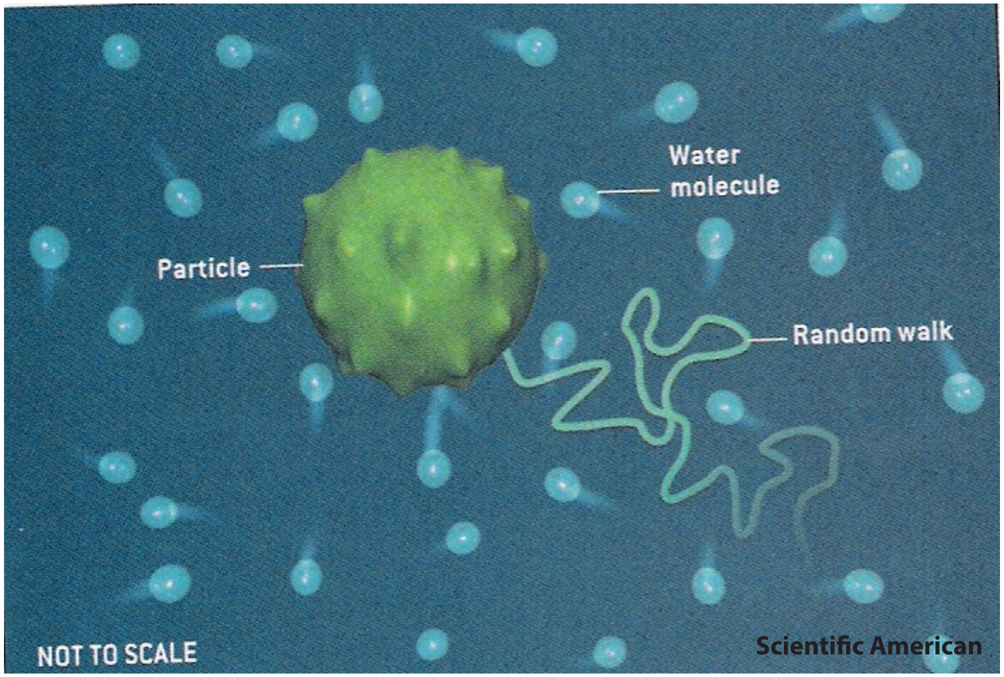
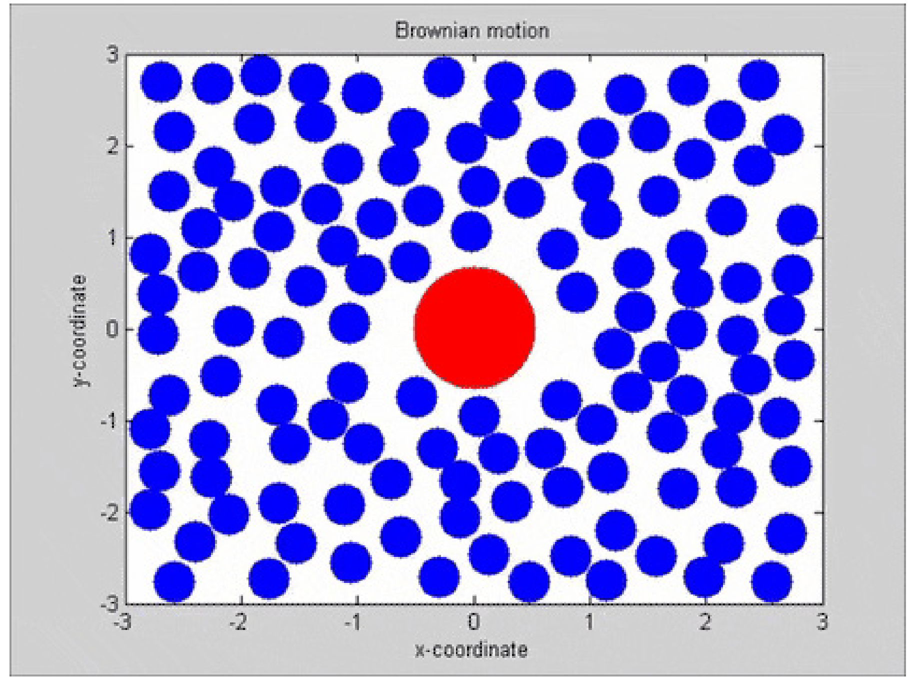
Brownian Motion and Diffusion Processes

Kenneth M. Golden
Department of Mathematics
University of Utah

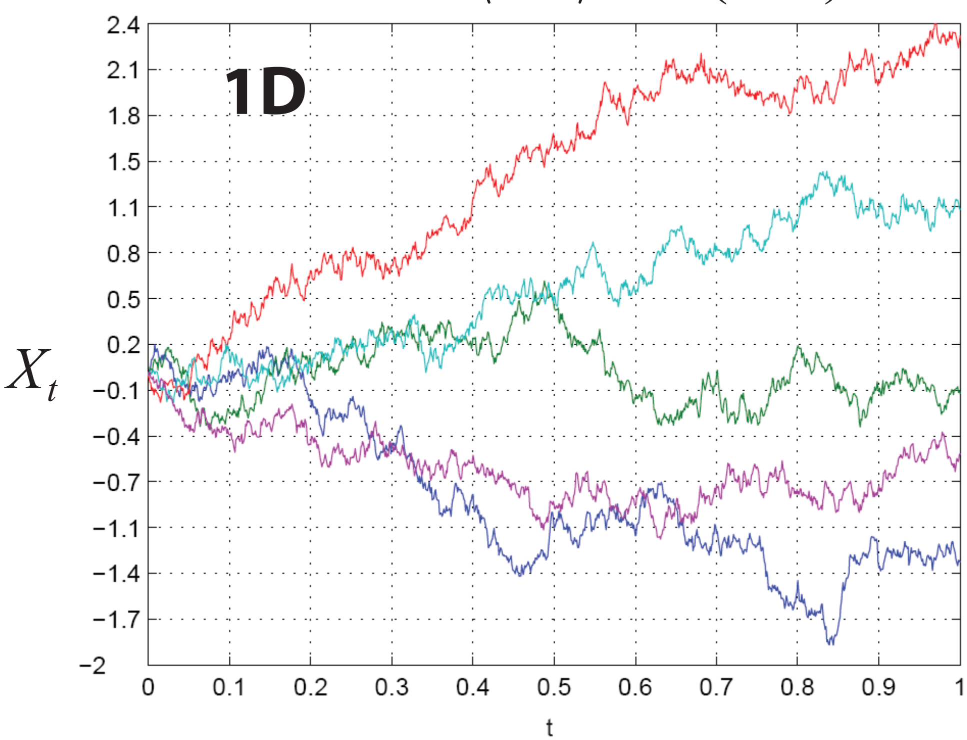
SLMath Summer School
UAF June 24, 2025

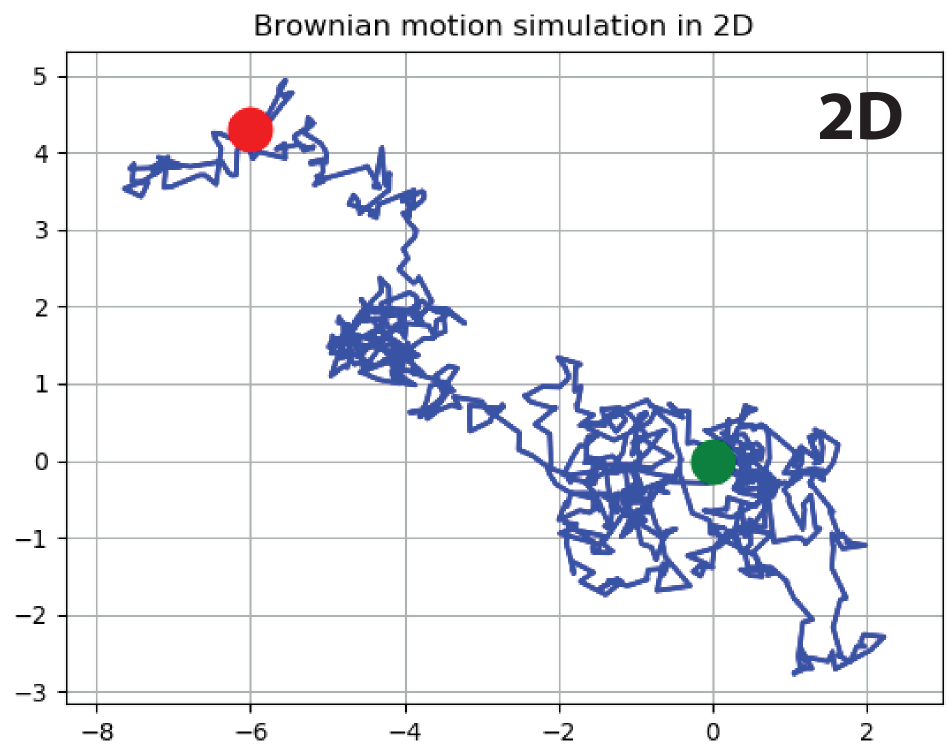
Brownian motion and diffusion





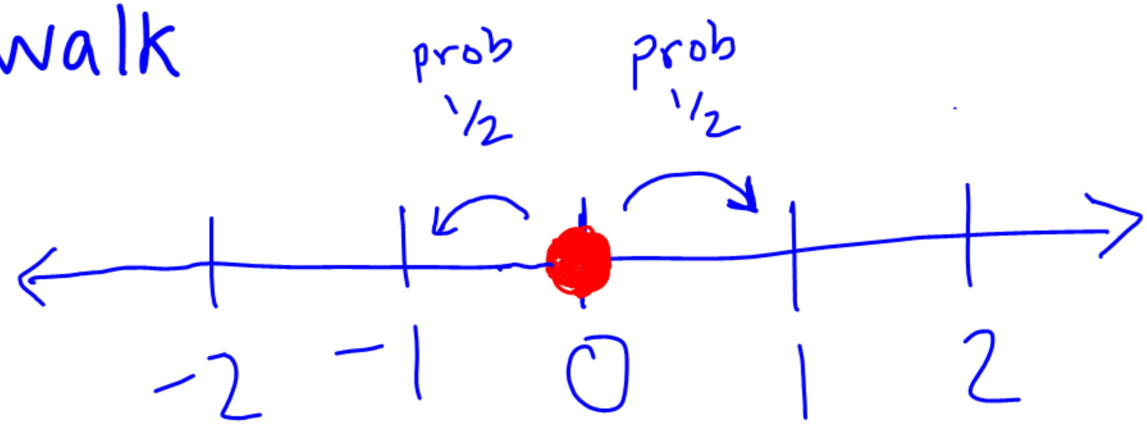
Einstein: $\langle X^2 \rangle \sim (\ ?\ )\, t$

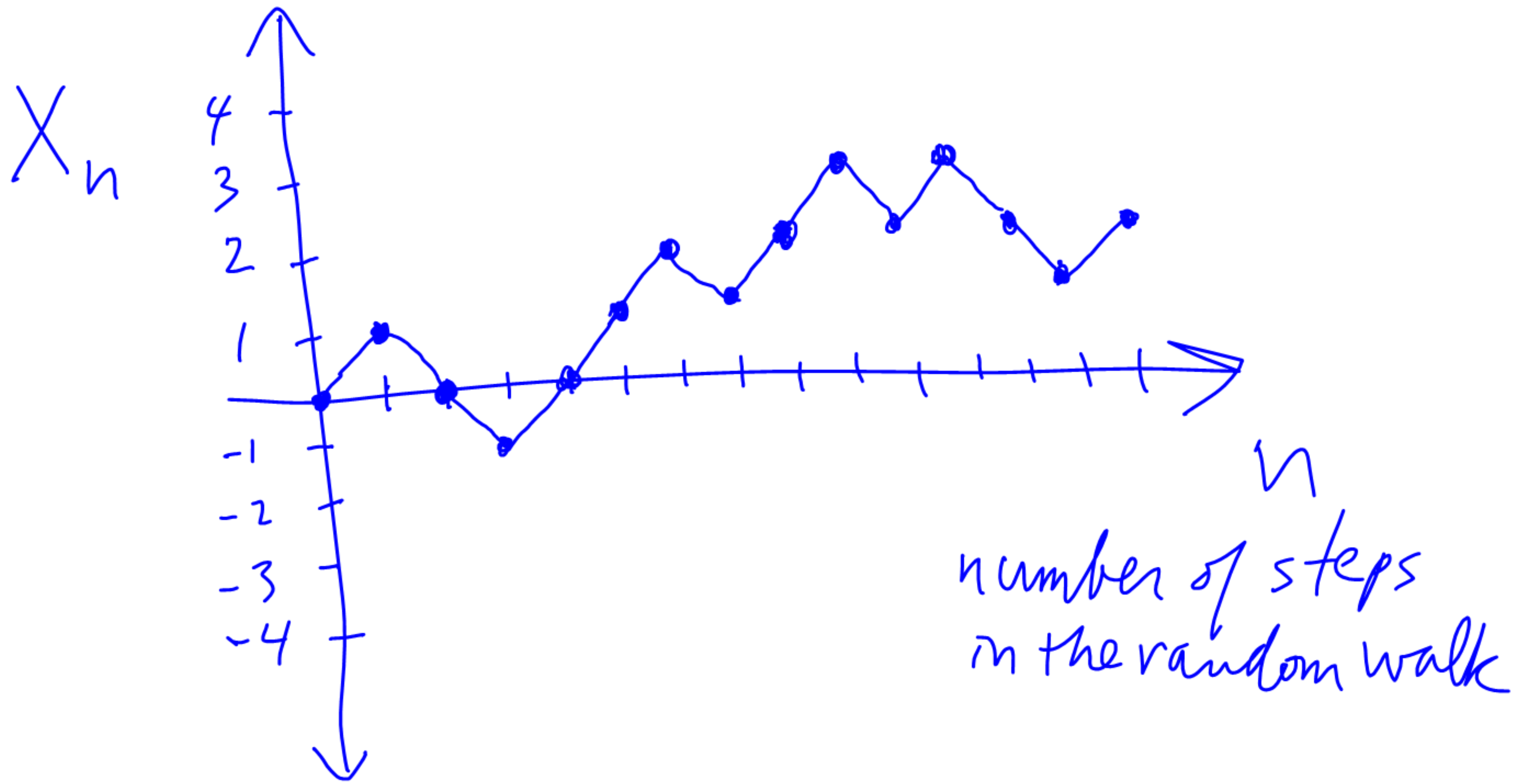


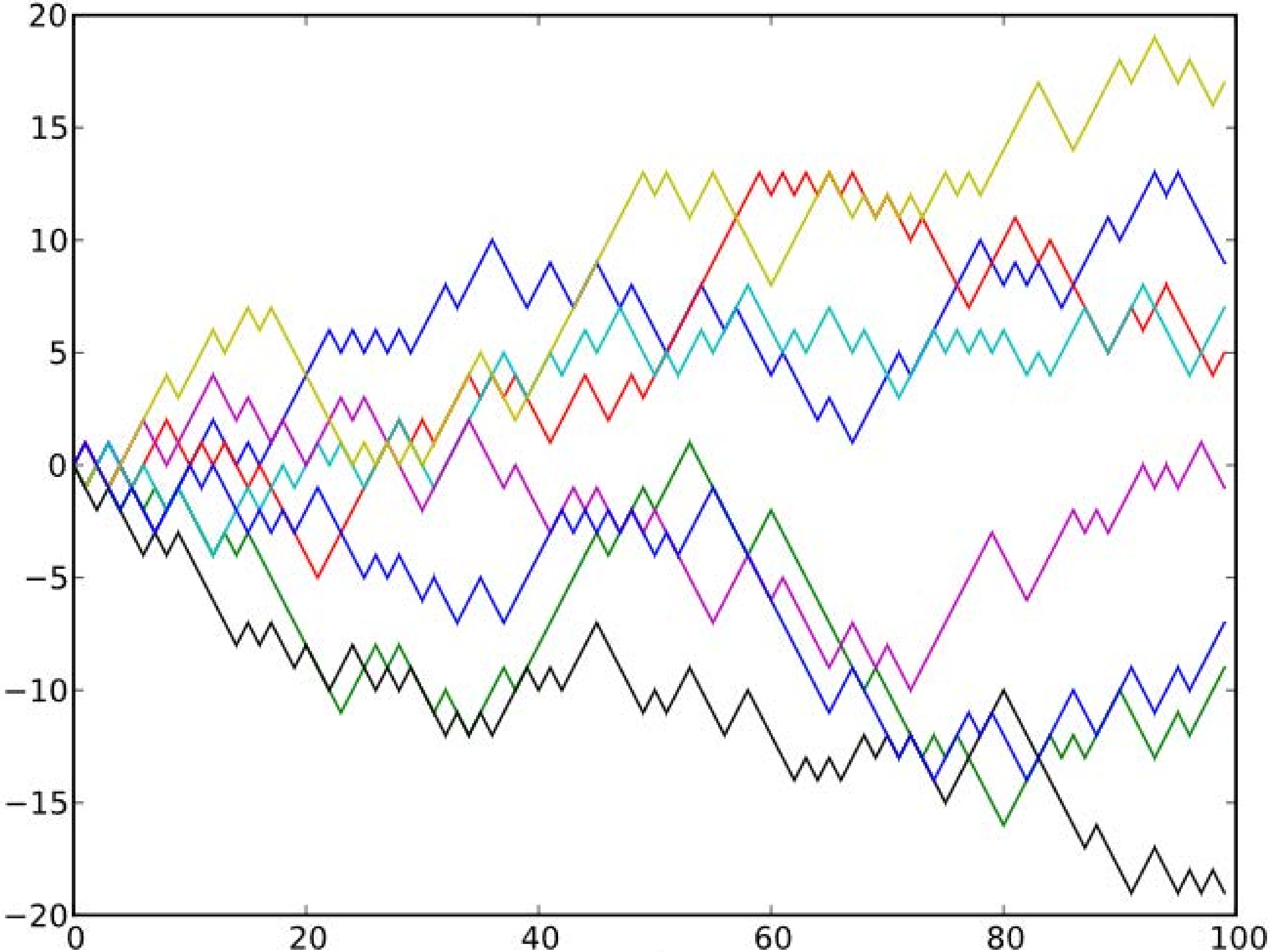


FSU

one dimensional random walk






20
15
10
5
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−5
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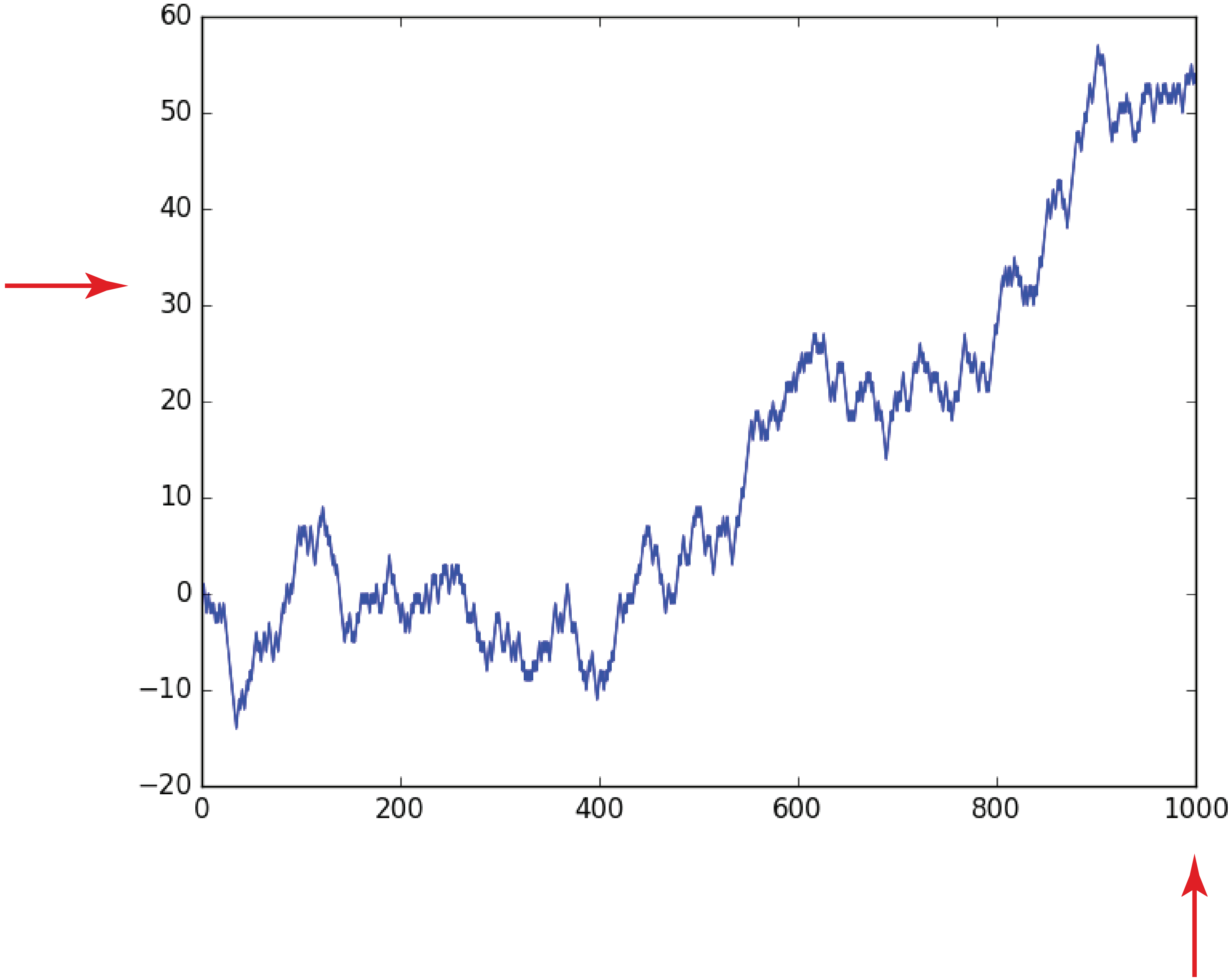
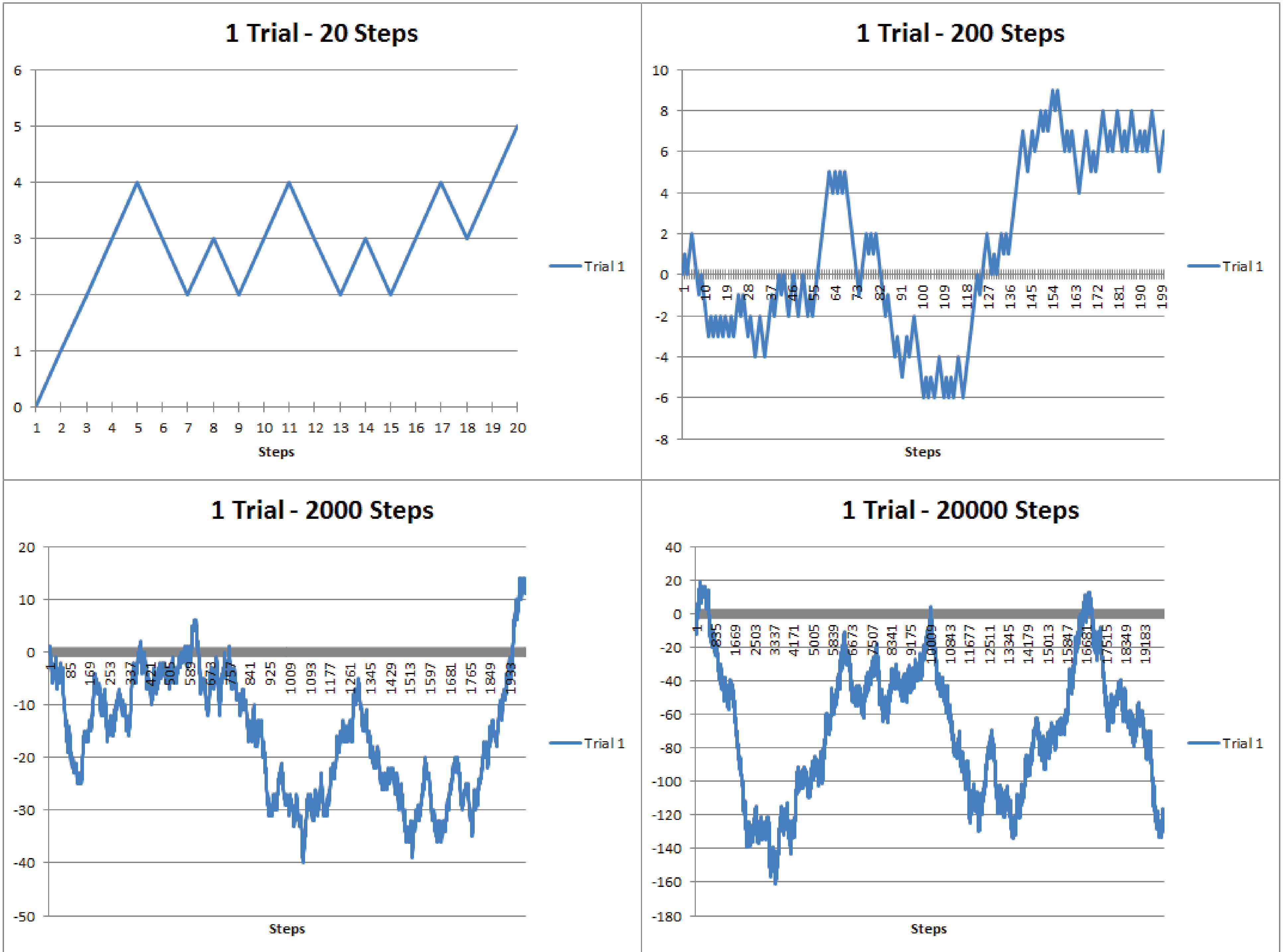
−20
0
20
40
60
80
100
t

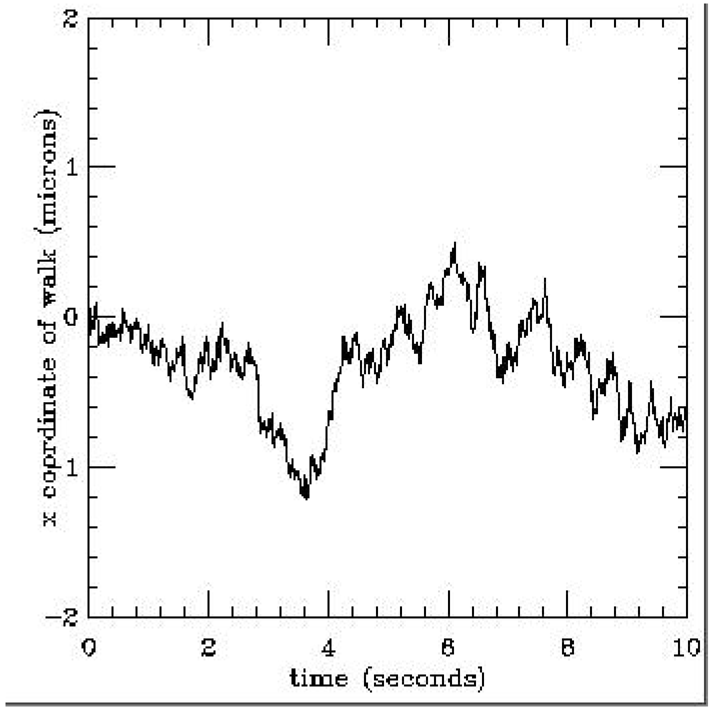
random walk with large n "looks like" brownian motion

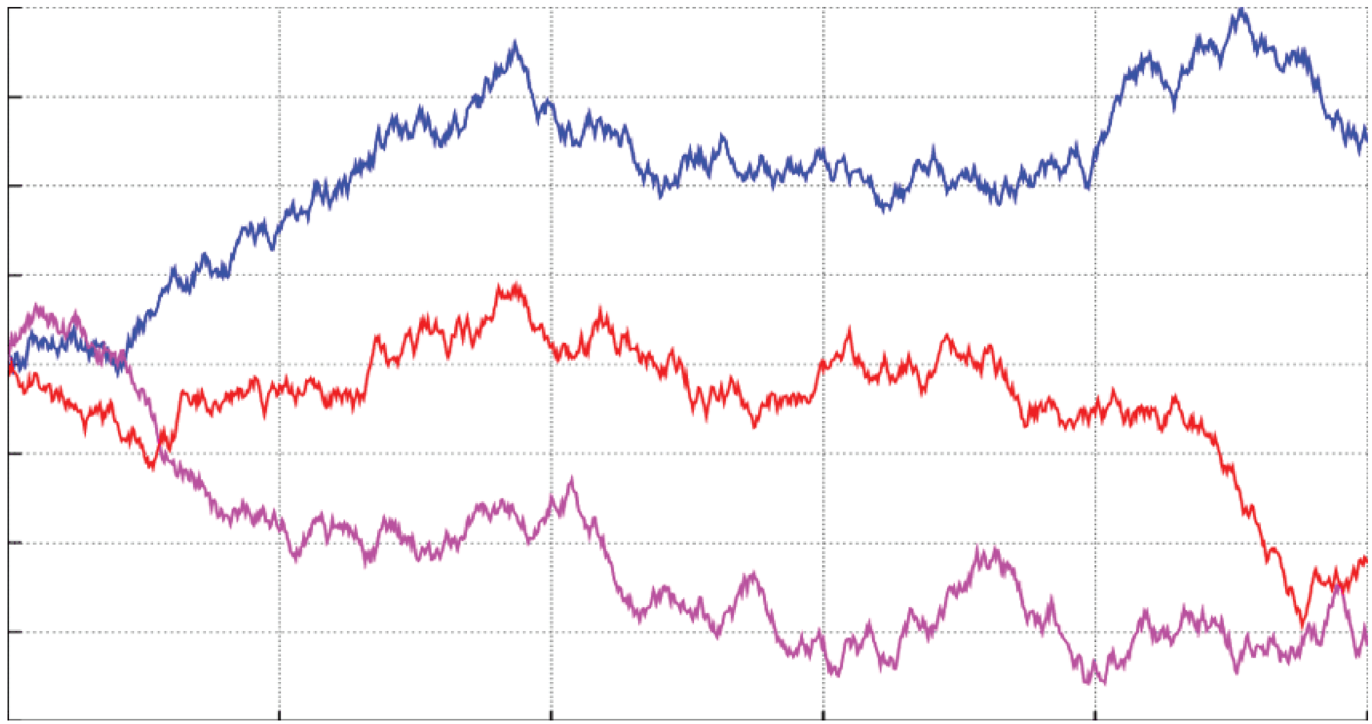


random walk converges to Brownian motion as space and time step sizes → 0

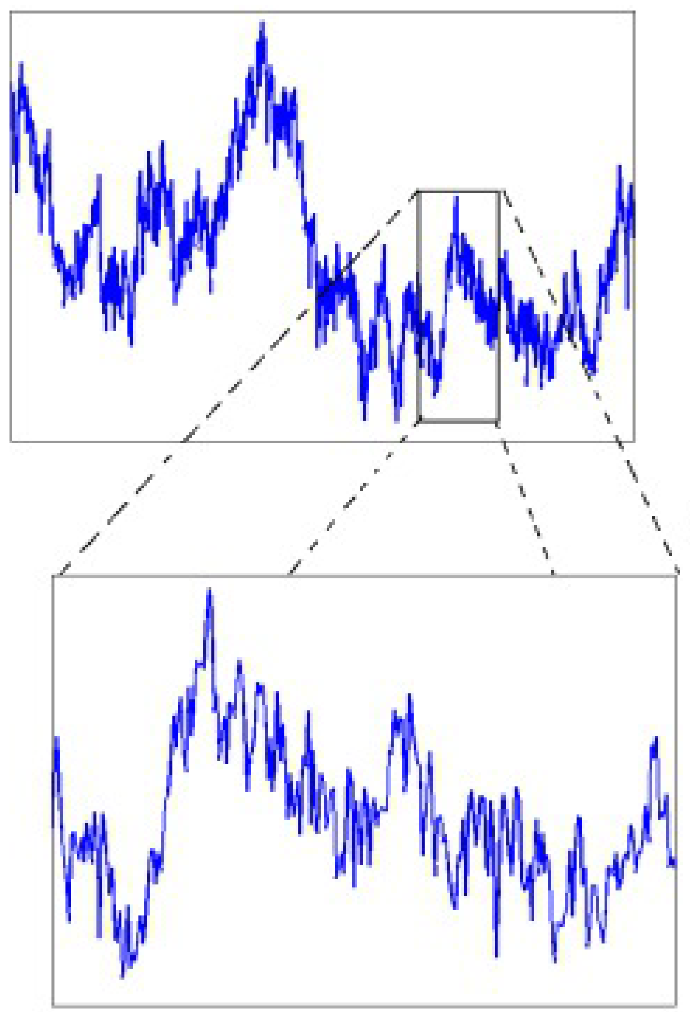


sample paths of 1D Brownian motion





Brownian motion paths are random, self-similar fractals



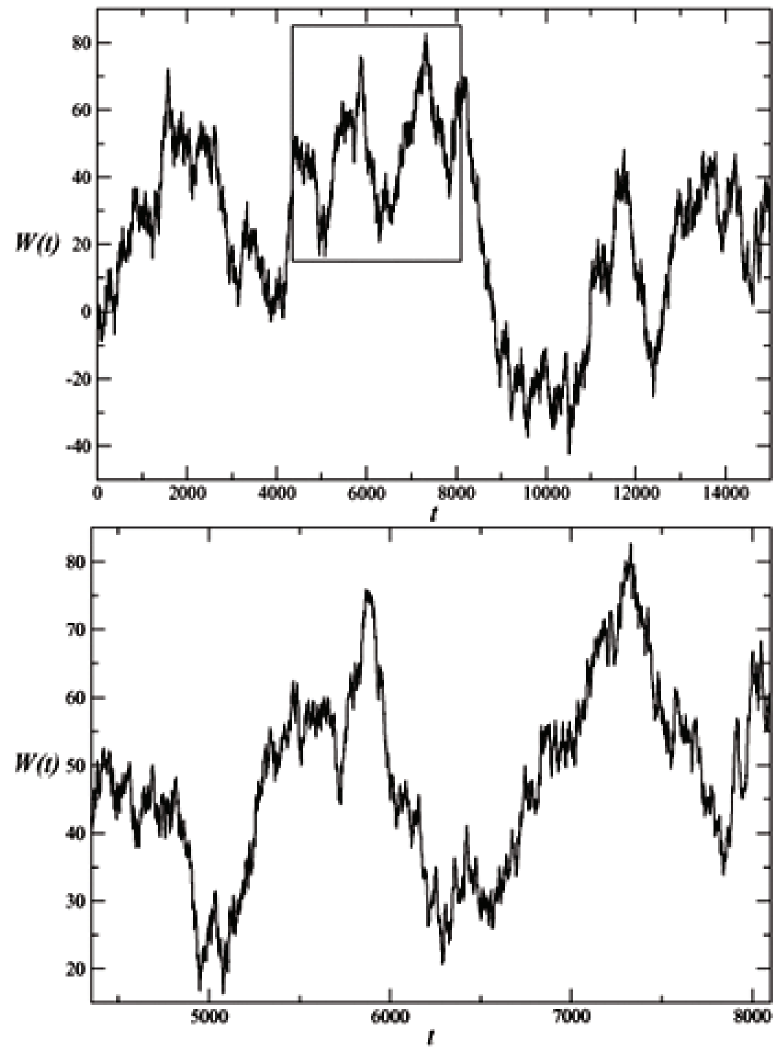
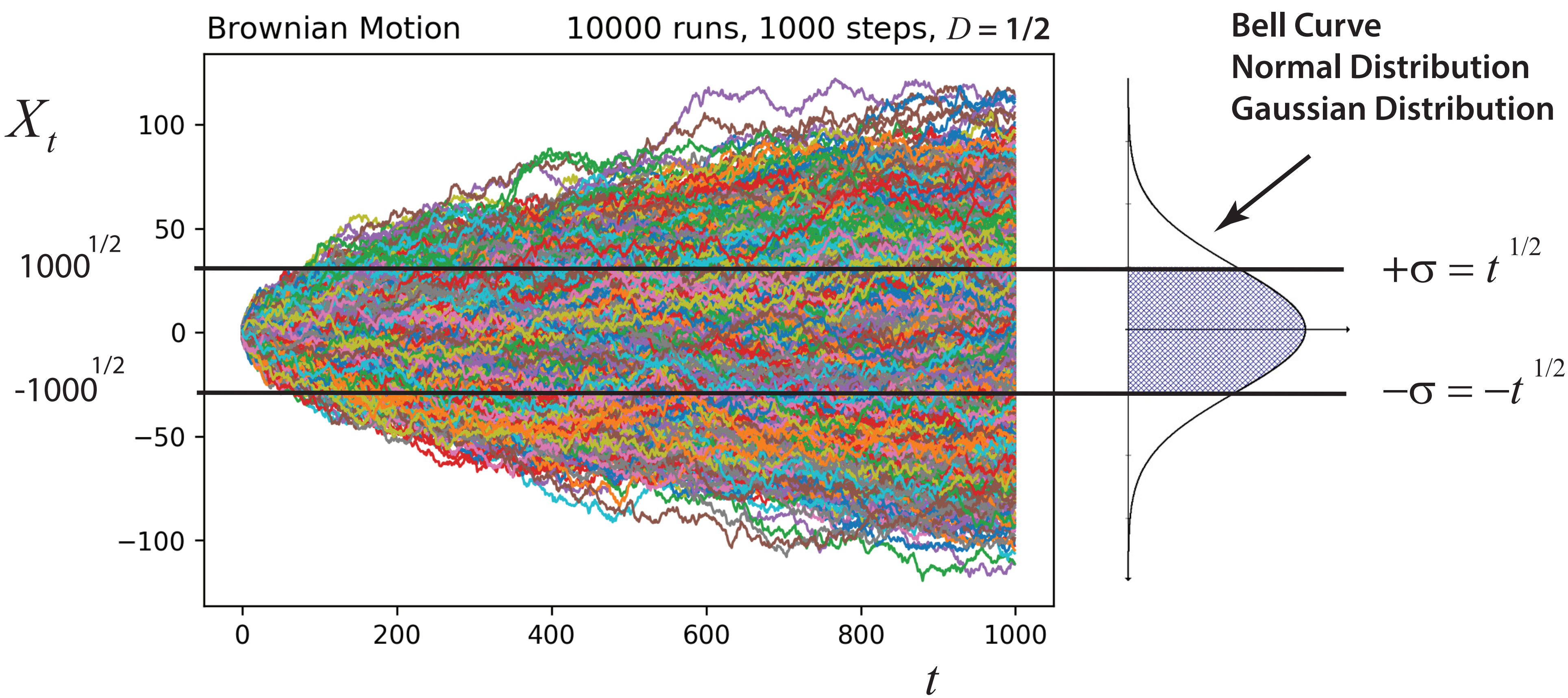


Figure 7. Self-similarity of a Brownian motion path. In (a) we plot a path of a Brownian motion with 15000 time steps. The curve in (b) is a blow-up of the region delimited by a rectangle in (a), where we have rescaled the x axis by a factor 4 and the y axis by a factor 2. Note that the graphs in (a) and (b) "look the same," statistically speaking. This process can be repeated indefinitely.

nowhere differentiable!

Brownian motion and the diffusion equation



diffusion equation $$\frac{\partial u}{\partial t} = D\,\nabla^2 u \qquad u(x,t) = \frac{1}{\sqrt{4\pi D t}}\, e^{-x^2/4Dt}$$

$$u(x,0) = \delta_0(x)$$

variance $\sigma^2 = 2Dt$ $\qquad D = \frac{1}{2}$

mean squared displacement $$\boxed{\langle X_t^2 \rangle = 2Dt = t^1}$$

$$\boxed{\langle |X_t| \rangle = t^{1/2}}$$

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A RANDOM WALK DOWN *Wall Street*

The Time-Tested Strategy *for* Successful Investing

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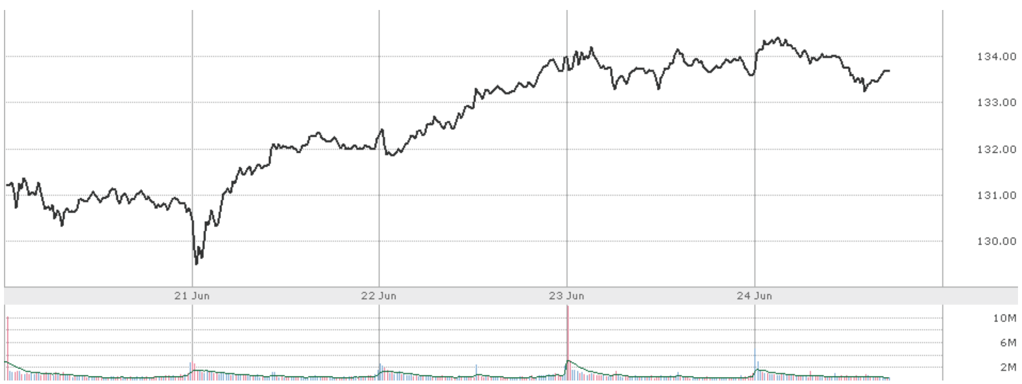
COMPLETELY REVISED *and* UPDATED

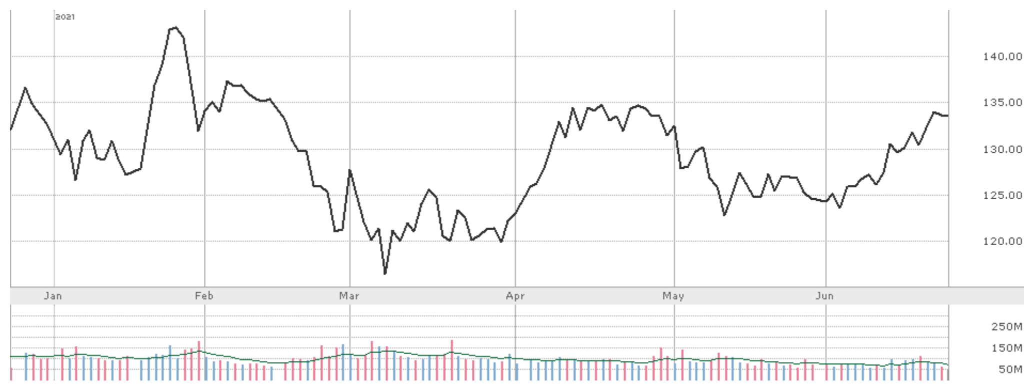
Do stock prices move randomly, like Brownian motion?

TSLA last 6 months

DOW June 24

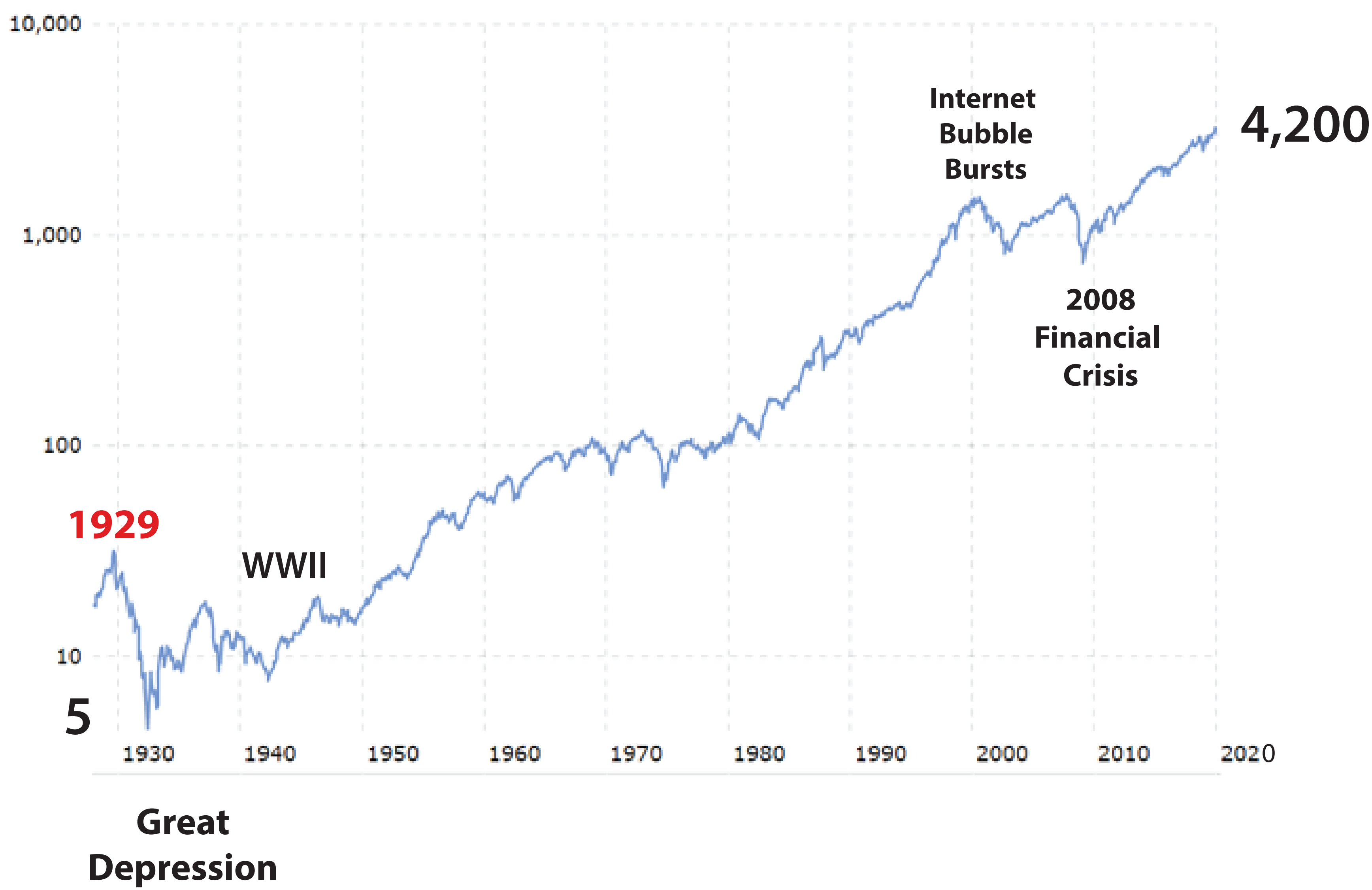


AAPL 5 day

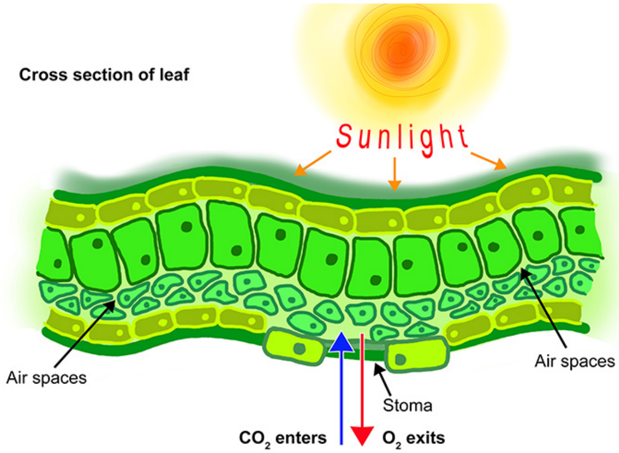


AAPL last 6 months

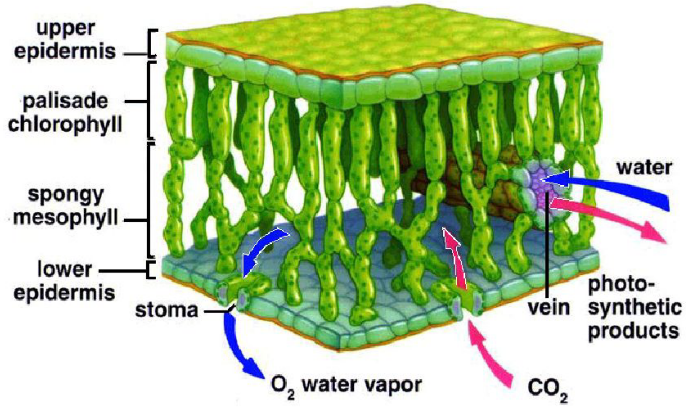
S&P 500 (broadest benchmark average) 1927 - present



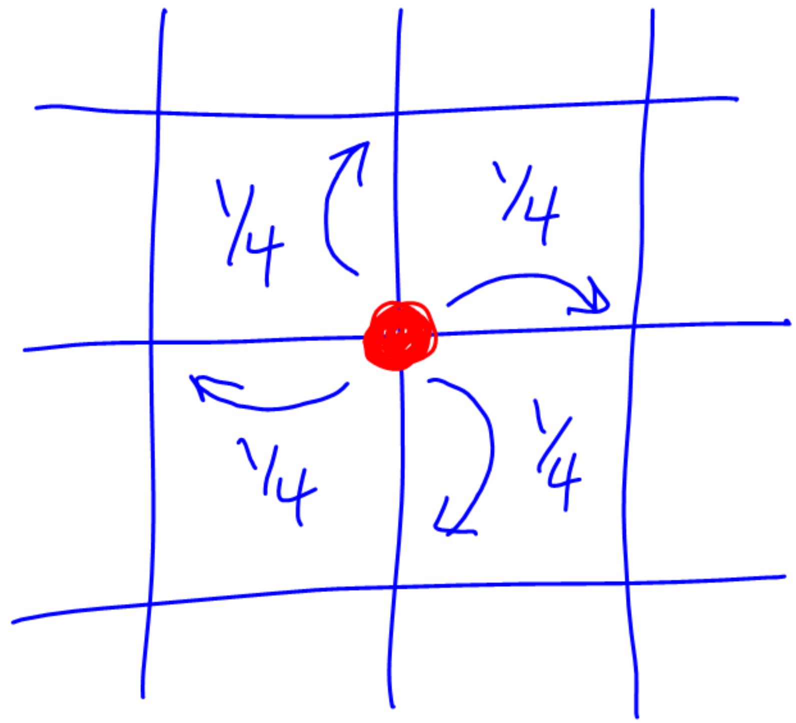
Examples of diffusion

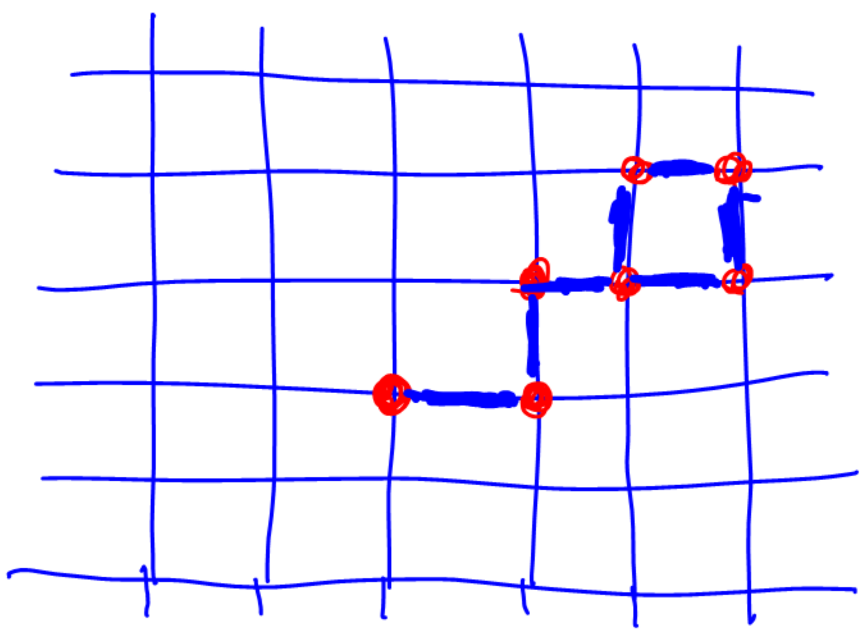


What role do the stomata play in the processes of photosynthesis?

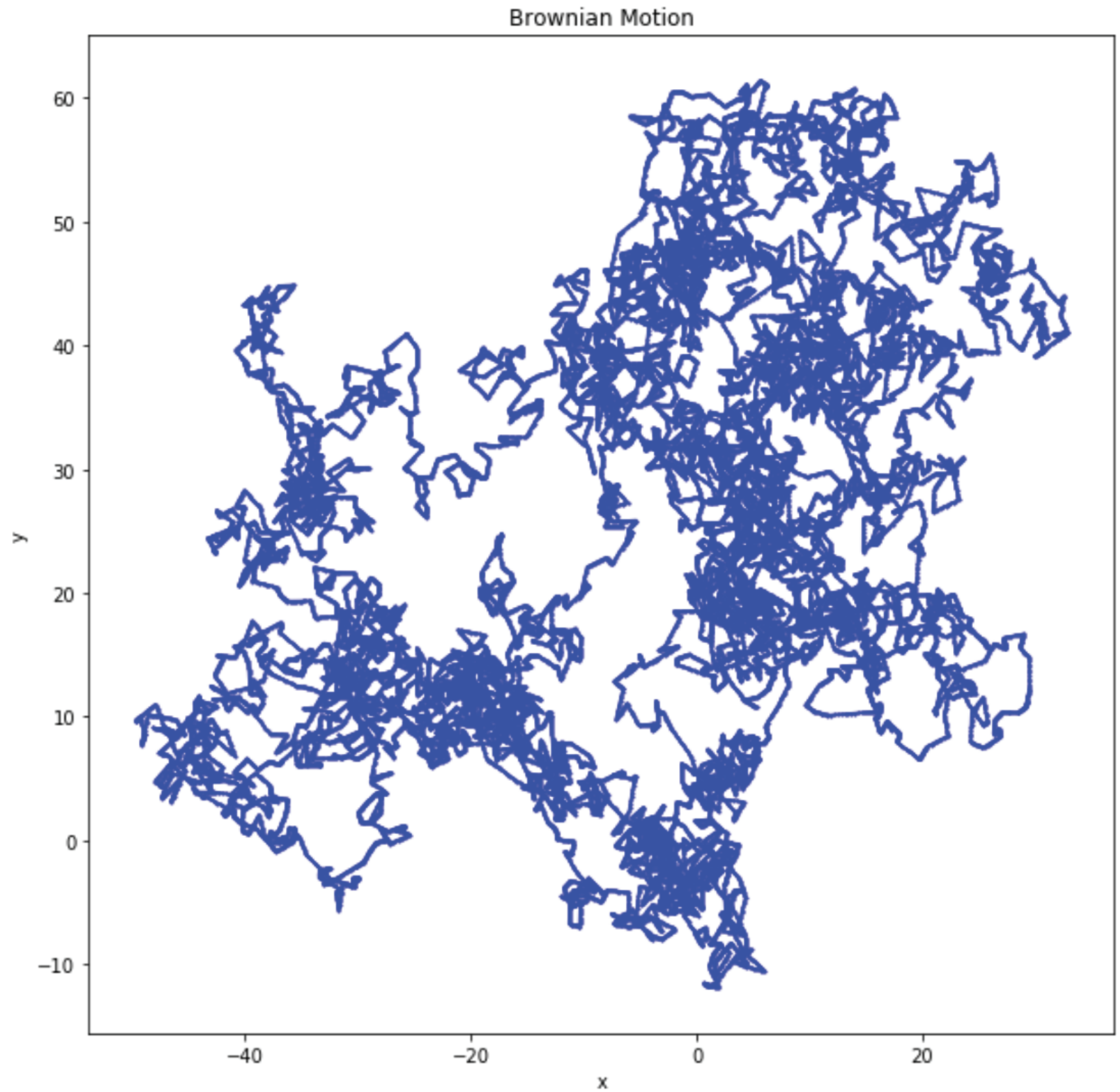


two dimensional random walk

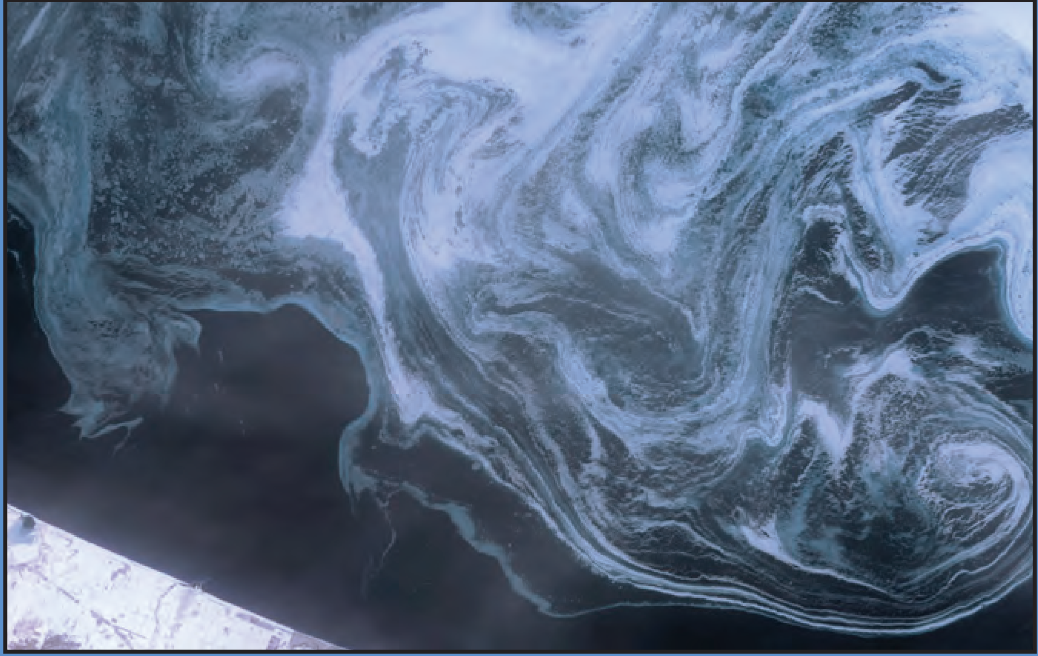




two dimensional brownian motion

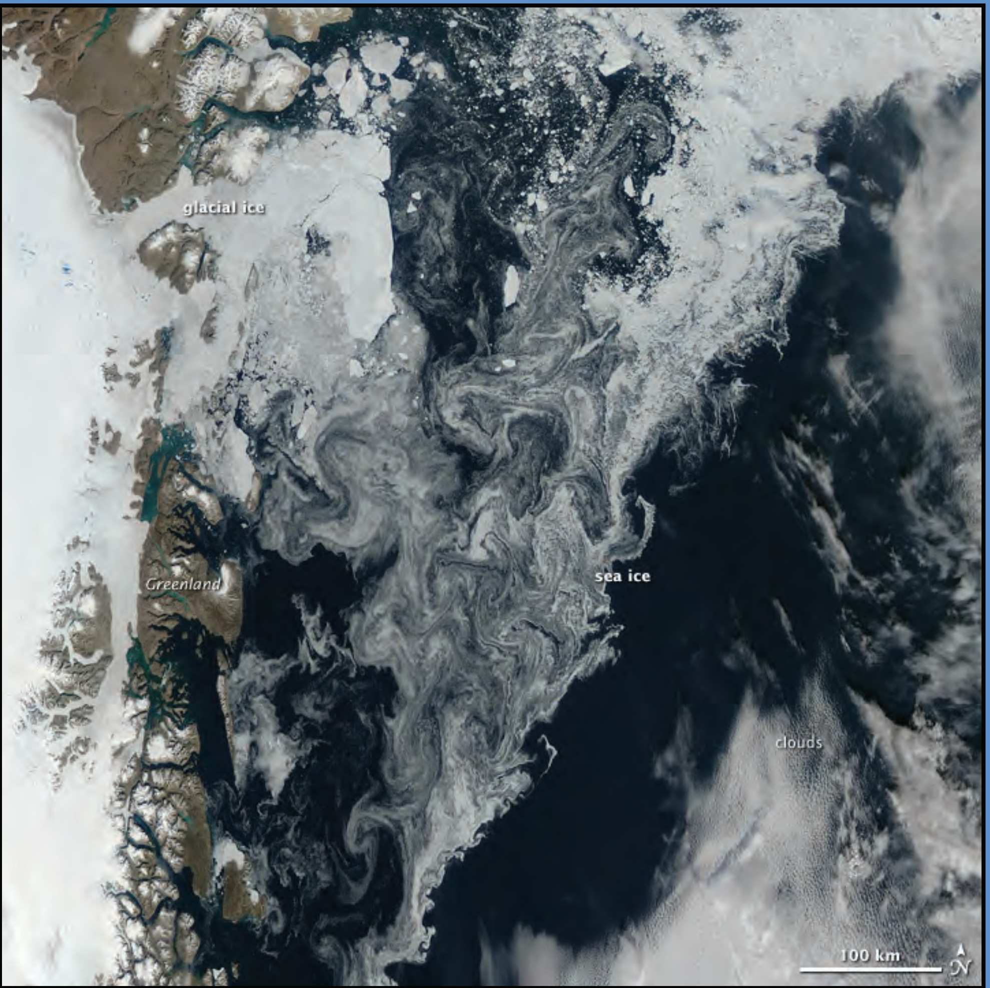


Advection-diffusion plays a key role in the transport of sea ice in atmospheric and oceanic flows.

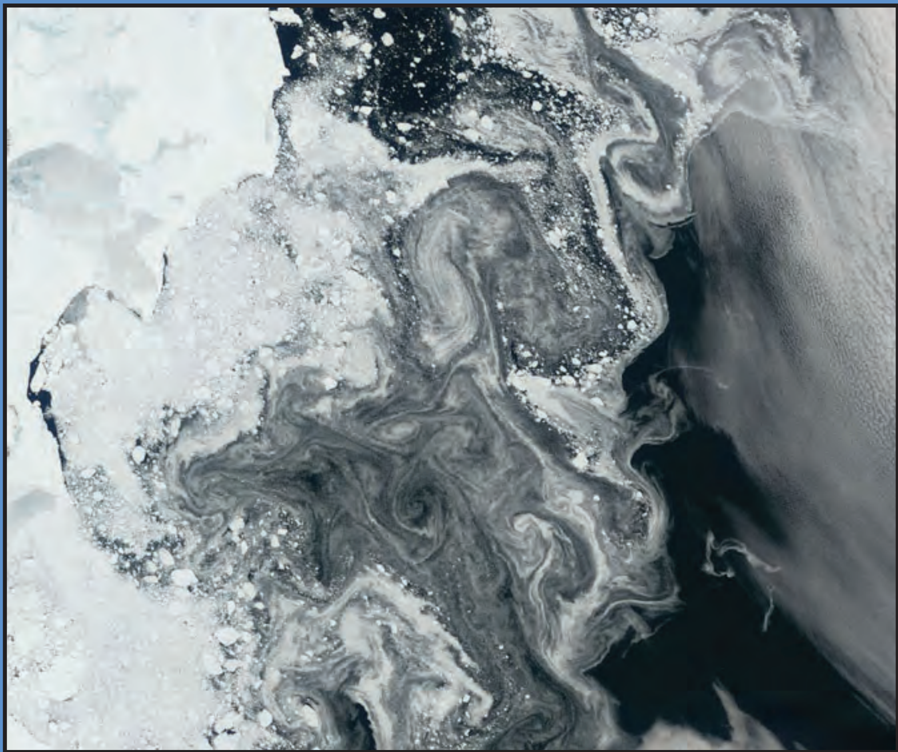


Sea of Okhotsk, NASA, 2009

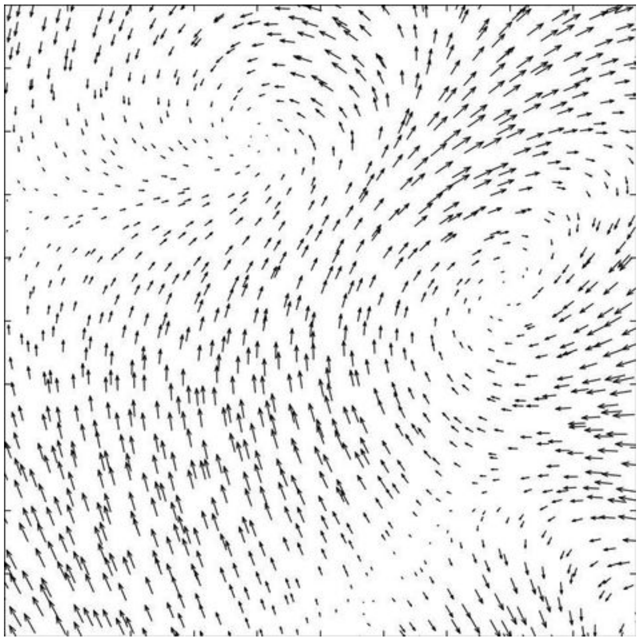
Ice in the Greenland Sea (77.5° N, 9° W), NASA, 2014

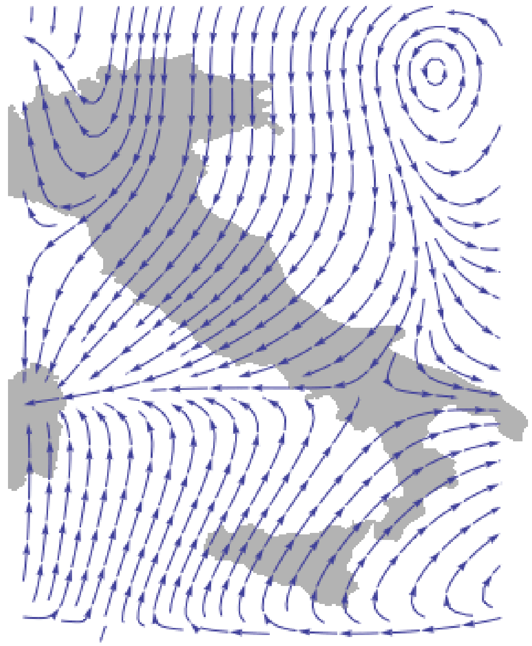


Off the northeastern coast of Greenland, NASA, 2006



Homogenization for advection diffusion





$$\frac{\partial u}{\partial t} = D\,\nabla^2 u - \mathrm{v}\cdot\nabla u \qquad \nabla\cdot\mathrm{v} = 0$$

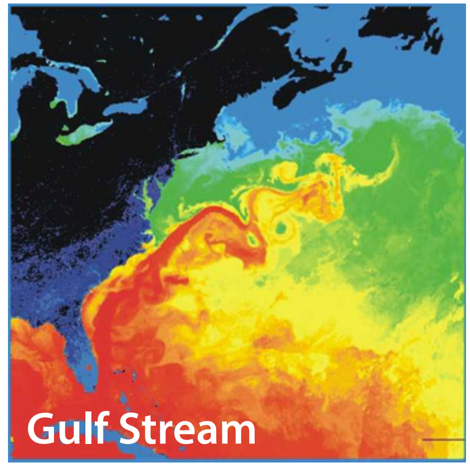
$\downarrow$ **homogenize** $\quad \langle X_t^2 \rangle \sim 2D^*\,t$

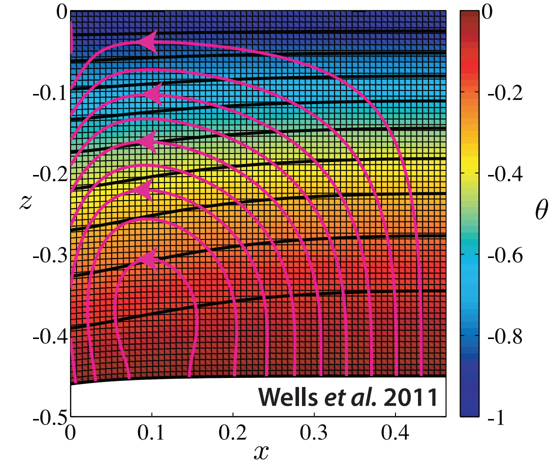
$$\frac{\partial\,\overline{u}}{\partial t} = D^*\,\nabla^2\,\overline{u} \qquad t \longrightarrow \infty$$

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere





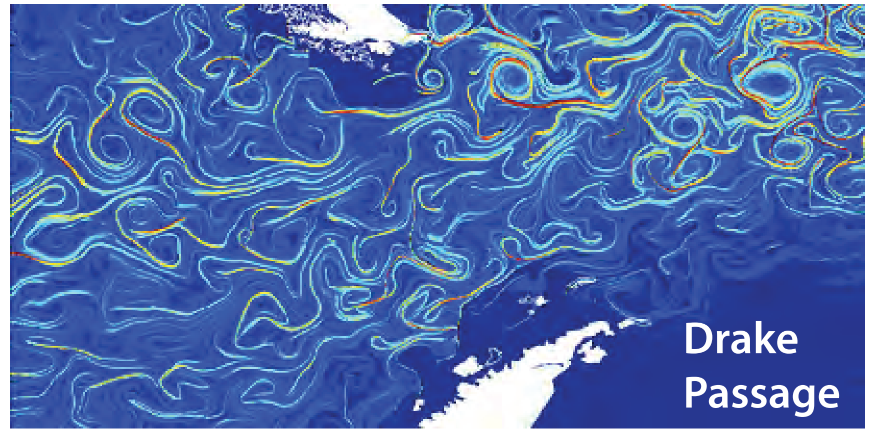
advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

homogenize

$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$



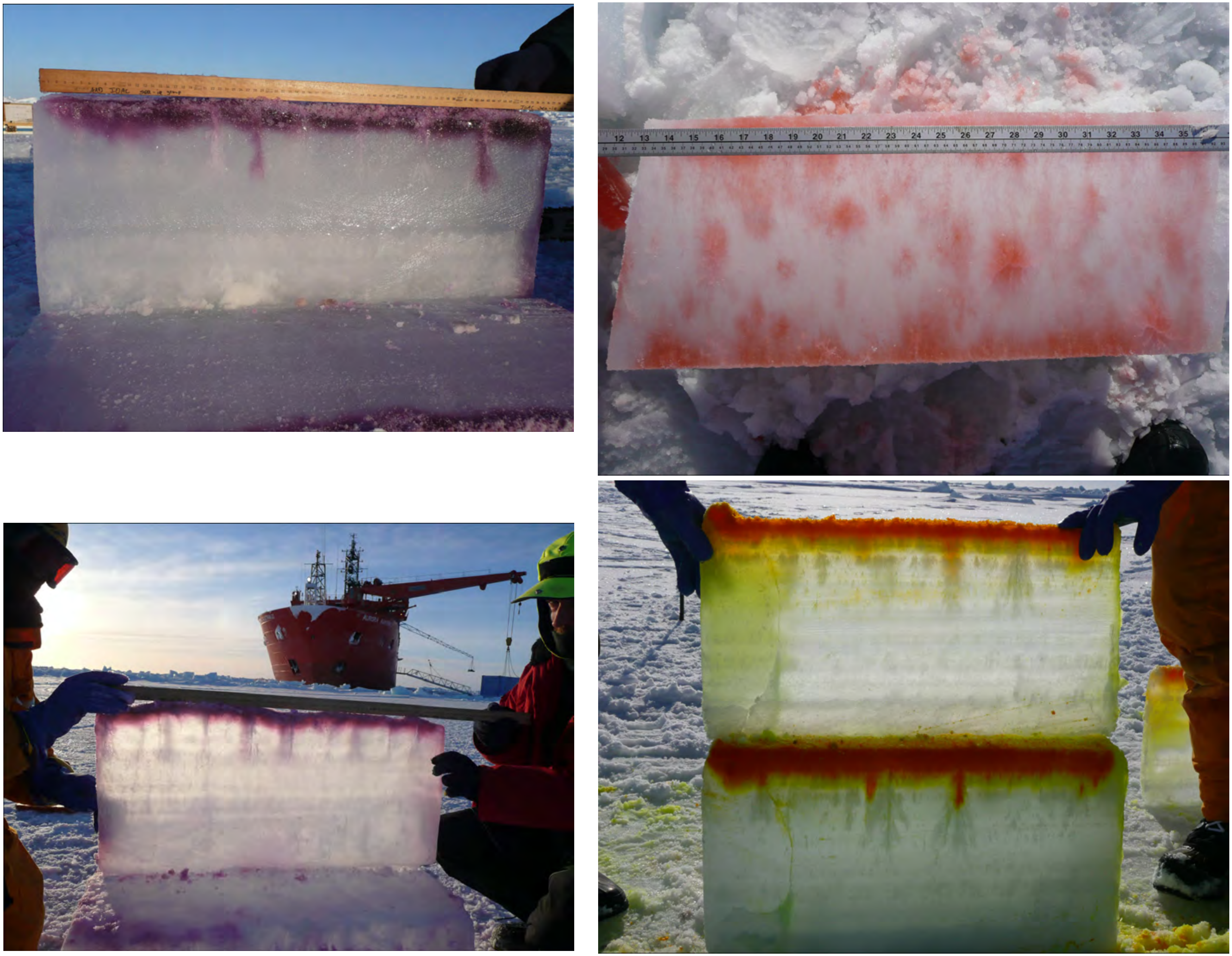
κ^* *effective diffusivity*

Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019

tracers flowing through inverted sea ice blocks



Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

$$\kappa^* = \kappa\left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2+\tau^2}\right), \quad F(\kappa) = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2+\tau^2}$$

- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H \Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$, Δ is the Laplace operator
- $i\Gamma H\Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ-plane

rigorous framework for numerical computations of spectral measures and effective diffusivity for model flows

new integral representations, theory of moment calculations

separation of material properties and flow field

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Homogenization for convection-enhanced thermal transport in sea ice

N. Kraitzman, R. Hardenbrook,
H. Dinh, N. B. Murphy, E. Cherkaev,
J. Zhu and K. M. Golden

August 2024

First rigorous mathematical theory of thermal conductivity of sea ice with convective fluid flow; captures data.

missing in climate models

advection diffusion equation with a velocity field u

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa_0 \Delta T$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\xrightarrow{\textit{homogenize}}$$

$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* effective diffusivity

Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91

composites

$$\frac{\epsilon^*}{\epsilon_2} = 1 - \int_0^1 \frac{d\mu(z)}{s - z}$$

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

μ spectral measure of $\Gamma\chi$

advection diffusion

$$\frac{\kappa^*}{\kappa_0} = 1 - \int_0^\infty \frac{d\rho(z)}{t - z}$$

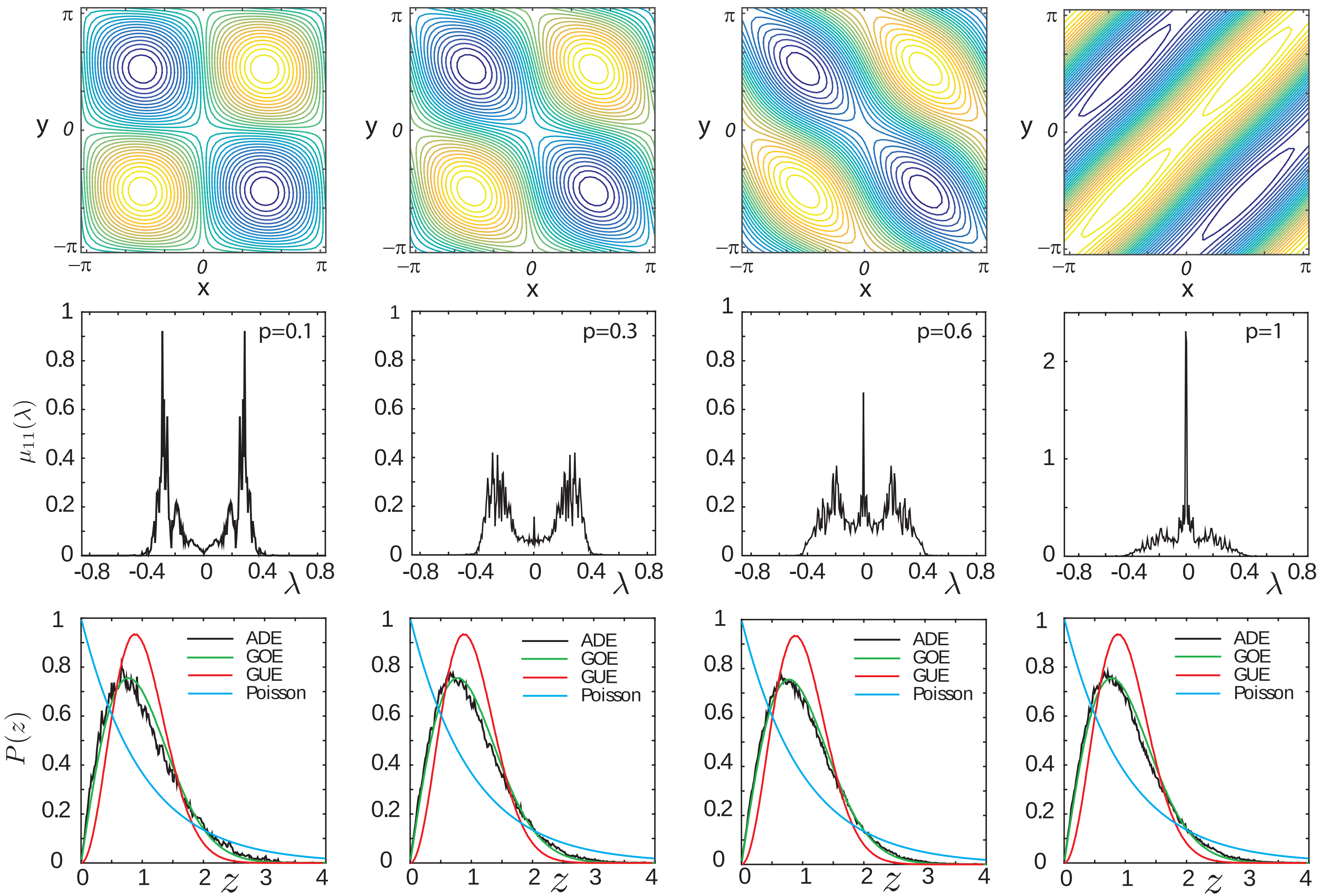
ξ = Péclet number $\quad t = -1/\xi^2$

ρ spectral measure of $\Gamma\mathbf{H}$

$\mathbf{u} = \nabla \cdot \mathbf{H} \quad \mathbf{H}$ antisymmetric vector potential

Spectral measures and eigenvalue spacings for cat's eye flow

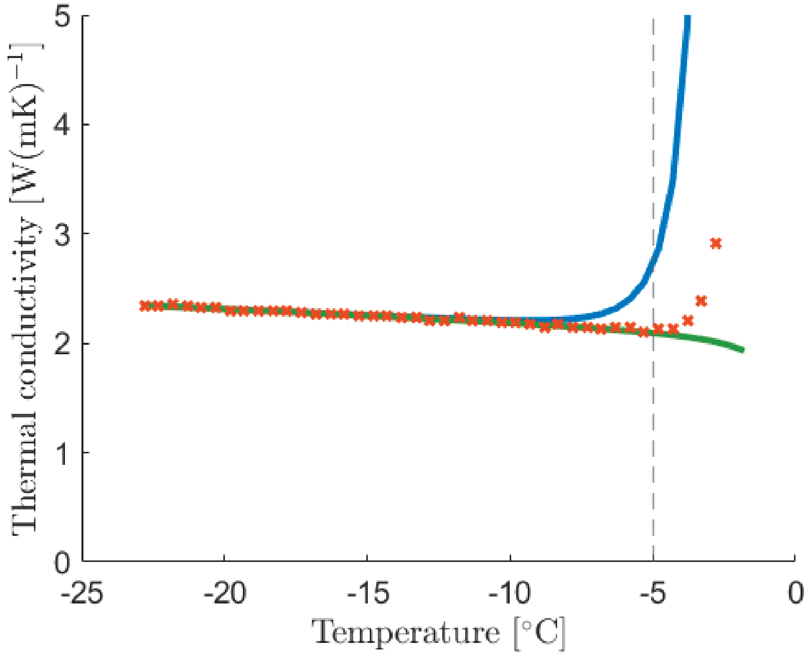
$H(x,y) = \sin(x)\sin(y) + A\cos(x)\cos(y), \quad A \sim U(-p,p)$



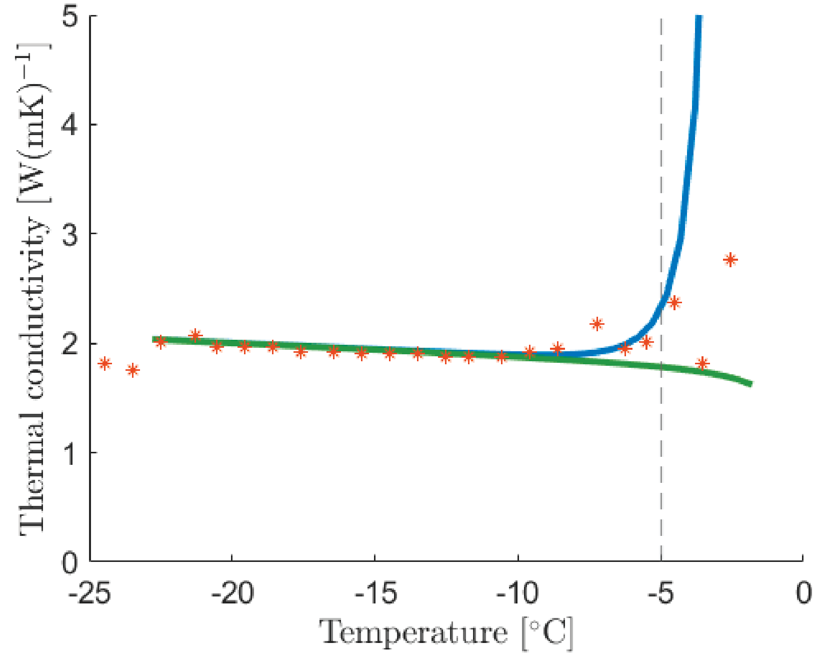
Murphy, Cherkaev, Xin, Golden, 2017

Bounds on Convection Enhanced Thermal Transport

simulations



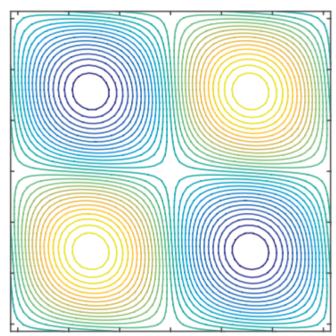
data [Trodahl et al., 2001]



Monte-Carlo simulations of SDE with temperature dependent Péclet number P

strength of advection B = κP/2π

Euler-Maruyama and subsampling methods for SDE



cat's eye flow model for brine convective flow

Rigorous Padé approximant bounds in terms of P using Stieltjes integral + analytic continuation method for the measure

Darcy velocity v = 0.5 [m/s]

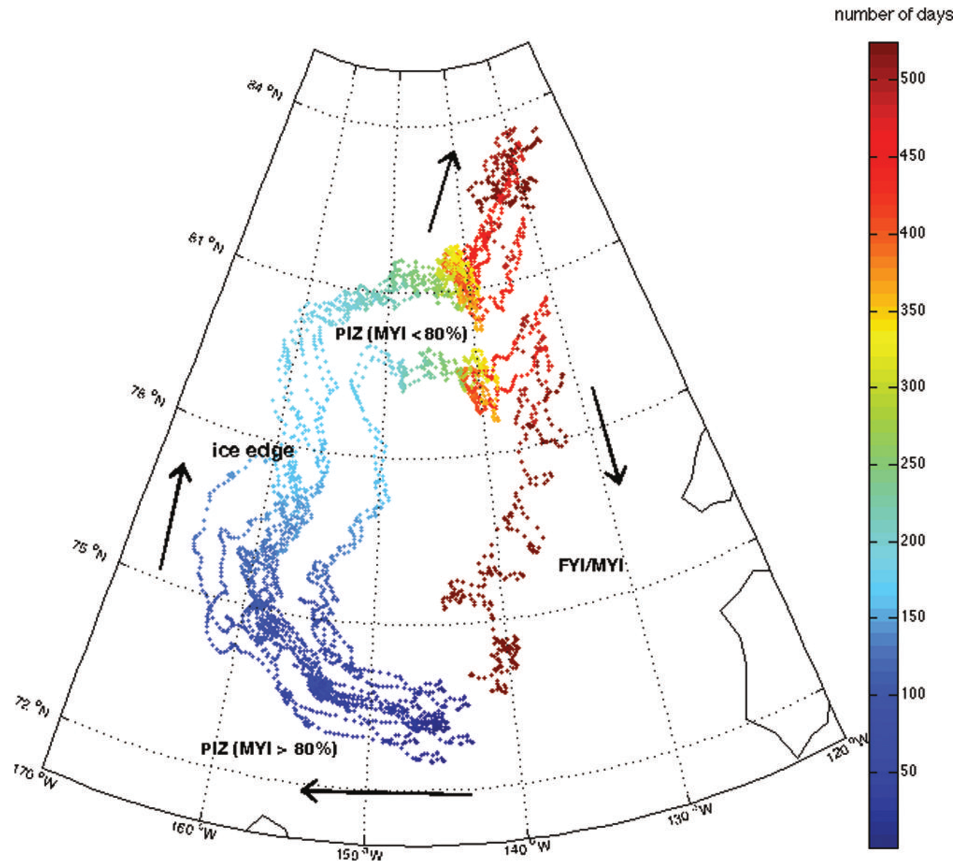
Kraitzman, Hardenbrook, Dinh, Murphy, Cherkaev, Zhu, & Golden

***Proc. Royal Soc. A*, 2024**

Anomalous diffusion in sea ice dynamics

Ice floe diffusion in winds and currents

Jennifer Lukovich, Jennifer Hutchings, David Barber, *Ann. Glac.* 2015



- On short time scales floes observed (buoy data) to exhibit Brownian-like behavior, but they are also being advected by winds and currents.

- Effective behavior is purely diffusive, sub-diffusive or super-diffusive depending on ice pack and advective conditions - Hurst exponent.

Floe Scale Model of Anomalous Diffusion in Sea Ice Dynamics

Huy Dinh, Elena Cherkaev, Court Strong, Ken Golden 2021

$$\left\langle \left| \mathbf{x}(t) - \mathbf{x}(0) - \langle \mathbf{x}(t) - \mathbf{x}(0) \rangle \right|^2 \right\rangle \sim t^{\alpha}$$

$\alpha =$ Hurst exponent, a measure of anomalous diffusion.

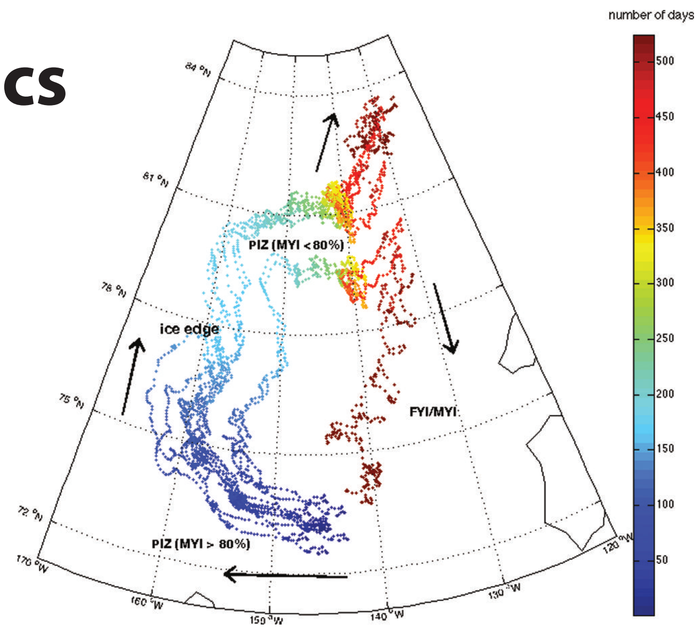
Measured from bouy position data. Detects ice pack crowding and advective forcing.

J.V. Lukovich, J.K. Hutchings, D.G. Barber ***Annals of Glaciology*** **2015**

diffusive	$\alpha = 1$	Sparse packing, uncorrelated advective field.
sub-diffusive	$\alpha < 1$	Dense packing, crowding dominates advection.
super-diffusive	$\alpha = 5/4$	Sparse packing, shear dominates advection.
	$\alpha = 5/3$	Sparse packing, vorticity dominates advection.

Goal: Develop numerical model to analyze regimes of transport in terms of ice pack crowding and advective conditions.

Anomalous diffusion in sea ice dynamics



Ice floe diffusion in winds and currents

observations from GPS data

Lukovich, Hutchings, Barber, *Ann. Glac.* 2015

Floe scale model of advection diffusion

Huy Dinh, Tyler Evans, Kaeden George, Ben Murphy, Elena Cherkaev, Ken Golden 2025

$$\left\langle |\mathbf{x}(t) - \mathbf{x}(0) - \langle \mathbf{x}(t) - \mathbf{x}(0) \rangle|^2 \right\rangle \sim t^{\alpha}$$

$\alpha =$ **Hurst exponent**

diffusive $\alpha = 1$

sub-diffusive $\alpha < 1$

super-diffusive $\alpha > 1$

Model Approximations

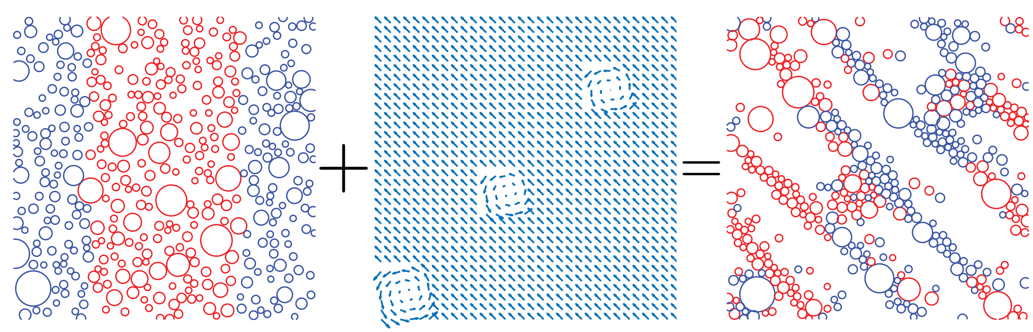
Power Law Size Distribution: $N(D) \sim D^{-k}$

D. A. Rothrock and A. S. Thorndike Journal of Geophysical Research 1984

Floe-Floe Interactions: Linear Elastic Collisions

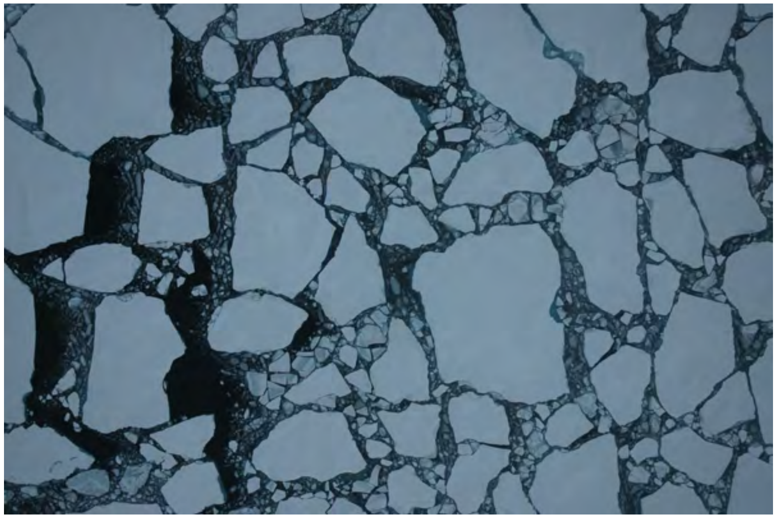
Advective Forcing: Passive, Linear Drag Law

Fractional PDE

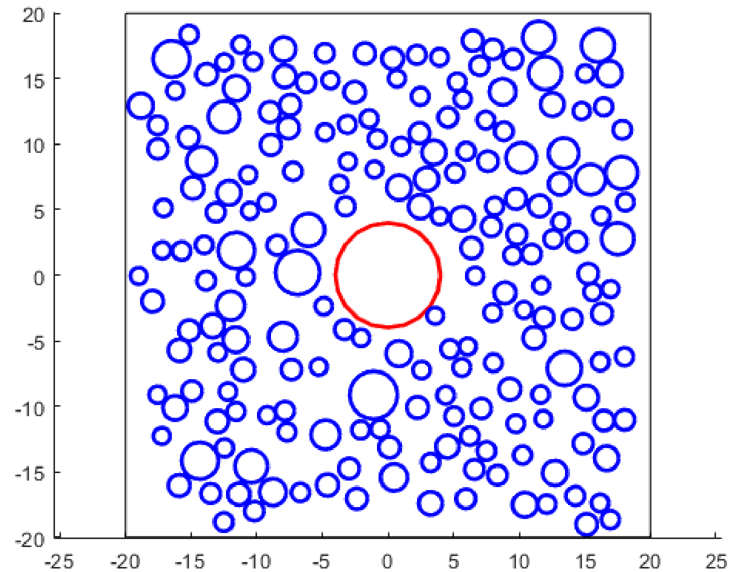


$\psi = 0.3$ **Vorticity Dominated Forcing**

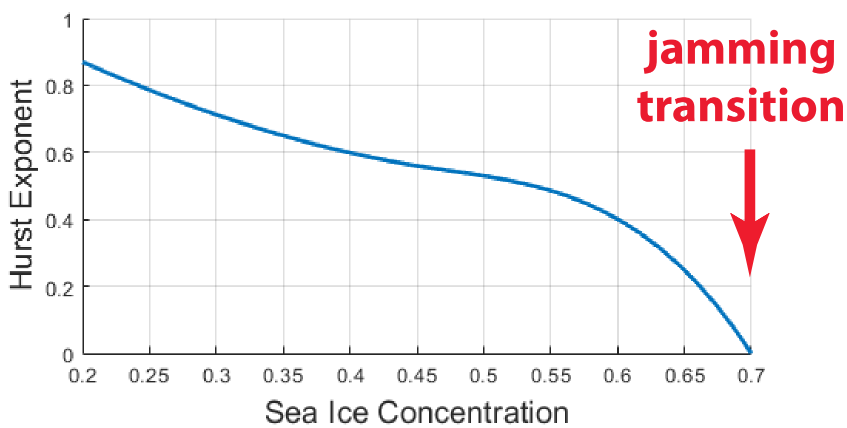
Arctic sea ice pack with tagged particle



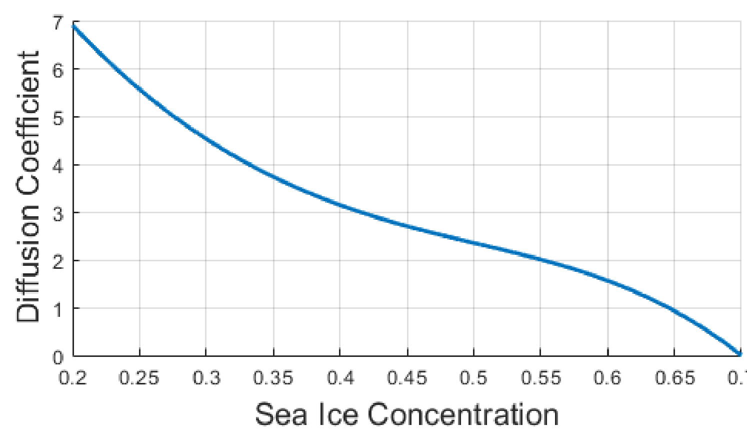
Einstein's pollen grain



Hurst exponent



diffusion coefficient



Home ranges in moving habitats: polar bears and sea ice

"diffusive" polar bear motion on drifting sea ice

Marie Auger-Méthé, Mark Lewis, Andrew Derocher, *Ecography*, 2016

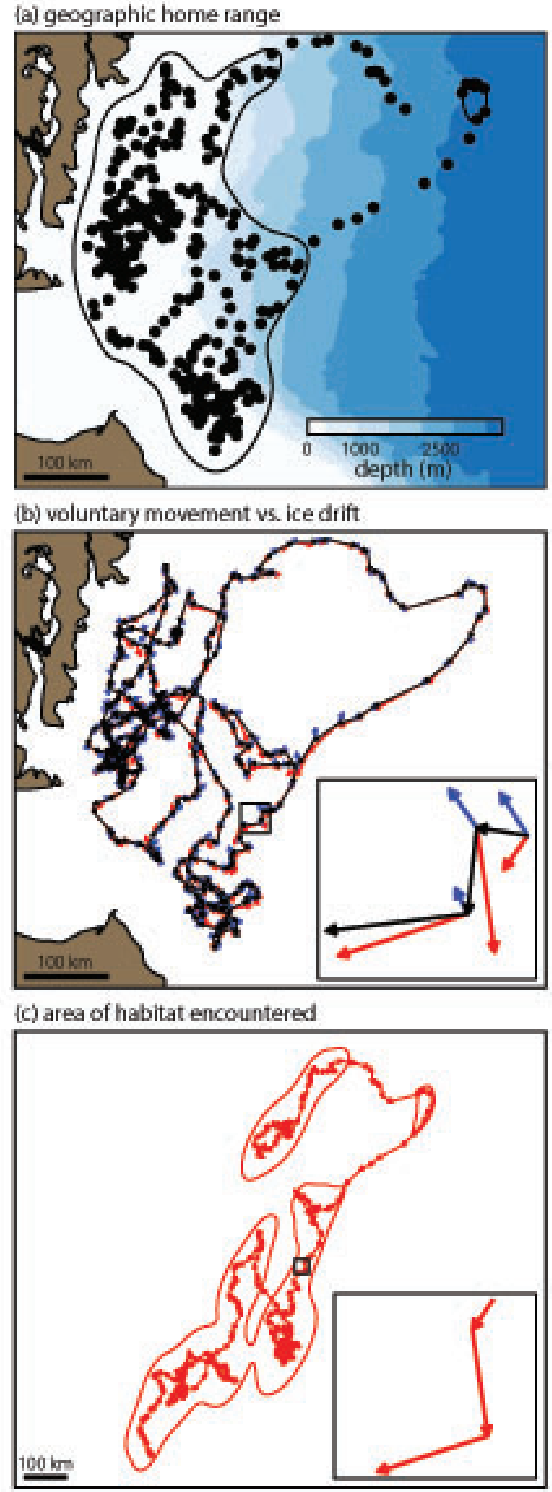
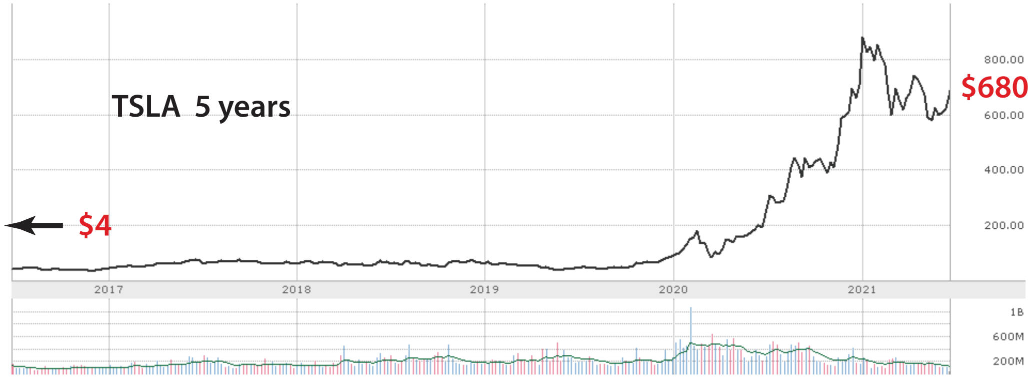
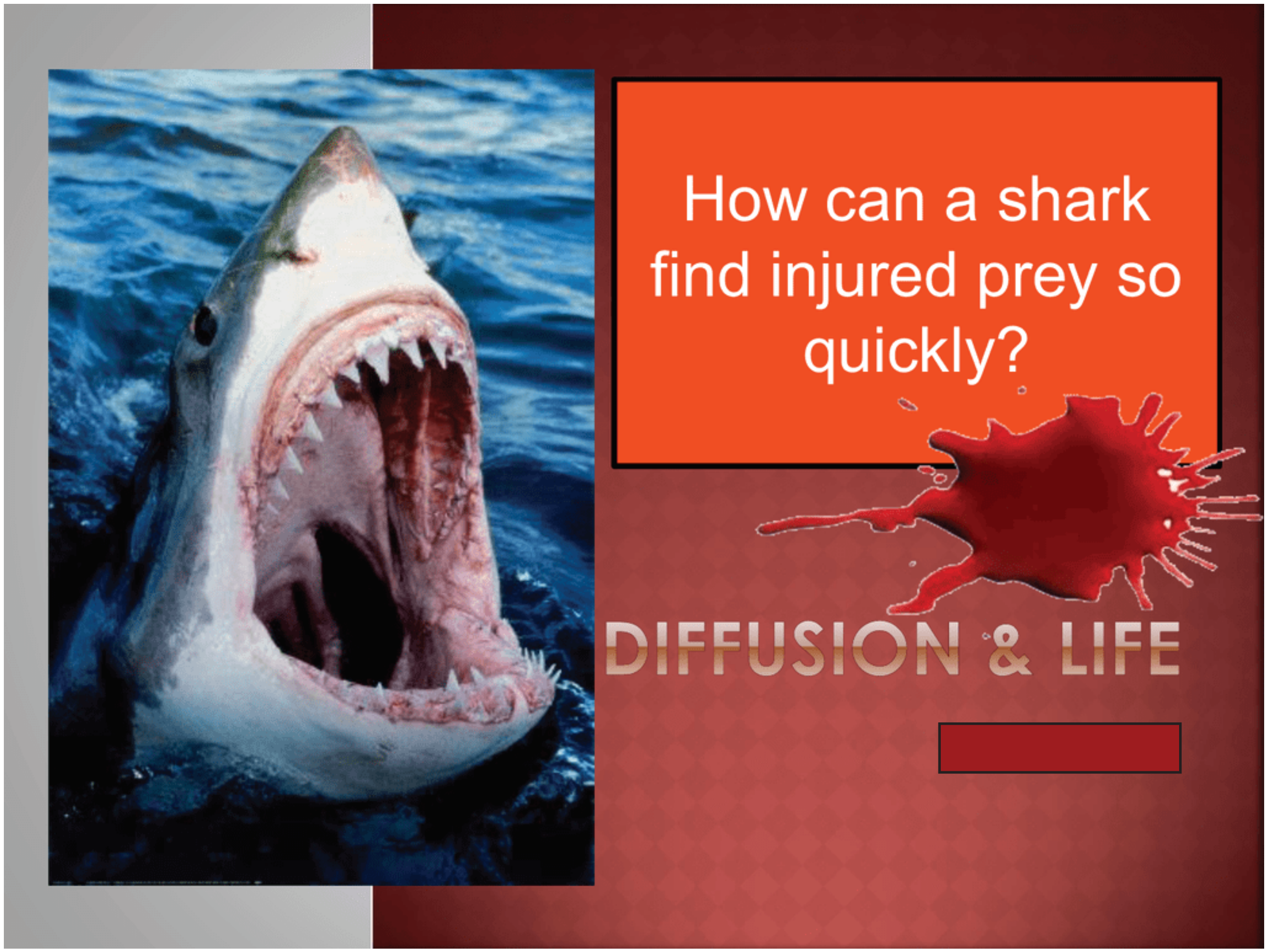



Photo (c) Joan Cambray

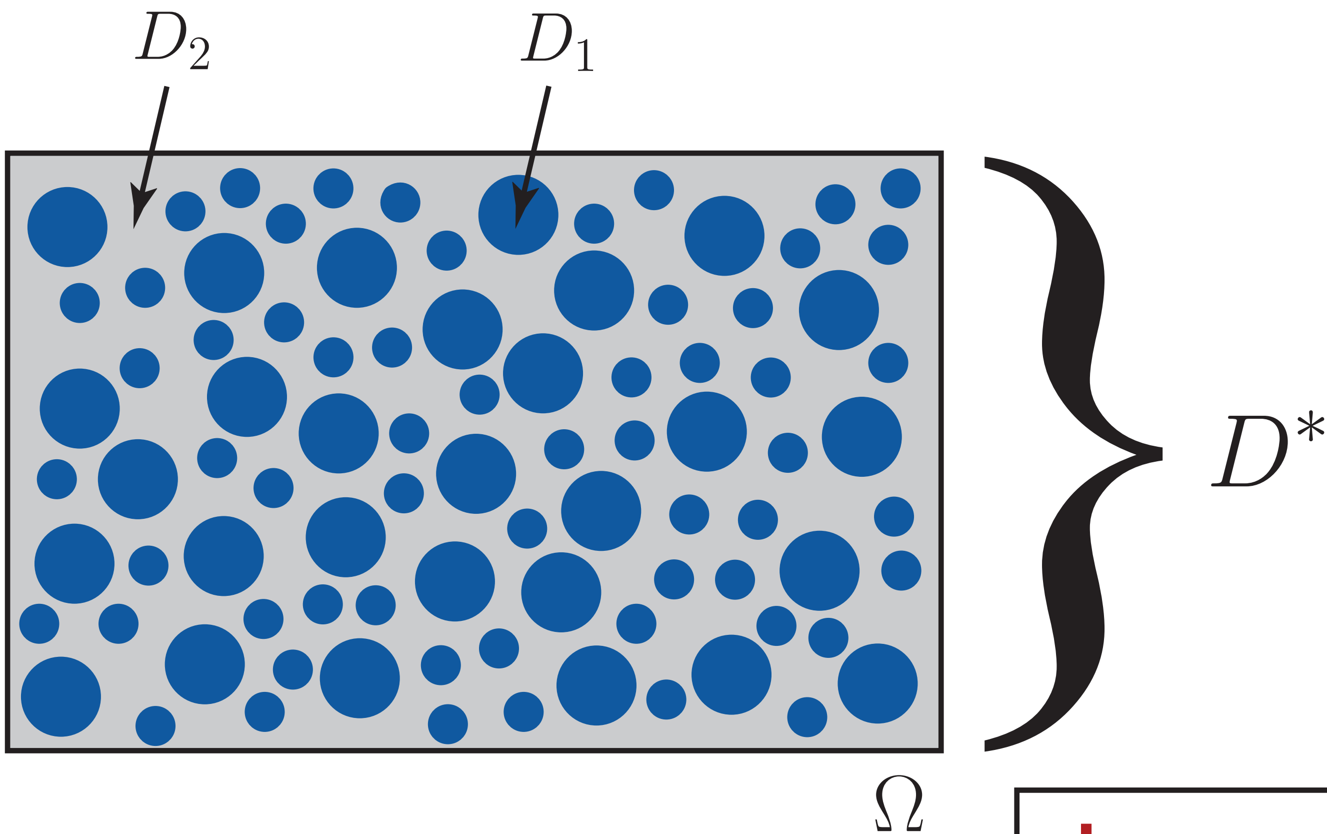
Brownian-like over shorter time periods, with an upward “drift” over time for growing companies
downward ... declining companies



diffusion vs. advection



Homogenization for diffusion in two phase media



effective diffusivity

local diffusivity

$$D(x) = \begin{cases} D_1 & x \in \Omega_1 \\ D_2 & x \in \Omega_2 \end{cases}$$

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(x) \nabla u)$$

homogenized parameter captures effective behavior

$$\langle X_t^2 \rangle \sim 2D^* \, t$$

$$t \longrightarrow \infty$$

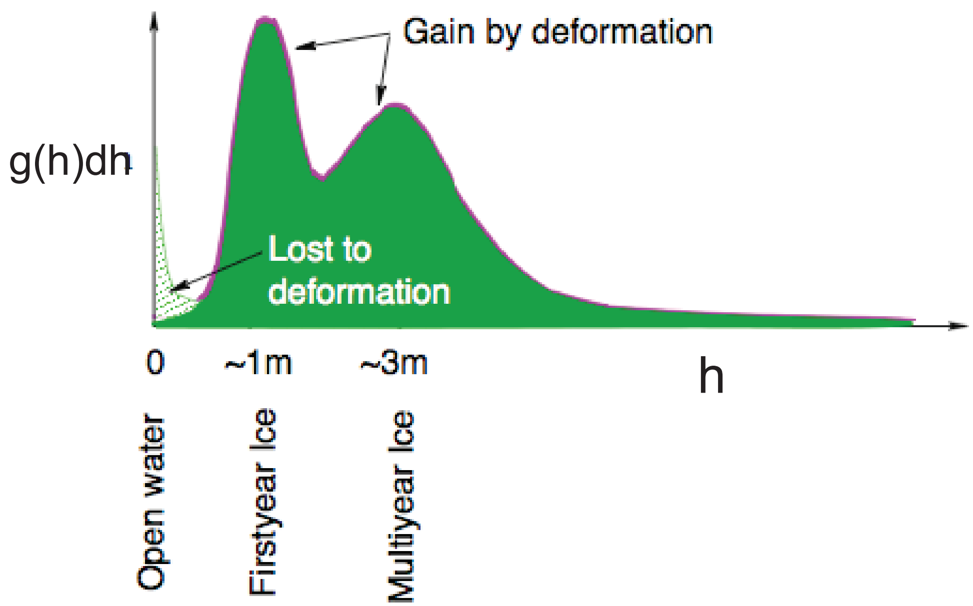
ice thickness distribution $g(x, y, h, t)$ evolution equation

$$\frac{Dg}{Dt} = -g\nabla \cdot \mathbf{u} + \Psi(g) - \frac{\partial}{\partial h}(fg) + \mathcal{L}$$

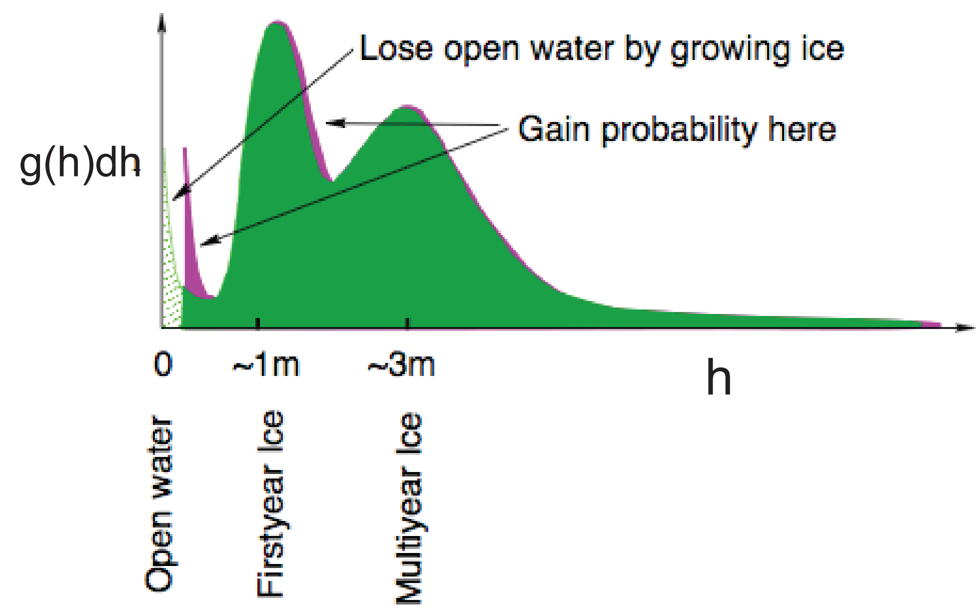
$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + \mathbf{u} \cdot \nabla g$	Lagrangian or convective derivative
$\mathbf{u}$	ice velocity field
h	ice thickness
$-g\nabla \cdot \mathbf{u}$	flux divergence
Ψ	mechanical redistribution opening and ridging
f	thermodynamic growth rate
$\frac{\partial}{\partial h}(fg)$	ice growth/melt results in *thickness advection*
$\mathcal{L}$	lateral melting

depend on g: Ψ, f, $\frac{\partial}{\partial h}(fg)$, $\mathcal{L}$

Ψ = Mechanical redistribution



Advection in thickness space from growth



Cecilia Bitz

transform ice thickness distribution equation to Fokker-Planck type equation; Boltzmann framework

Toppaladoddi and Wettlaufer, *PRL*, 2015

thickness h is a diffusion process with probability density g(h,t)

"microscopic" mechanical processes that influence
ice thickness distribution— rafting, ridging, and open water formation
occur over very rapid time scales relative to geophysical-scale changes of *g(h)*

$$\Psi = \int_0^\infty [g(h+h')w(h+h',h') - g(h)w(h,h')]dh'$$

w = ***transition probability***

moments k_1 , k_2

Fokker-Planck

$$\frac{\partial g}{\partial t} = -\frac{\partial}{\partial h}\left[\left(\frac{\epsilon}{h} - k_1\right) g\right] + \frac{\partial^2}{\partial h^2}(k_2 g)$$

Langevin

$$\frac{dh}{dt} = \left(\frac{\epsilon}{h} - k_1\right) + \sqrt{2k_2}\,\xi(t)$$

$\xi(t)$ = ***Gaussian white noise***

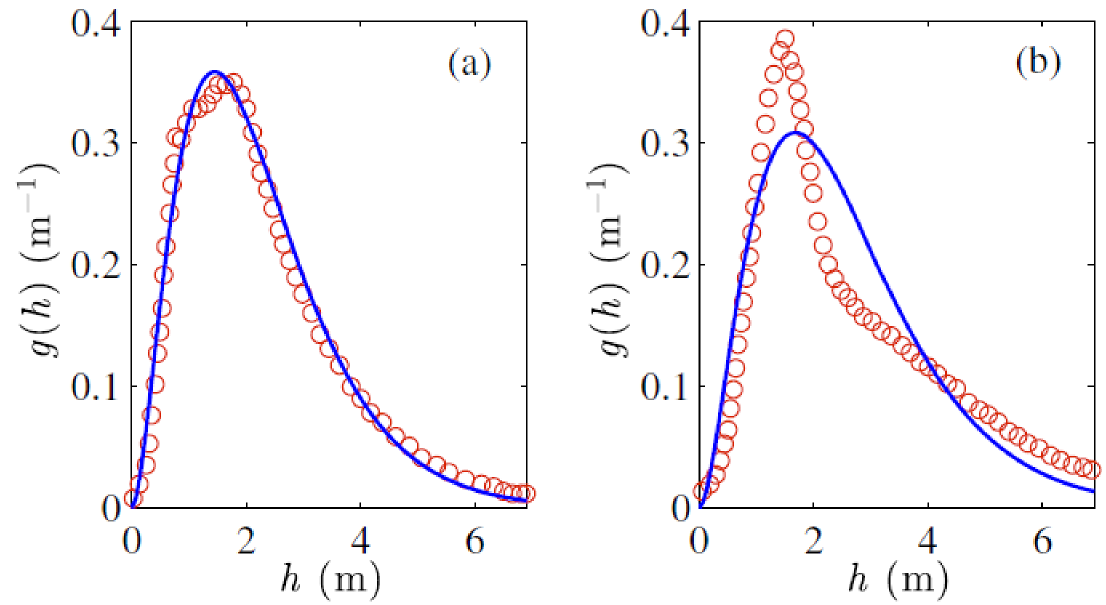
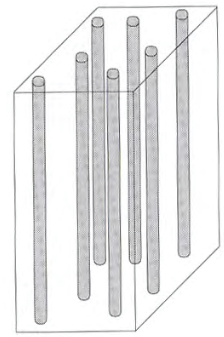


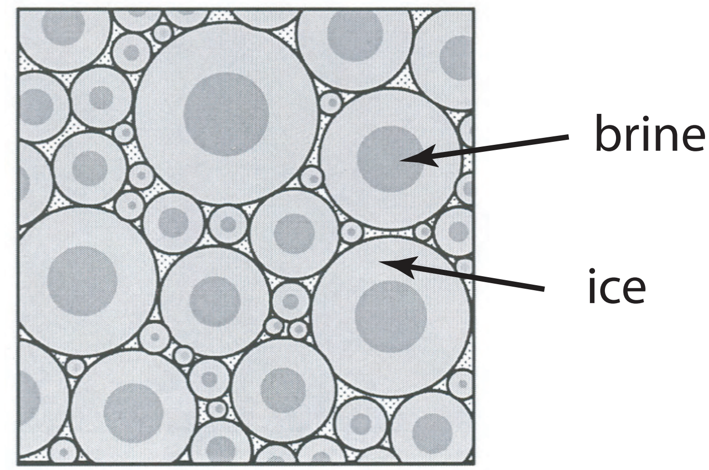
FIG. 1 (color online). Comparison of our theory with satellite measurements for February through March (F-M) of (a) 2008 and (b) 2004. Circles are the distribution functions from ICESat [9] and lines are the fits using Eq. (10). In (a), $q = 1.849$ and $H = 0.783$ m, and in (b), $q = 1.848$ and $H = 0.910$ m.

PIPE BOUNDS on vertical fluid permeability k

Golden, Heaton, Eicken, Lytle, Mech. Materials 2006
Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophys. Res. Lett. 2007

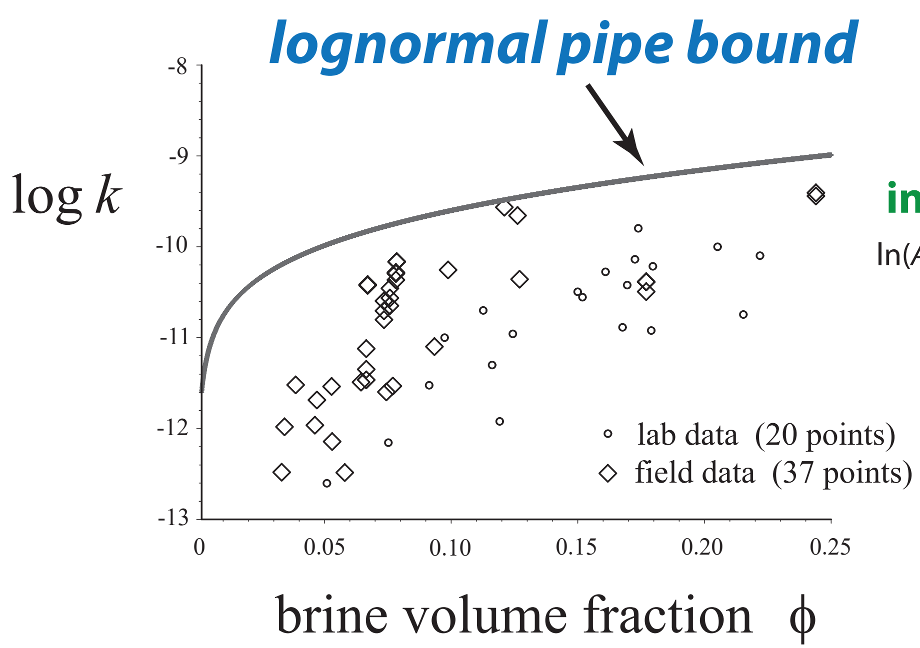
vertical pipes
with appropriate radii
maximize k





optimal coated cylinder geometry

fluid analog of arithmetic mean upper bound for effective conductivity of composites (Wiener 1912)



Golden et al., Geophys. Res. Lett. 2007

$$k \leq \frac{\phi \langle R^4 \rangle}{8 \langle R^2 \rangle} = \frac{\phi}{8} \langle R^2 \rangle e^{\sigma^2}$$

inclusion cross sectional areas A lognormally distributed

ln(A) normally distributed, mean μ (increases with T) variance σ^2 (Gow and Perovich 96)

get bounds through variational analyis of ***trapping constant*** γ for diffusion process in pore space with absorbing BC

Torquato and Pham, *PRL* 2004

$$\mathbf{k} \leq \gamma^{-1} \mathbf{I}$$

for any ergodic porous medium (Torquato 2002, 2004)

diffusing tracer with concentration $c(x,t)$

reacts with traps on pore boundaries

$$\frac{\partial c}{\partial t} = D\Delta c + G, \qquad x \in \Omega_b$$

D = diffusion constant

G = reactant source rate

κ = surface reaction rate constant

$$D\frac{\partial c}{\partial n} + \kappa c = 0, \qquad x \in \partial\Omega_b$$

boundary condition

trapping constant: $\gamma^{-1} = \langle u \rangle$

$u(x)$ is scaled concentration field in steady state with absorbing b.c. (diffusion-controlled limit)

$$\Delta u = -1, \;\; x \in \Omega_b \qquad u = 0, \;\; x \in \partial\Omega_b$$

variational inequality $\gamma \geq \langle \nabla v \cdot \nabla v \rangle^{-1}$

$$\forall v \in \{\text{ergodic } v(x) : \Delta v = -1, \;\; x \in \Omega_b\}$$

mean survival time: $\tau = \dfrac{1}{\gamma \phi D}$

$$\mathbf{k} \leq \gamma^{-1}\, \mathbf{I}$$

variational analysis using trial fields yields VOID BOUND:

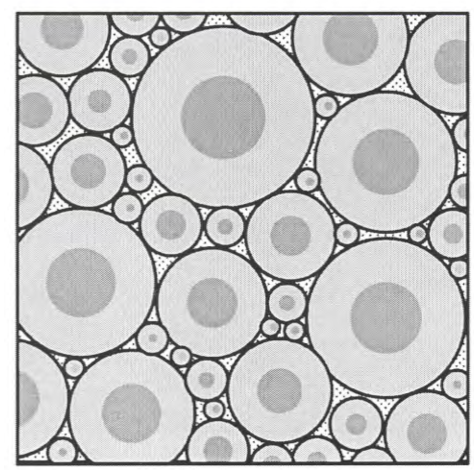
$$\gamma \geq \frac{(1-\phi)^2}{\ell_P^2}$$

where ℓ_P is a pore length scale defined by

$$\ell_P^2 = -\int_0^\infty (S_2(r) - \phi^2)\, r \ln r \; dr, \quad d = 2, \qquad \ell_P^2 = \frac{1}{d-2}\int_0^\infty (S_2(r) - \phi^2)\, r \; dr, \quad d \geq 3$$

EVALUATE void bound for Hashin-Shtrikman coated spheres (d=3) and cylinders (d=2)

optimal geometry



$$\gamma \geq \frac{8\langle R_I^2 \rangle}{\phi \langle R_I^4 \rangle}, \quad d = 2$$

$$\mathbf{k} \;\leq\; \gamma^{-1}\mathbf{I}$$

for any ergodic porous medium (Torquato, 2002)

NSF Research Training Grant (RTG) with 15 Applied Math faculty:

optimization and inverse problems

July 2022 - June 2027

Overall goal: Build an advanced, competitive U.S. STEM workforce.

- Strengthen our graduate and postdoctoral programs in applied math to attract top students in the nation, and place them in top jobs.
- Provide transformative experiences that draw students into math.

Arctic Mathpeditions - May 2024 & 2026

OPEN POSITIONS:
Postdoctoral, Ph.D., Undergraduate

Model Approximations

Floes $\approx$ Discs

Forces on Disc $= \quad F_{drag} + F_{collision}$

A. Herman ***Physical Review E*** **2011**

Floe-Floe Interactions: Linear Elastic Collisions

$F_{collision}$ follows Hooke's Law.

Advective Forcing: Passive, Linear Drag Law

v is the advective velocity field.

F_{drag} is proportional to relative velocity.

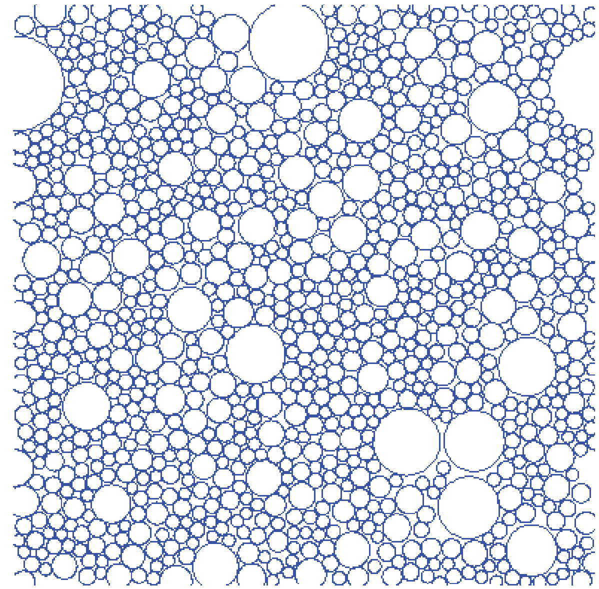
Ice Pack Characteristics

ϕ = sea ice concentration (floe area fraction)

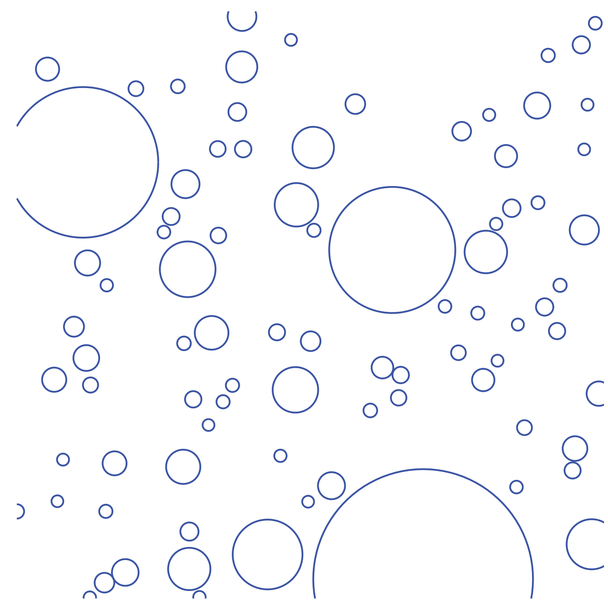
Power Law Size Distribution: $N(D) \sim D^{-k}$

T. Toyota, S. Takatsuji, M. Nakayama Geophysical Review Letters 2006

k = floe diameter exponent



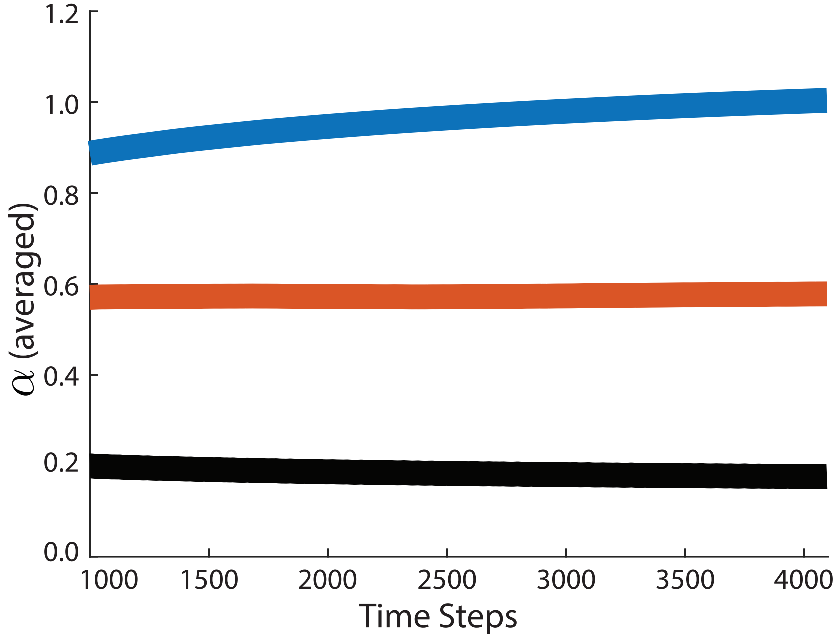
k = 2.9, ϕ = 0.8

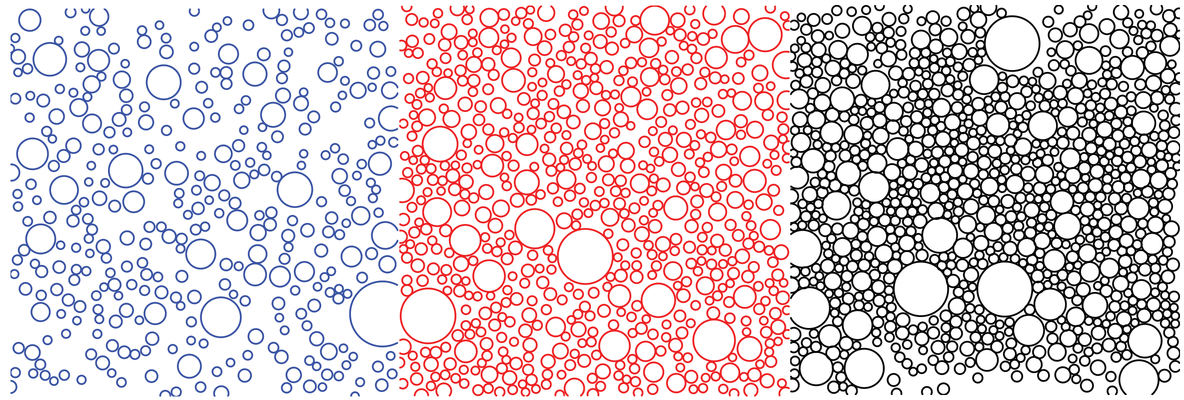


k = 1.7, ϕ = 0.1

Model Results

Crowding in random advective forcing.



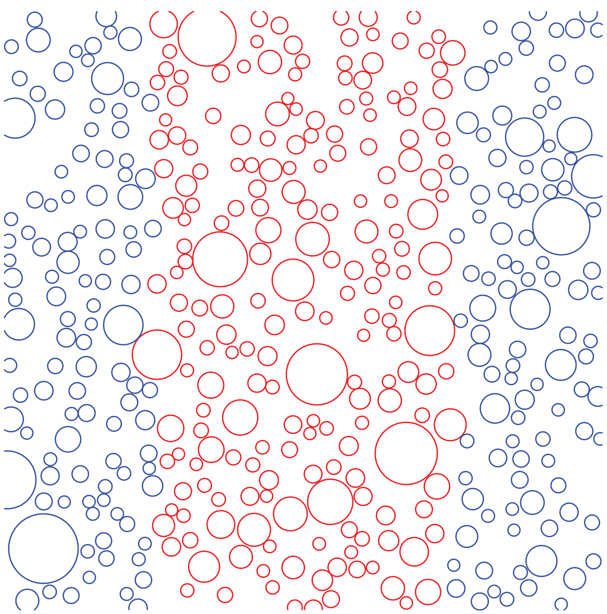


Concentration = 0.3
k = 2.9

Concentration = 0.5
k = 2.9

Concentration = 0.7
k = 2.9

Concentration = 0.3 k = 2.9



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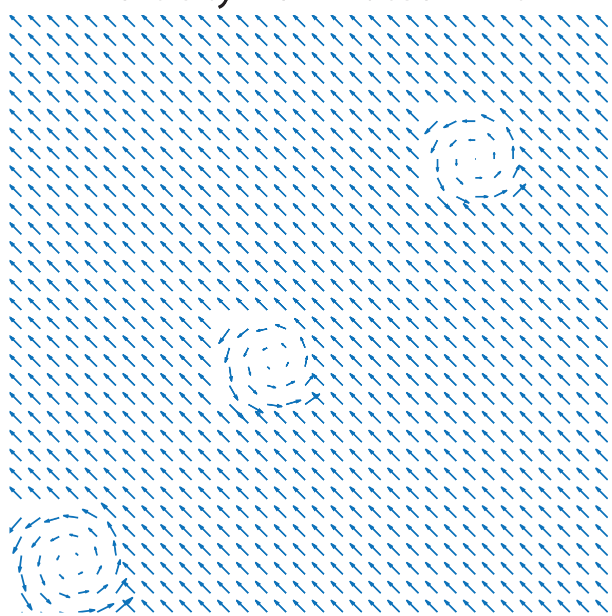
Vorticity Dominated Drift



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$\alpha = 5/3$