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# Fundamental problems in statistical physics of jammed packings

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**Abstract**

For packed i.e., "jammed", hard and rough objects kinetic energy is a minor and ignorable quantity, as is elastic strain. Hence in the static case, the stress equations need supplementing by "missing equations" depending solely on configurations. A different pathway of analysis is the calculation of the probability distribution of interparticle forces. This paper presents the mini-review of recently obtained results in this field and poses a number of fundamental problems which are yet to be solved.


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## 1. The problem

The crucial granular system concerns packed, hard and rough objects: packed means no kinetic energy, hard means no elastic deformation and rough means all motion is confined to sliding and rolling after friction threshold is overcome. A granular material can have different packing fractions according to its history of preparation, and the application of external forces causes forces to exist between the grains. It has been suggested that in appropriate circumstances the central concept of statistical mechanics, namely entropy, can be applied to analysis of jammed granular packings [1–3]. Experiments done by Chicago [4,5] and Rennes groups [6] show that columns of (both spherical and irregular shape) grains tapped with a fixed number of taps but with an

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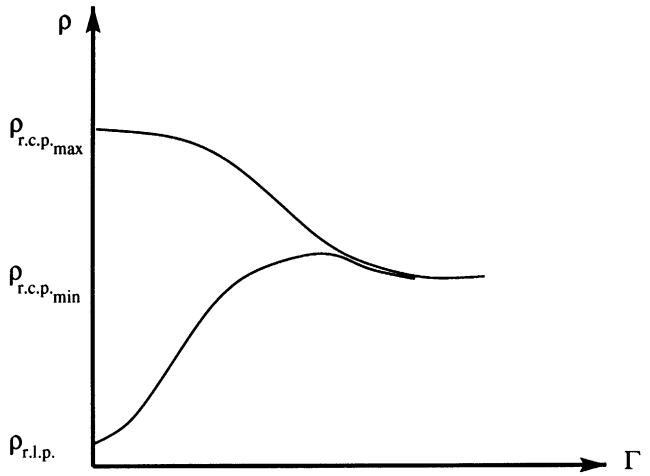


Fig. 1. Dependence of the packing fraction on the history of tapping amplitude. The parameter $\Gamma = A/g$ is the ratio of the recorded peak acceleration during a single tap $A$ to the gravitational acceleration $g$. In the Chicago experiments, $\Gamma$ was varied by changing the amplitude of excitation $A$ at fixed frequency $\omega = 30$ Hz.

increasing magnitude of tap remove initial voids and take one into the low density packed limit, $\phi = 0.59$, but repeating the tapping now decreasing the magnitude takes one to the maximum density, $\phi = 0.64$ (see Fig. 1. Thereafter the curve is reversible and confirms an "ergodic" condition, or at least we believe it does [7]. Recent experiments in Cavendish, in which the electrical conductivity fraction of vibrated irregularly shaped graphite grains is measured as a function of the tapping amplitude, confirm the existence of the reversible branch [8]. Computer simulations and analytical models seem to provide support for this point of view [9–14].

## 2. Probability distributions

For infinitely hard bodies there is no enthalpy and external forces have no effect on distribution functions. This means that if $P$ is the probability distribution of configurations and of intergranular forces, it must separate into two parts

$$P = P_c(\text{configurations})P_f(\text{forces})\,. \tag{1}$$

We now consider $P_c$ which is the function of geometrical characteristics of the system. Let us assume that the set of contact points $\mathbf{C}^{\alpha\beta}$ is the total geometrical specification for a static packing. We define the centroid of contacts of grain $\alpha$ (see Fig. 2)

$$\mathbf{R}^{\alpha} = \frac{\sum_{\beta} \mathbf{C}^{\alpha\beta}}{z^{\alpha}}\,, \tag{2}$$

where $z^{\alpha}$ is the coordination number of grain $\alpha$. The distance between particles $\alpha$ and $\beta$ is defined as the distance between their centroids of contacts

$$\mathbf{R}^{\alpha\beta} = \mathbf{R}^{\beta} - \mathbf{R}^{\alpha} = \mathbf{r}^{\alpha\beta} - \mathbf{r}^{\beta\alpha}\,, \tag{3}$$

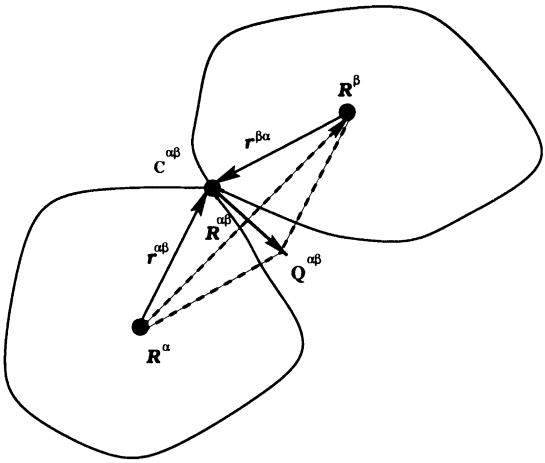


Fig. 2. The centroids of two neighbouring grains in contact.

where $\mathbf{r}^{\alpha\beta}$ is the vector joining the centroid of contact with the contact point (see Fig. 2) i.e.,

$$\sum_{\beta} \mathbf{r}^{\alpha\beta} = 0 \,. \tag{4}$$

To complete the geometrical description of the first coordination shell we introduce vector $\mathbf{Q}^{\alpha\beta}$. This vector which characterises the relative position of neighbouring centroid with respect to the contact point is defined by (see Fig. 2)

$$\mathbf{Q}^{\alpha\beta} = -(\mathbf{r}^{\alpha\beta} + \mathbf{r}^{\beta\alpha}) \,. \tag{5}$$

For a spherical particle this vector characterises the deviation of the centroid of contacts position from the geometrical centre of a sphere (e.g. for a honeycomb array of discs $\mathbf{Q}^{\alpha\beta} = 0$). One can see that these vectors defined in terms of contact points represent the relative orientation of grains with respect to their nearest neighbours. The set of tensors which describe topology and connectivity of the contact network can be introduced as follows

$$F_{ij}^{\alpha} = \sum_{\beta} R_i^{\alpha\beta} R_j^{\alpha\beta} \,, \tag{6}$$

$$G_{ij}^{\alpha} = \sum_{\beta} Q_i^{\alpha\beta} R_j^{\alpha\beta} \,, \tag{7}$$

$$H_{ij}^{\alpha} = \sum_{\beta} Q_i^{\alpha\beta} Q_j^{\alpha\beta} \,. \tag{8}$$

The number of contact positions for a configuration of $N$ grains in contact with an average coordination number $\langle z\rangle$ is $Nd\langle z\rangle/2$. It follows then from counting (which neglects correlations between nearest neighbours) the number of independent components of configuration tensors that Eqs. (6)–(8) uniquely determine the positions of contact points if $\langle z\rangle = d(d+1)$ which approximately corresponds to the RCP limit of spherical particles. These tensors characterise the relative arrangement of nearest neighbours in the first coordination shell of the reference particle $\alpha$. In particular tensor $F_{ij}^{\alpha}$ tells us something about the second shell of contacts, and also about the volume $V^{\alpha}$ occupied by the grain $\alpha$.

### 2.1. Configurations and entropy

In statistical mechanics entropy is defined by

$$S = k \log \int \delta(E - H(p,q))\, \mathscr{D}p\, \mathscr{D}q\ , \tag{9}$$

where $E$ is the energy and $H(p,q)$ the Hamiltonian. Entropy $S$ is function of internal energy $E$, the number of particles $N$ and volume $V$. The analogue of (9) for jammed configurations is

$$S = \log \int \delta(V - W(\zeta))\Theta(\zeta)\, \mathscr{D}\zeta\ , \tag{10}$$

where entropy is now function of number of particles $N$ and volume $V$. For the granular system with zero kinetic energy we *postulate* a phase space defined by collective coordinates $\zeta$, which are basically functions of the points where grains touch, via a volume function $W(\zeta)$ which takes the place of the Hamiltonian and the condition $E = H$ of the thermal system is replaced by $V = W$ of the granular system. This system is said to be "jammed". The function $\Theta(\zeta)$ insures that all grains are in locked positions by touching their neighbours. For jammed configurations we find $S(N,V)$ from (10), and so we define a canonical ensemble via $Y$ which we call the effective volume, the analogue of free energy:

$$Y = V - XS = V + X\,\frac{\partial Y}{\partial X}\ , \tag{11}$$

where

$$\mathrm{e}^{-(Y/X)} = \int \mathrm{e}^{-(W(\zeta)/X)}\Theta(\zeta)\, \mathscr{D}\zeta\ . \tag{12}$$

In general there are configurations of the powder which will not be described by this ensemble, for example arched holes with "rattlers" (i.e., free grains which are not in a stable contact with their nearest neighbours) in them. Nevertheless, the fruitful area for commencing study is when (10) applies and we confine ourselves to such a case. The contact points are the total specification for static packings, thus their set must define $W$.

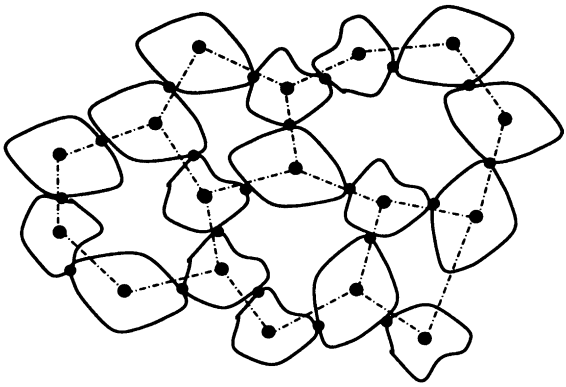

Fig. 3. Two-dimensional continuous random network: a sketch of a packing of three-fold coordinated grains of irregular shape. Black circles denote the contact points and dashed lines joining the centroids of contacts, form the continuous random network.

How can one analytically express the volume function in terms of contact points positions? Thus $W$ is $W(\mathbf{C}^{\alpha\beta})$ where $\mathbf{C}^{\alpha\beta}$ is the point where grain $\alpha$ touches grain $\beta$. One can see from Fig. 3 that vector $\mathbf{R}^{\alpha\beta} = \mathbf{R}^{\alpha} - \mathbf{R}^{\beta}$ joins the centroids of contact of grains $\alpha$ and $\beta$ and the packing can be represented by the continuous random network of connected branch vectors. Ball and Blumenfeld have shown [15] that the packing area in two dimensions can be related to the antisymmetric part of the tensor $C_{ij}^{\alpha}$ of a grain $\alpha$

$$C_{ij}^{\alpha} = \sum_{l} R_i^{\alpha l} r_j^{\alpha l} \tag{13}$$

where vector $R_i^{\alpha l}$ joins the centroid of contacts of the grain $\alpha$ with the centroid of loop $l$ and vector $r_j^{\alpha l}$ joins the contact points of the grain $\alpha$ which belong to this loop (see Fig. 4). In two dimensions an exact theory of $W$ has been given by Blumenfeld and Edwards [16], but we continue with a cruder theory wherein a "first coordination shell" theory is offered, knowing the success of conventional statistical mechanics when much can be achieved with a Hamiltonian, $\mathscr{H} = \sum \mathscr{H}^{\alpha}$ that is additive over particles.

Since tensor $F_{ij}^{\alpha}$ describes the configuration of the first coordination shell of a reference grain $\alpha$, its invariants play a special role in our formalism. An approximation to the volume function of a packing of $N$ grains is

$$W = \sum_{\alpha}^{N} W^{\alpha} = \sum_{\alpha}^{N} \sqrt{\mathrm{Det}\, F_{ij}^{\alpha}}\,. \tag{14}$$

This $W^{\alpha}$ is the area or volume of the first coordination shell, produced by grain $\alpha$ with its nearest neighbours, but it does not accommodate complex topologies with multiple contacts between the pairs of neighbouring grains. The addition of terms describing the correlation between coordination shells, to the volume function, makes the integration

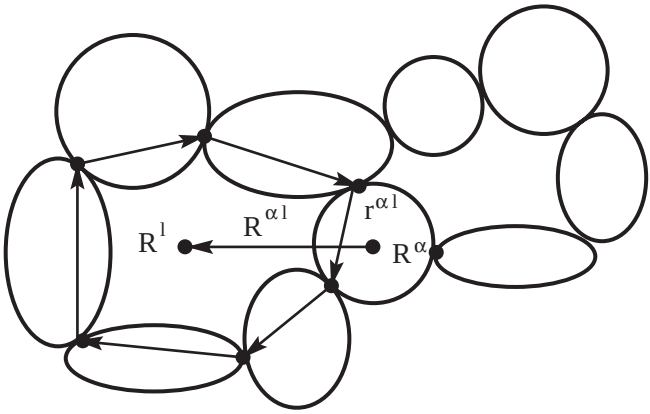


Fig. 4. Vectors join contact points in neighbouring grains and form the loop $l$: vector $R_i^{\alpha l}$ joins the centroid of contacts of a grain $\alpha$ with the centroid of loop $l$ and vector $r_j^{\alpha l}$ joins the contact points of the grain $\alpha$ which belong to the loop $l$.

of (12) very difficult to accomplish. Given that the volume function can be written in terms of the eigenvalues of $F_{ij}^{\alpha}$, one can immediately see that integration with respect to $\mathscr{D}\zeta$ actually implies integration in the space of eigenvalues. We argue that the function $\Theta(\zeta)$ can be written in terms of the second invariant of $F_{ij}^{\alpha}$, i.e., its trace so that it provides the limits of integration in (12). The trace of the configuration tensor $F_{ij}^{\alpha}$ is

$$\mathrm{Tr}\{F_{ij}^{\alpha}\} = \sum_{\beta} |\mathbf{R}^{\alpha\beta}|^2 \,, \tag{15}$$

so that $\mathrm{Tr}\{F_{ij}^{\alpha}\}/z^{\alpha}$ gives the square of an average distance between the centroids of contact within the first coordination shell of grain $\alpha$. The distance between the centroids of contact of grains in contact has upper and lower bounds because particles are impenetrable and are "jammed" in the cage formed by their nearest neighbours. It is easy to see then that the presence of a step-function $\Theta(F_{ii}^{\alpha})$ insures that in a packing of hard grains in contact configurations which contain overlapping and "rattling" particles are indeed forbidden. A crude theory (which has some justification but we do not give here) is

$$\mathrm{e}^{-Y/X} = \left( \int_{V_0}^{V_1} \mathrm{e}^{-W/X} \, \mathrm{d}W \right)^N \tag{16}$$

which gives as a good approximation, exact at $X = 0$ and $\infty$:

$$V = \frac{V_0(V_0 - V_1) - X(V_0 + V_1)}{V_0 - V_1 - 2X} \,. \tag{17}$$

Note that the minimum volume is $V_0$, and the maximum $\frac{1}{2}(V_0 + V_1)$ as is expected.

### 2.2. Forces and stress tensor

Newton's laws of force and couple balance for every grain give us the system of $Nd(d+1)/2$ equations for $\sum_{\alpha}^{N} z^{\alpha} d/2$ interparticle forces $\{\mathbf{f}^{\alpha\beta}\}$

$$\sum_{\beta} f_i^{\alpha\beta} + g_i^{\alpha} = 0 \,, \tag{18}$$

$$f_i^{\alpha\beta} + f_i^{\beta\alpha} = 0 \,, \tag{19}$$

$$\sum_{\beta} \varepsilon_{ikl} f_k^{\alpha\beta} r_l^{\alpha\beta} + c_i^{\alpha} = 0 \,, \tag{20}$$

where $i = 1, \ldots, d$ is the Cartesian index, $g_i^{\alpha}$ is the external body force acting on grain $\alpha$ and $c_i^{\alpha}$ is the external body couple which we take to be zero.

The simplest statically determinate problem of stress transmission in a static granular material is that where grains are considered to be perfectly hard, perfectly rough and each grain $\alpha$ has a coordination number $z^{\alpha} = d+1$ (a more general case exists when an average coordination number is $\bar{z} = d+1$). A problem is said to be statically determinate if the state of stress can be determined without knowledge of the displacement field. Let us now derive equations of stress propagation which depend only on the geometrical characteristics of the packing. We construct the contact force probability functional

$$P\{\mathbf{f}^{\alpha\beta}\} = \mathcal{N} \prod_{\alpha=1,\,\beta=n.n.}^{N} \delta\left(\sum_{\beta} f_i^{\alpha\beta} + g_i^{\alpha}\right) \delta\left(\sum_{\beta} \varepsilon_{ikl} f_k^{\alpha\beta} r_l^{\alpha\beta}\right) \delta(f_i^{\alpha\beta} + f_i^{\beta\alpha}) \,, \tag{21}$$

where the normalisation $\mathcal{N}$, which is a function of a configuration, is defined as

$$\mathcal{N}^{-1} = \int \prod_{\alpha=1\,\beta=n.n.}^{N} P\{\mathbf{f}^{\alpha\beta}\} \mathscr{D}\mathbf{f}^{\alpha\beta} \,, \tag{22}$$

and $\prod_{\alpha=1,\beta=n.n.}^{N}$ means the product of Newton's equations for all grains; $\beta = n.n.$ is a nearest neighbour label for a reference grain $\alpha$. It is still not clear how to construct the macroscopic force field given the set of intergranular contact forces (though we present a crude model for the distribution of contact forces in Section 2.3) because of the sign-oscillating nature of the latter. In order to pursue an analytical approach we introduce a tensor which couples the set of intergranular contact forces $\{\mathbf{f}^{\alpha\beta}\}$ to the set of contact points $\{\mathbf{C}^{\alpha\beta}\}$. The tensorial force moment $S_{ij}^{\alpha}$ for grain $\alpha$ is defined as

$$S_{ij}^{\alpha} = \sum_{\beta} f_i^{\alpha\beta} r_j^{\alpha\beta} \,. \tag{23}$$

Since the external body couple $c_i^{\alpha} = 0$ tensor $S_{ij}^{\alpha}$ is symmetric and has $d(d+1)/2$ independent components. Tensor $S_{ij}^{\alpha}$ has a well-defined macroscopic

analogue

$$\sigma_{ij}(\mathbf{r}) = \left\langle \sum_{\alpha=1}^{N} S_{ij}^{\alpha} \delta(\mathbf{r} - \mathbf{R}^{\alpha}) \right\rangle , \tag{24}$$

which is a smooth differentiable function. The method offered by the authors in [17] was to consider the probability functional for the set $\{S_{ij}^{\alpha}\}$

$$P\{S_{ij}^{\alpha}\} = \mathscr{M} \int \prod_{\alpha,\beta}^{N} \delta \left( S_{ij}^{\alpha} - \sum_{\beta} f_i^{\alpha\beta} r_j^{\alpha\beta} \right) \tag{25}$$

$$\times P\{\mathbf{f}^{\alpha\beta}\} \mathscr{D}\mathbf{f}^{\alpha\beta} , \tag{26}$$

where the normalisation $\mathscr{M}$ is a function of contact network configuration. The main idea of this formalism (see Ref. [17] for mathematical details) is to transform (26) into

$$P\{S_{ij}^{\alpha}\} = P\{S_{ij}^{\alpha}|force\} P\{S_{ij}^{\alpha}|geometry\} , \tag{27}$$

where $P\{S_{ij}^{\alpha}|force\}$ gives the set of $Nd$ stress–force equations

$$\sum_{\beta} S_{ij}^{\alpha} M_{jl}^{\alpha} R_l^{\alpha\beta} - \sum_{\beta} S_{ij}^{\beta} M_{jl}^{\beta} R_l^{\beta\alpha} + g_i^{\alpha} = 0 . \tag{28}$$

The macroscopic version of these discrete equations is

$$\nabla_j \sigma_{ij}(\mathbf{r}) + g_i(\mathbf{r}) = 0 , \tag{29}$$

$P\{S_{ij}^{\alpha}|geometry\}$ is given by

$$\int \prod_{\alpha}^{N} \delta \left( S_{ij}^{\alpha} - \frac{1}{2} \sum_{\beta} \left( \phi_i^{\beta} r_j^{\alpha\beta} + \phi_j^{\beta} r_i^{\alpha\beta} \right) \right) \mathscr{D}\phi^{\alpha} \tag{30}$$

so that it provides the $Nd(d-1)/2$ constraints on $\{S_{ij}^{\alpha}\}$ through the integration over the set of $Nd$ vector point fields $\phi_i^{\alpha}$ [17]. The missing $d(d-1)/2$ macroscopic stress-geometry equations are hidden in the set of $Nd(d-1)/2$ constraints on $\{S_{ij}^{\alpha}\}$

$$S_{ij}^{\alpha} = \frac{1}{2} \sum_{\beta} \left( \phi_i^{\beta} r_j^{\alpha\beta} + \phi_j^{\beta} r_i^{\alpha\beta} \right) . \tag{31}$$

After eliminating $\{\phi^{\alpha}\}$ in Eq. (31) one will obtain a set of coupled $Nd(d-1)/2$ constraints on $\{S_{ij}^{\alpha}\}$. This can be accomplished analytically by writing diagonal and off-diagonal components of $N$ tensors $\{S_{ij}^{\alpha}\}$ and vector field $\{\phi^{\alpha}\}$ as $N$-dimensional vectors and inverting matrices corresponding to the diagonal elements of $\{S_{ij}^{\alpha}\}$ and substituting the results into the equation for the off-diagonal components. These linear constraints on $\{S_{ij}^{\alpha}\}$ can then be appropriately decoupled and averaged into $d(d-1)/2$

stress-geometry equations [17]. In the first coordination shell approximation one can write

$$\phi_i^\beta = \psi^\alpha r_i^{\alpha\beta} + \chi^\alpha r_i^{\beta\alpha} \ . \tag{32}$$

This gives us configuration tensors which we introduced in Section 2 (6–7), namely $F_{ij}^\alpha$, $H_{ij}^\alpha$ and the symmetric part of tensor $G_{ij}^\alpha$

$$G_{ij}^{\alpha S} = \sum_\beta \frac{1}{2}\left(R_i^{\alpha\beta} Q_j^{\alpha\beta} + R_j^{\alpha\beta} Q_i^{\alpha\beta}\right) \ . \tag{33}$$

Thus we have

$$S_{ij}^\alpha = \psi^\alpha (F_{ij}^\alpha - 2G_{ij}^{\alpha S} + H_{ij}^\alpha) + \chi^\alpha (H_{ij}^\alpha - F_{ij}^\alpha) \ , \tag{34}$$

where $G_{ij}^{\alpha S}$ is the symmetric part of the tensor $G_{ij}^\alpha$. After eliminating of $\psi^\alpha$ and $\chi^\alpha$ we obtain

$$\begin{vmatrix} S_{11}^\alpha & H_{11}^\alpha - G_{11}^{\alpha S} & H_{11}^\alpha - F_{11}^\alpha \\ S_{12}^\alpha & H_{12}^\alpha - G_{12}^{\alpha S} & H_{12}^\alpha - F_{12}^\alpha \\ S_{22}^\alpha & H_{22}^\alpha - G_{22}^{\alpha S} & H_{22}^\alpha - F_{22}^\alpha \end{vmatrix} = 0 \ . \tag{35}$$

The simplest homogeneous form is a uniform powder under gravity for which the missing equation gives average values of:

$$\sigma_{xx} = \sigma_{yy}, \sigma_{xy} = 0 \ , \tag{36}$$

$$\sigma_{xx} = \rho g x = \sigma_{yy} \ , \tag{37}$$

which is hydrostatic pressure. Thus we have found that $P_c$ is $P_c(F, r)$ and the whole distribution is, in a schematic notation

$$P = \mathcal{N}\delta(V - W)\Theta P\{S_{ij}^\alpha\} \ . \tag{38}$$

Here $\mathcal{N}$ is a normalization and the $\delta$ and $\Theta$ functions represent the constraints that the volume $W$ of the system is equal to $V$; $P\{S_{ij}^\alpha\}$ indicates that the stress obeys Eqs. (28) and (31).

### 2.3. Distribution of forces

Although $S_{ij}^\alpha$ can be averaged into a mean stress tensor, which does lead to important experimental findings, at the grain level stress is perhaps not a helpful quantity; it is forces at the surface that matter, and can be measured. Experiments at the Cavendish Laboratory [18] use confocal microscopy to image a packing of emulsion particles which have been compressed by ultracentrifugation. In Fig. 5 the particles are shown in their initial packing under gravity (below the random close packing regime) and after compression, i.e., at the onset of particle deformation away from spherical. We have found the refractive index matched combination of the continuous and the

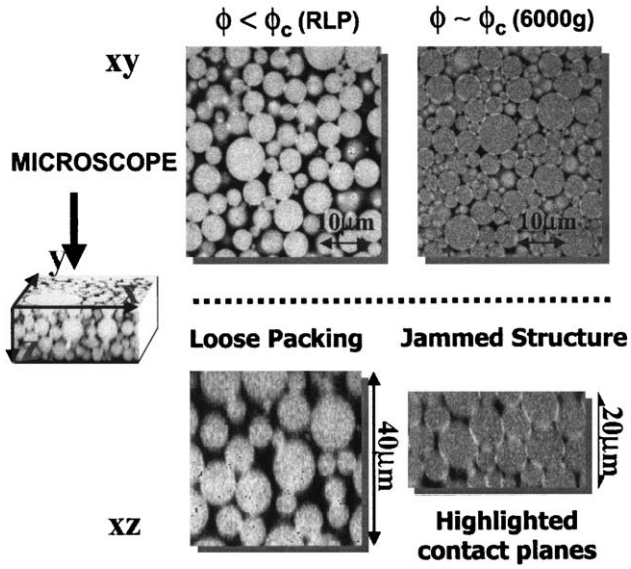


Fig. 5. Confocal microscopy images of the emulsion system under gravity and after centrifugation. Upper images are in the horizontal $(x, y)$ plane; lower images are reconstructions of a slice in $x, z$.

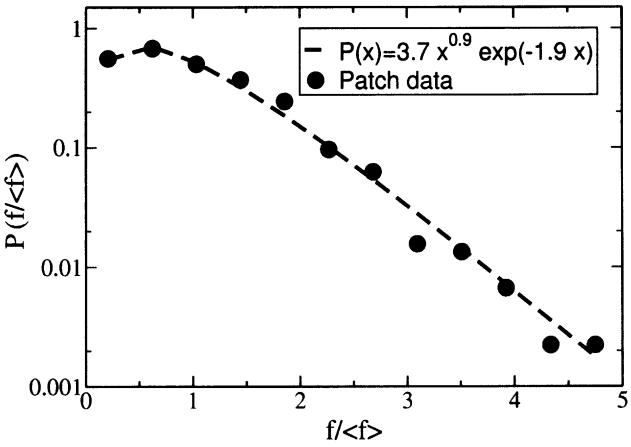


Fig. 6. Probability distribution of contact forces in the compressed emulsion system (see Fig. 5).

dispersed phase appropriate for use in the confocal microscope. Due to the increased fluorescence of the deformed surfaces of particles in contact, the interparticle forces could be extracted using the Princen model [19]. The probability distribution of the forces is presented in Fig. 6 and is consistent with previous studies by Liu et al. [20] which find an exponential distribution of forces.

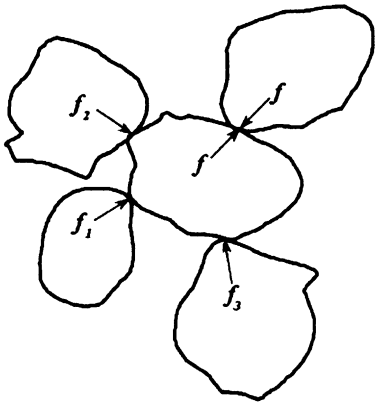


Fig. 7. 2D Schematic of contact forces between packed grains.

Recent simulations also support this picture. The geometric complexity is such that a full statistical 'transport' equation is very complex [21], but it is possible to produce a simple but convincing equation for the distribution (Edwards and Grinev, 2003). To be simple one wants a probability in one variable alone. A suitable variable is the magnitude of the force between two grains (see Fig. 7).

Ignoring gravity we find the relation

$$\mathbf{f} = -(\mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3). \tag{39}$$

Consider the scalar $f$

$$f = |\mathbf{f}| = \mathbf{f} \cdot \frac{\mathbf{f}}{|\mathbf{f}|} \,. \tag{40}$$

We now look at the components of $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ in the direction of $\mathbf{f}$, and denote these $\lambda_1 f_1$, etc.. We now argue that the probability of finding $f$, $P(f)$, is related by Newton's second law to the other forces acting, hence (in two dimensions for simplicity)

$$P(f) = \int P(f_1, \lambda_1) P(f_2, \lambda_2) \tau(\lambda_1, \lambda_2) \delta(f - \lambda_1 f_1 - \lambda_2 f_2) \, \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \, \mathrm{d}f_1 \, \mathrm{d}f_2 \,, \tag{41}$$

where $\tau$ is the weight factor which contains angles and does not allow grains 1,2 and the grain studied, to overlap. The factor $\tau$ is a tiresomely complicated function and its behaviour is well modelled by just integrating $\lambda_1$, $\lambda_2$ between 0 and 1:

$$P(f) = \int_0^1 \mathrm{d}\lambda_1 \int_0^1 \mathrm{d}\lambda_2 \int_0^\infty \mathrm{d}f_1 P(f_1) \int_0^\infty \mathrm{d}f_2 P(f_2) \delta(f - \lambda_1 f_1 - \lambda_2 f_2) \,. \tag{42}$$

This equation can be solved analytically by Fourier transform and gives

$$P(f) = \frac{f}{p}\, e^{-f/p}\,, \tag{43}$$

where $\int P\, df = 1$ and $\int Pf\, df = p^2$. In three dimensions the front factor is replaced by $f^{1/2}$. A full analysis will need the insertion of (as yet unknown) correlation functions, but could in principle discover how forces are correlated across the material, where it has been claimed that force chains exist which resemble percolation paths. However this simple treatment bears out the simplest experimental results and both experiment and theory have obvious extensions. This concludes our discussion of static problems and now we turn to the much more difficult analysis of slow dynamics.

## 3. Slow dynamics

The basic movement is when two grains are subjected to sufficient force to overcome friction when sliding and rolling result. The picture in Fig. 9 is a standard problem in Ninteenth century dynamics textbooks and is very complex. Without friction it is resolved by the Gibbs–Appell equations (Desloge [22], Pars [23]; Whittaker [24]); with friction it involves putting Gibbs–Appell in Rayleighan form which we have not seen in the textbooks. A progression of these ideas is given in Fig. 8.

The equations are much more complicated than statics (see Fig. 10) because:

(1) They involve tangents and normals to the surface which are often not well defined and also require some law of friction which could be non-linear. We will assume

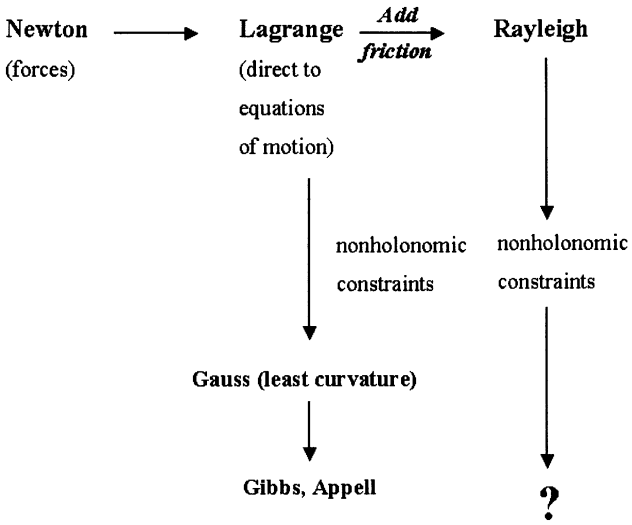


Fig. 8. Ninteenth century resolutions to standard dynamics problems.

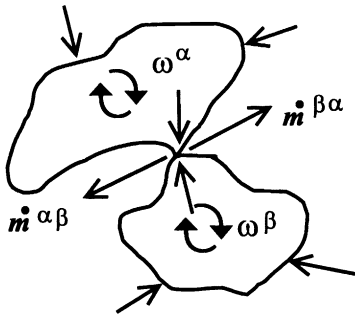


Fig. 9. A model for grain dynamics.

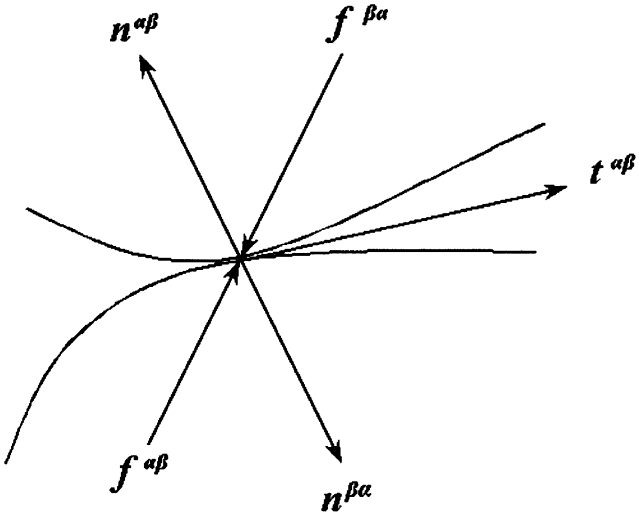


Fig. 10. Representation of normal and tangential forces between grains.

$t$ and $n$ exist and use the crudest friction law by replacing all normal forces by an average so that the sliding force is $\mu\mathbf{v}$.

(2) The description of dynamics involves both slipping and rotation i.e., velocities $\mathbf{v}^\alpha$ and rotations $\omega^\alpha$. However only $\mathbf{v}^\alpha$ can give rise to a macroscopic variable $\mathbf{v}(r)$, for $\omega^\alpha$ will vary from grain to grain, indeed for non slipping motion $\omega^\alpha = -\omega^\beta$ where $\alpha$ and $\beta$ are neighbours. It is like magnetism in an antiferromagnet. Although it is straightforward to include it as the antisymmetric complement of Eq. (49) below, we omit it for this reason, and the cruder reason that the size of algebra is vast, so we simply omit $\omega$'s altogether.

(3) In general, forces cause particles to accelerate, but we can argue that inertia is not significant at a microscopic level and hence argue that the equation

(now dropping vector notation)

$$m\ddot{x} = f_{ext} - f_{reaction} = f_{ext} - \mu\dot{x} \tag{44}$$

will become $f_{ext} = f_{reaction}$ and $f_{reaction} = \mu\dot{x}$, i.e.,

$$\mu(v^{\alpha} - v^{\beta}) = \mu v^{\alpha\beta} = f^{\alpha\beta} - n^{\alpha\beta}(f^{\alpha\beta} \cdot n^{\alpha\beta}) - f^{\beta\alpha} + n^{\beta\alpha}(n^{\beta\alpha} \cdot f^{\beta\alpha}) \tag{45}$$

$$f^{\alpha\beta} \cdot n^{\alpha\beta} = f^{\beta\alpha} \cdot n^{\beta\alpha} , \tag{46}$$

hence

$$v^{\alpha\beta} \cdot n^{\alpha\beta} = v^{\beta\alpha} \cdot n^{\beta\alpha} = 0 , \tag{47}$$

i.e., grains stay in contact along the normal but slide along the tangent.

Thus the equation for the forces $f^{\alpha\beta}$ are as before sliding, but now the velocity difference $v^{\alpha\beta}$ along the tangential directions is given by Eq. (45). (As we have recorded one should supplement these equations for $f$ and $v$ with the equations for $\omega$ which we here omit.) The basic variable in dynamics which takes the central position of $\sigma_{ij}$ in statics has to be the flow tensor which we shall label $\psi_{ij}$ (having already used $f$). In normal rheology this is written as

$$\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{48}$$

but with our modification and simplification of the full equations we prefer to directly find an equation for $v(r)$. Thus given by Eq. (45) we define averages which are related to the static definition of stress

$$\psi_{ij}^{\alpha} = \frac{1}{2}\sum_{\beta}(R_i^{\alpha\beta} v_j^{\alpha\beta} + R_j^{\alpha\beta} v_i^{\alpha\beta}) , \tag{49}$$

which is the analogue of Eq. (23) since $v \propto f$ and $r \propto R$. Substituting for $v$ from Eq. (45) we get

$$\psi_{ij}^{\alpha} = \frac{1}{2\mu}\sum_{\beta}(R_i^{\alpha\beta}(f_j^{\alpha\beta} - n_j^{\alpha\beta}(f_k^{\alpha\beta} n_k^{\alpha\beta}) - f_j^{\beta\alpha} + n_j^{\beta\alpha}(n_k^{\beta\alpha} f_k^{\beta\alpha})) + (i \leftrightarrow j) . \tag{50}$$

Proceeding now as in Ref. [17], covering the same ground that produced Eqs. (28) and (23) we reach

$$\frac{\partial \psi_{ij}}{\partial r_j} = \Lambda_{ik} f_k^{(ext)} , \tag{51}$$

where

$$\Lambda_{ik} = \frac{1}{2\mu}(\delta_{ik} - \langle n_{i\alpha} n_{k\alpha} \rangle) \tag{52}$$

but a new 'missing equation' also emerges whose form is

$$\sum_{ij} L^m_{ij}\psi_{ij} = 0 \tag{53}$$

with $m = 1$ in two dimensions and 1,2,3 in three. As before $\psi_{ij}$ has $d(d+1)/2$ components and the velocity vs. force equation has only $d$ components, so there must be $d(d-1)/2$ missing equations.

These equations are purely geometric in origin for incompressible grains. They have a similar structure to Eq. (35), but new tensors appear in place of $F$, $H$ and $G$ that have averages of the normals built into them. The details will appear in a subsequent publication, but we must catalogue a large number of crudities in the present work.

### 3.1. Unresolved problems

Many problems are unresolved at the time of this conference.

- *Thresholds*: friction depends on normal forces and movement only starts when force passes a threshold. In this paper we are far below or far above that threshold, but many real problems have both situations, in say layered flow.
- The viscosities that arise are related in a complex renormalized way to the bare values, and the averaging will involve the compactivity and also the accelerations cannot be omitted.
- Rolling can be inserted, and resolves the fact that, in the treatment above, our 'flow tensor' is not the classical symmetrical velocity gradient.
- We have not discussed fluctuation and not developed transport equations, like the Boltzmann or Fokker–Planck equations, so we do not calculate the time and space dependence of the compactivity in terms of the flow.
- The shape of the grains, for example elliptical grains, lead to a pulsation of the modulus of $R^{\alpha\beta}$ as one grain slides over another, and there will be no flow without Reynolds dilatancy, but this has been omitted so far.
- Grain contacts are always being made and lost.

We offer here the simplest picture of the flow of otherwise jammed systems. Although there is no difficulty in visualizing the rolling and sliding of packed particles, and indeed this is much the most familiar many body problem in everyday life, it is much more difficult to handle mathematically and requires a clean break with conventional statistical mechanics based ultimately on mean free path ideas. We hope this paper is a start on the problem.

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## References

[1] A. Mehta (Ed.), Granular Matter: An Interdisciplinary Approach, Springer, New York, 1993.
[2] A. Liu, S.R. Nagel, Jamming and Rheology: Constrained Dynamics on Microscopic Scales, Taylor & Francis, London, 2001.
[3] S.F. Edwards, R.B.S. Oakeshott, Physica A 157 (1989) 1080.
[4] E.R. Nowak, J.B. Knight, M.L. Povinelli, H.M. Jaeger, S.R. Nagel, Powder Technol. 94 (1997) 79.
[5] E.R. Nowak, J.B. Knight, E. BenNaim, H.M. Jaeger, S.R. Nagel, Phys. Rev. E 57 (1998) 1971.
[6] P. Philippe, Ph.D. Thesis, University of Rennes 1, 2002. See also P. Philippe, D. Bideau, Europhys. Lett. 60 (2002) 677.
[7] S.F. Edwards, D.V. Grinev, Adv. Phys. 51 (2002) 1669.
[8] A. Chakravarty, S.F. Edwards, D.V. Grinev, M. Mann, T.E. Phillipson, A.J. Walton, Proceedings of the Workshop on Quasi-static Deformations of Particulate Materials, Budapest University of Technology and Economics Press, Budapest, 2003.
[9] A. Barrat, J. Kurchan, V. Loreto, M. Sellitto, Phys. Rev. E 63 (2001) 051301, see also Phys. Rev. Lett. 85 (2000) 5034.
[10] A. Prados, J.J. Brey, B. Sanchez-Rey, Physica A 284 (2000) 277.
[11] A. Coniglio, M. Nicodemi, Physica A 296 (2001) 451.
[12] M. Nicodemi, A. Fierro, A. Coniglio, Europhys. Lett. 60 (2002) 684.
[13] D.S. Dean, Alexandre Lefevre, Phys. Rev. E 64 (2001) 046110, see also ArXiv cond-matt/0111331.
[14] H.A. Makse, J. Kurchan, Nature 415 (2002) 614.
[15] R.C. Ball, R. Blumenfeld, Phys. Rev. Lett. 88 (2002) 115505-1.
[16] R. Blumenfeld, S.F. Edwards, Phys. Rev. Lett. 90 (2003) 114303-1.
[17] S.F. Edwards, D.V. Grinev, Physica A 302 (2001) 162.
[18] J. Brujić, Edwards, S.F. Grinev, D.V. Hopkinson, I. Brujić, D. Makse, H.A., Faraday Discuss. 123 (2003) 207.
[19] H.M. Princen, J. Colloid. Interface Sci. 71 (1979) 55;
H.M. Princen, M.P. Aronson, J.C. Moser, J. Colloid. Interface Sci. 75 (1980) 246.
[20] C.H. Liu, S.R. Nagel, D. Shecter, S. Coppersmith, Science 269 (1995) 513.
[21] J.-P. Bouchaud, P. Claudin, D. Levine, M. Otto, Eur. J. Phys. E 4 (2001) 451.
[22] E.A. Desloge, Classical mechanics, Vols. 1 and 2, reprinted edition, Krieger, Florida, 1989.
[23] L.A. Pars, A Treatise on Analytical Dynamics, Heinemann, London, 1968 (Chapter XII).
[24] E.T. Whittaker, A treatise on the analytical dynamics of particles and rigid bodies, 4th ed., Cambridge Univ. Press, Cambridge, 1989.