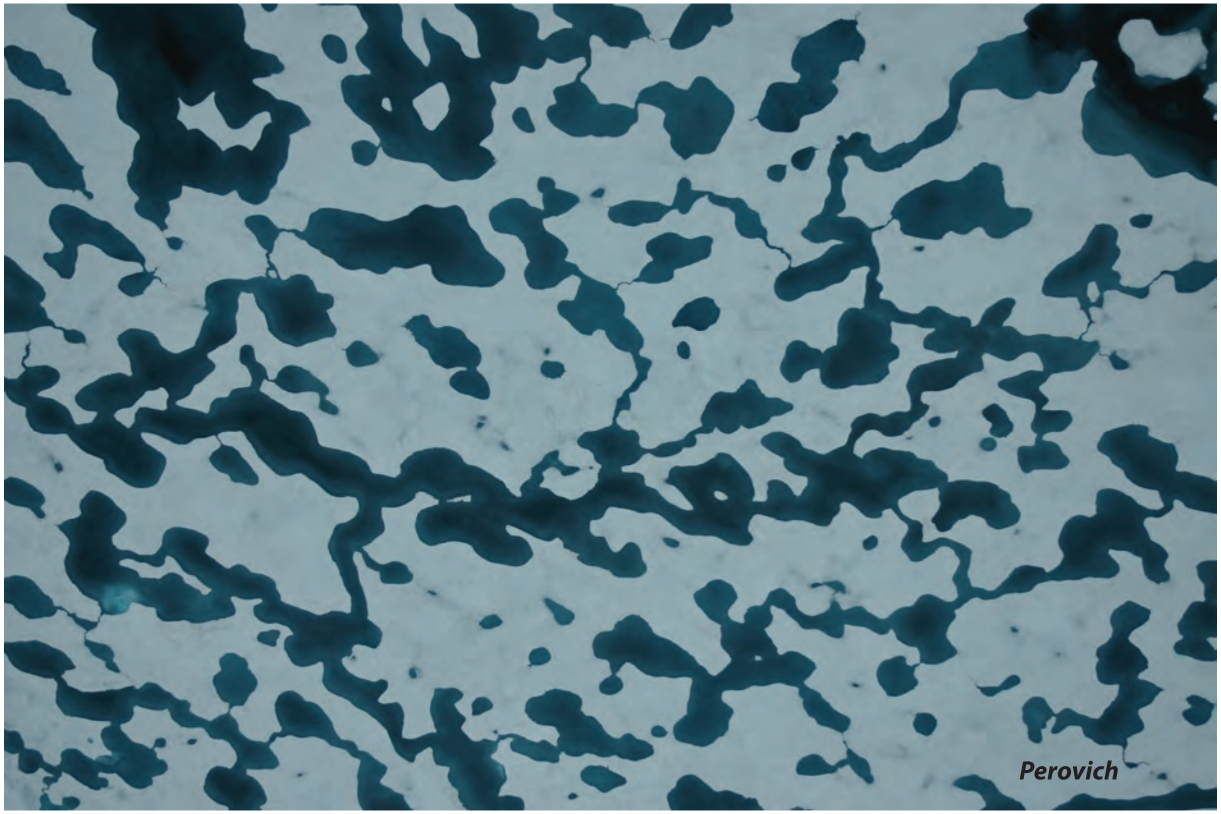
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

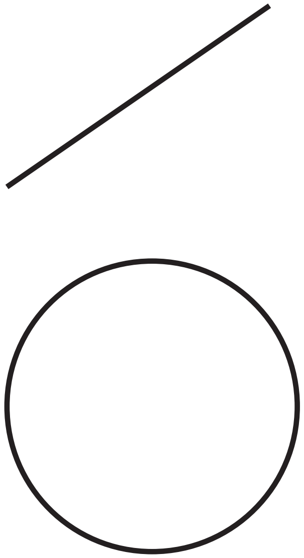
Flocco, Feltham, Hunke 2012



Are there universal features of the evolution similar to phase transitions in statistical physics?

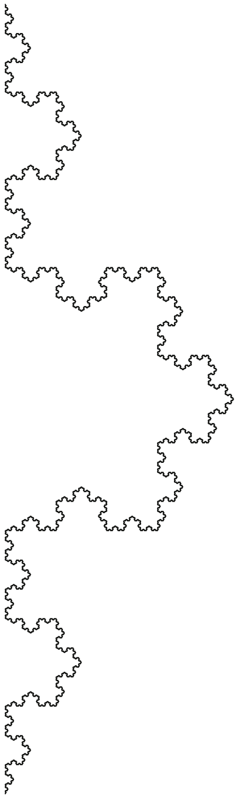
fractal curves in the plane

they wiggle so much that their dimension is >1



simple curves

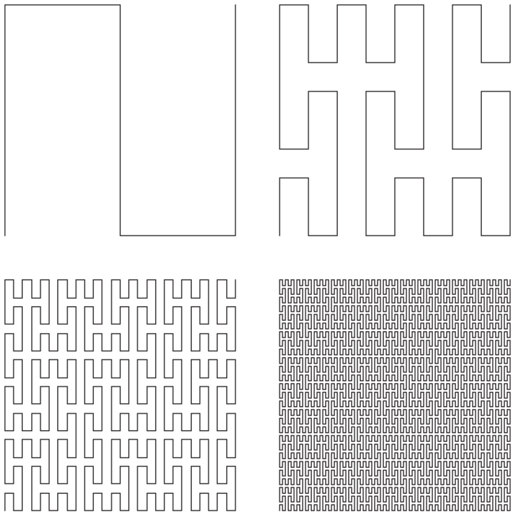
$D = 1$

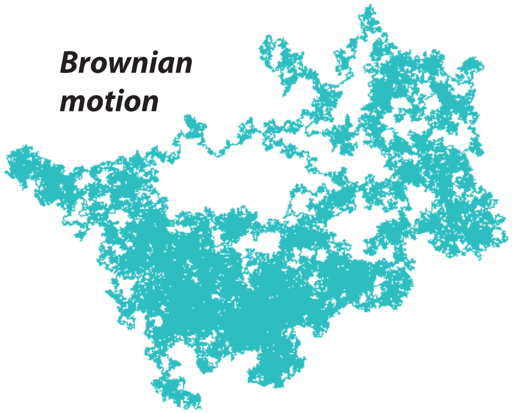


Koch snowflake

$D = 1.26$

Peano curve





space filling curves

$D = 2$

30th Congressional District, Texas, 1991-1996

clouds exhibit fractal behavior from 1 to 1000 km

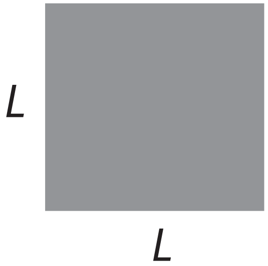
use ***perimeter-area*** data to find that cloud and rain boundaries are fractals

$$D \approx 1.35$$

S. Lovejoy, Science, 1982

$$P \sim \sqrt{A}$$

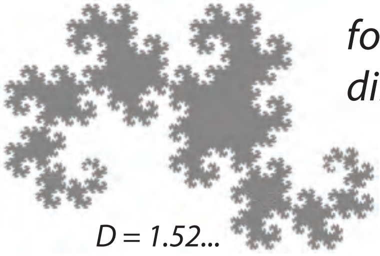
simple shapes



$A = L^2$

$P = 4L = 4\sqrt{A}$

$$P \sim \sqrt{A}^{D}$$

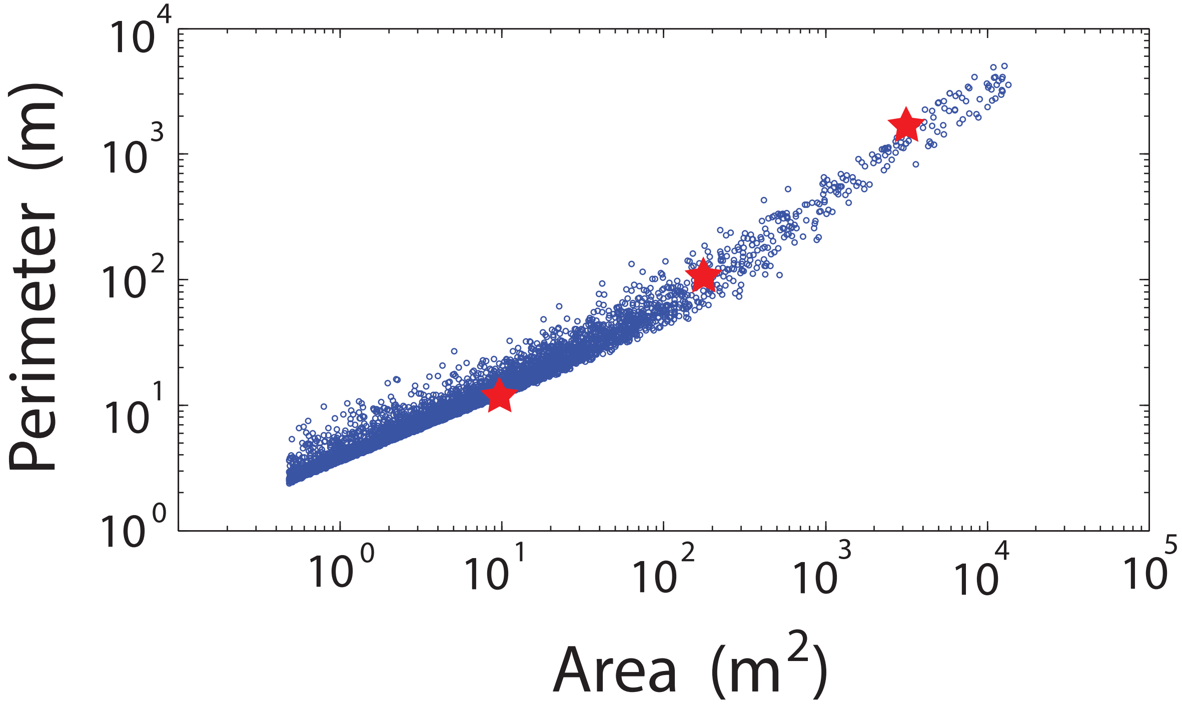


for fractals with dimension D

Transition in the fractal geometry of Arctic melt ponds

The Cryosphere, 2012

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

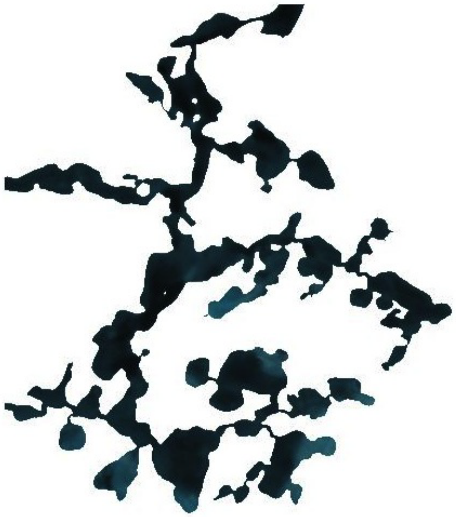


data from
5269 Arctic
melt ponds

simple pond

~ 30 m

transitional pond



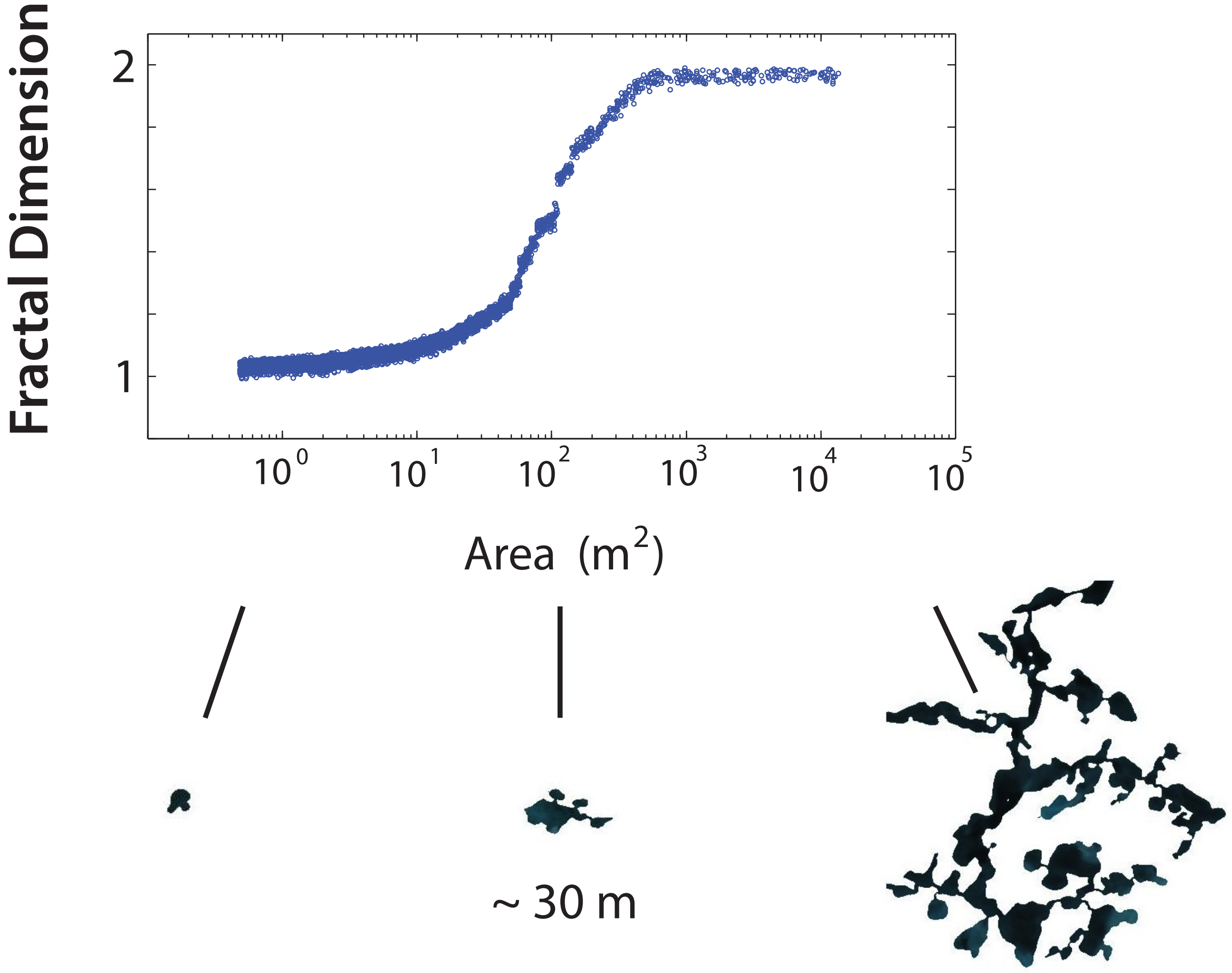
complex pond

Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

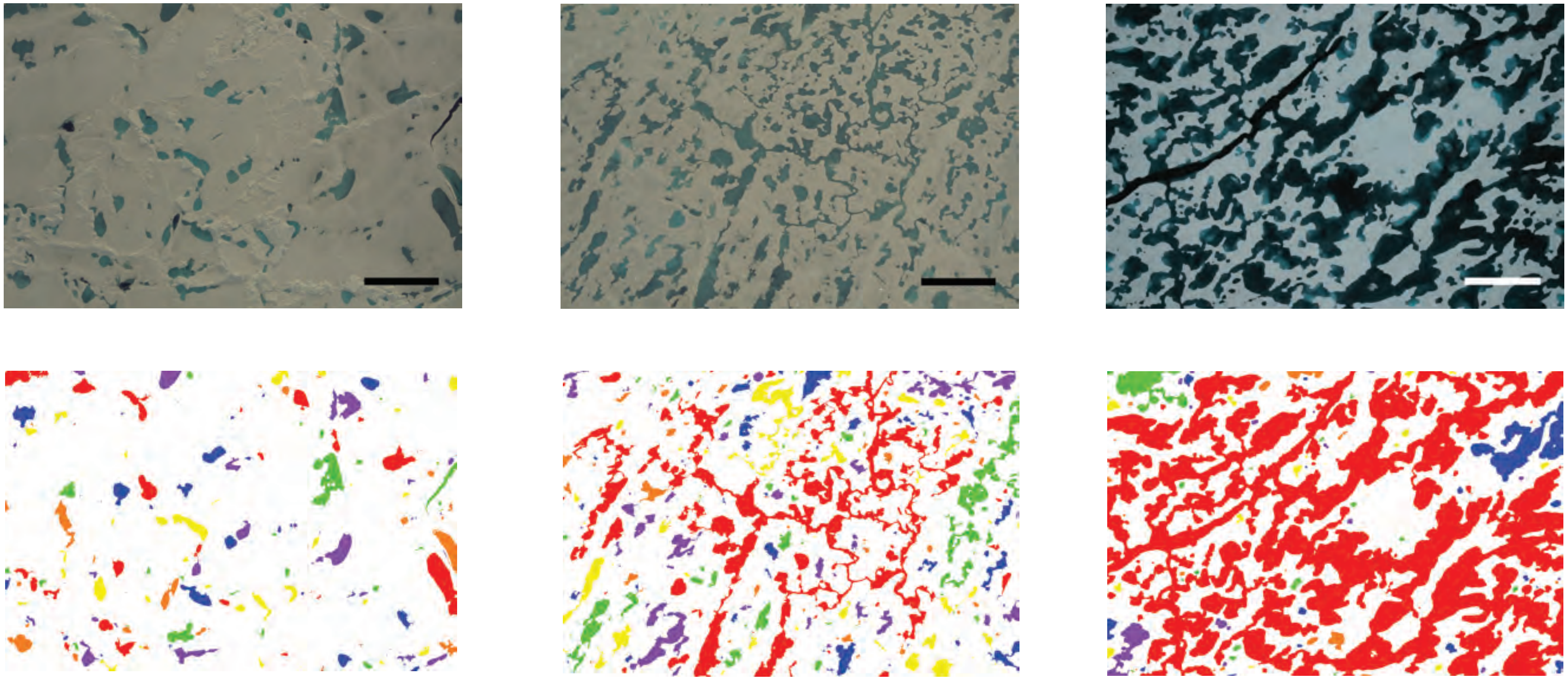
The Cryosphere, 2012

complexity grows with length scale



simple pond ***transitional pond*** ***complex pond***

small simple ponds coalesce to form large connected structures with complex boundaries



melt pond percolation

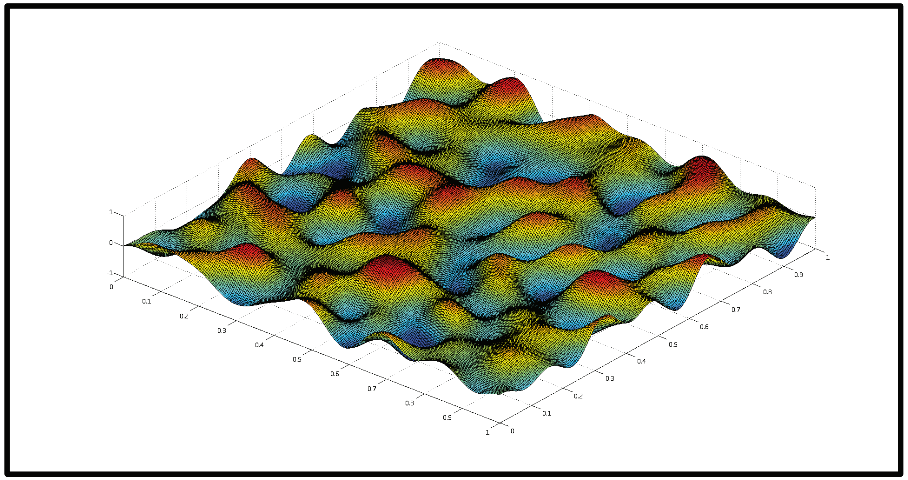
results on percolation threshold, correlation length, cluster behavior

Anthony Cheng (Hillcrest HS), Dylan Webb (Skyline HS), Court Strong, Ken Golden

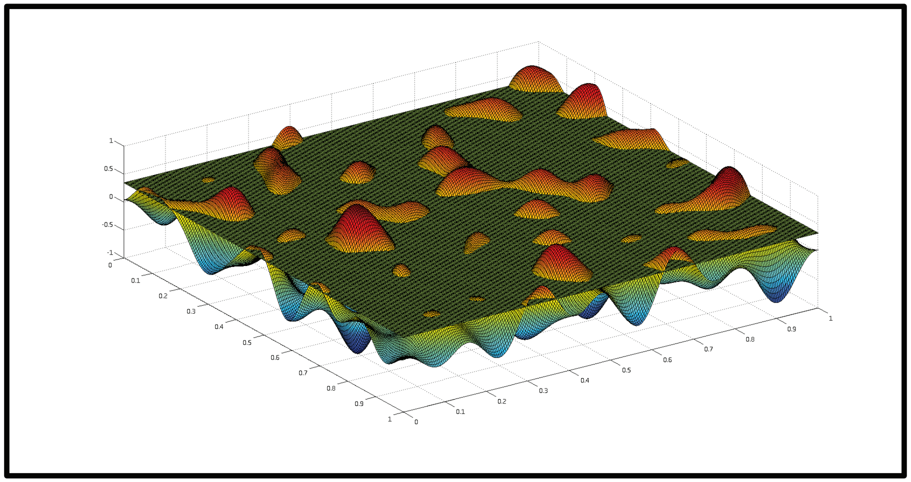
Continuum percolation model for melt pond evolution

level sets of random surfaces

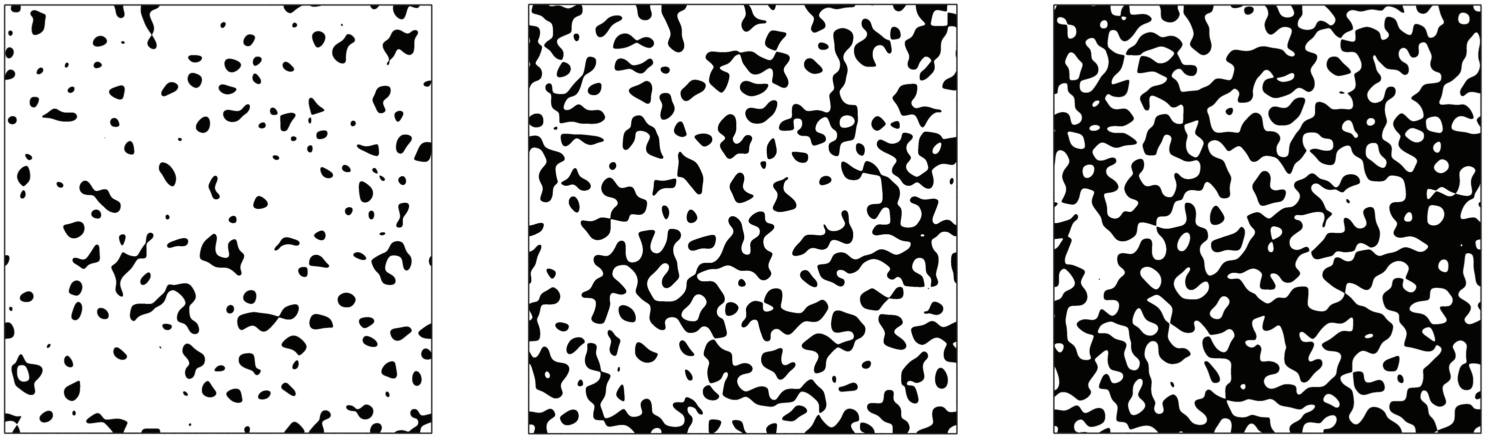
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018



random Fourier series representation of surface topography



intersections of a plane with the surface define melt ponds

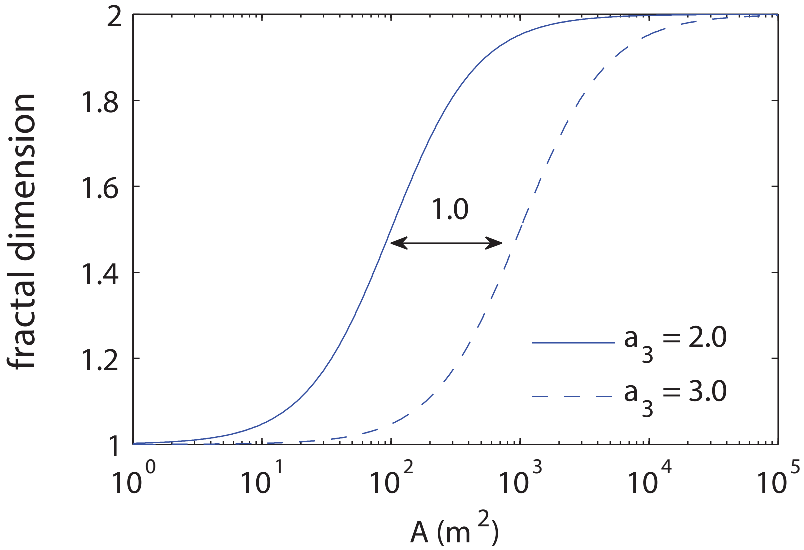


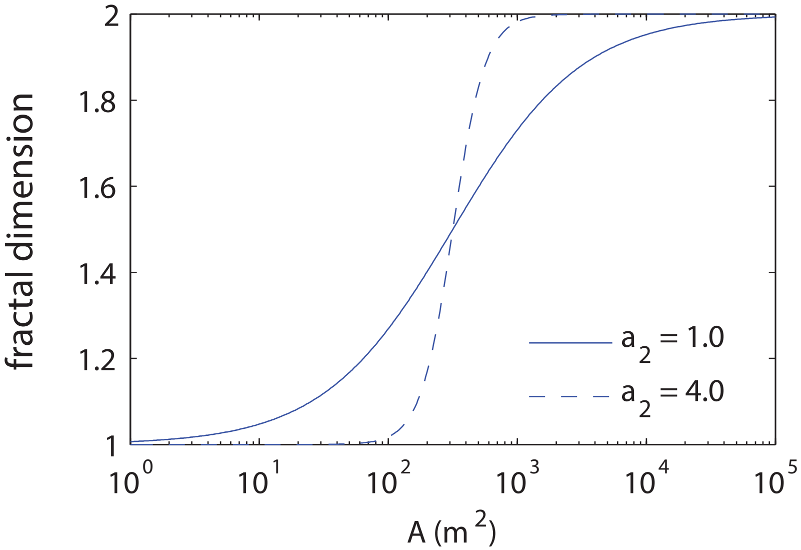
electronic transport in disordered media

diffusion in turbulent plasmas

Isichenko, Rev. Mod. Phys., 1992

fractal dimension curves depend on statistical parameters defining random surface





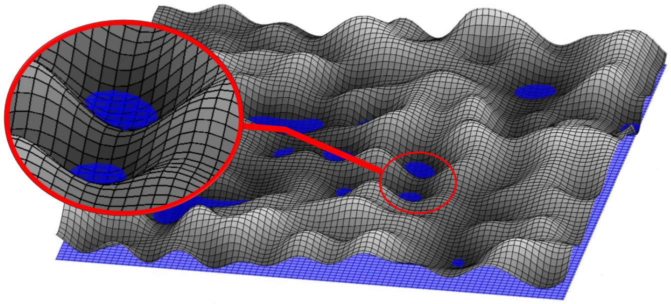
Topology of the sea ice surface and the fractal geometry of Arctic melt ponds

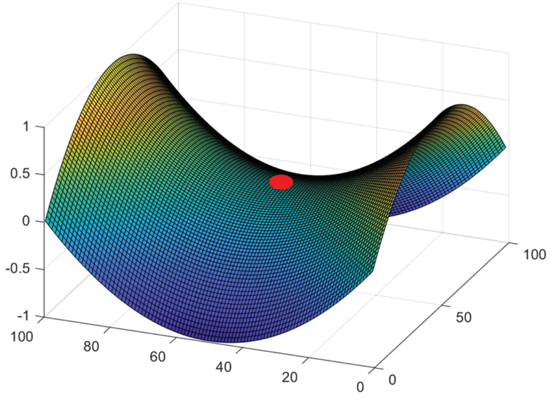
Physical Review Research (invited, under revision)

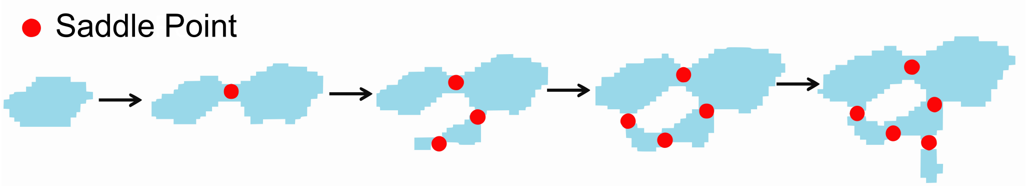
Ryleigh Moore, Jacob Jones, Dane Gollero,
Court Strong, Ken Golden

Several models replicate the transition in fractal dimension, but none explain how it arises.

We use Morse theory applied to the random surface model to show that saddle points play the critical role in the fractal transition.



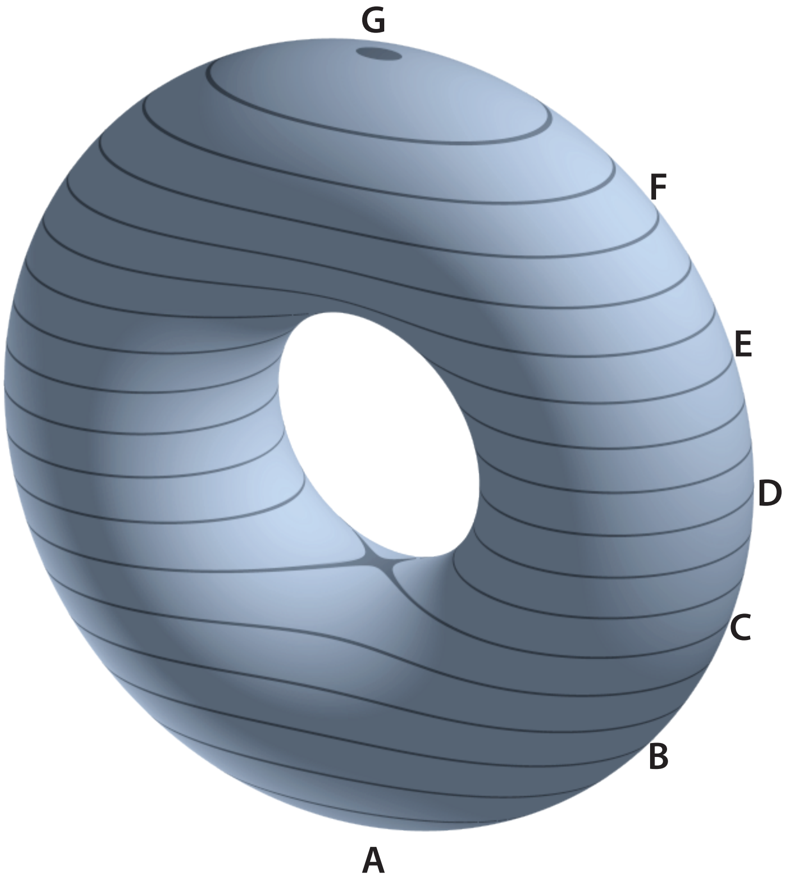




saddles drive the fractal transition

ponds coalesce
(change topology) and
complexify at saddle points

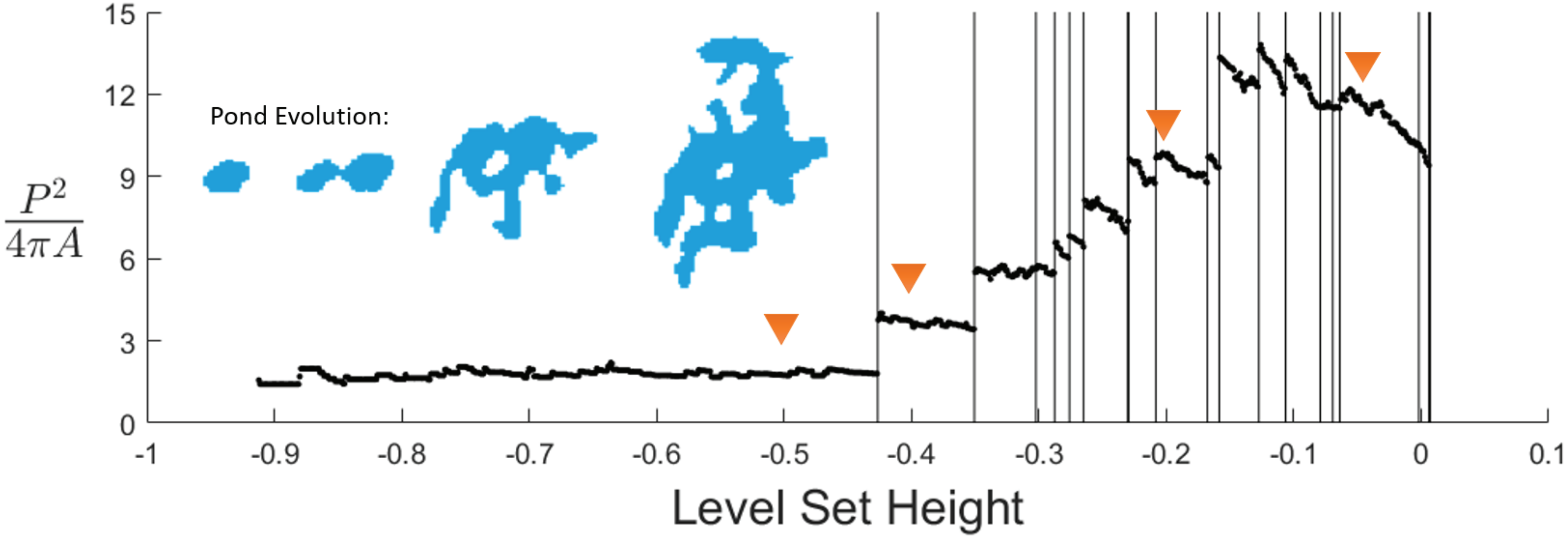
Morse theory



Morse theory tells us that changes in the topology of a surface occur at critical points of smooth functions on the surface: maxima, minima, and saddles.

Main results

Isoperimetric quotient - as a proxy for fractal dimension - increases in discrete jumps when ponds coalesce at saddle points.



Horizontal fluid permeability “controlled” by saddles ~ electronic transport in 2D random potential.

drainage processes, seal holes

melt pond evolution depends also on large-scale “pores” in ice cover

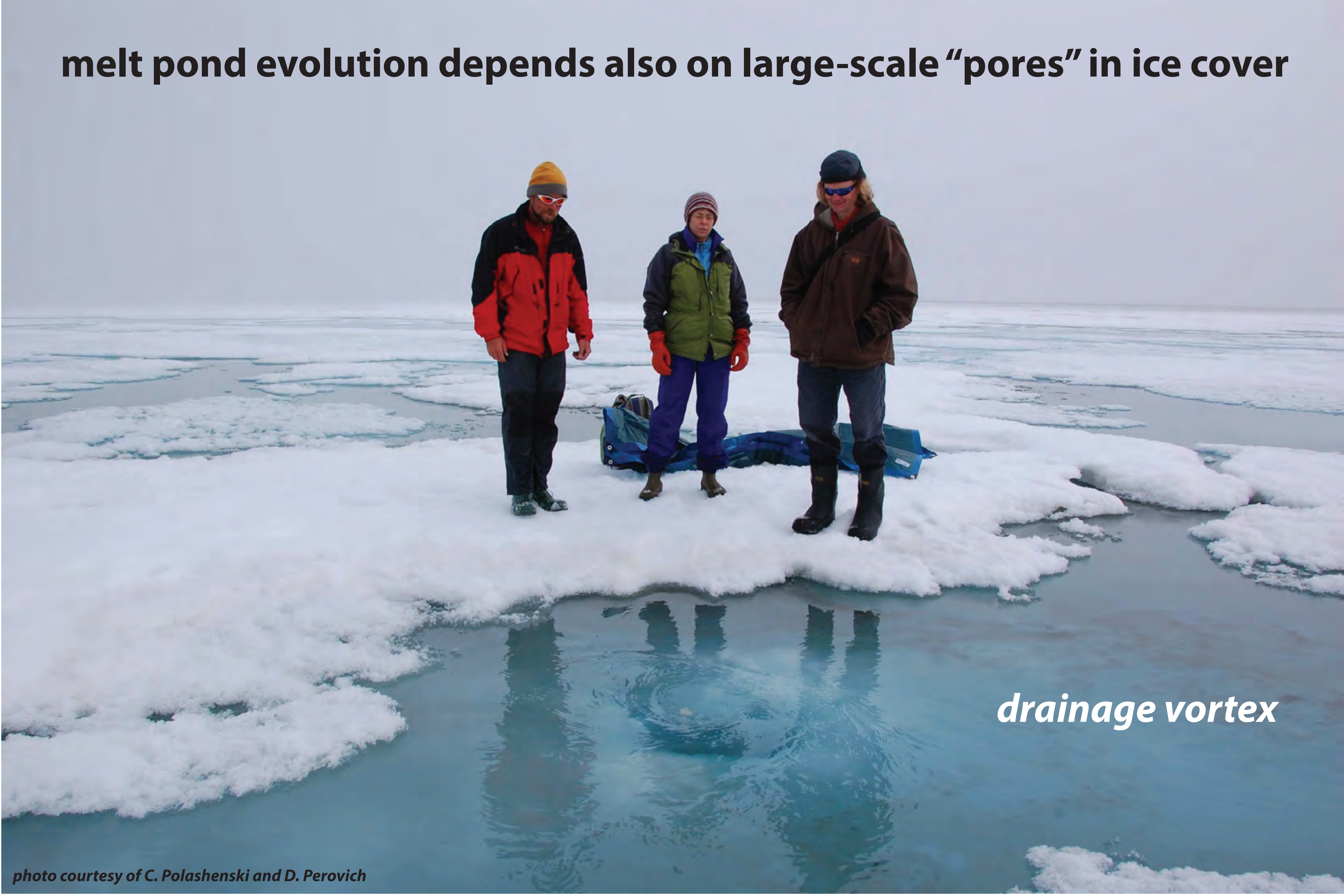


photo courtesy of C. Polashenski and D. Perovich

Melt pond connectivity enables vast expanses of melt water to drain down seal holes, thaw holes, and leads in the ice.

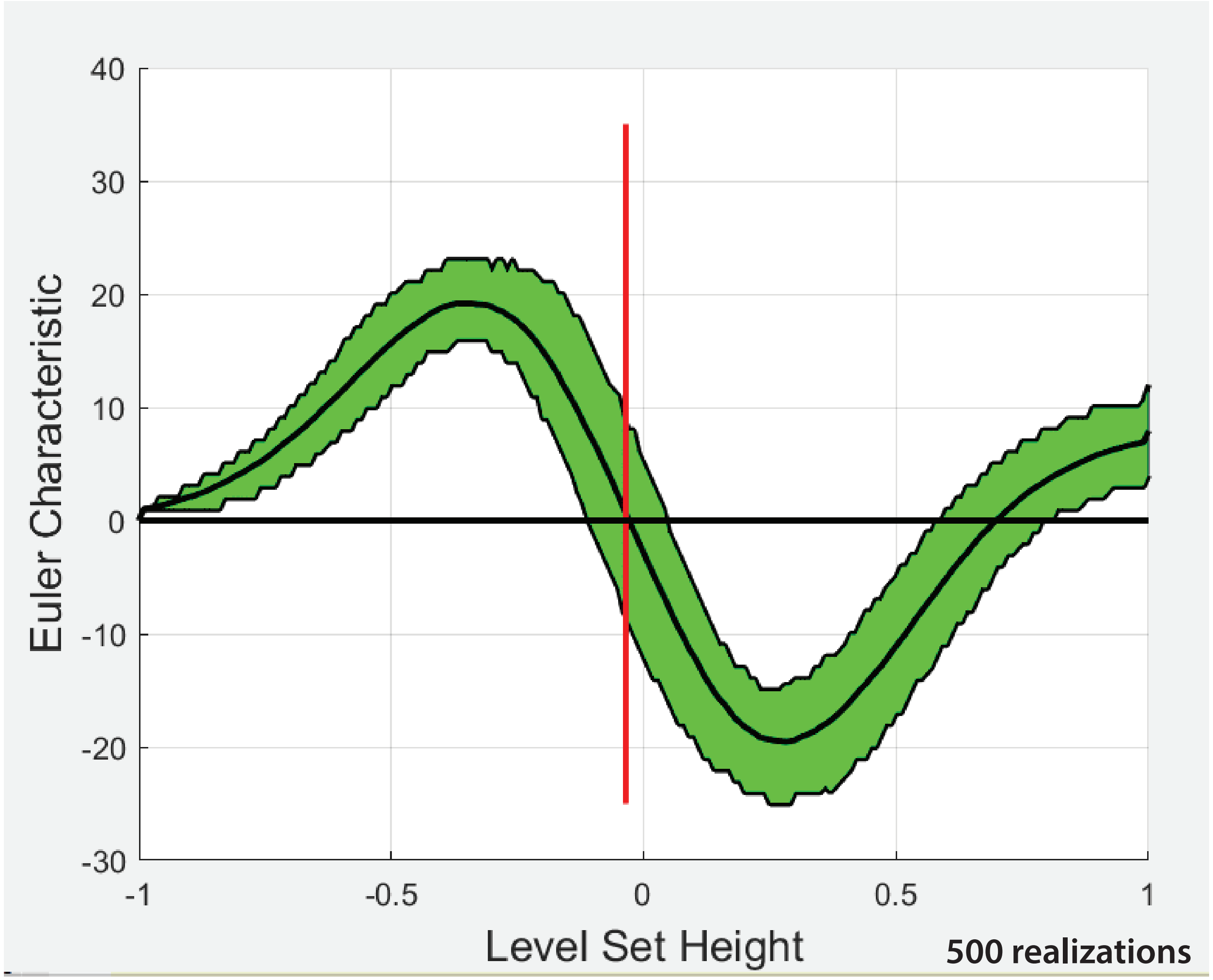
Euler characteristic = # maxima + # minima - # saddles

Topological Data Analysis

topological invariant

persistent homology

filtration - sequence of nested topological spaces, indexed by water level



Expected
Euler Characteristic Curve (ECC)

tracks the evolution of the EC of the flooded surface as water rises

zero of ECC ~ percolation

percolation on a torus
creates a giant cycle

Bobrowski & Skraba, 2020

Carlsson, 2009

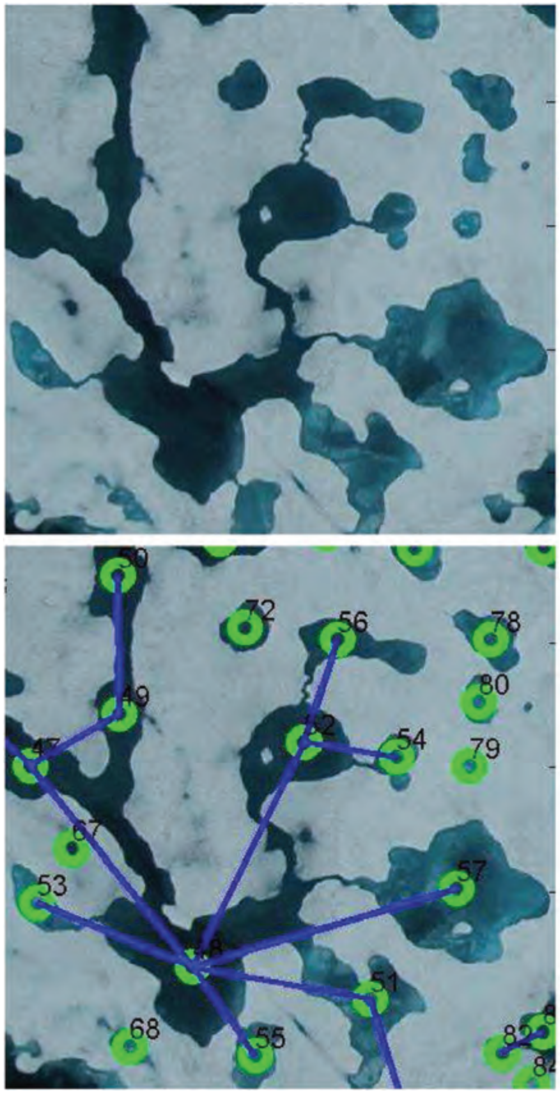
Vogel, 2002 GRF

image analysis
porous media
cosmology
brain activity

melt pond donuts

Network modeling of Arctic melt ponds

Barjatia, Tasdizen, Song, Sampson, Golden
Cold Regions Science and Tecnology, 2016



develop algorithms to map images of melt ponds onto

random resistor networks

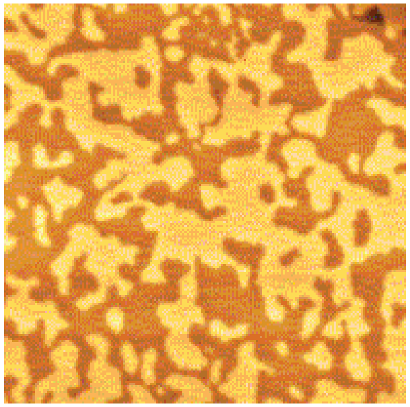
graphs of nodes and edges with edge conductances

edge conductance ~ neck width

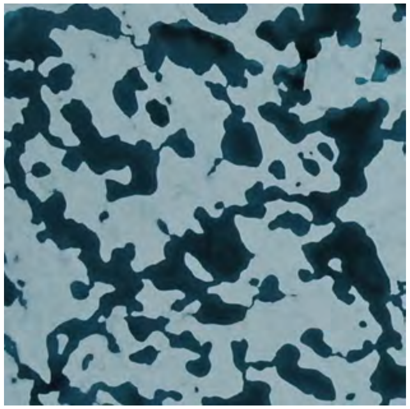
compute effective horizontal fluid conductivity

From magnets to melt ponds

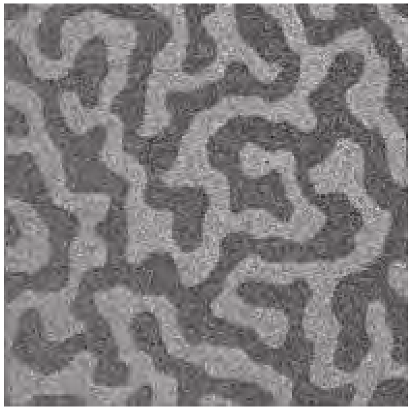
100 year old model for magnetic materials used to explain melt pond fractal geometry



magnetic domains
cobalt



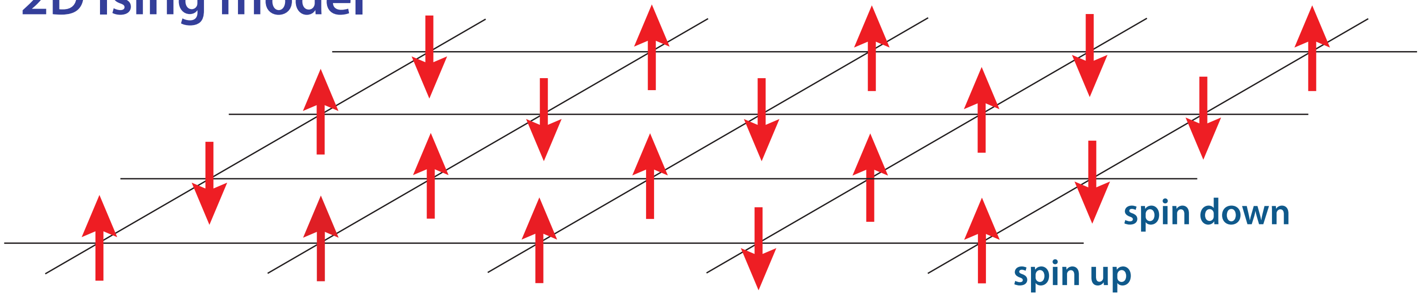
Arctic melt ponds

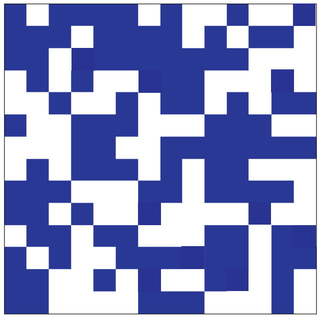


magnetic domains
cobalt-iron-boron

Arctic melt ponds

2D Ising model

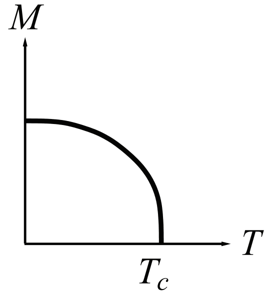




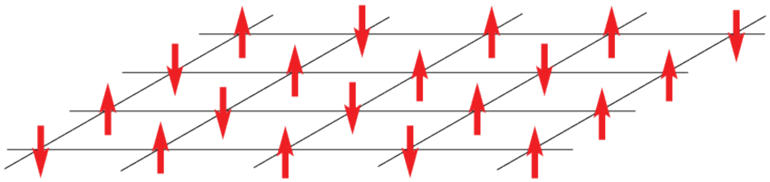
Ma, Sudakov, Strong, Golden, *New J. Phys.* 2019

Golden, Ma, Strong, Sudakov, *SIAM News* 2020

Ising Model for a Ferromagnet

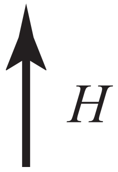


Curie point critical temperature



$$s_i = \begin{cases} +1 & \text{spin up} \quad \textbf{blue} \\ -1 & \text{spin down} \quad \textbf{white} \end{cases}$$

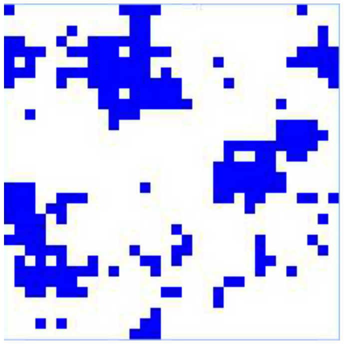
applied magnetic field



$$\mathcal{H} = -H\sum_{i} s_i - J\sum_{<i,j>} s_i s_j$$

nearest neighbor Ising Hamiltonian

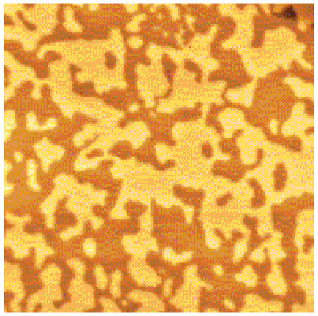
islands of like spins



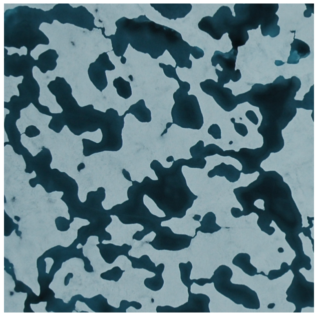
$$M(T,H) = \lim_{N\to\infty}\frac{1}{N}\left\langle \sum_{j} s_j \right\rangle$$

effective magnetization

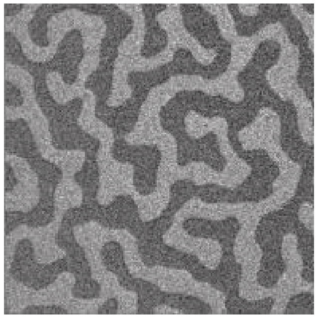
energy is lowered when nearby spins align with each other, forming **magnetic domains**



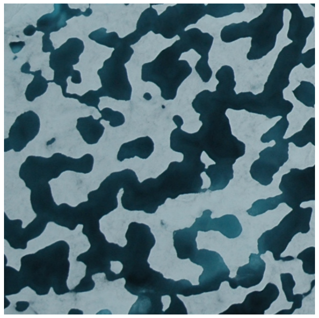
magnetic domains in cobalt



melt ponds (Perovich)



magnetic domains in cobalt-iron-boron



melt ponds (Perovich)

Ising model for ferromagnets ⟶ Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.*, 2019

$$\mathcal{H} = -\sum_{i}^{N} H_i\, s_i - J \sum_{<i,j>}^{N} s_i s_j \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

random magnetic field represents snow topography

magnetization M

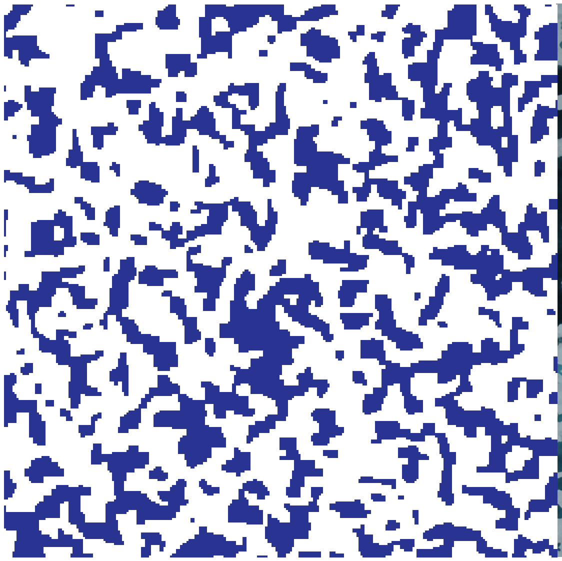
pond area fraction $F = \dfrac{(M+1)}{2}$

~ albedo

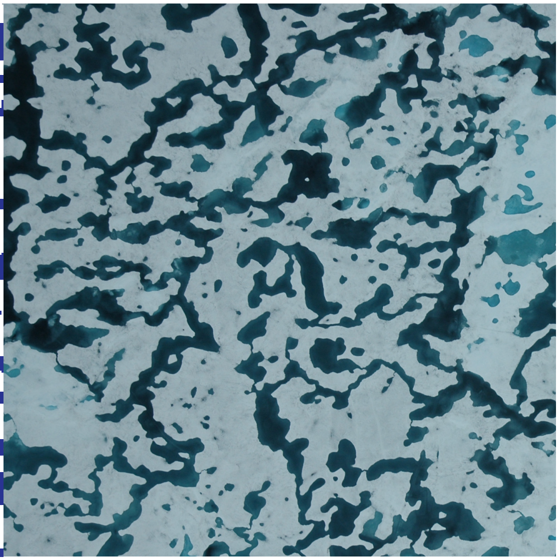
only nearest neighbor patches interact

Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system "flows" toward metastable equilibria.

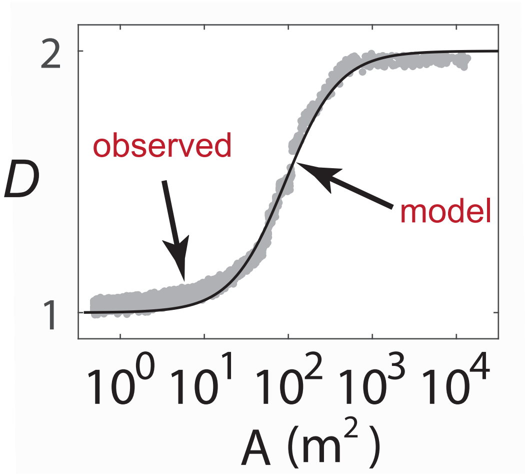
Order from Disorder



Ising model



melt pond photo (Perovich)



pond size distribution exponent

observed	-1.5
	(Perovich, *et al.* 2002)
model	-1.58

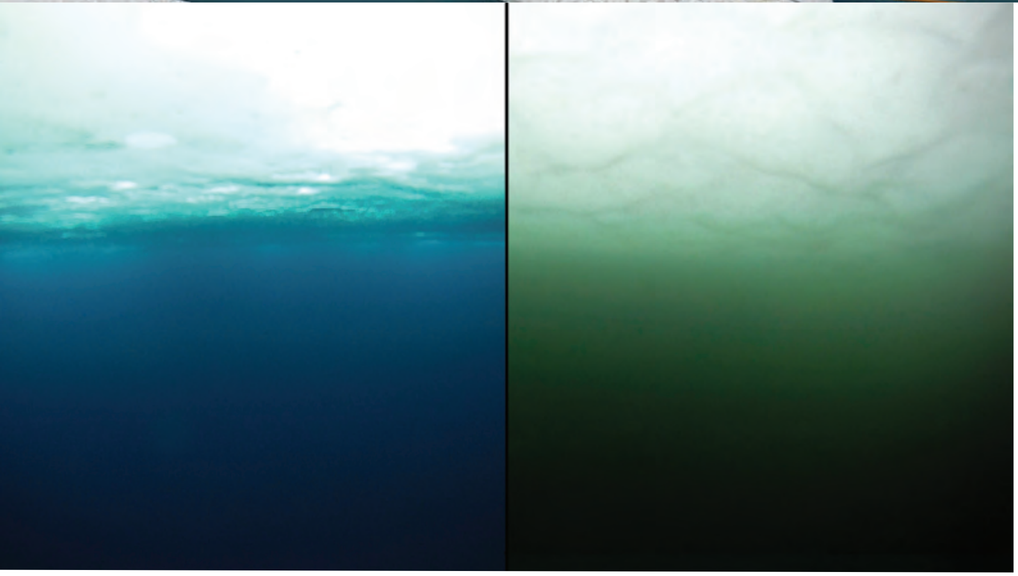
Scientific American
EOS, PhysicsWorld, ...

ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data

Melt ponds control transmittance of solar energy through sea ice, impacting upper ocean ecology.

WINDOWS

Perovich



no bloom bloom

massive under-ice algal bloom

Arrigo et al., *Science* 2012

Have we crossed into a new ecological regime?

The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

Horvat, Rees Jones, Iams, Schroeder,
Flocco, Feltham, ***Science Advances*** 2017

The effect of melt pond geometry on the distribution of solar energy under first year sea ice

Horvat, Flocco, Rees Jones, Roach, Golden
Geophys. Res. Lett. 2019

(2015 AMS MRC)

Ising model

partition function

$$Z_N(z) = a_N \prod_{n=1}^{N} (z - z_n), \qquad |z_n| = 1$$

free energy

$$f(T,H) = \frac{-1}{\beta} \int_{|t|=1} \log(z - t)\, d\nu(t)$$

order parameter

$$M(T) = -\frac{\partial f}{\partial H}$$

$$\frac{\partial^2 M}{\partial H^2} \leq 0$$

G.H.S. inequality

Griffiths, Hurst, Sherman *JMP* 1970

transport in composites

$$\mathcal{Z}_N(s) = \prod_{n=1}^{N} (s - s_n), \qquad s_n \in [0,1]$$

$$\Phi(p,s) = \int_0^1 \log(s - t)\, d\mu(t)$$

$$F(p,s) = \frac{\partial \Phi}{\partial s}$$

$$\frac{\partial^2 m}{\partial h^2} \leq 0$$

Golden, *JMP* 1995; *PRL* 1997

Stieltjes integral representation for magnetization (~ albedo)

and scaling relations for critical exponents Baker, *Phys. Rev. Lett.* 1968

$$M(\tau) = \tau + \tau(1-\tau^2)G(\tau^2) \qquad \tau = \tanh(\beta H)$$

$$G(\tau^2) = \int_0^\infty \frac{d\psi(y)}{1+\tau^2 y}$$

Herglotz **(Lee-Yang 1952)**

---

parallel Herglotz structure for transport in composites
analogous critical behavior and scaling relations hold near *p*$_c$

Golden, *J. Math. Phys.* 1995 **(C. Newman)**
***Phys Rev. Lett.* 1997**

$$F(s) = 1 - m(h) = \int_0^1 \frac{d\mu(w)}{s-w}$$

$$m(h) = \frac{\sigma^*}{\sigma_2} \qquad h = \frac{\sigma_1}{\sigma_2} \to 0 \qquad \sigma^*(p,h)$$

effective conductivity
of two phase composite
lattice or continuum

$$m(h) = 1 + (h-1)g(h) \qquad g(h) = \int_0^\infty \frac{d\phi(y)}{1+hy}$$

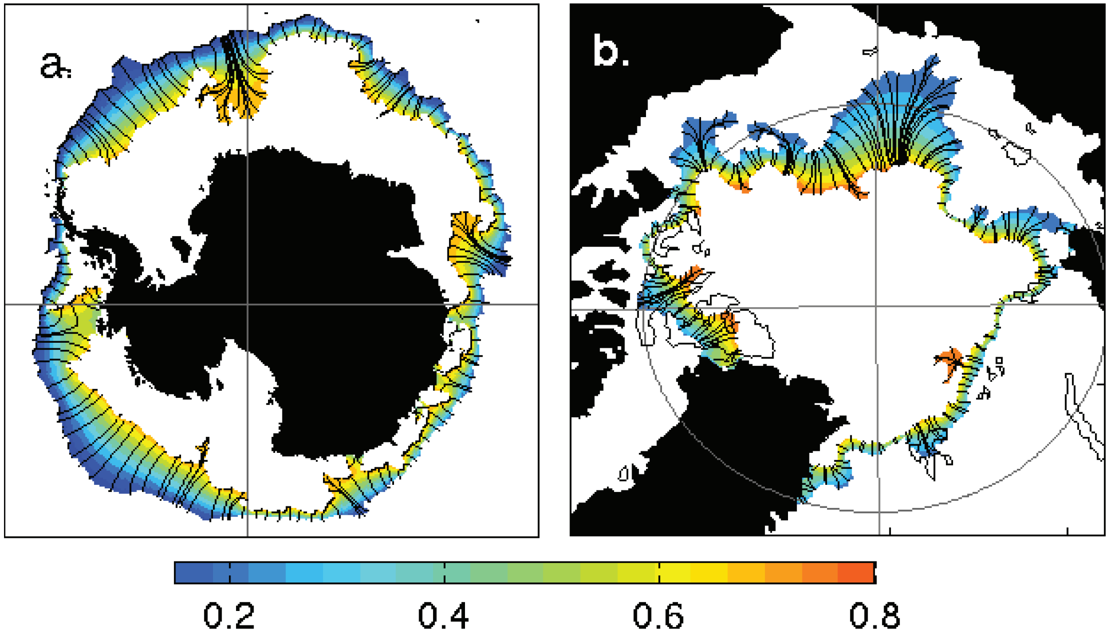
Herglotz

$$w = \frac{y}{y+1}$$

Marginal Ice Zone

MIZ

- biologically active region
- intense ocean-sea ice-atmosphere interactions
- region of significant wave-ice interactions



transitional region between dense interior pack ($c > 80\%$) sparse outer fringes ($c < 15\%$)

MIZ WIDTH

fundamental length scale of ecological and climate dynamics

Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

How to objectively measure the "width" of this complex, non-convex region?

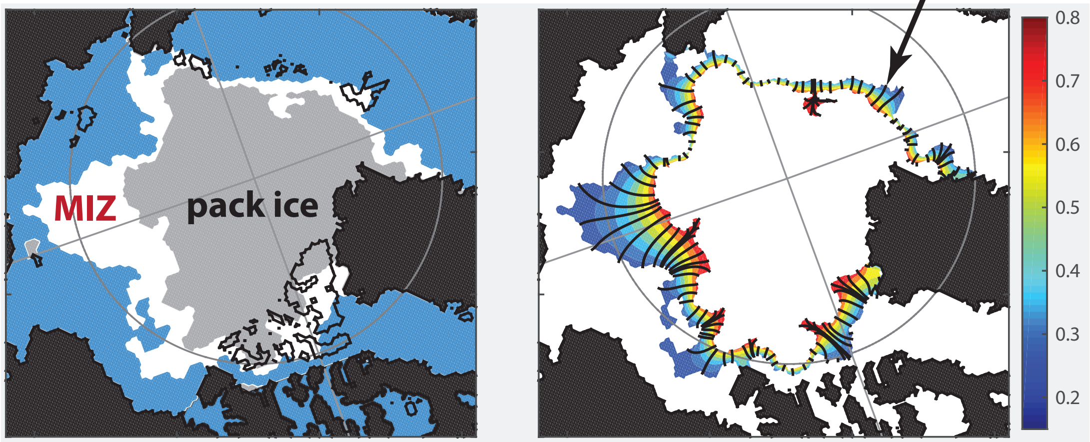
Objective method for measuring MIZ width motivated by medical imaging and diagnostics

Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

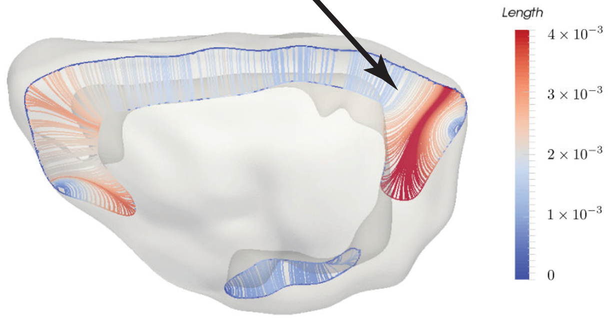
39% widening
1979 - 2012

streamlines of a solution to Laplace's equation

"average" lengths of streamlines



Arctic Marginal Ice Zone



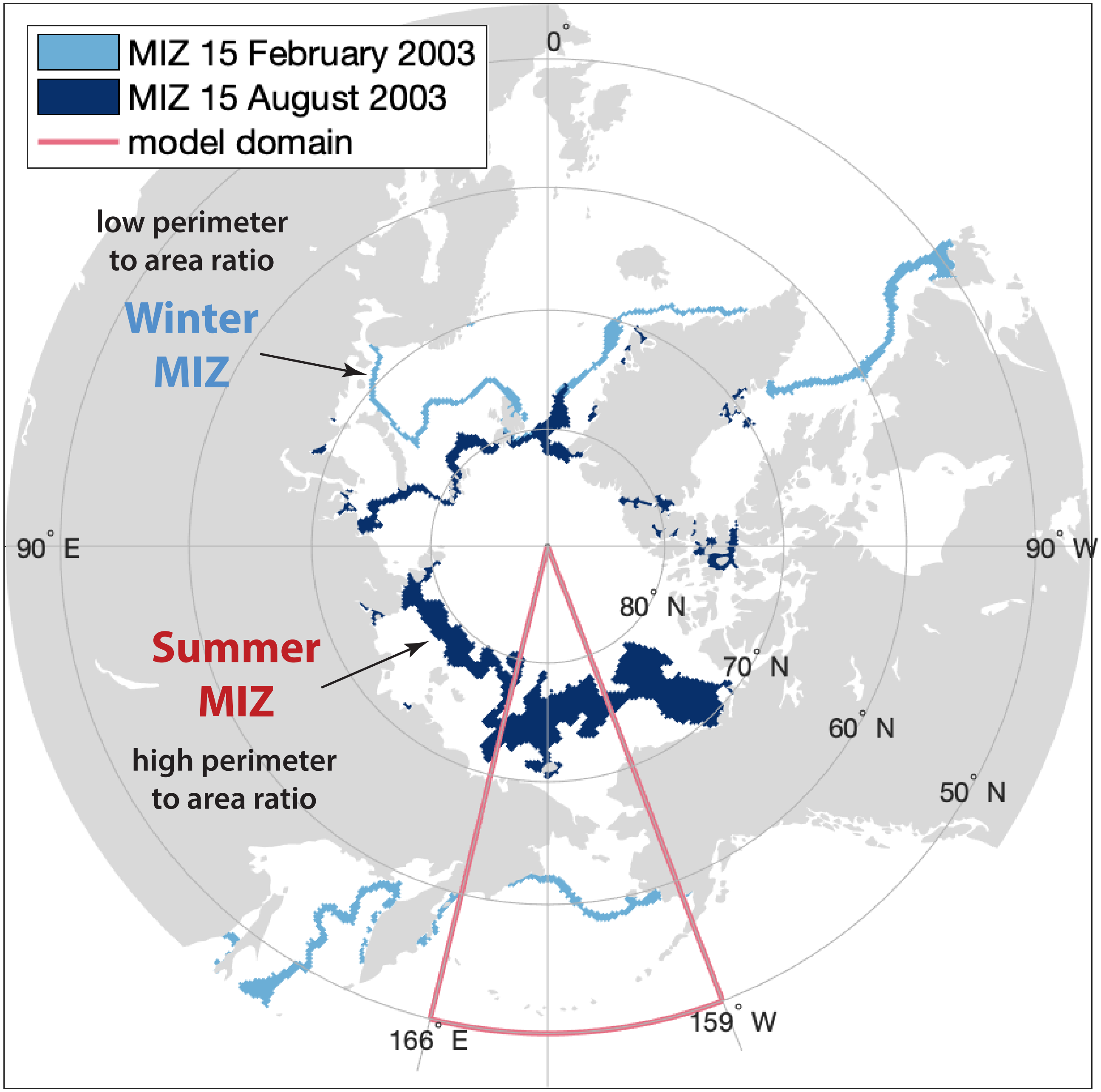
crossection of the cerebral cortex of a rodent brain

analysis of different MIZ WIDTH definitions

Strong, Foster, Cherkaev, Eisenman, Golden
J. Atmos. Oceanic Tech. 2017

Strong and Golden
Society for Industrial and Applied Mathematics News, April 2017

Observed Arctic MIZ



low fractal
dimension

low perimeter
to area ratio

**Winter
MIZ**

**Summer
MIZ**

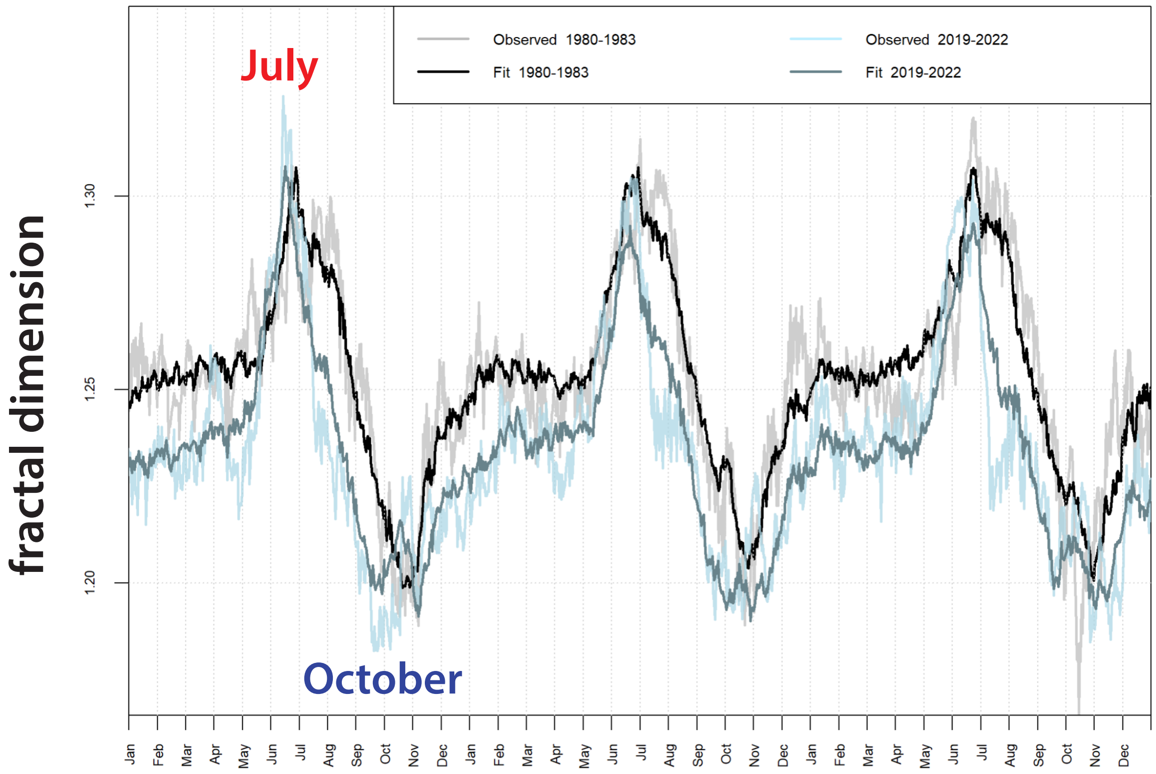
high perimeter
to area ratio

high fractal
dimension

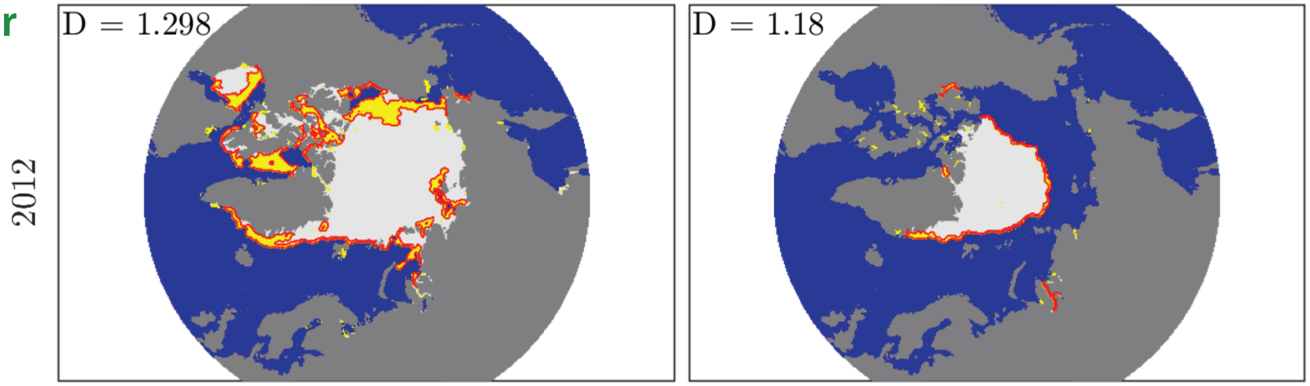
Identifying Fractal Geometry in Arctic Marginal Ice Zone Dynamics

Julie Sherman, Court Strong, Ken Golden, *Environ. Res. Lett.* 2025

Compute the fractal dimension of the boundary of the Arctic MIZ by boxcounting methods; analyze seasonal cycle and long term trends.



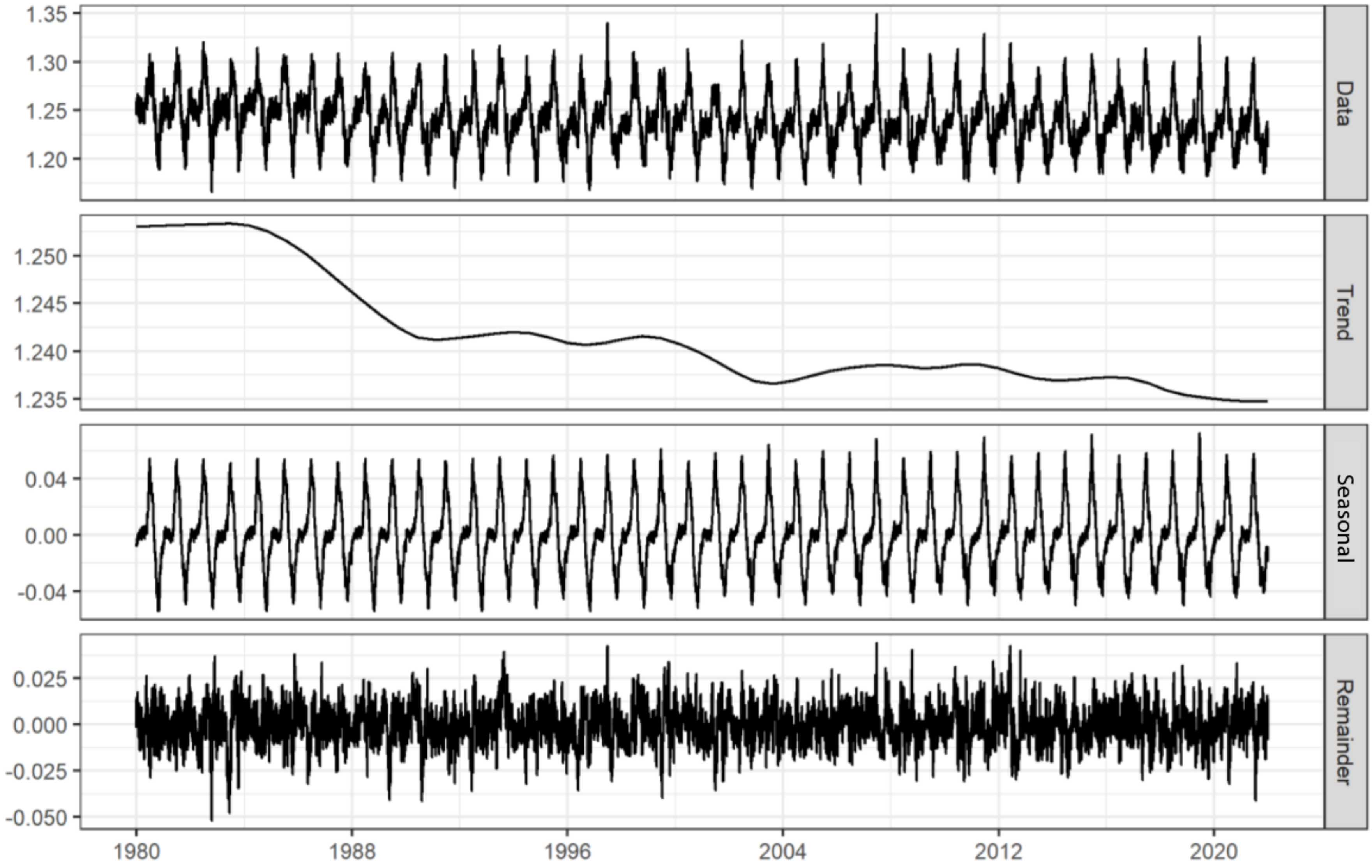
early summer



early autumn

wave and thermal interactions with fractal boundary

Arctic MIZ fractal dimension from 1980 to 2021



Geographical distribution of average fractal dimension