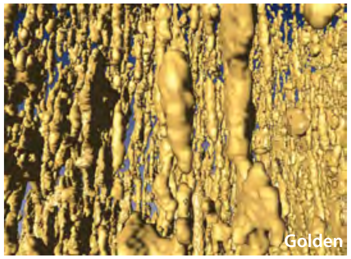
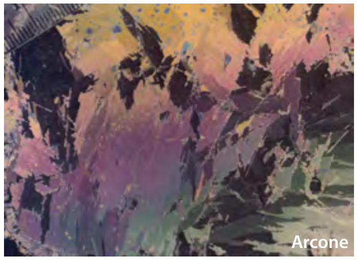
millimeters


Golden

centimeters


Arcone

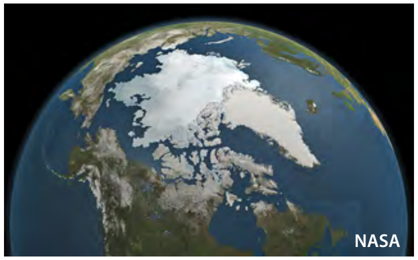
meters


Darelius

kilometers

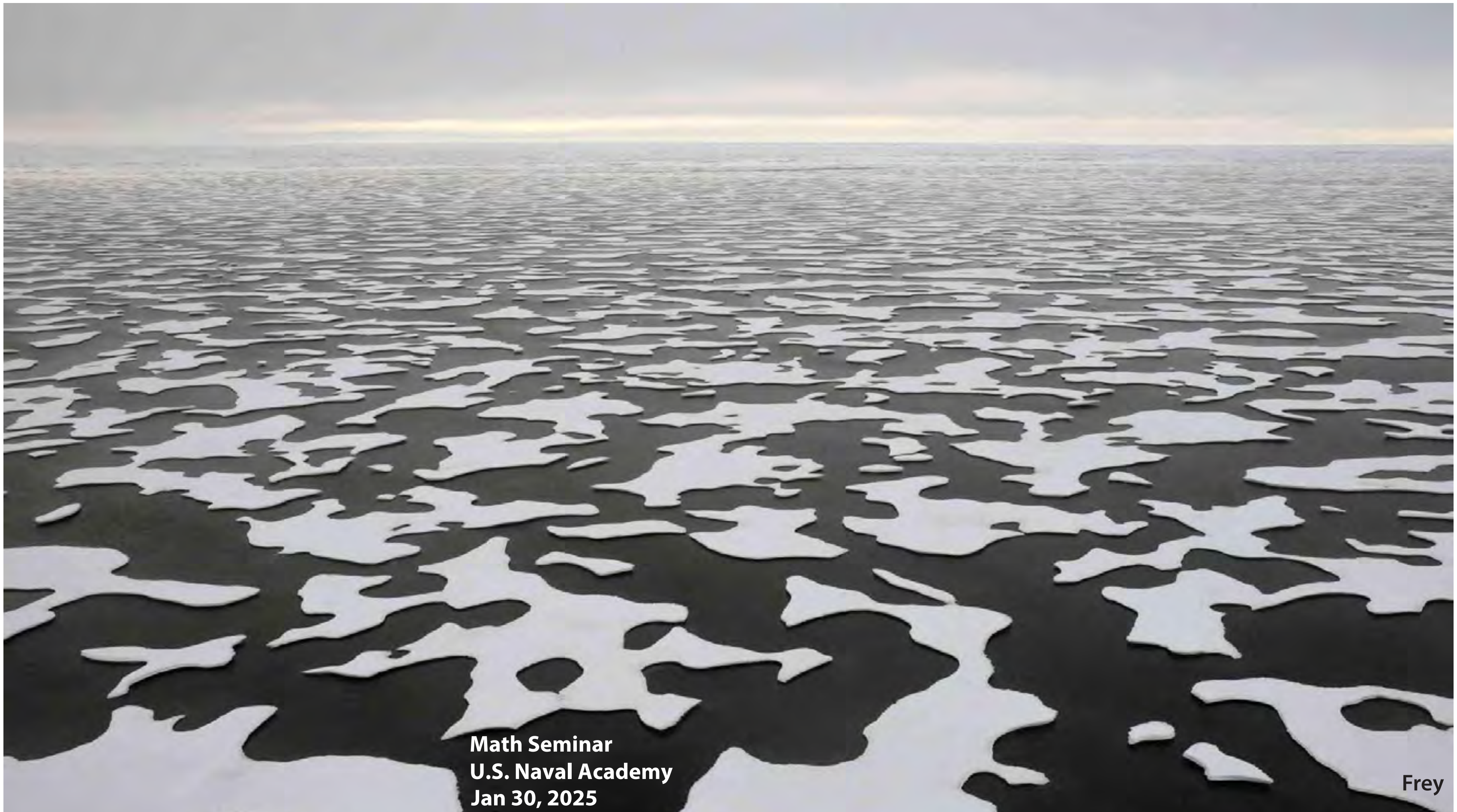

NASA

10^3 kilometers


NASA

From Micro to Macro in Modeling Sea Ice

Ken Golden, University of Utah


Math Seminar
U.S. Naval Academy
Jan 30, 2025

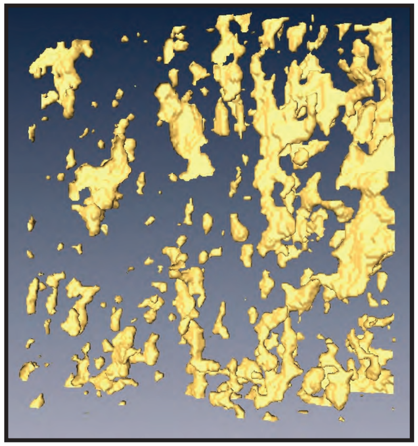
Frey

Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

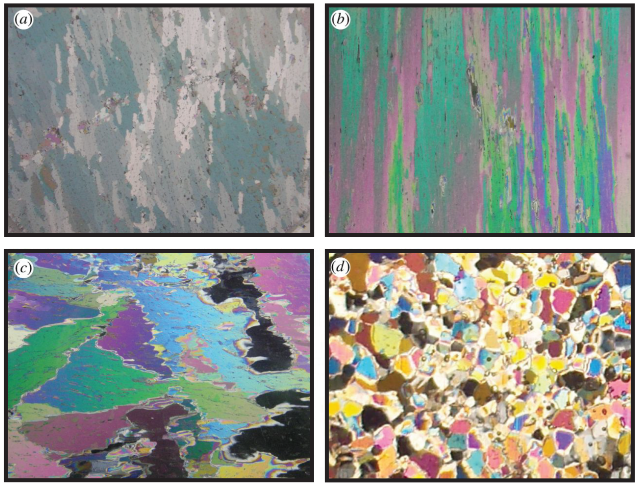
Weeks & Assur 1969



H. Eicken

Golden *et al. GRL* 2007

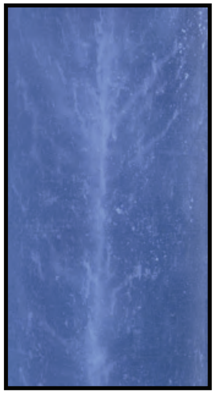
polycrystals



Gully *et al. Proc. Roy. Soc. A* 2015

brine channels

D. Cole



K. Golden

millimeters

centimeters

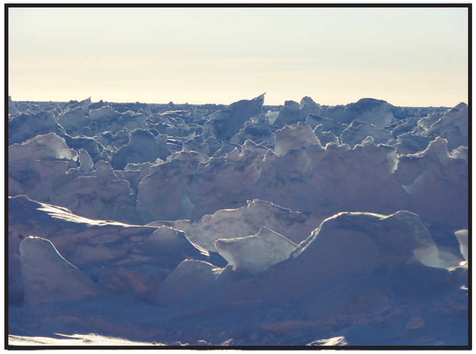
---

mesoscale

Arctic melt ponds

K. Frey

Antarctic pressure ridges



K. Golden

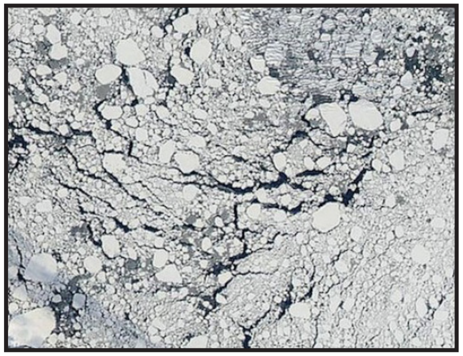
meters

macroscale

sea ice floes

J. Weller

sea ice pack



NASA

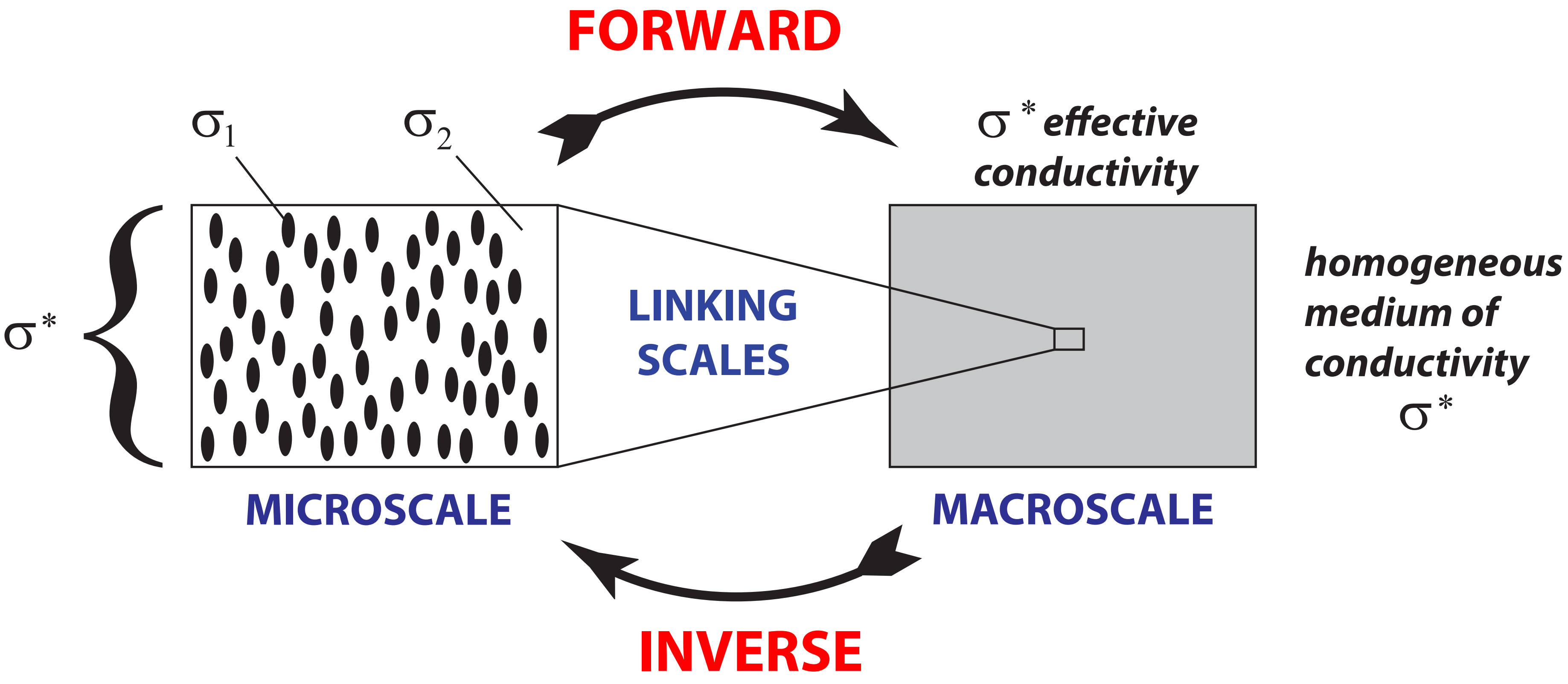
kilometers

Central theme:

How do we use "small scale" information
to find effective behavior on larger scales
relevant to climate and ecological models?

OBJECTIVE: advance how sea ice is represented in climate models
improve projections of fate of SEA ICE and its ECOSYSTEMS

HOMOGENIZATION for Composite Materials



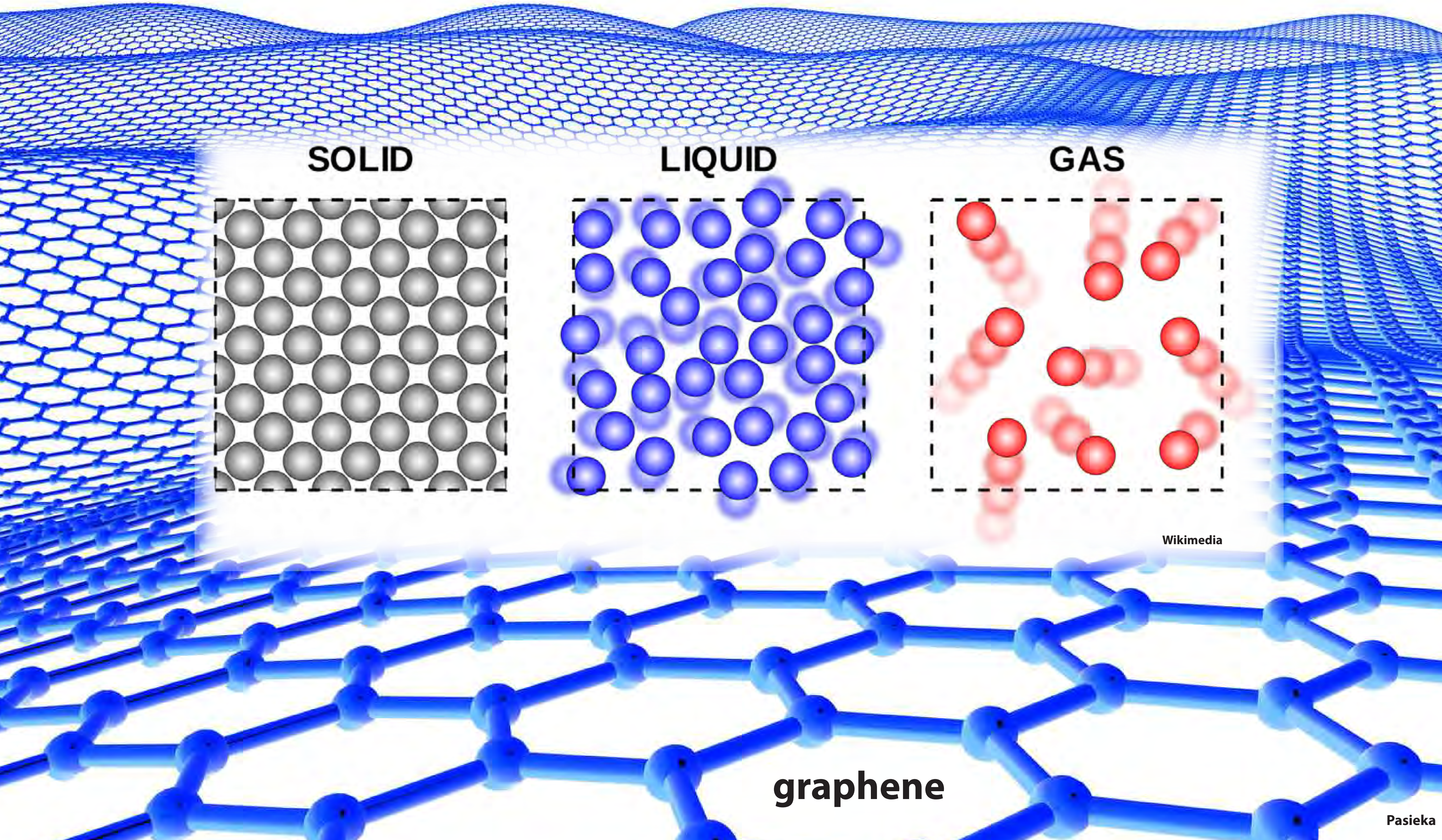
Maxwell 1873, Einstein 1906
Wiener 1912, Hashin and Shtrikman 1962

STATISTICAL PHYSICS

percolation, phase transitions
solid state, semiconductors

How do microscopic laws determine macroscopic behavior?

Banwell, Burton, Cenedese, Golden, Astrom, Physics of the Cryosphere, *Nature Reviews Physics* 2023



Wikimedia

graphene

Pasieka

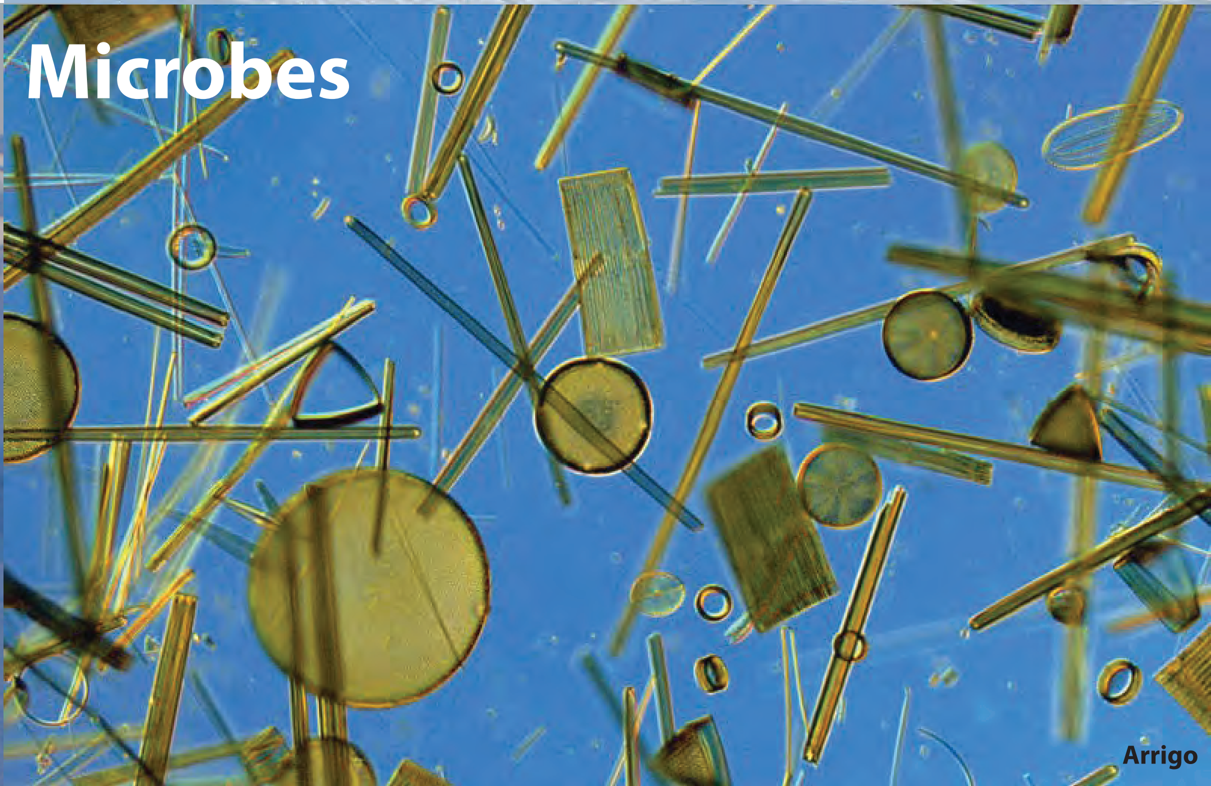
Polar Ecology and the Physics of Sea Ice

How do sea ice properties affect the life it hosts?

How does life in and on sea ice affect its physical properties?

Pekelo/iStockphoto


Golden



Arrigo

Tour a few examples of multiscale modeling of physical and biological processes in the sea ice system.

microscale

mesoscale

macroscale

Take-away for mathematicians and physicists - our sea ice studies lead us into:

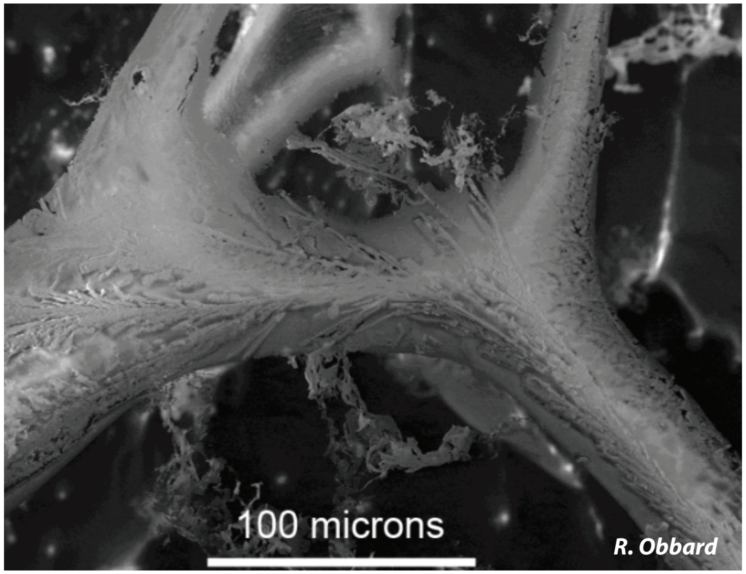
spectral analyis, random matrix theory, topological data analysis,
UQ, anomalous diffusion, dynamical systems

percolation, Anderson localization, mushy layers, phase transitions
semiconductors, Ising models, quasicrystals

+ fractal geometry

microscale

brine inclusions in sea ice (mm)

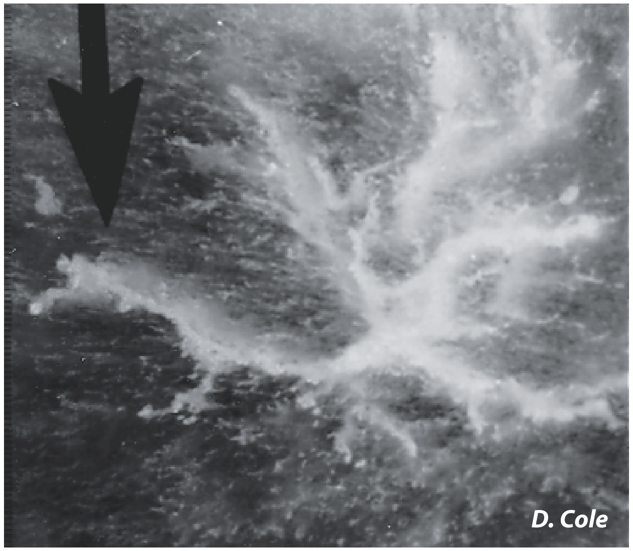


micro - brine channel (SEM)

sea ice is a porous composite

pure ice with brine, air, and salt inclusions

brine channels (cm)

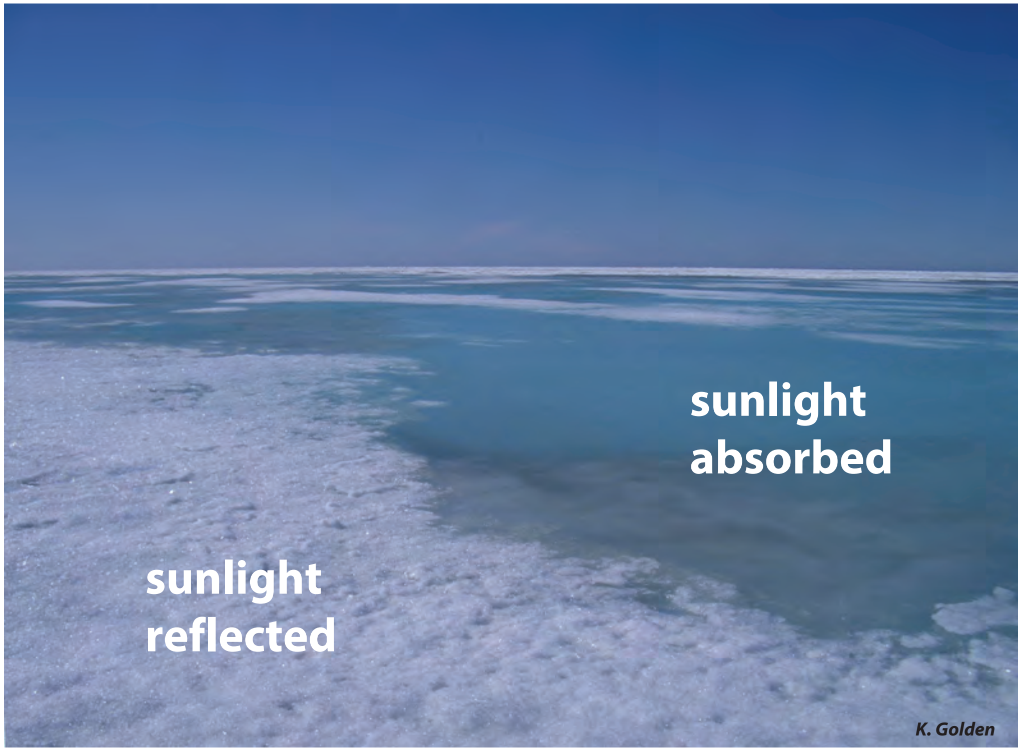


horizontal section

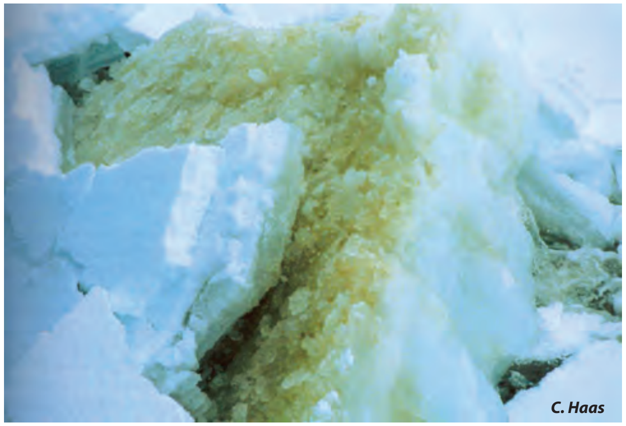
vertical section

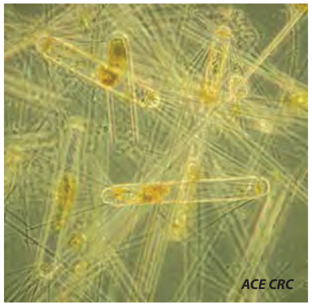
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

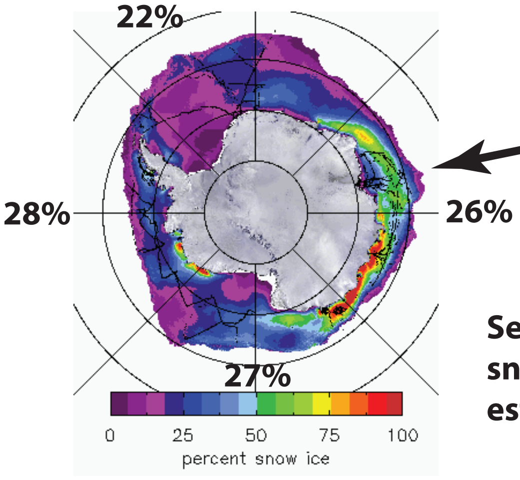
evolution of Arctic melt ponds and sea ice albedo



nutrient flux for algal communities







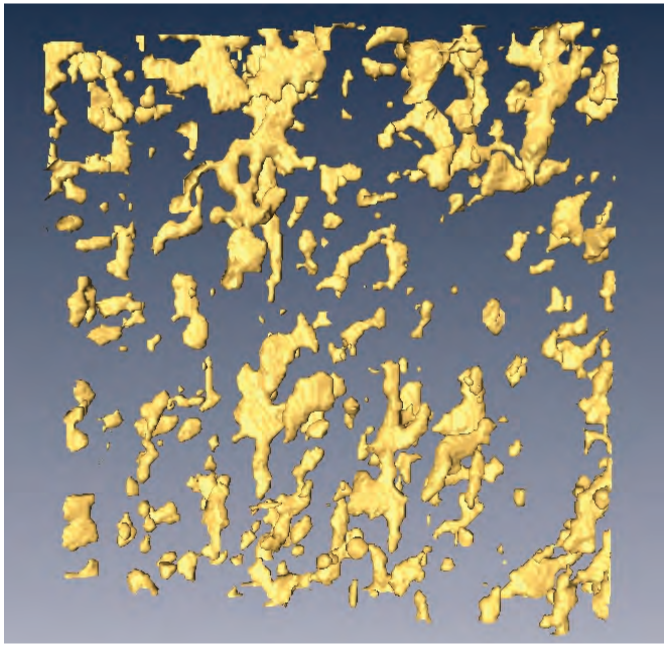
T. Maksym and T. Markus, 2008

September snow-ice estimates

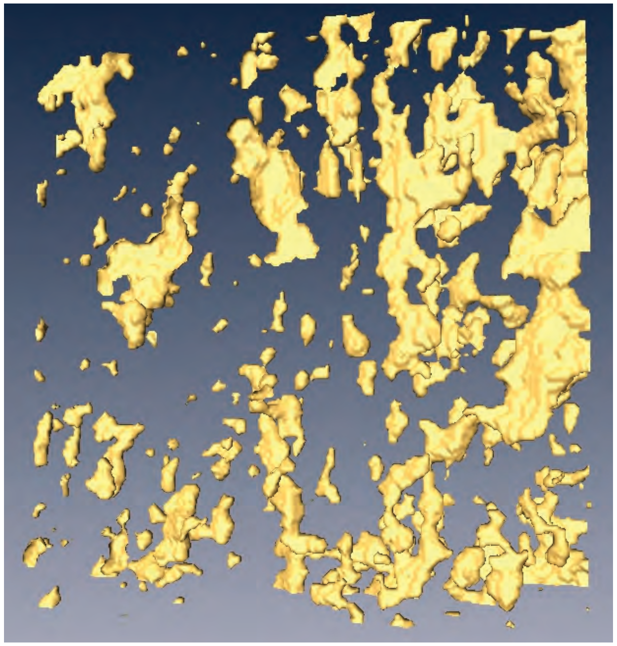
Antarctic surface flooding and snow-ice formation

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO_2*

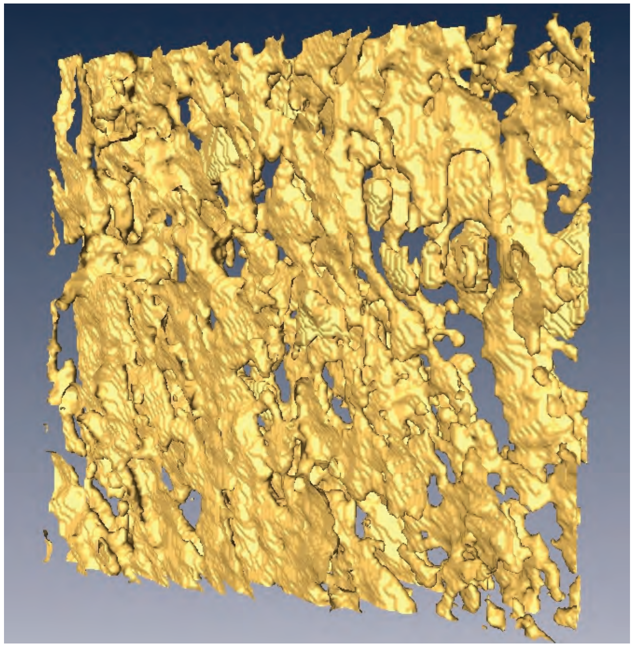
brine volume fraction and ***connectivity*** increase with temperature



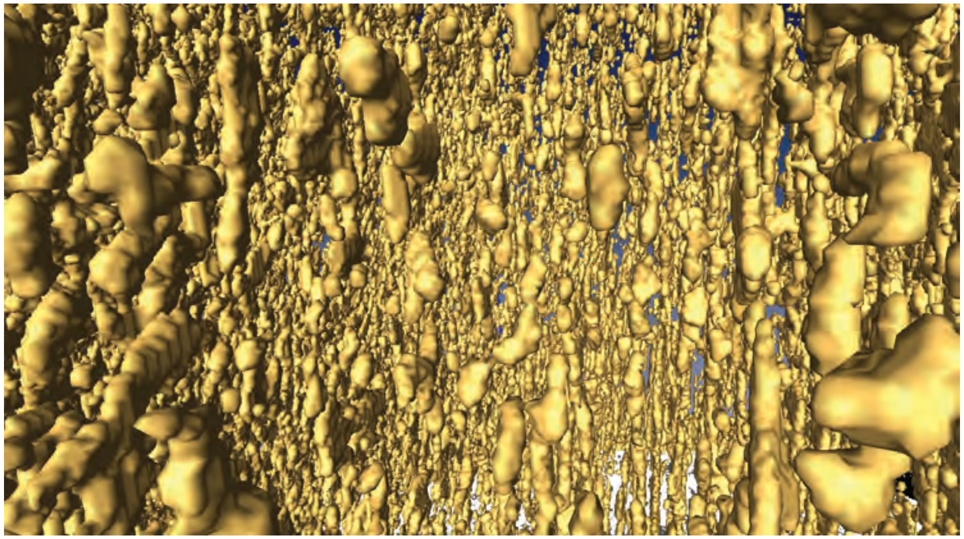
$T = -15\,^\circ\mathrm{C}, \ \phi = 0.033$



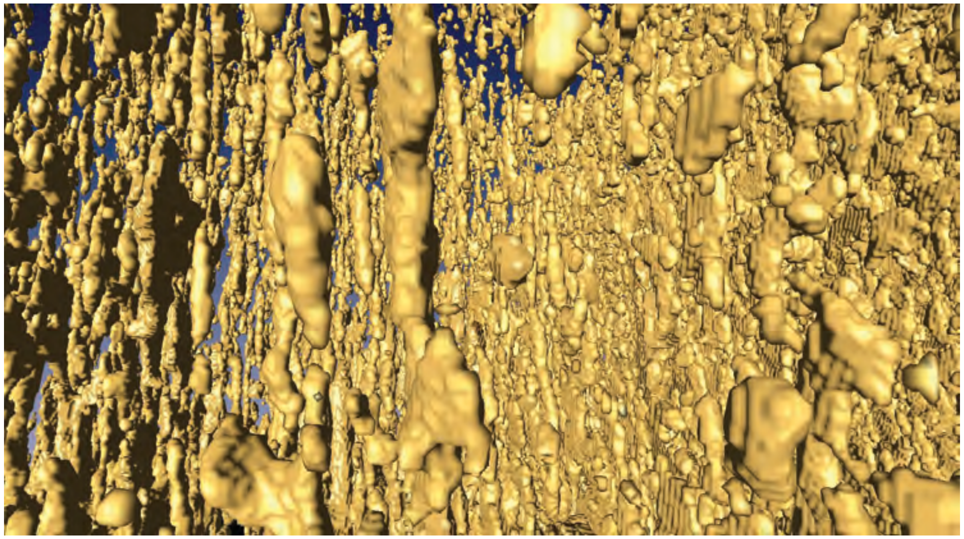
$T = -6\,^\circ\mathrm{C}, \ \phi = 0.075$



$T = -3\,^\circ\mathrm{C}, \ \phi = 0.143$



$T = -8^\circ\mathrm{C}, \ \phi = 0.057$



$T = -4^\circ\mathrm{C}, \ \phi = 0.113$

X-ray tomography for brine in sea ice

Golden *et al.*, *Geophysical Research Letters*, 2007

Critical behavior of fluid transport in sea ice

Arctic field data

vertical fluid permeability k (m^2)

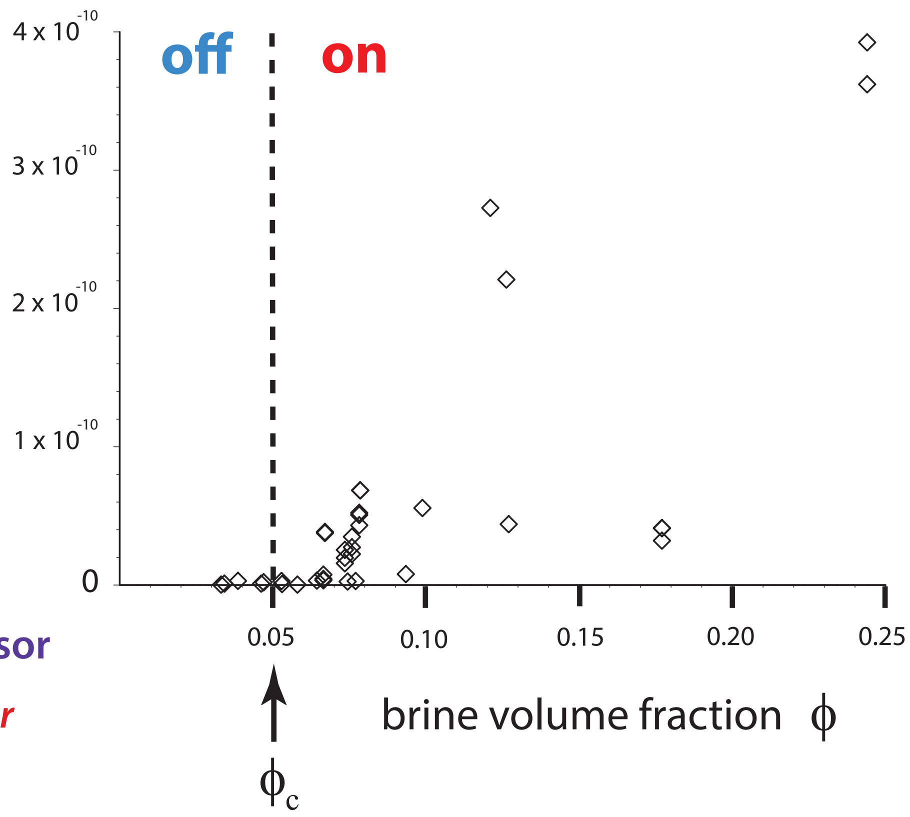
Darcy's Law

$$v = -\frac{\mathbf{k}}{\eta} \nabla p$$

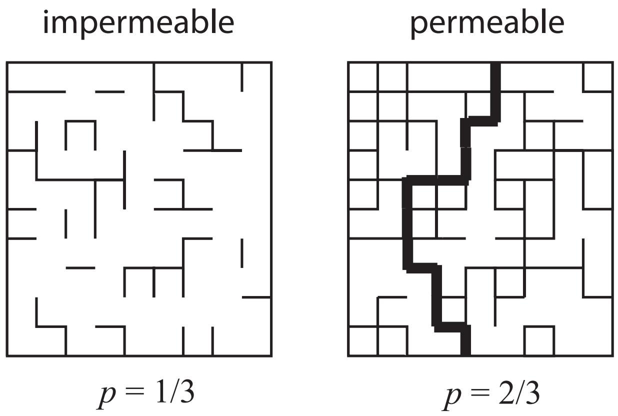
$\mathbf{k}$ = fluid permeability tensor

homogenized parameter

impermeable permeable



"on - off" switch for bulk fluid flow



lattice percolation

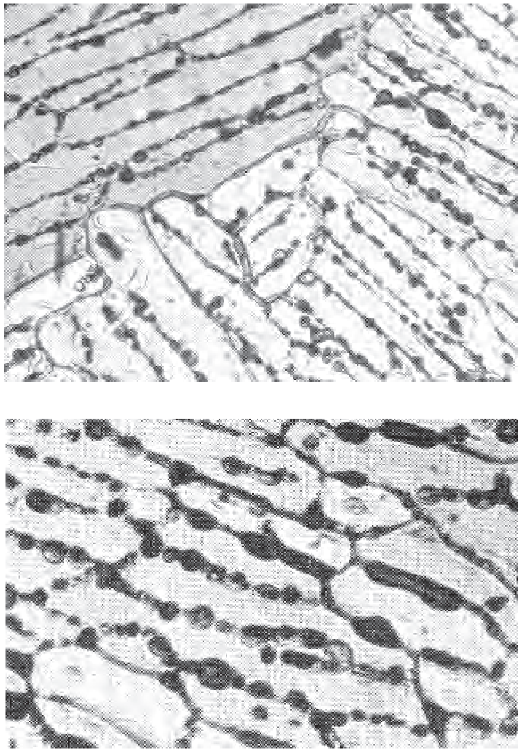
PERCOLATION THRESHOLD $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ C,\ S \approx 5$ ppt

RULE OF FIVES

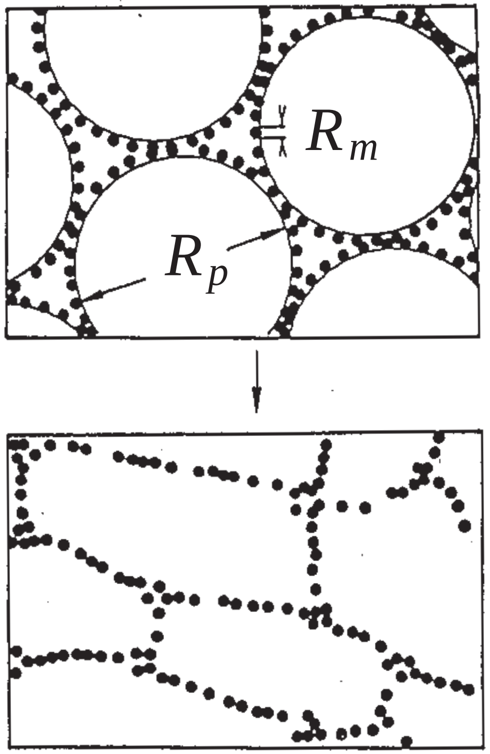
Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

Continuum percolation model for ***stealthy*** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

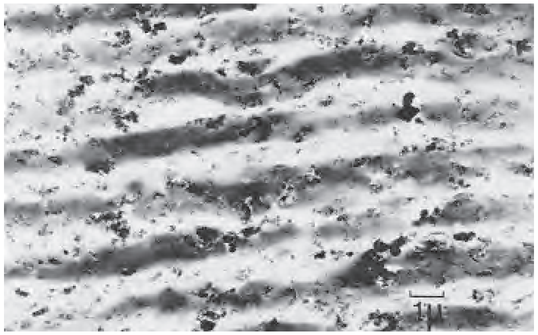
$\phi_c \approx 5\ \%$ Golden, Ackley, Lytle, *Science*, 1998



sea ice



compressed powder



radar absorbing composite

sea ice is radar absorbing

transport in percolation theory

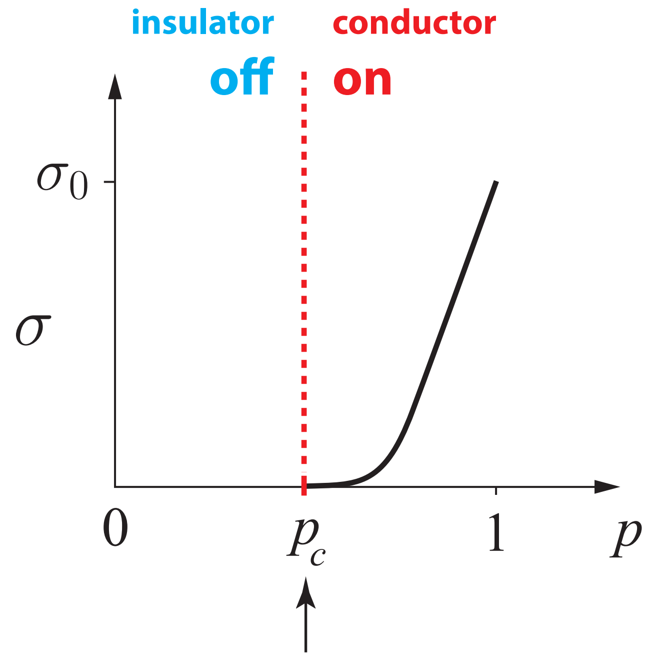
MICRO $\xrightarrow{\text{lattice homogenization}}$ **MACRO**

local conductivity (electrical or fluid)

effective conductivity or fluid permeability

$$\textbf{bond} \longrightarrow \begin{cases} \sigma_0 & \text{probability } p \\ 0 & \text{probability } 1 - p \end{cases}$$

consider local conductivities
1 and *h* > 0
smooths, softens transition



percolation threshold

$$\sigma(p) \sim \sigma_0\,(p - p_c)^t \quad p \to p_c^+$$

UNIVERSAL critical exponents for lattices -- depend only on dimension

1 ≤ ***t*** ≤ **2** (for idealized model), Golden, *Phys. Rev. Lett.* 1990 ; *Comm. Math. Phys.* 1992

non-universal behavior in continuum

sea ice algal communities

D. Thomas 2004

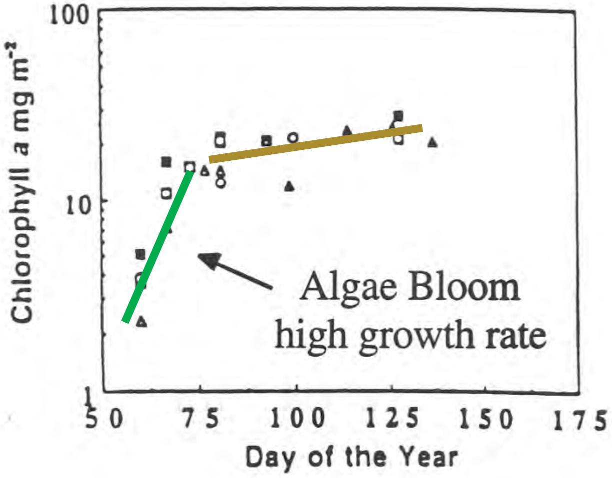
nutrient replenishment controlled by ice permeability

biological activity turns on or off according to *rule of fives*

Golden, Ackley, Lytle ***Science 1998***

Fritsen, Lytle, Ackley, Sullivan ***Science 1994***

critical behavior of microbial activity



Convection-fueled algae bloom
Ice Station Weddell

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007

Geophysical
Research
Letters

28 AUGUST 2007
Volume 34 Number 16
American Geophysical Union

A unified approach to understanding permeability in sea ice • Solving the mystery of booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

melt pond evolution

microscale

governs

mesoscale processes

percolation theory for fluid permeability

$$k(\phi) = k_0(\phi - 0.05)^2$$

critical exponent t

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

from critical path analysis in hopping conduction

hierarchical model

rock physics

network model

rigorous bounds

X-ray tomography for brine inclusions

confirms rule of fives

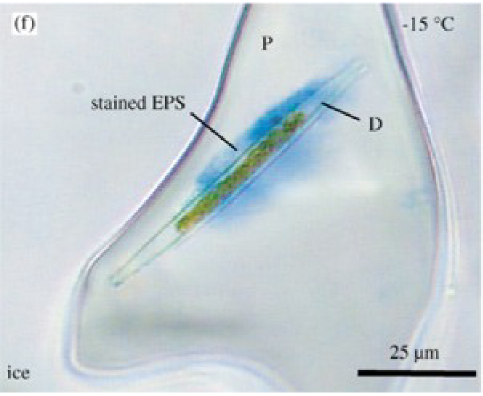
brine percolation threshold of $\phi = 5\%$ for bulk fluid flow

Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009

theories agree closely with field data

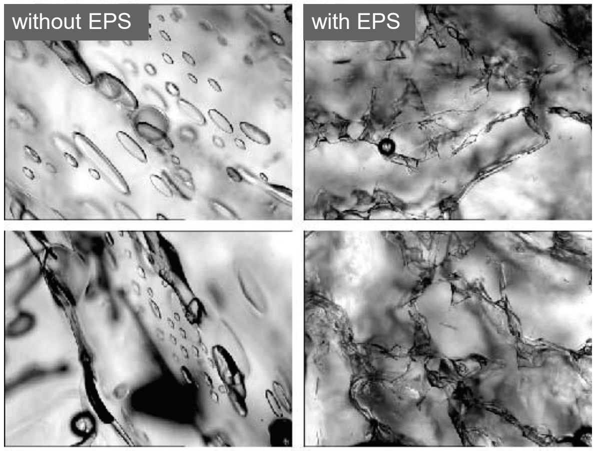
Sea ice algae secrete exopolymeric substances (EPS) affecting evolution of brine microstructure.

How does EPS affect fluid transport? **How does the biology affect the physics?**

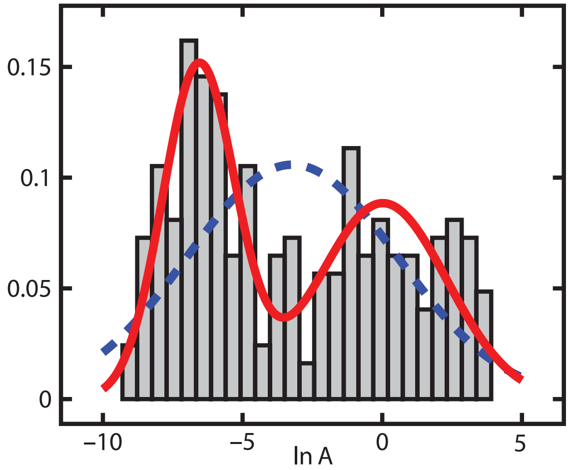


Krembs

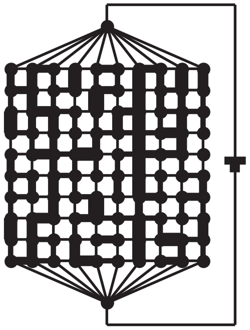
FRACTAL

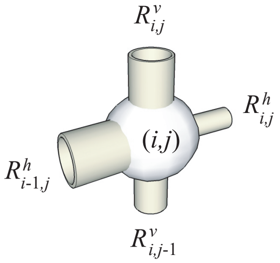


Krembs, Eicken, Deming, PNAS 2011



RANDOM PIPE MODEL





Zhu, Jabini, Golden,
Eicken, Morris
***Ann. Glac.* 2006**

- 2D random pipe model with bimodal distribution of pipe radii
- Rigorous bound on permeability *k;* results predict observed drop in *k*

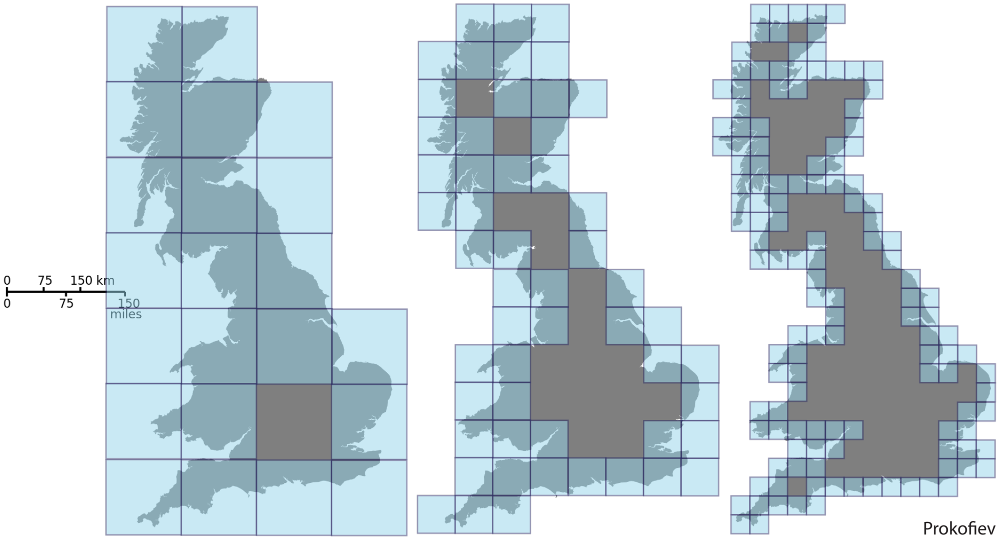
Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
***Multiscale Modeling and Simulation*, 2018**

EPS - Algae Model Jajeh, Reimer, Golden, 2024

SIAM News
June 2024

Thermal Evolution of Brine Fractal Geometry in Sea Ice

Nash Ward, Daniel Hallman, Benjamin Murphy, Jody Reimer,
Marc Oggier, Megan O’Sadnick, Elena Cherkaev and Kenneth Golden, 2024

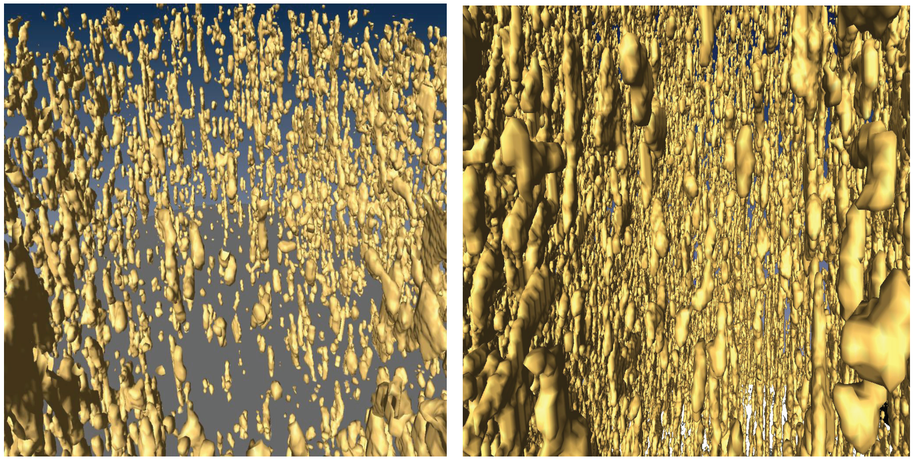


Prokofiev

fractal dimension of the coastline of Great Britain by box counting

$$N(\epsilon) \sim \epsilon^{-D}$$

brine channels and inclusions “look” like fractals (from 30 yrs ago)



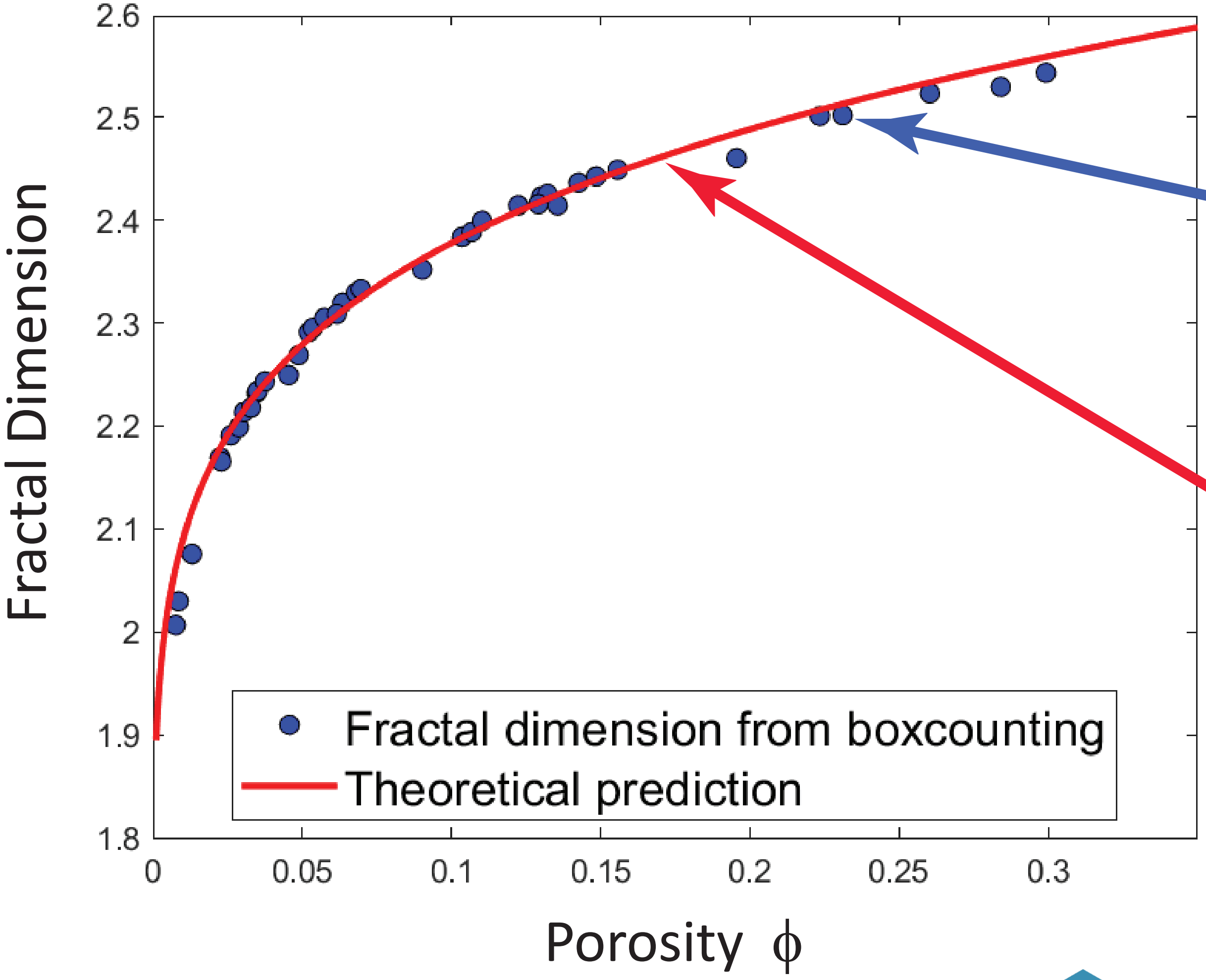
T = -12° C, $\phi = 0.033$ T = -8° C, $\phi = 0.057$

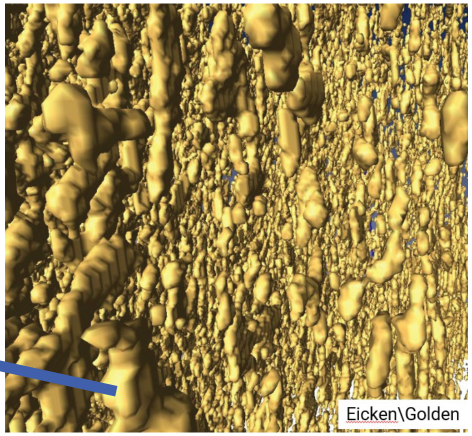
Golden, Eicken, et al. *GRL*, 2007

X-ray computed tomography of brine in sea ice

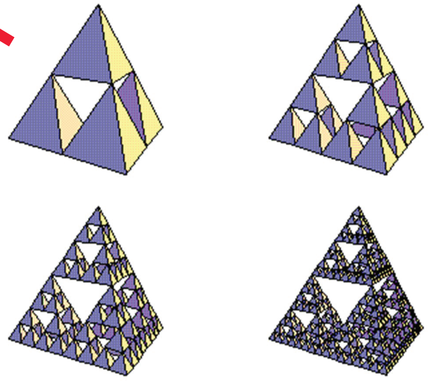
columnar and granular

The first quantitative study of the fractal dimension of brine in sea ice and its strong dependence on temperature and porosity.





Follows same curve as exactly self-similar Sierpinski tetrahedron



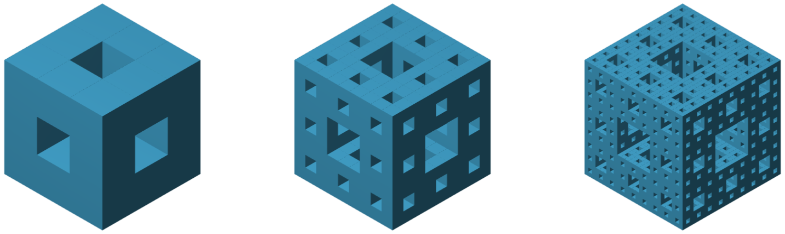
D. Eppstein

red curve

$$F_d = d_E - \frac{\ln \phi}{\ln(\lambda_{min}/\lambda_{max})}$$

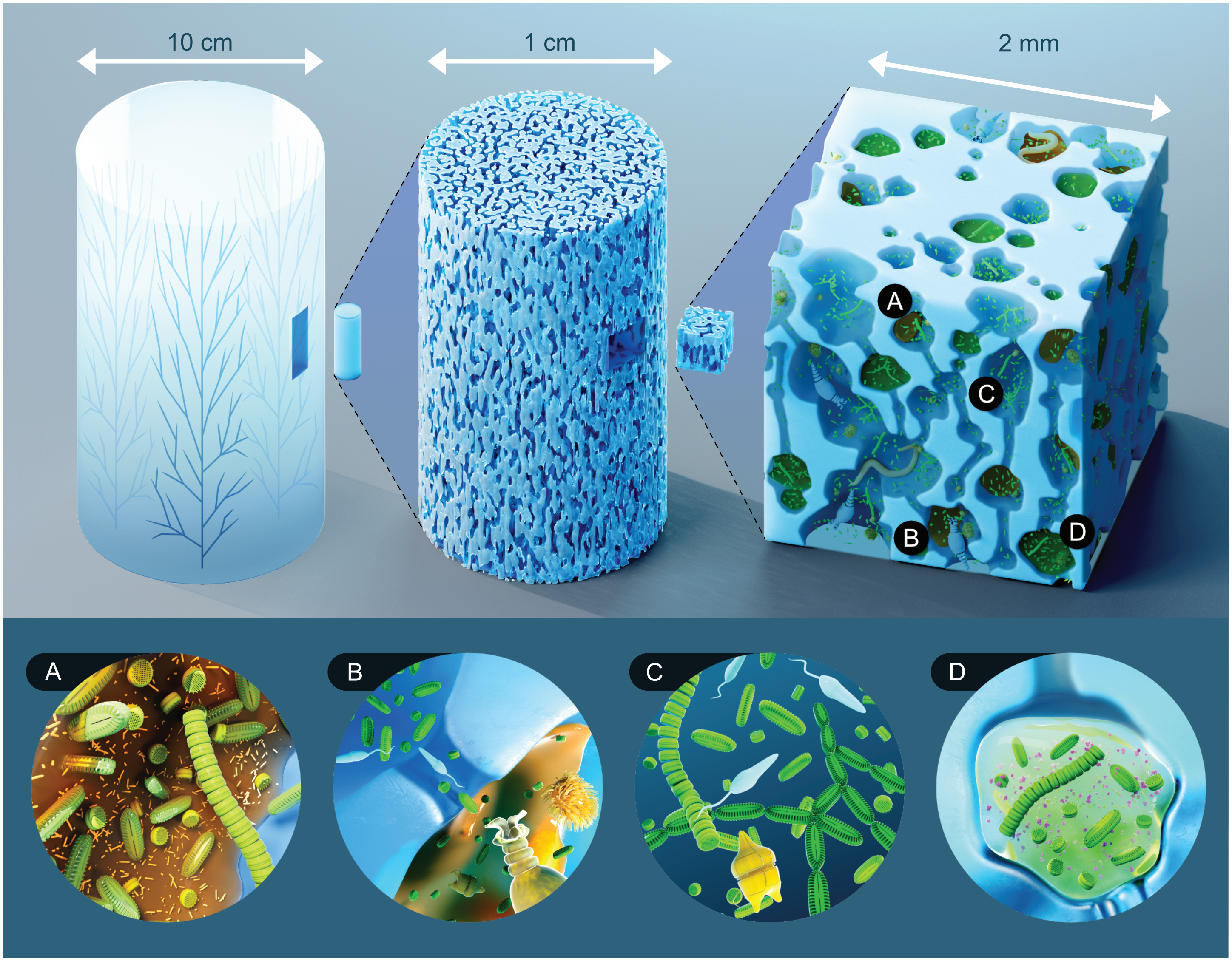
Katz and Thompson,1985; Yu and Li, 2001

discovered for sandstones
statistically self-similar porous media



Fractal geometry of brine in sea ice, Ward, *et al.* 2024

Implications of brine fractal geometry on sea ice ecology and biogeochemistry



Brine inclusions are home to ice endemic organisms, e.g., bacteria, diatoms, flagellates, rotifers, nematodes.

The habitability of sea ice for these organisms is inextricably linked to its complex brine geometry.

(A) Many sea ice organisms attach themselves to inclusion walls; inclusions with a higher fractal dimension have greater surface area for colonization.

(B) Narrow channels prevent the passage of larger organisms, leading to refuges where smaller organisms can multiply without being grazed, as in (C).

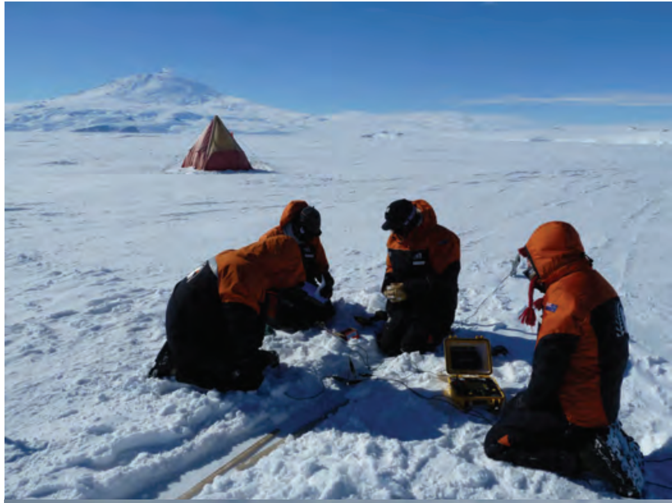
(D) Ice algae secrete extracellular polymeric substances (EPS) which alter incusion geometry and may further increase the fractal dimension.

Arctic and Antarctic field experiments

develop electromagnetic methods of monitoring fluid transport and microstructural transitions

extensive measurements of fluid and electrical transport properties of sea ice:

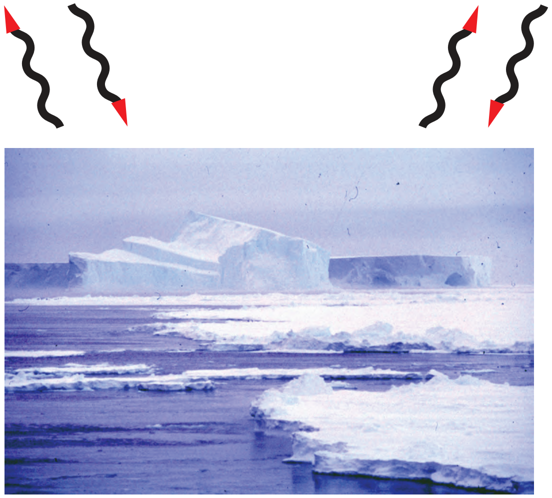
2007	*Antarctic*	*SIPEX*
2010	*Antarctic*	*McMurdo Sound*
2011	*Arctic*	*Barrow AK*
2012	*Arctic*	*Barrow AK*
2012	*Antarctic*	*SIPEX II*
2013	*Arctic*	*Barrow AK*
2014	*Arctic*	*Chukchi Sea*



Remote Sensing of Sea Ice

with radar, microwaves, ...

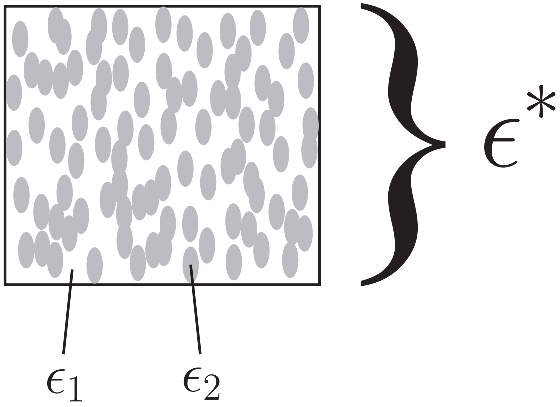
interaction of EM waves with brine and polycrystalline microstructures, rough surfaces



INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data ε^*

Effective complex permittivity of a composite in the quasistatic (long wavelength) limit



$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

electrical conductivity
thermal conductivty
magnetic permeability
diffusivity

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \ , \text{composite geometry} \right)$$

What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?

Analytic Continuation Method for Homogenization

Bergman 1978, Milton 1979, Golden & Papanicolaou 1983, Milton 2002

Stieltjes integrals for homogenized parameters separate component parameters from geometry

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z} \qquad s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

material parameters

- spectral measure of self adjoint operator $\Gamma\chi$ (matrix)

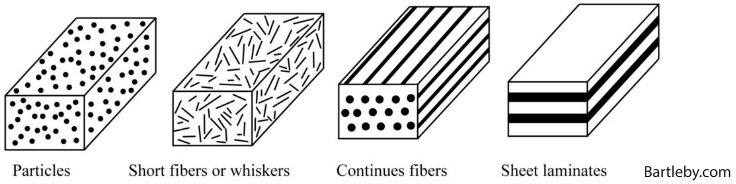
  $$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

  $\chi =$ characteristic function of the brine phase

- $E = s\,(s + \Gamma\chi)^{-1} e_k$ **resolvent**

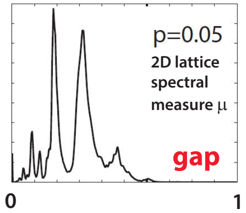
- **bounds in the complex plane; approximations inverse bounds to recover porosity, connectivity**

complexities of mixture geometry



distilled

distilled



spectral measure mass = p_1

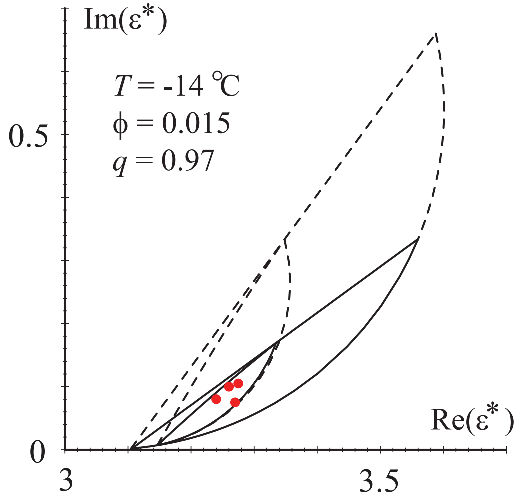
spectral properties of operator (matrix) ~ quantum states, energy levels for atoms

eigenvectors **eigenvalues**

$\Gamma\chi$: microscale $\longrightarrow$ macroscale

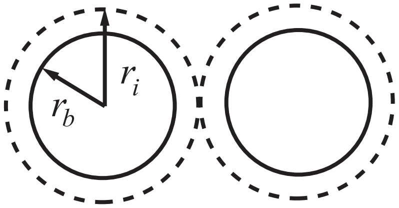
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle

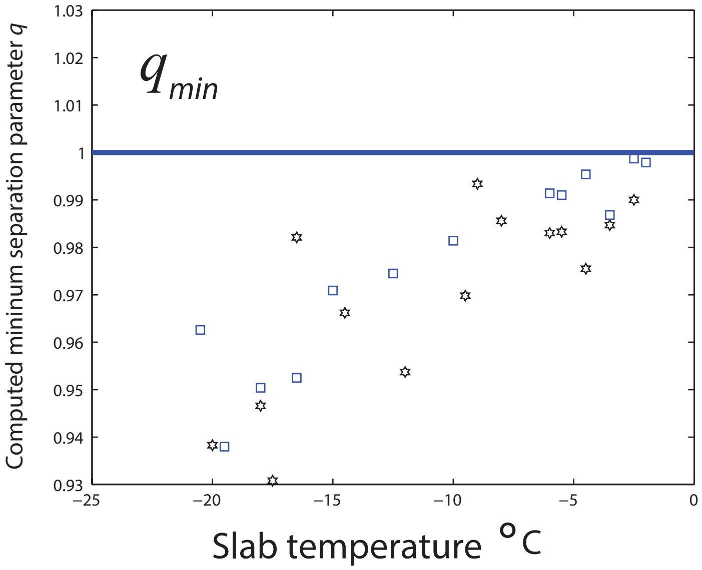


$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

inverse bounds



Inverse Homogenization

Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001), McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

ε^* ⟶ ***composite geometry***
(spectral measure μ)

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

rigorous inverse bound
on spectral gap

construct algebraic curves which bound admissible region in (p,q)-space

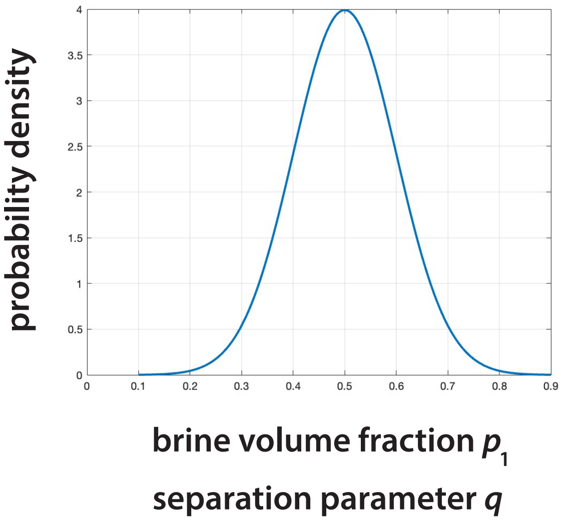
Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

Uncertainty Quantification for Homogenization via Stieltjes Integral Representations

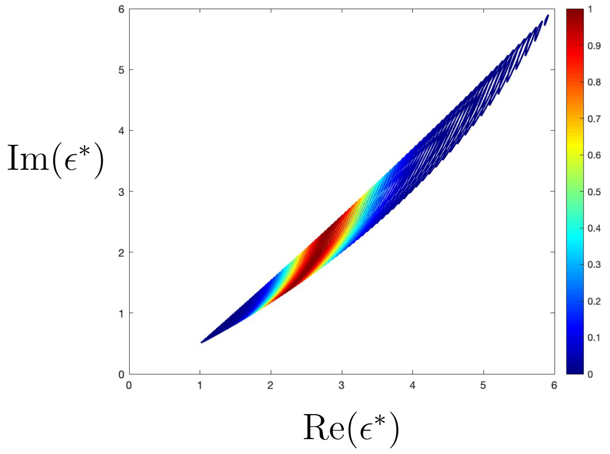
Clara Platt, Elena Cherkaev, Akil Narayan, Debdeep Bhattacharya, Ken Golden 2025

Classical bounds in the analytic continuation method assume fixed microstructural parameters, such as porosity, local permittivities, or inclusion separations.

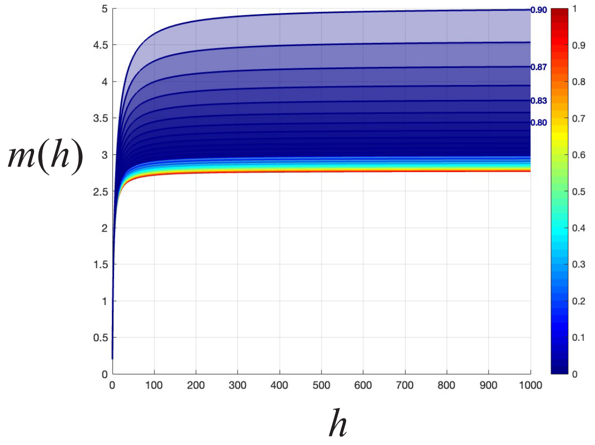
But what if there is uncertainty, and they are random variables?



Hashin-Shtrikman bounds



matrix-particle bounds



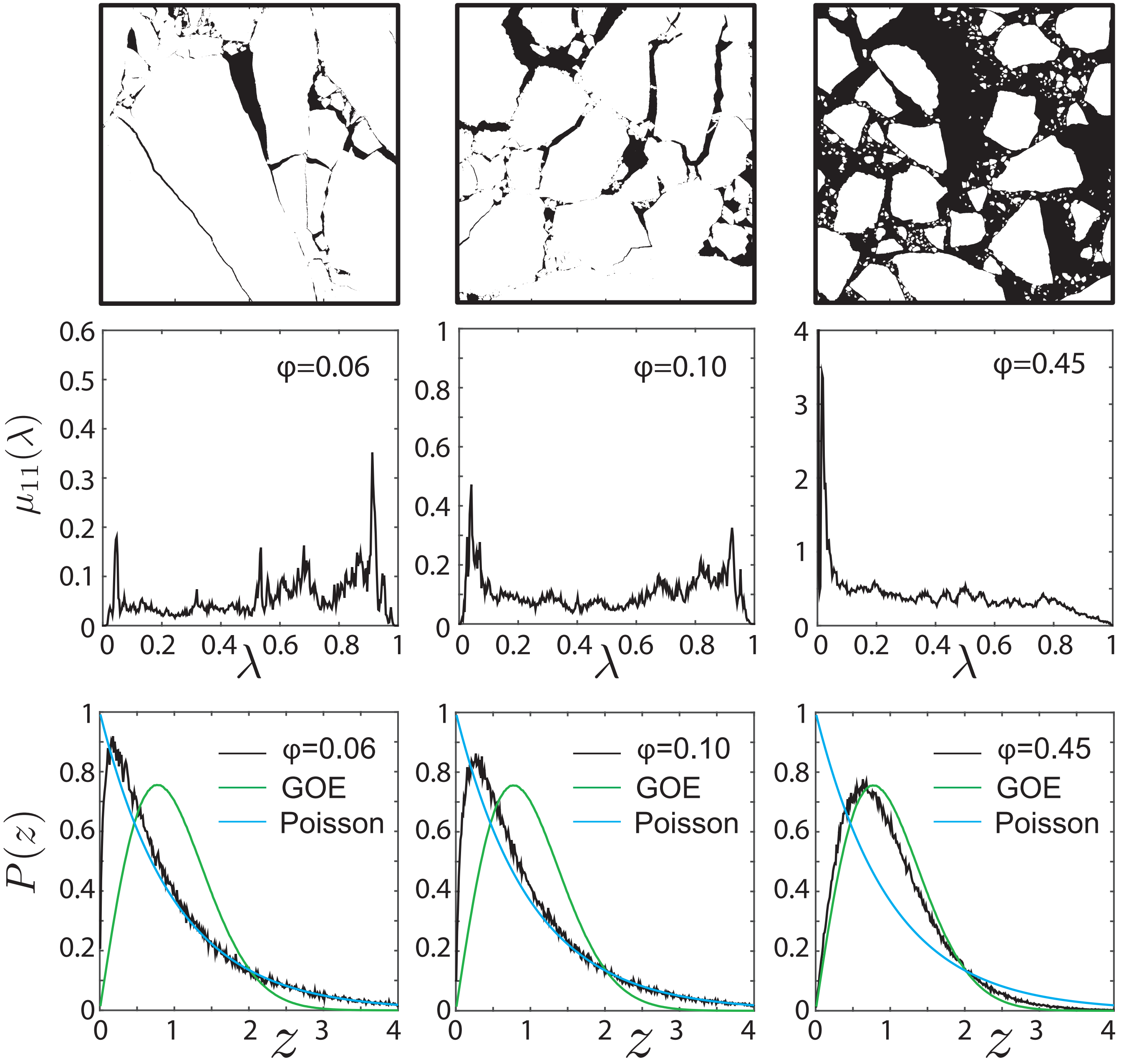
UQ for complex permittivity & thermal conductivity of sea ice

Spectral computations for sea ice floe configurations

spectral measures

eigenvalue spacing distributions

RANDOM MATRIX THEORY



UNIVERSAL Wigner-Dyson distribution

Anderson localization transition

uncorrelated → **level repulsion**

Murphy, Cherkaev, Golden, *Phys. Rev. Lett.* 2017; Murphy, Cherkaev, Hohenegger, Golden, *Comm. Math. Sci.* 2015

Eigenvalue Statistics of Random Matrix Theory

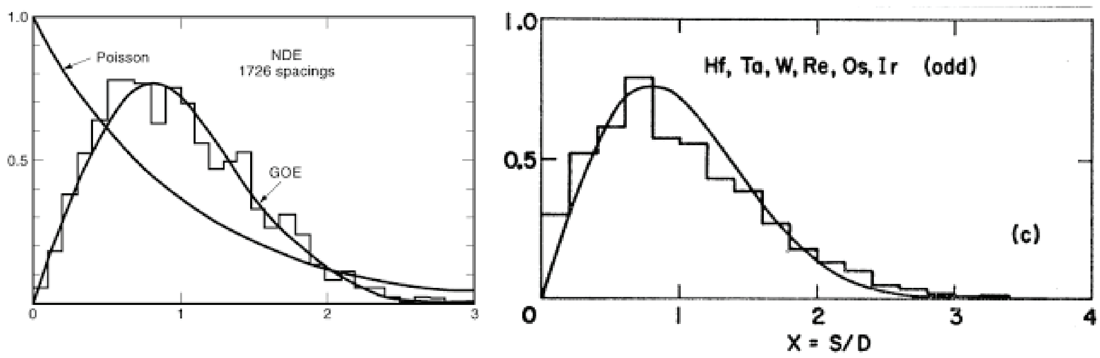
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[\mathbf{N}]_{ij} \sim N(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^{T})/2$ ***Gaussian orthogonal ensemble (GOE)***

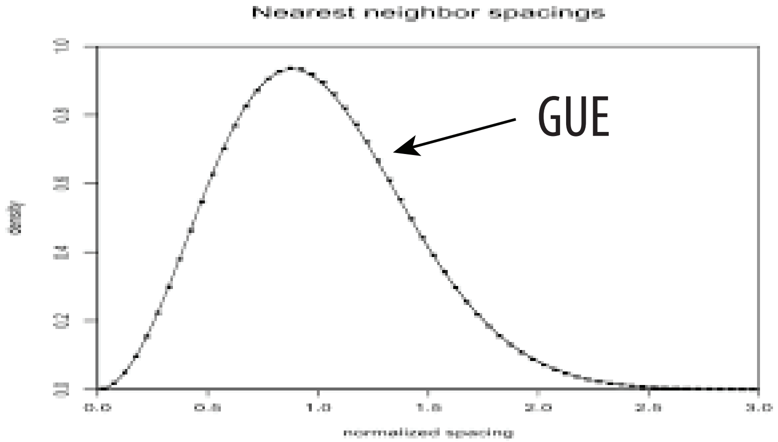
$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^{\dagger})/2$ ***Gaussian unitary ensemble (GUE)***

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics.

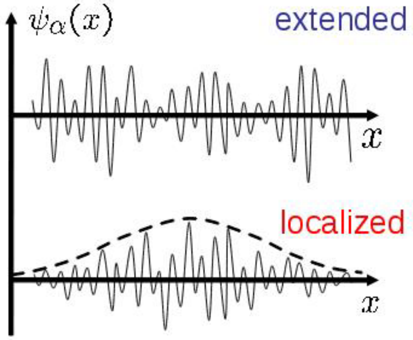
Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



Universal eigenvalue statistics arise in a broad range of "unrelated" problems!



Anderson localization

disorder-driven

metal / insulator transition

Anderson 1958
Mott 1949
Evangelou 1992
Shklovshii et al 1993

Wave equations

propagation vs. localization in wave physics:
quantum, optics, acoustics, water waves

Laplace + Diffusion
equations

we find percolation-driven

Anderson transition for classical transport in composites

mobility edges, localization, universal spectral statistics

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

but no wave interference or scattering effects at play!

local conductivity in 1D inhomogeneous material

$$\sigma(x) = 3 + \cos x + \cos kx$$

effective conductivity

$$\sigma^*(k) = \begin{cases} \text{constant} & k \ \text{irrational} \qquad \textbf{quasiperiodic} \\ f(k) & k \ \text{rational} \qquad \textbf{periodic} \end{cases}$$

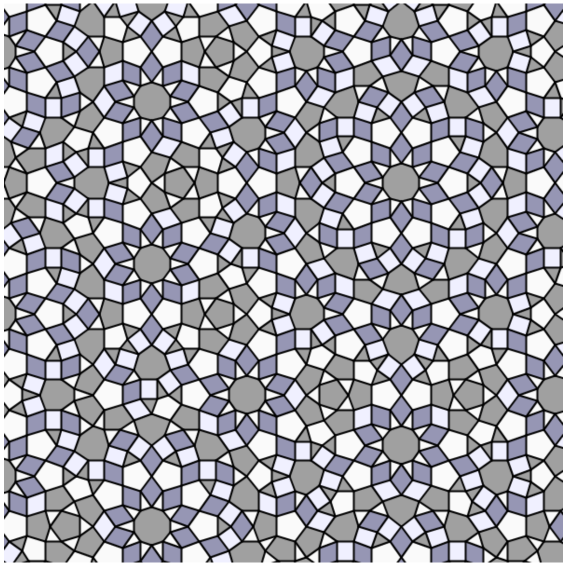
Golden, Goldstein, Lebowitz, Phys. Rev. Lett. 1985

Order to Disorder in Quasiperiodic Composites

D. Morison (Physics), N. B. Murphy, E. Cherkaev, K. M. Golden, *Communications Physics* 2022

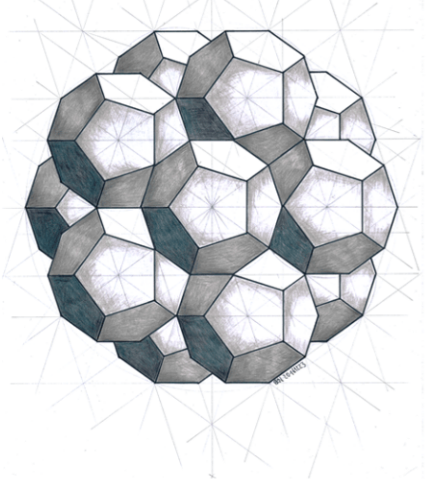
quasiperiodic crystal

quasicrystal



quasiperiodic checkerboard

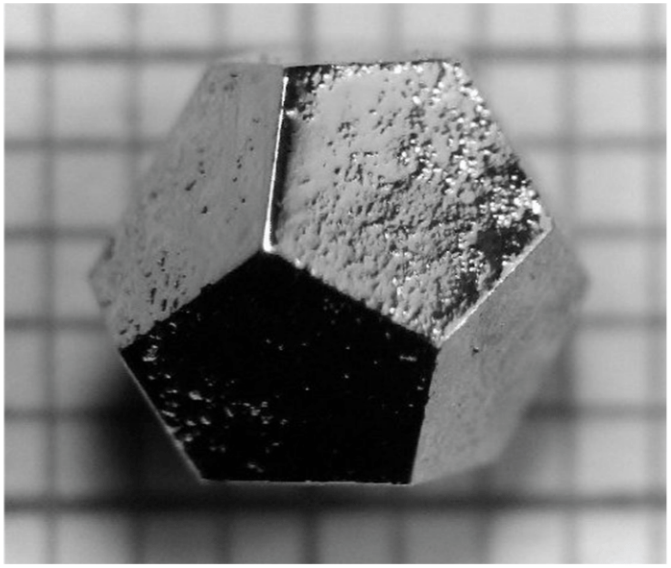
Stampfli, 2013



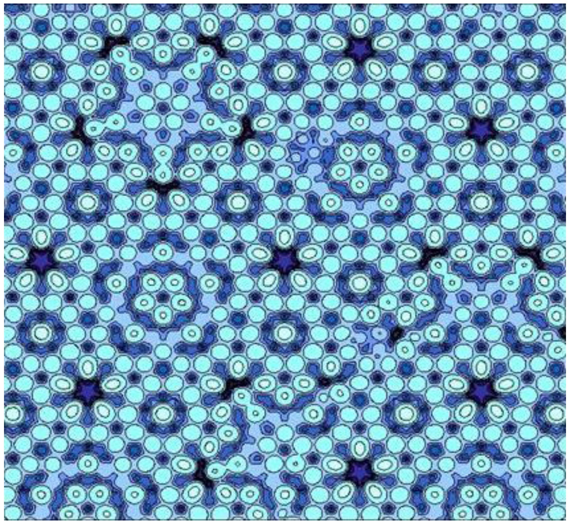
dense packing of dodecahedra

3D Penrose tiling

Tripkovic, 2019



Holmium–magnesium–zinc quasicrystal



energy surface Al-Pd-Mn quasicrystal

Unal et al., 2007

ordered but aperiodic

lacks translational symmetry

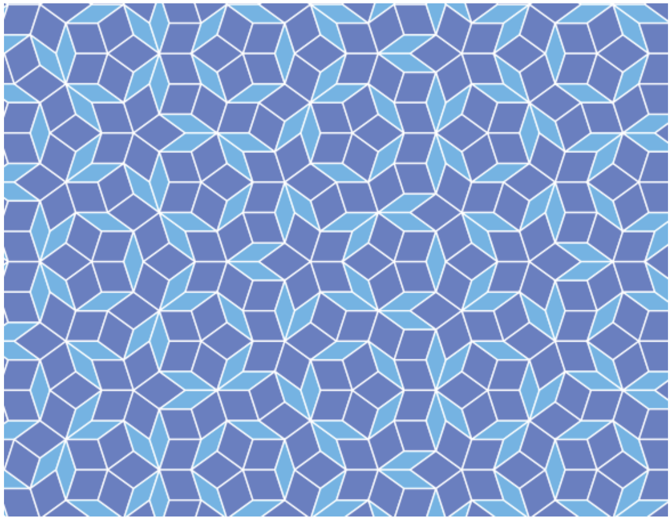
Shechtman et al., *Phys. Rev. Lett.*, 1984
Levine & Steinhardt, *Phys. Rev. Lett.*, 1984

---

classical transport in quasiperiodic media

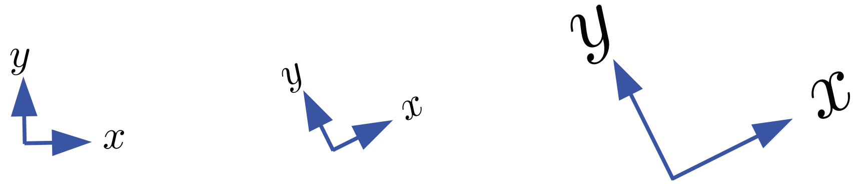
Golden, Goldstein & Lebowitz, *Phys. Rev. Lett.*, 1985
Golden, Goldstein & Lebowitz, *J. Stat. Phys.*, 1990

⋮



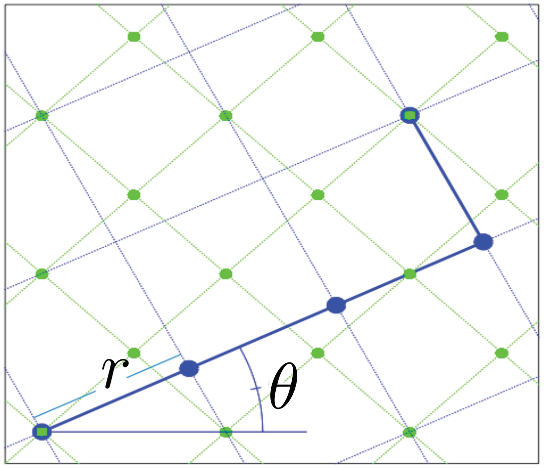
aperiodic tiling of the plane - R. Penrose 1970s

Moiré patterns generate two component composites on any scale



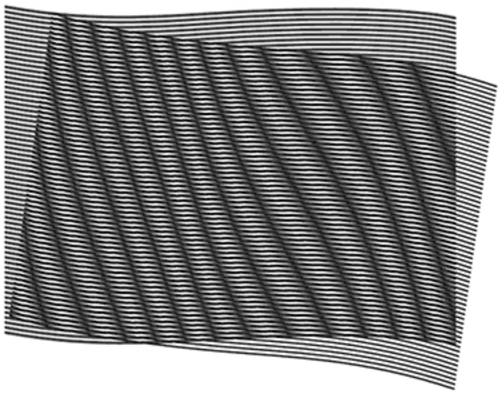
rotation
dilation

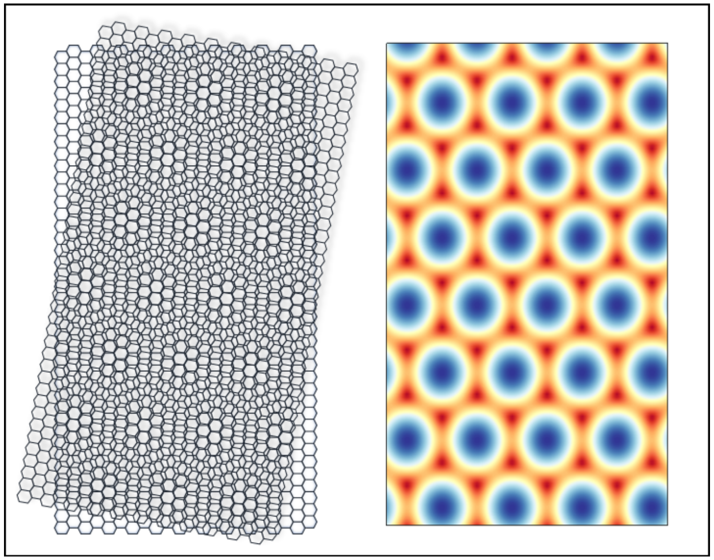
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



$$\psi\left(x', y'\right) = \cos 2\pi x' \cos 2\pi y'$$

$$\chi = \begin{cases} 1, & \psi \geq 0 \\ 0, & \psi < 0 \end{cases}$$





quantum dots
artificial atoms

Tran et al.
Nature 2019

Order to disorder in quasiperiodic composites

Morison, Murphy, Cherkaev, Golden, Comm. Phys. 2022

sea ice inspired - twisted bilayer composites

tunable quasiperiodic composites with exotic properties

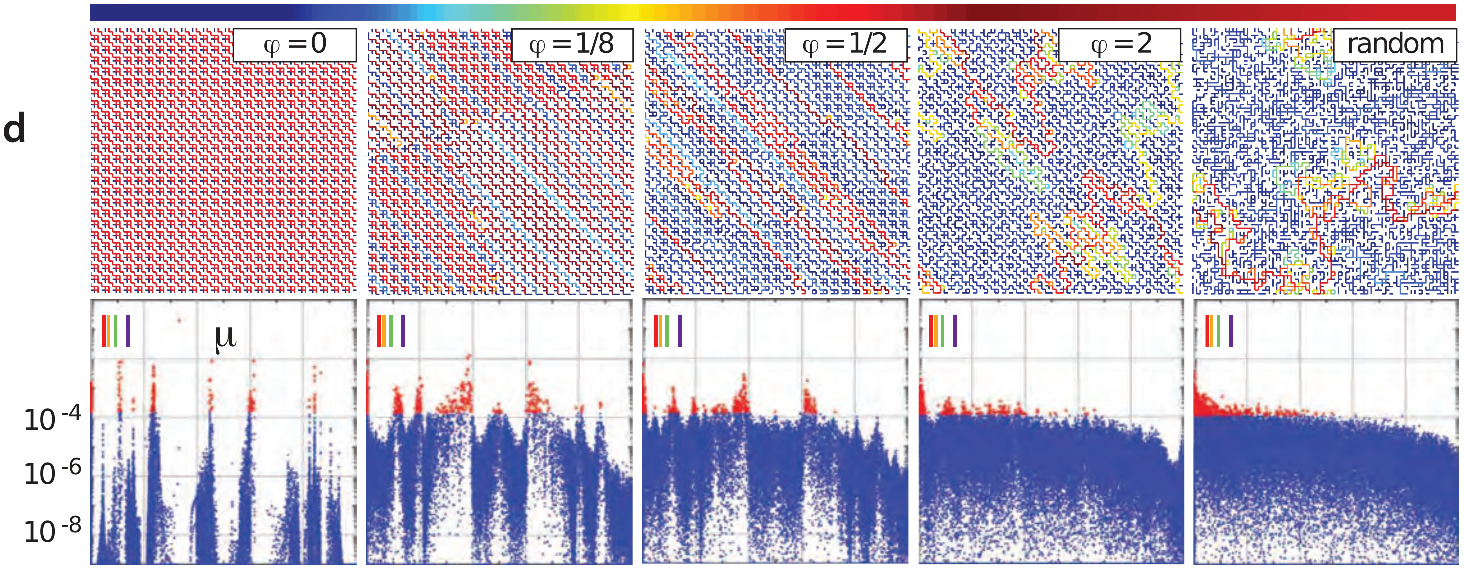
(optical, electrical, thermal) Anderson localization; our Moiré patterned geometries are similar to twisted bilayer graphene

increasing twist angle between two lattices

periodic → quasiperiodic

electric field strength

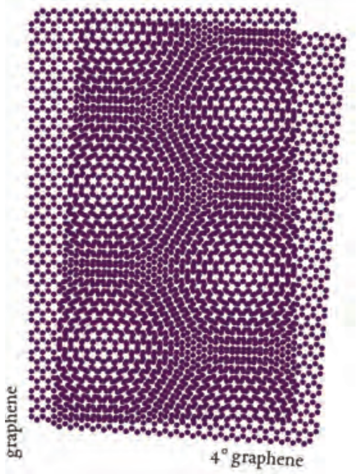
spectral measure

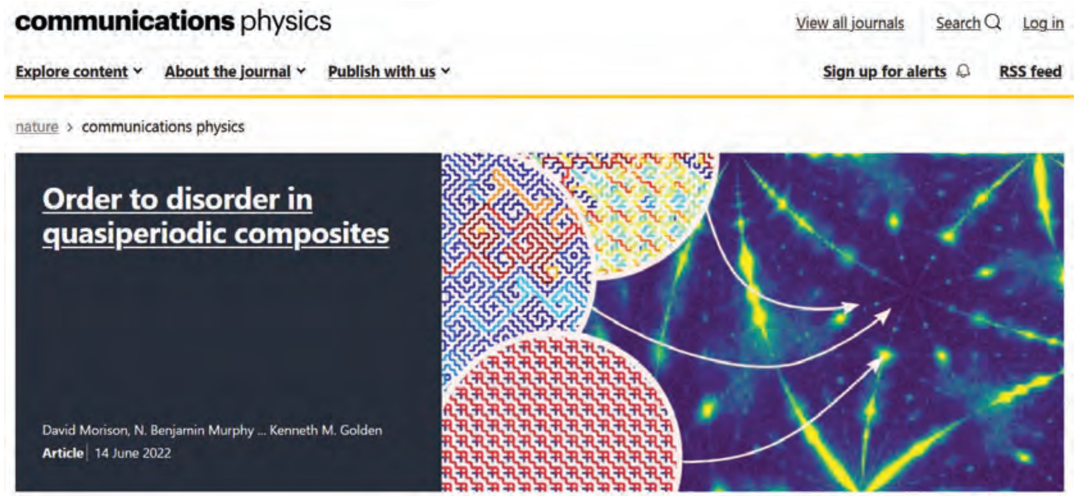


RRN at percolation threshold

twisted bilayer graphene

superconducting magic twist angle





communications physics

Order to disorder in quasiperiodic composites

David Morison, N. Benjamin Murphy ... Kenneth M. Golden

Article | 14 June 2022

constellation of periodic systems in a sea of randomness

Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

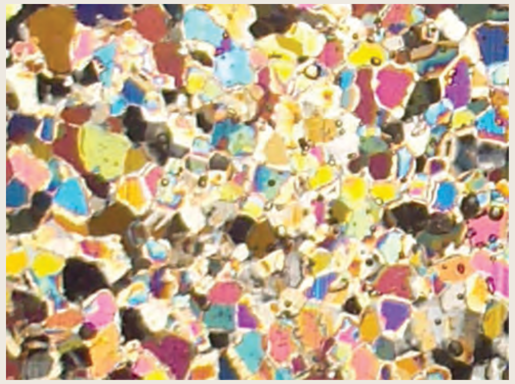
  Milton (1981, 2002), Barabash and Stroud (1999), ...

- **Forward and inverse bounds**

  orientation statistics

- **Applied to sea ice using two-scale homogenization**

- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A 8 Feb 2015

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS
OF SCIENTIFIC
PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

THE
ROYAL
SOCIETY
PUBLISHING

21/01/15 1:47 PM

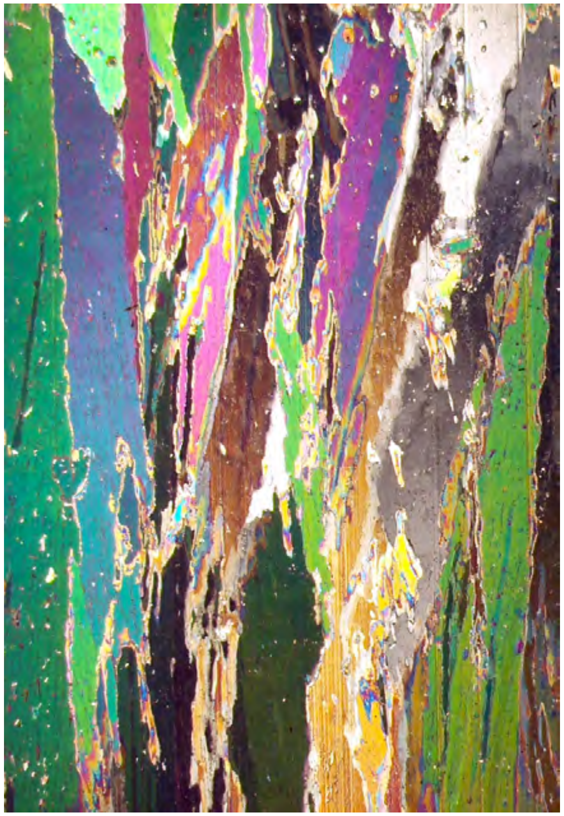
higher threshold for fluid flow in granular sea ice

microscale details impact "mesoscale" processes

nutrient fluxes for microbes
melt pond drainage
snow-ice formation

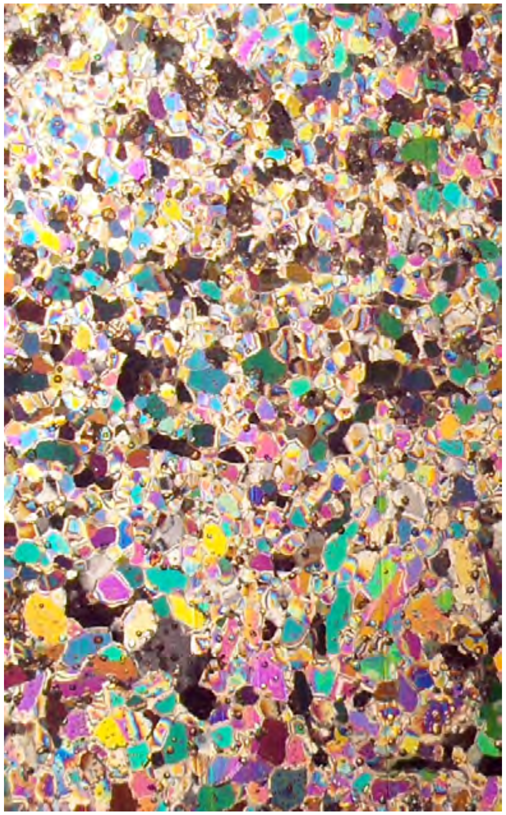
columnar

5%



granular

10%



Golden, Furse, Gully, Lin, Mosier, Sampson, Tison 2025

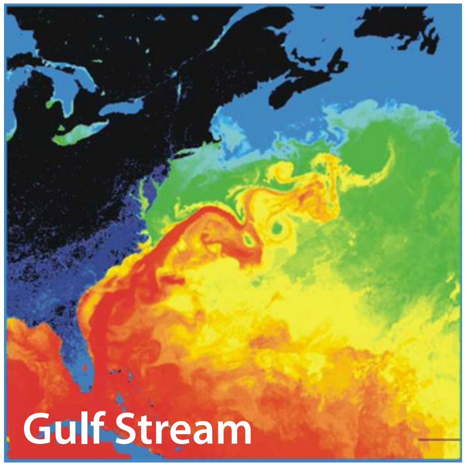
electromagnetically distinguish ice types
inverse homogenization for polycrystals

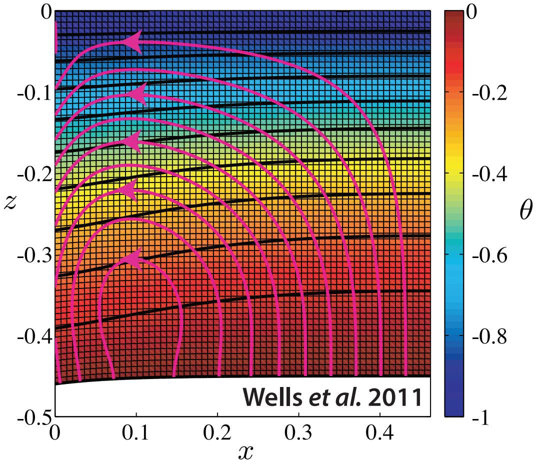
mesoscale

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\Big\downarrow \quad \textit{homogenize}$$

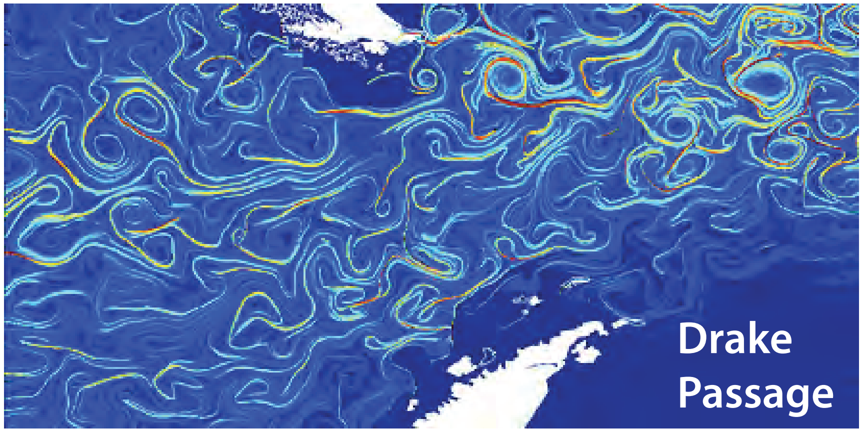
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* ***effective diffusivity***

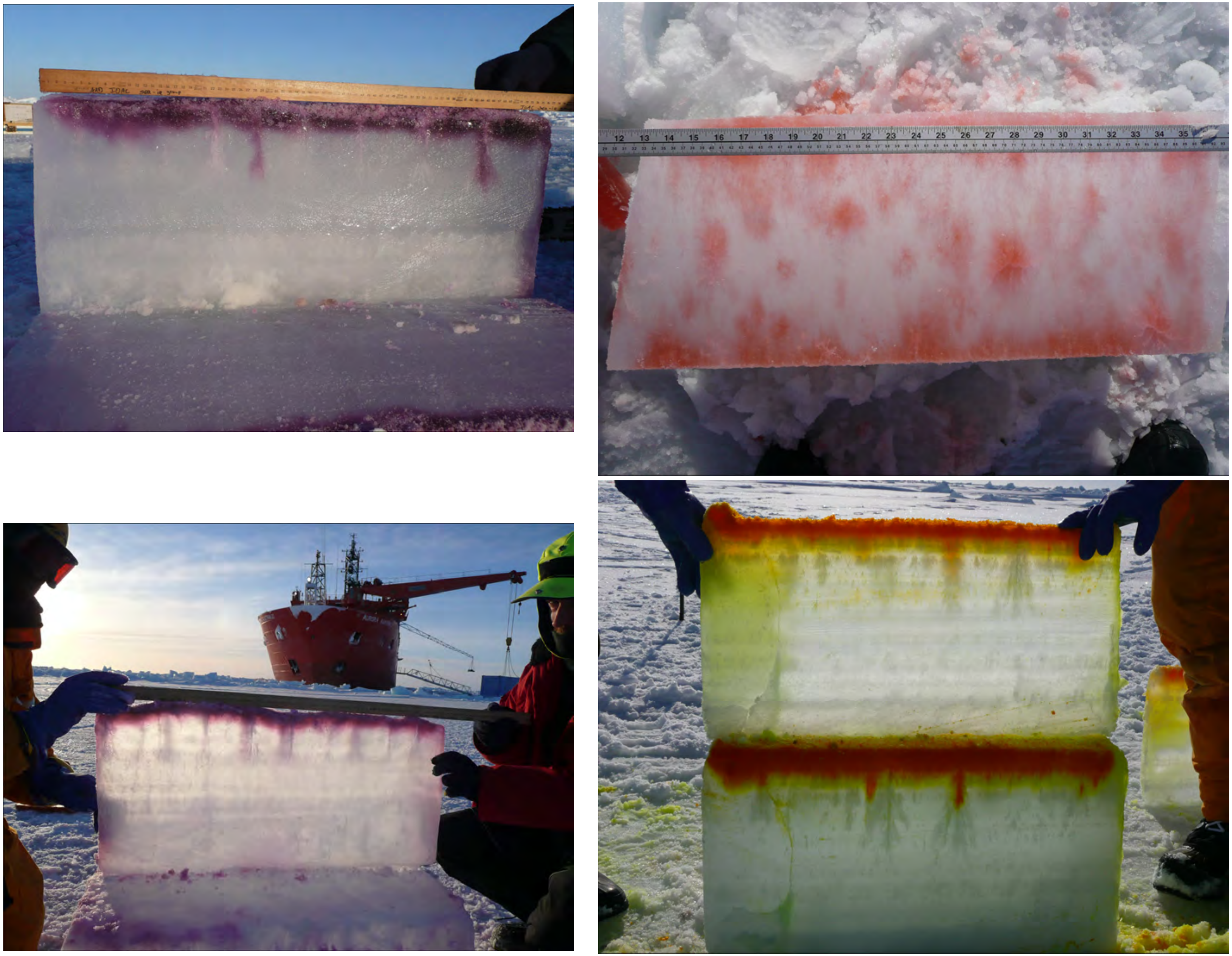
Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020



tracers flowing through inverted sea ice blocks



Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

$$\kappa^* = \kappa\left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}\right), \quad F(\kappa) \; = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}$$

- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H\Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$, Δ is the Laplace operator
- $i\Gamma H\Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ-plane

rigorous framework for numerical computations of
spectral measures and effective diffusivity for model flows

new integral representations, theory of moment calculations

separation of material properties and flow field

PROCEEDINGS
OF THE ROYAL SOCIETY A

MATHEMATICAL, PHYSICAL AND ENGINEERING SCIENCES

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Homogenization for convection-enhanced thermal transport in sea ice

N. Kraitzman, R. Hardenbrook,
H. Dinh, N. B. Murphy, E. Cherkaev,
J. Zhu and K. M. Golden

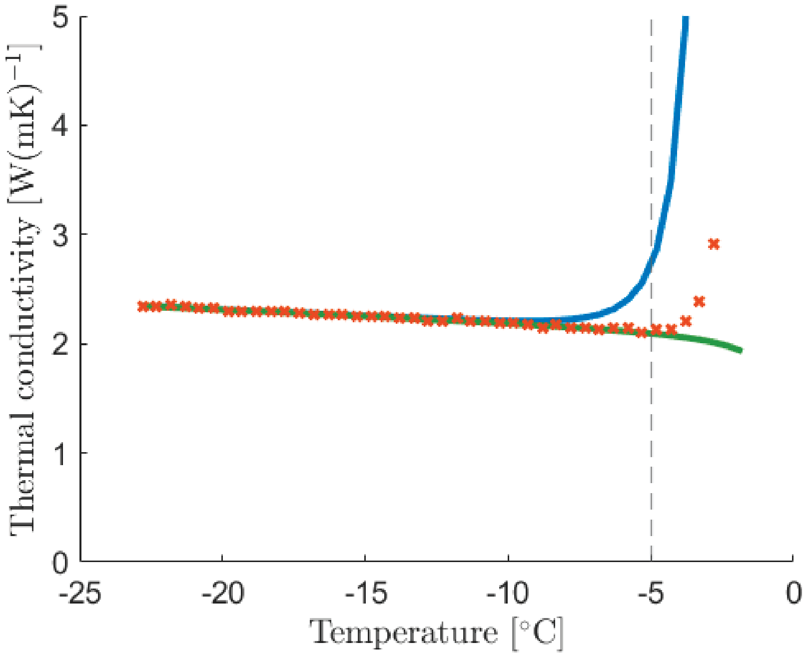
August 2024

First rigorous mathematical theory of thermal conductivity of sea ice with convective fluid flow; captures data.

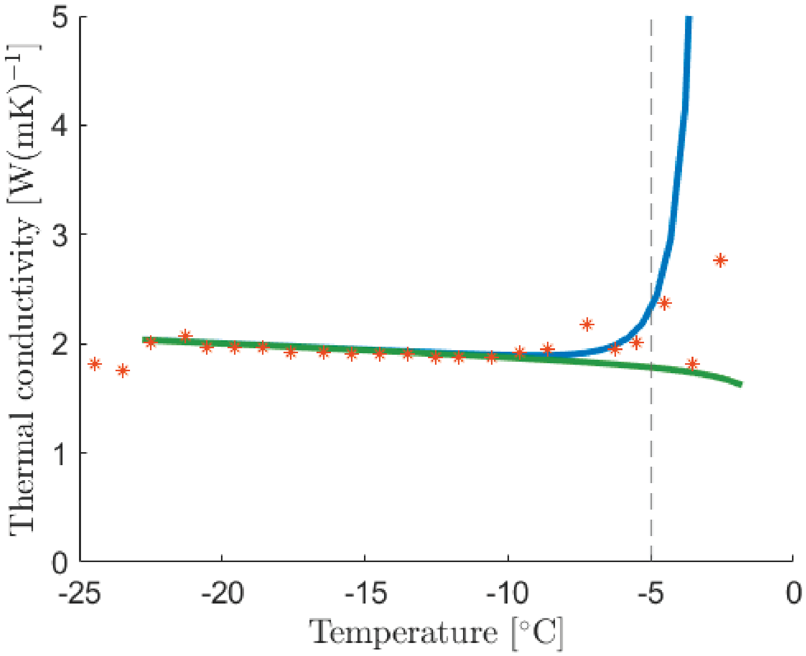
missing in climate models

Bounds on Convection Enhanced Thermal Transport

simulations



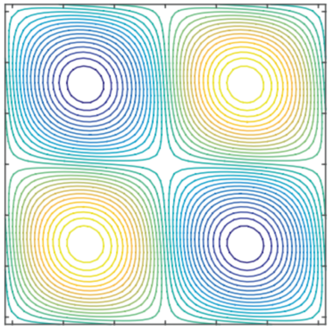
data [Trodahl et al., 2001]



Monte-Carlo simulations of SDE with temperature dependent Péclet number P

strength of advection B = κP/2π

Euler-Maruyama and subsampling methods for SDE



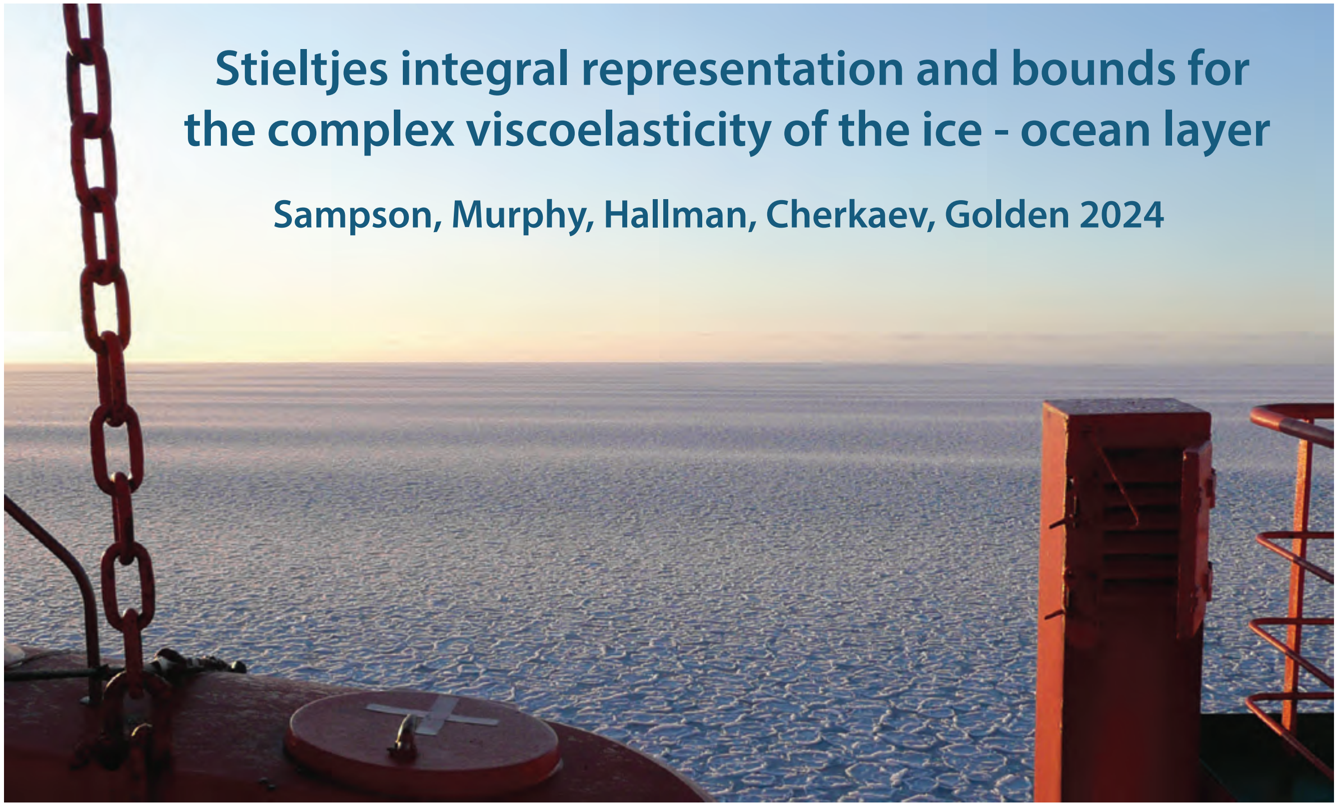
cat's eye flow model for brine convective flow

Rigorous Padé approximant bounds in terms of P using Stieltjes integral + analytic continuation method for the measure

Darcy velocity v = 0.5 [m/s]

Kraitzman, Hardenbrook, Dinh, Murphy, Cherkaev, Zhu, & Golden
***Proc. Royal Soc. A*, 2024**

ocean wave propagation through the sea ice pack



Stieltjes integral representation and bounds for the complex viscoelasticity of the ice - ocean layer

Sampson, Murphy, Hallman, Cherkaev, Golden 2024

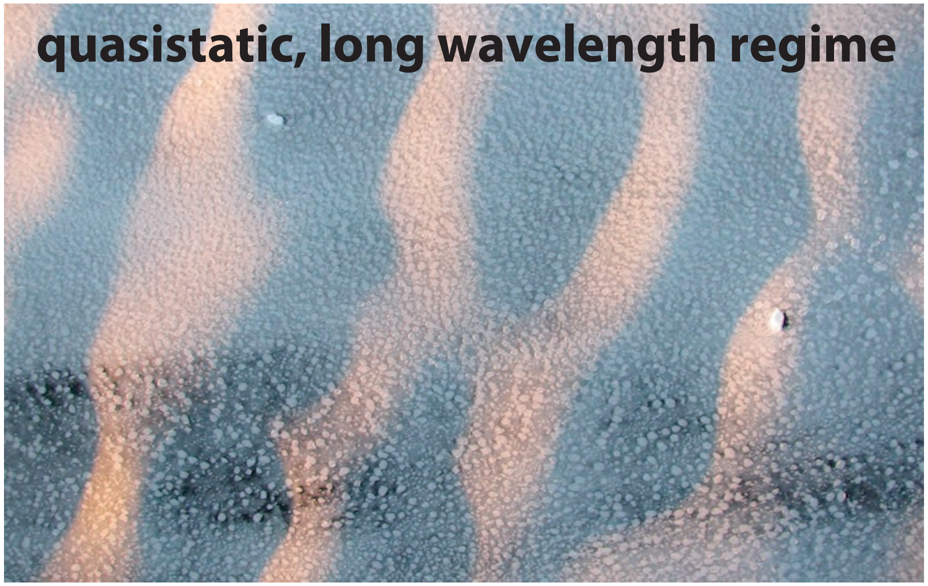
- wave-ice interactions critical to growth and melting processes
- break-up; pancake promotion floe size distribution

effective layer parameter previously fit to wave data

Keller 1998
Mosig, Montiel, Squire 2015
Wang, Shen 2012

Analytic Continuation Method

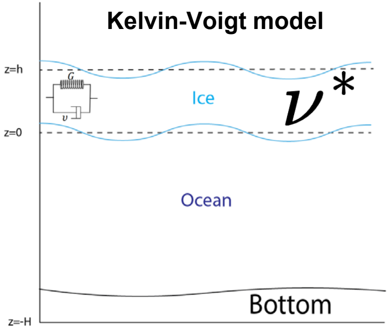
Bergman 1978, Milton 1979
Golden and Papanicolaou 1983
Milton, *Theory of Composites* 2002



quasistatic, long wavelength regime

homogenized parameter depends on sea ice concentration and ice floe geometry

like EM waves



Single effective rheological parameter (Mosig et al. 2015)

$$\nu^* = G - i\omega\rho v$$

Effective complex viscoelasticity

Integral representation

$$\frac{\nu^*}{\nu_2} = ||\epsilon_s^0||^2(1 - F(s))$$

$$F(s) = \int_0^1 \frac{d\mu(\lambda)}{s-\lambda}$$

divergence-free deviatoric stress

$$\nabla \cdot \sigma_s = 0$$

microscale

$$\sigma_s = 2\nu\epsilon_s$$

$$\nu(\vec{x}) = \chi_1\nu_1 + \chi_2\nu_2$$

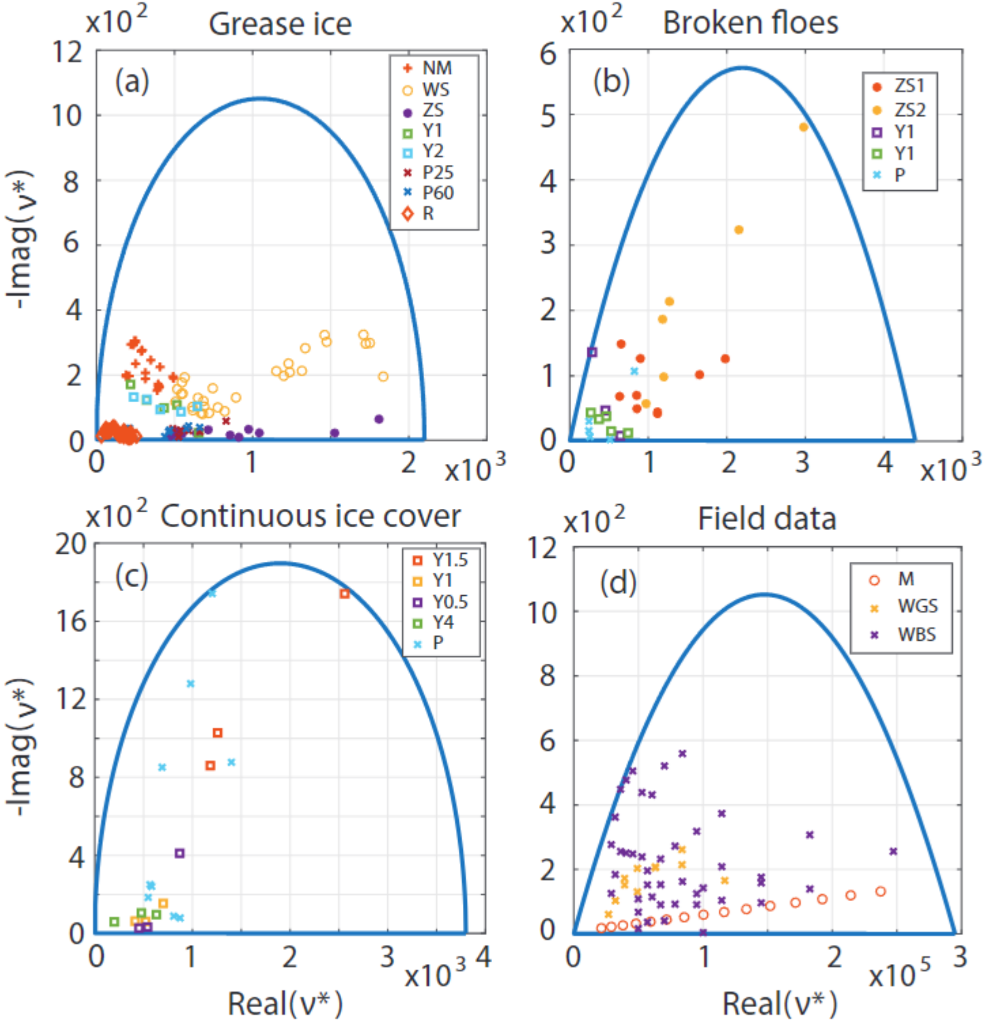
macroscale

$$\langle\sigma_s\rangle = 2\nu^*\epsilon_s^0$$

$$\langle\epsilon_s\rangle = \epsilon_s^0$$

Forward bounds for the effective viscoelasticity are fitted to well known wave-ice datasets, including *Wadhams et al. 1988, Newyear & Martin 1997, Wang & Shen 2010, Meylan et al. 2014*, and several others!

G shear modulus
P pressure
C elasticity
ρ density
v kinematic viscosity
g gravity
u displacement



Waves in sea ice and solid state physics

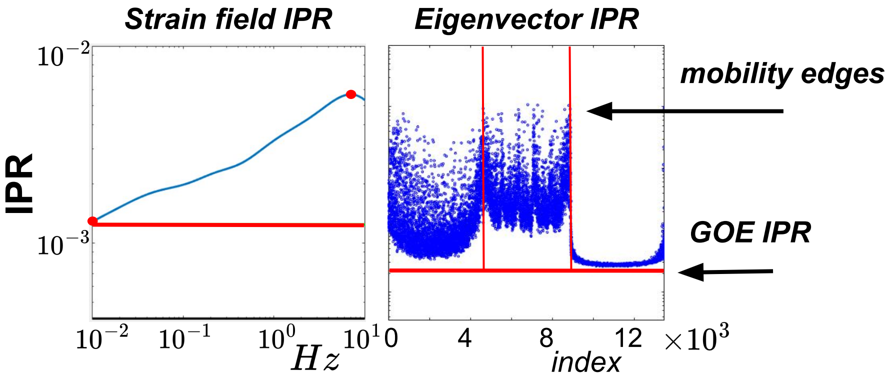
Resolvent representation of the deviatoric strain field

$$\chi\epsilon_s = s\left(sI - \chi\Gamma^S\chi\right)^{-1}\chi\epsilon_s^0$$

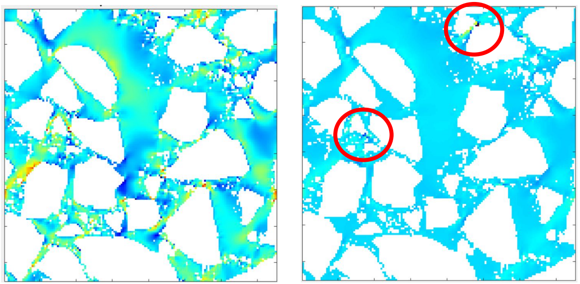
Stieltjes integral representation of effective complex viscoelasticity

$$\frac{\nu^*}{\nu_2} = \int_0^1 \frac{d\mu(\lambda)}{s-\lambda}$$

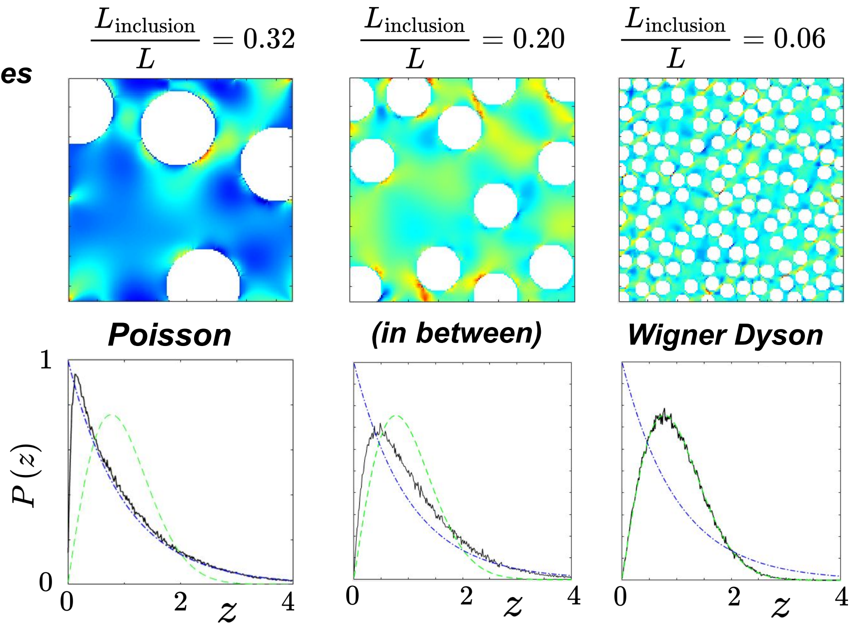
Increasing geometric order →



Extended strain ***Localized strain***



$f = 0.01Hz$ $f = 7.079Hz$



Transition in the eigenvalue spacing distribution of $\chi\Gamma^S\chi$

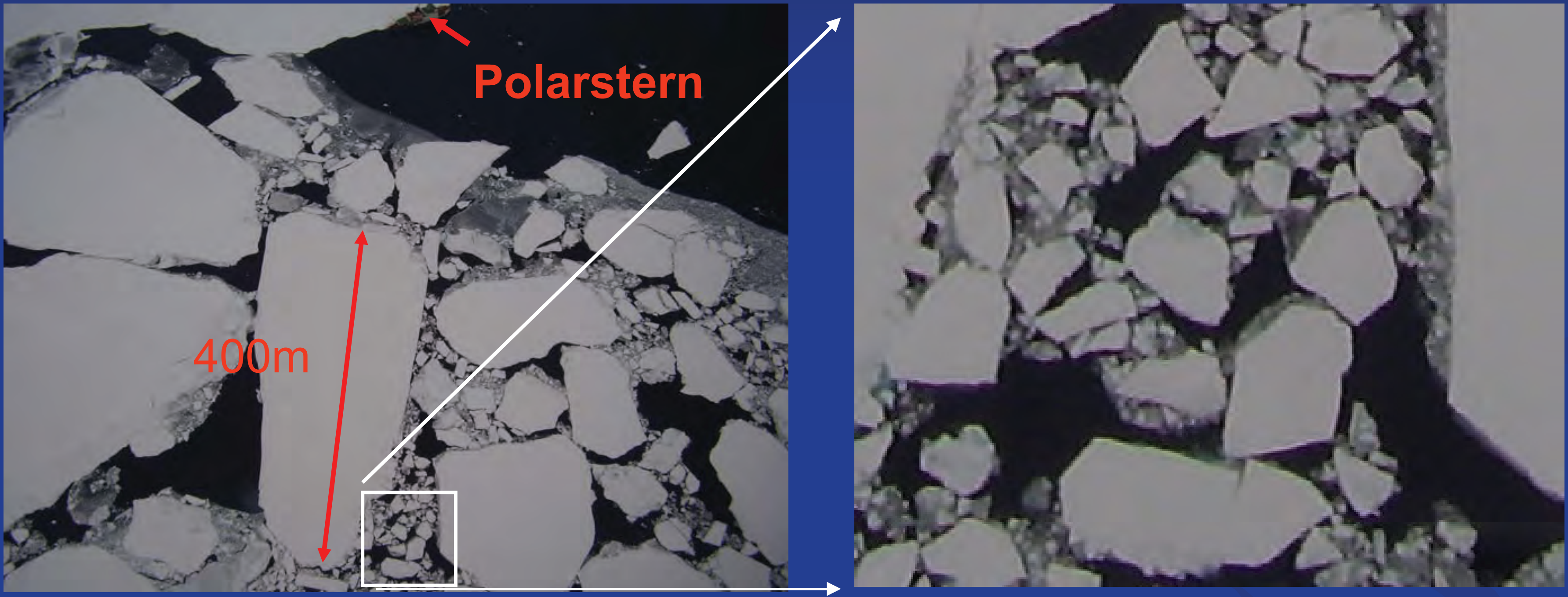
Large inclusions → ***Small inclusions***

"Local to collective deformation transition"

The sea ice pack has fractal structure.

Self-similarity of sea ice floes

Weddell Sea, Antarctica



fractal dimensions of Okhotsk Sea ice pack
smaller scales D~1.2, larger scales D~1.9

fractal dim. *vs.* floe size exponent
Adam Dorsky, Nash Ward, Ken Golden 2024

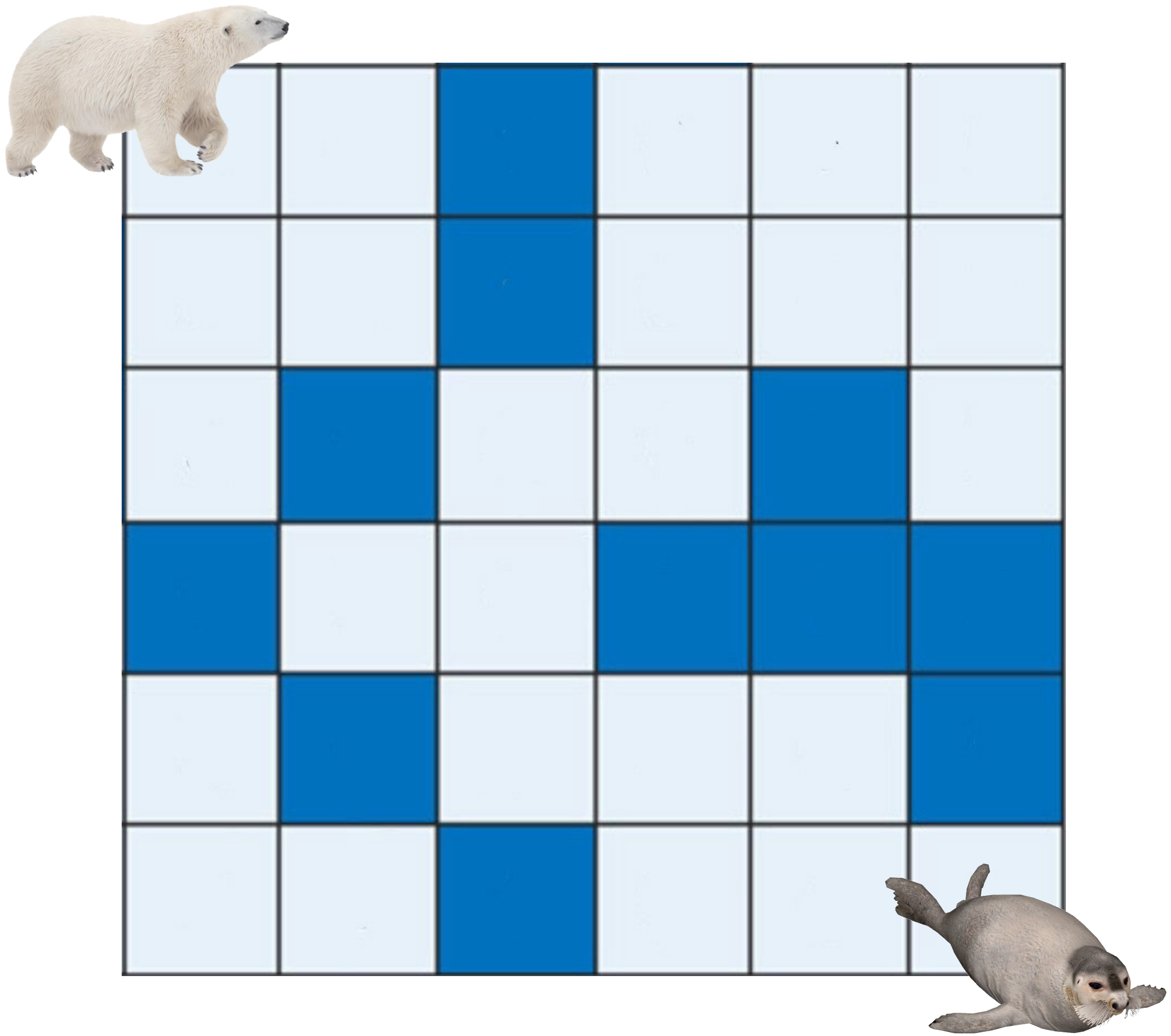
Toyota, *et al. Geophys. Res. Lett.* 2006
Rothrock and Thorndike, *J. Geophys. Res. 1984*

Optimal Movement of a Polar Bear in a Heterogenous Icescape

Nicole Forrester, Jody Reimer, Ken Golden 2024

Polar bears expend 5X more energy swimming than walking on sea ice.

As sea ice is lost, how do polar bears optimize their movement to save energy and survive?

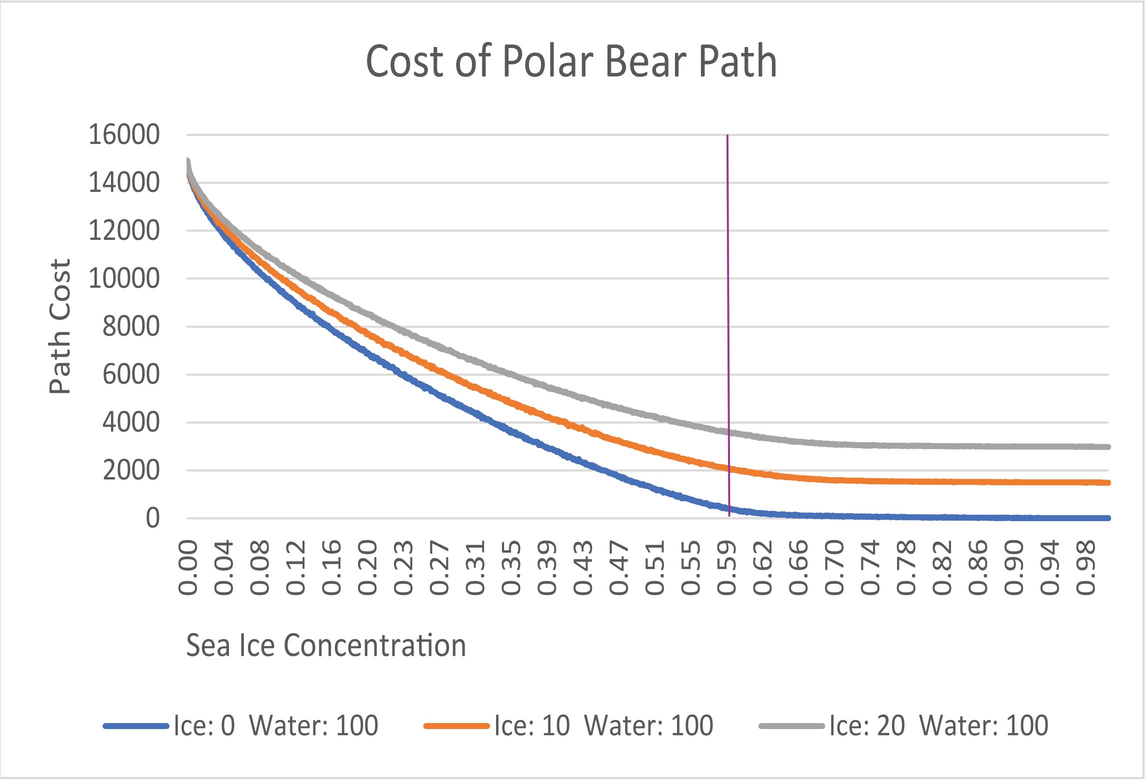


https://pluspng.com/img-png/polar-bear-png-polar-white-bear-png-1000.png
https://zt2downloadlibrary.fandom.com/wiki/Bearded_Seal_(Tamara_Henson)

Polar Bear Percolation

To study the importance of ice connectedness, we exaggerate the data by setting the cost of walking on ice to 0 with the cost of swimming still at 5.

$C(p)$



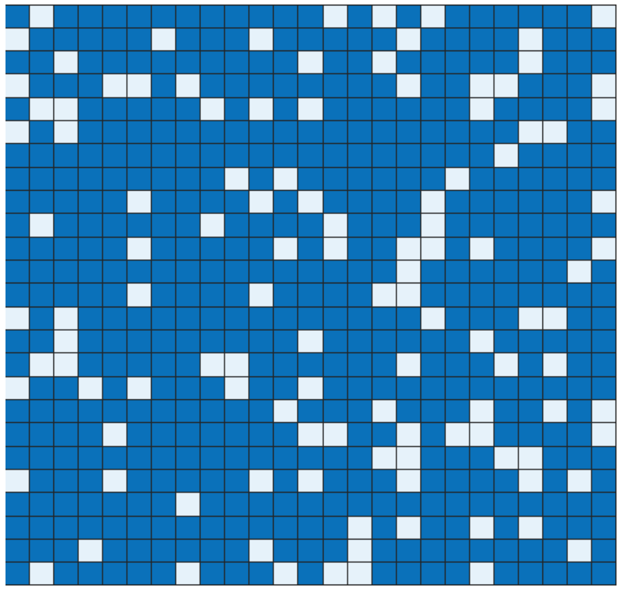
← $h = 0.2$

← $h = 0.1$

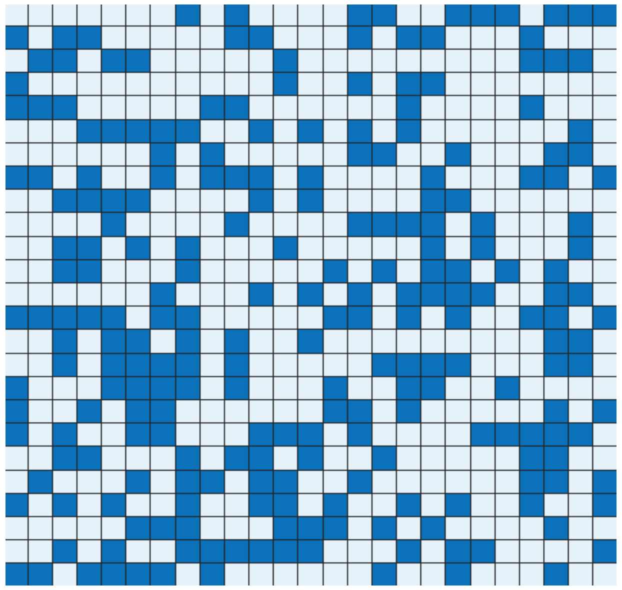
← $h = 0$

$$h = \frac{c_i}{c_w}$$

ratio of local “conductivities”



20% Ice

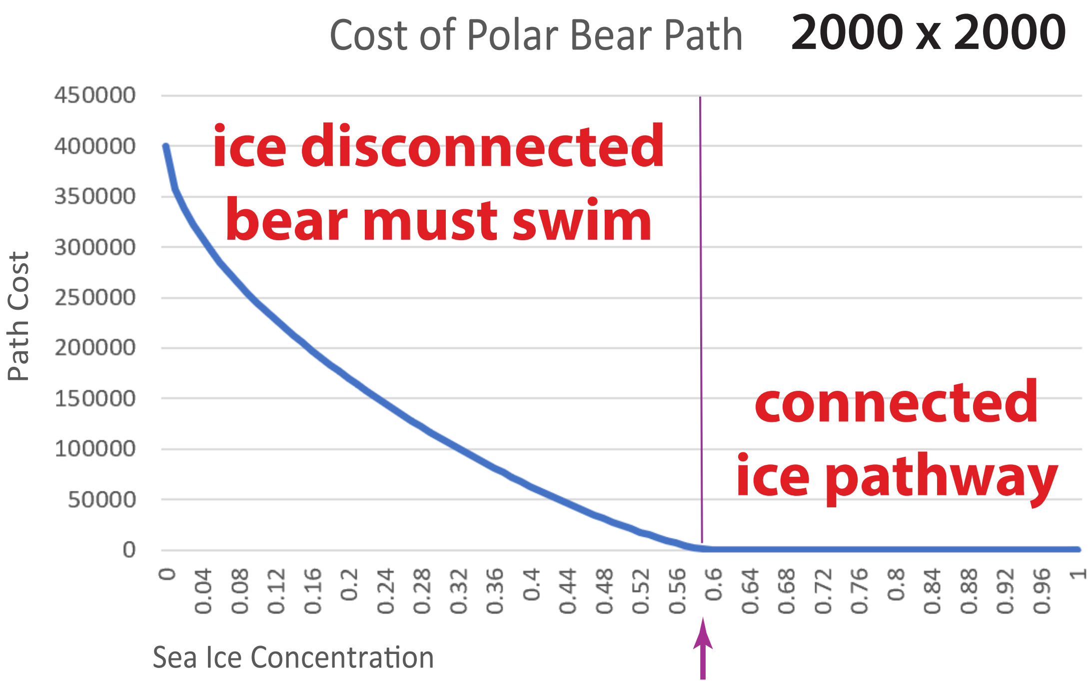


60% Ice

site percolation threshold

$p_c = 0.59 \;\; for \;\; d = 2$

$C(p)$



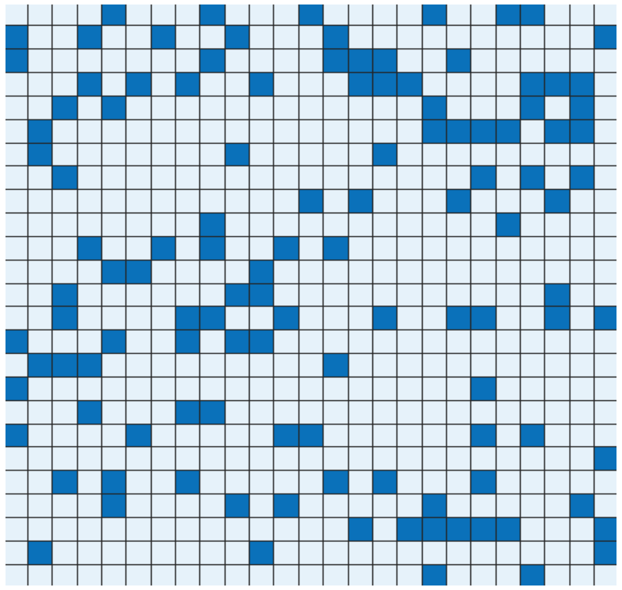
ice disconnected bear must swim

connected ice pathway

← $h = 0$

↑ p_c

Polar Bear Critical Exponent



80% Ice

SEA ICE ALGAE high level of local heterogeneity

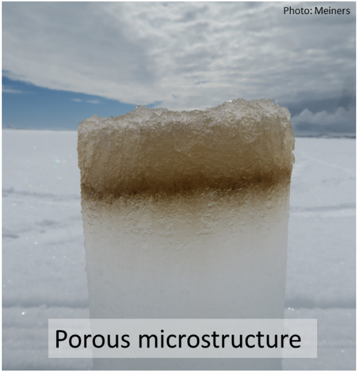


Photo: Meiners

Porous microstructure

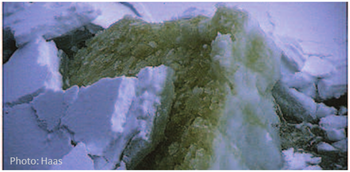
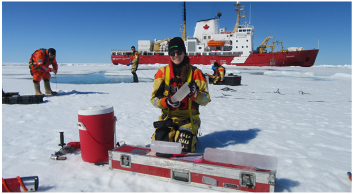


Photo: Haas



Can we improve agreement between algae models and data?

80% of polar bear diet can be traced to ice algae*.

---

* Brown TA, et al. (2018). *PloS one*, 13(1), e0191631

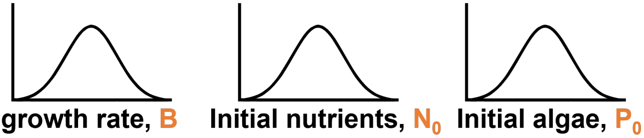
HETEROGENEITY in PARAMETERS & CONDITIONS

At each location within a larger region, consider

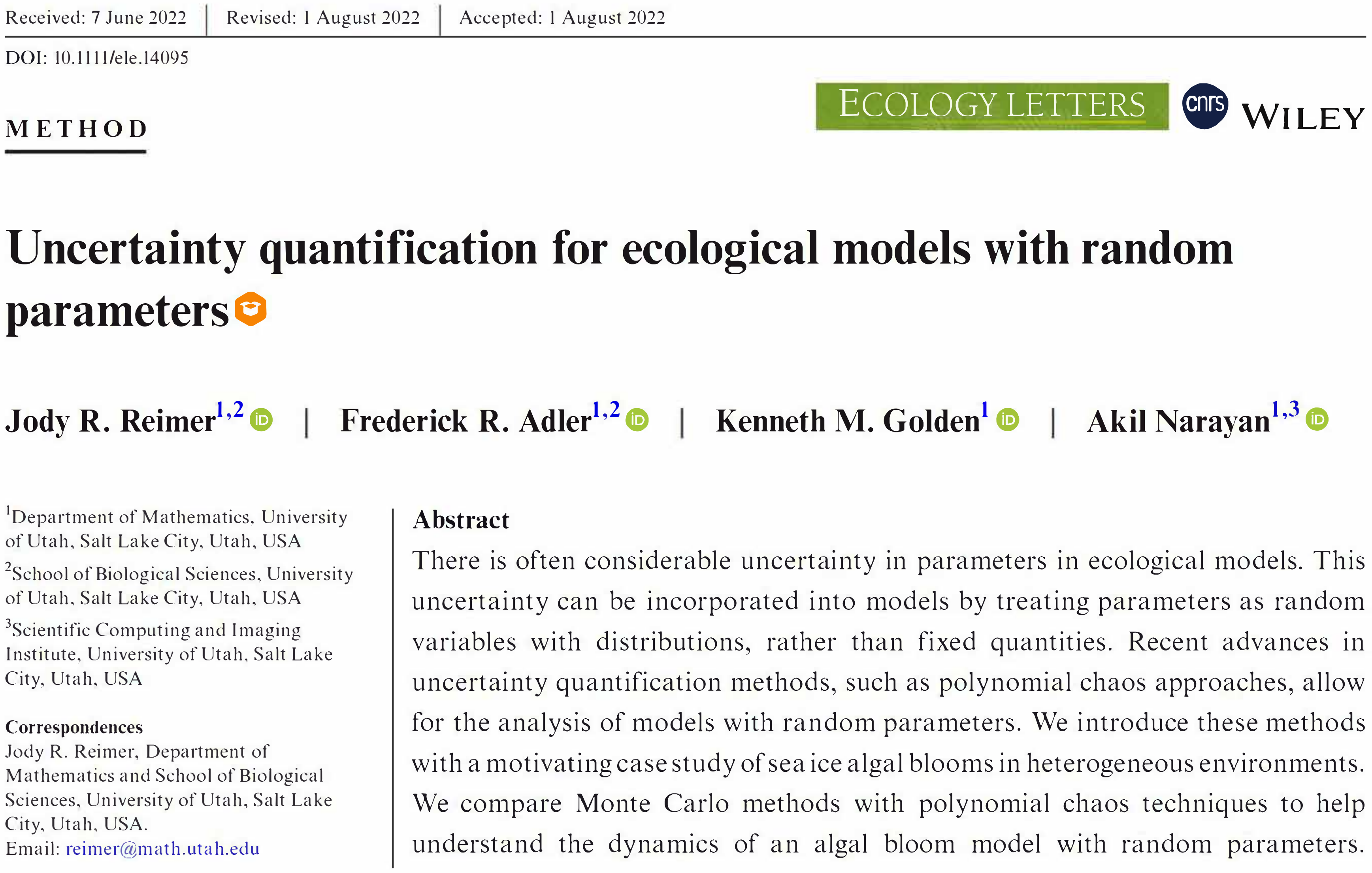
$$\frac{dN}{dt} = \alpha - BNP - \eta N$$
$$\frac{dP}{dt} = \gamma BNP - \delta P$$

treating parameters as random variables

$$N(0) = N_0, \qquad P(0) = P_0$$



But, Monte Carlo for Full Algae Model: 8 hours X 10,000



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DOI: 10.1111/ele.14095

ECOLOGY LETTERS cnrs WILEY

METHOD

Uncertainty quantification for ecological models with random parameters

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Abstract

There is often considerable uncertainty in parameters in ecological models. This uncertainty can be incorporated into models by treating parameters as random variables with distributions, rather than fixed quantities. Recent advances in uncertainty quantification methods, such as polynomial chaos approaches, allow for the analysis of models with random parameters. We introduce these methods with a motivating case study of sea ice algal blooms in heterogeneous environments. We compare Monte Carlo methods with polynomial chaos techniques to help understand the dynamics of an algal bloom model with random parameters.

N-P Model

Introduce polynomial chaos approach to widely used ecological ODE models, but with random parameters.

POLYNOMIAL CHAOS EXPANSIONS

$$N(t; B, P_0, N_0) \approx N_V(t; B, P_0, N_0) \coloneqq \sum_{j=1}^{n} \widetilde{N}_j(t)\phi_j(B, P_0, N_0),$$

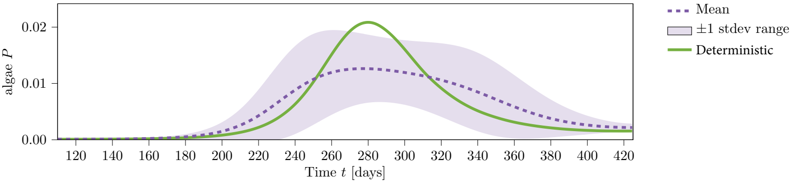
$$P(t; B, P_0, N_0) \approx P_V(t; B, P_0, N_0) \coloneqq \sum_{j=1}^{n} \widetilde{P}_j(t)\phi_j(B, P_0, N_0),$$

where

- $V \coloneqq \mathrm{span}\{\phi_j\}_{j=1}^{n}$
- ϕ_j are orthogonal polynomials that form a basis for V
- $(\widetilde{N}_j, \widetilde{P}_j)$ need to be computed

Xiu, D. (2010). Numerical methods for stochastic computations. Princeton university press.

ECOLOGICAL INSIGHTS



- lower peak bloom intensity
- longer bloom duration
- able to compare variance to data

Inverse Problem: given algal and nutrient data, recover growth rate distribution
Anthony Lee, Jody Reimer, Akil Narayan, Ken Golden 2025

melt pond formation and albedo evolution:

- ***major drivers in polar climate***
- ***key challenge for global climate models***

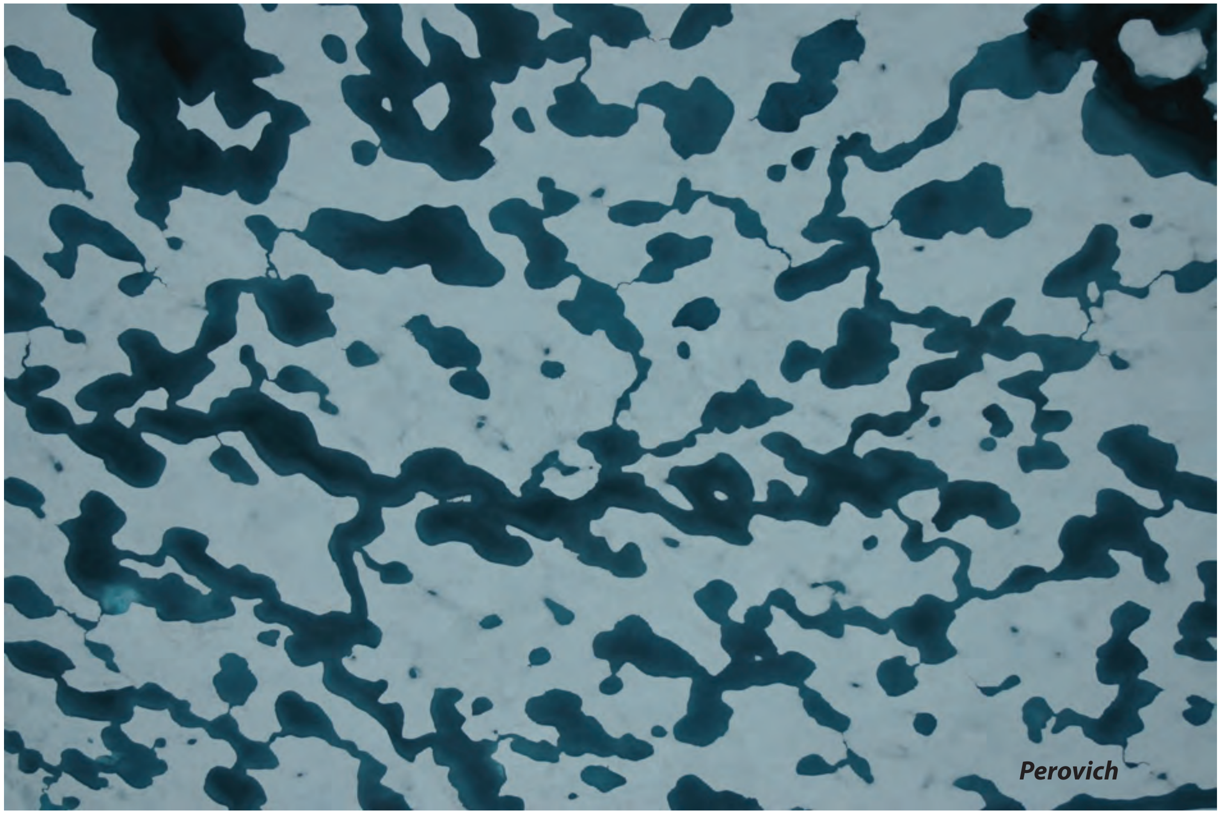
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

Flocco, Feltham, Hunke 2012


Perovich

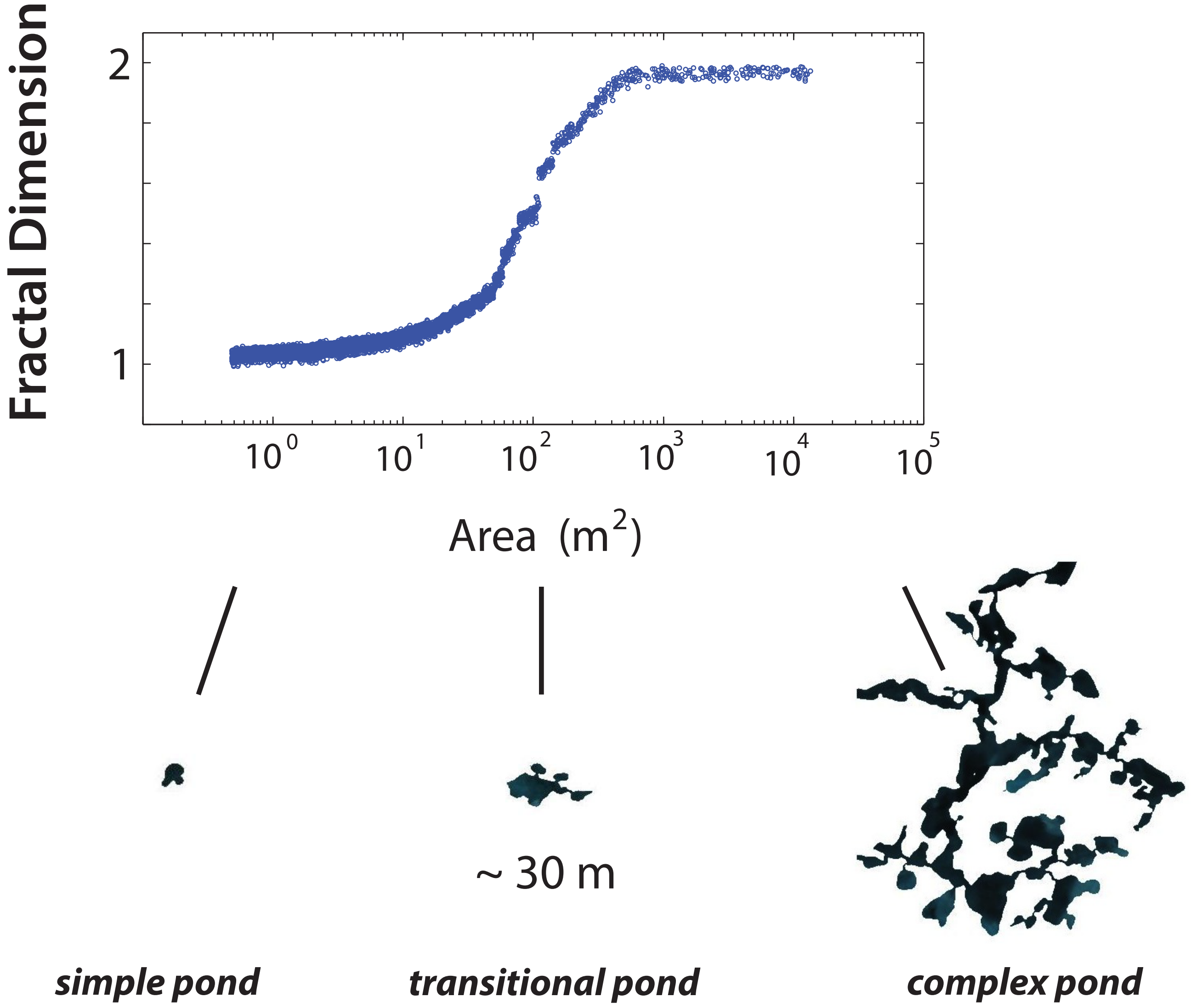
Are there universal features of the evolution similar to phase transitions in statistical physics?

Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

The Cryosphere, 2012

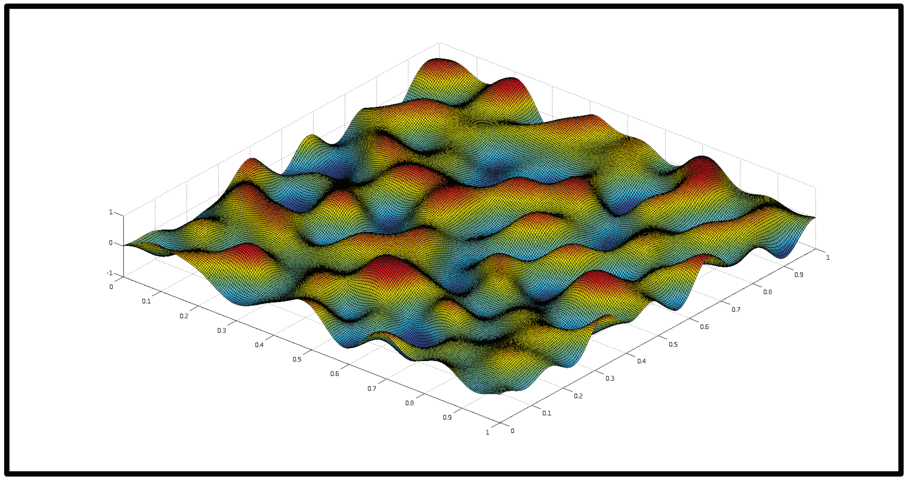
complexity grows with length scale



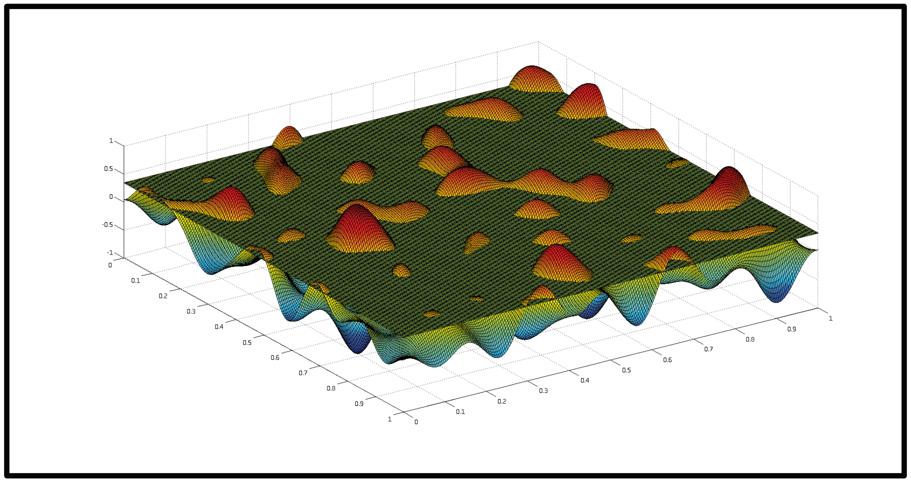
Continuum percolation model for melt pond evolution

level sets of random surfaces

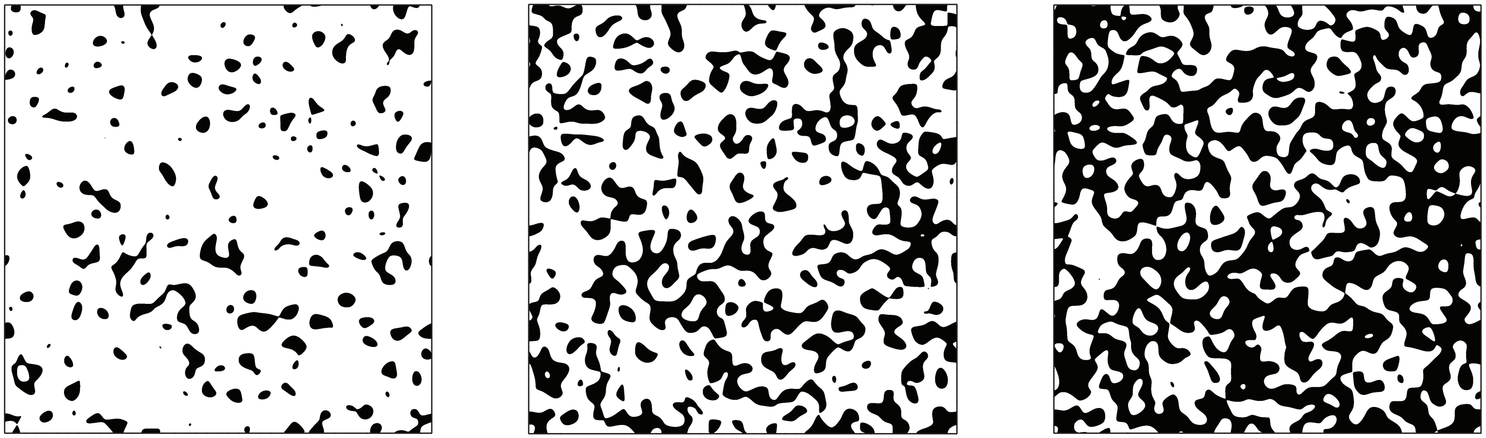
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018



random Fourier series representation of surface topography



intersections of a plane with the surface define melt ponds



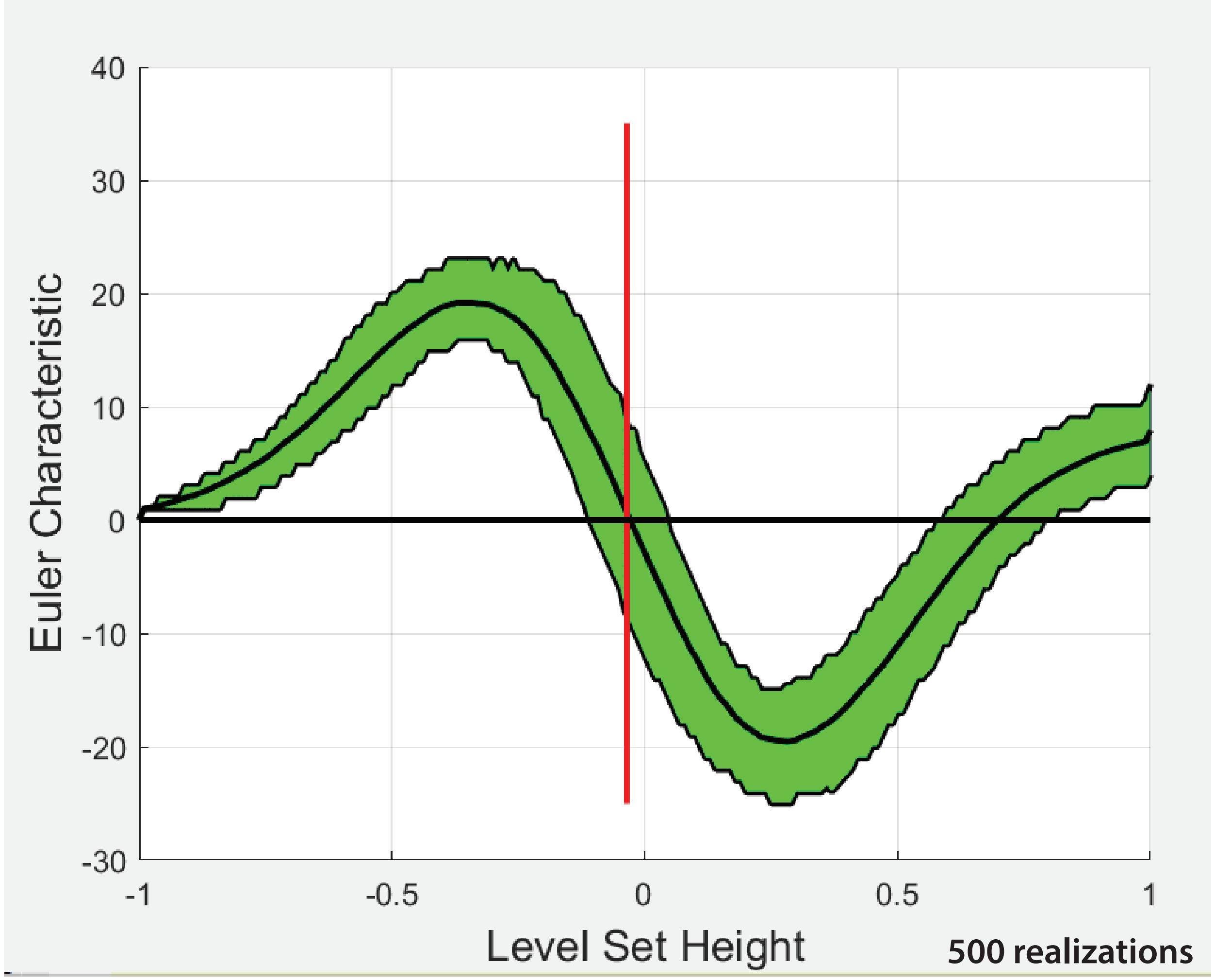
electronic transport in disordered media *diffusion in turbulent plasmas* *Isichenko, Rev. Mod. Phys., 1992*

Topological Data Analysis

Euler characteristic = # maxima + # minima - # saddles

topological invariant

filtration - sequence of nested topological spaces, indexed by water level



Expected
Euler Characteristic Curve (ECC)

tracks the evolution of the EC of the flooded surface as water rises

zero of ECC ~ percolation

percolation on a torus
creates a giant cycle

Bobrowski & Skraba, 2020

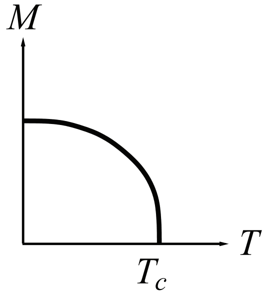
Carlsson, 2009

Vogel, 2002 GRF

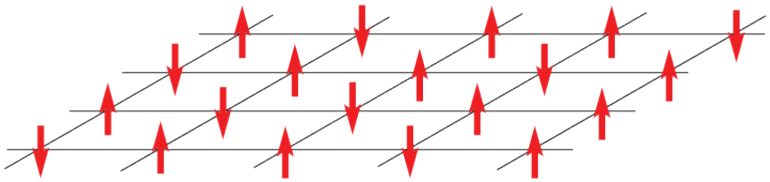
image analysis
porous media
cosmology
brain activity

Physical Review Research (invited, in revision), R. Moore, J. Jones, D. Gollero, R. Hardenbrook, C. Strong, K. M. Golden 2024

Ising Model for a Ferromagnet

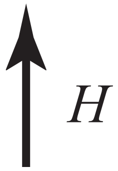


Curie point
critical temperature



$$s_i = \begin{cases} +1 & \text{spin up} \quad \textbf{blue} \\ -1 & \text{spin down} \quad \textbf{white} \end{cases}$$

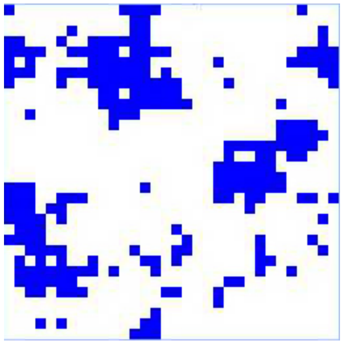
applied
magnetic
field



$$\mathcal{H} = -H\sum_{i} s_i - J\sum_{<i,j>} s_i s_j$$

nearest neighbor Ising Hamiltonian

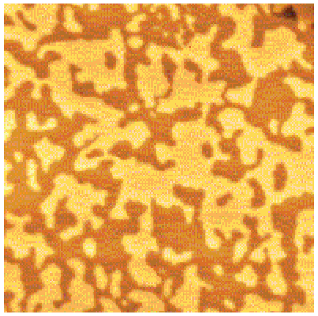
islands of like spins



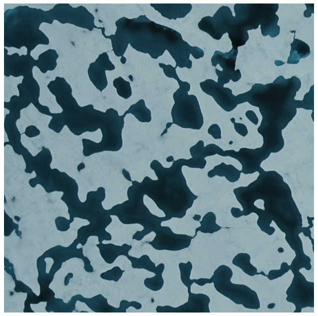
$$M(T,H) = \lim_{N\to\infty}\frac{1}{N}\left\langle \sum_{j} s_j \right\rangle$$

effective magnetization

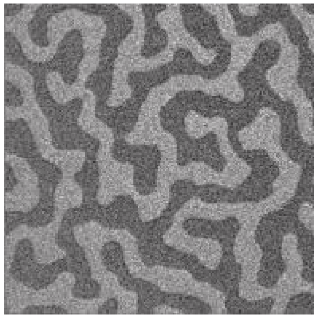
energy is lowered when nearby spins align with each other, forming **magnetic domains**



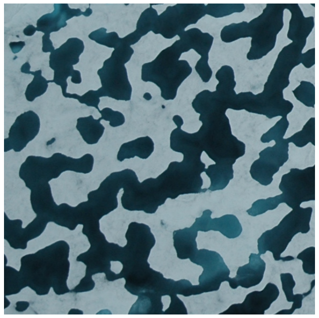
magnetic domains in cobalt



melt ponds (Perovich)



magnetic domains in cobalt-iron-boron



melt ponds (Perovich)

Ising model for ferromagnets ⟶ Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.*, 2019

$$\mathcal{H} = -\sum_{i}^{N} H_i\, s_i - J \sum_{<i,j>}^{N} s_i s_j \qquad s_i = \begin{cases} \uparrow \quad +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow \quad -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

random magnetic field represents snow topography

magnetization M

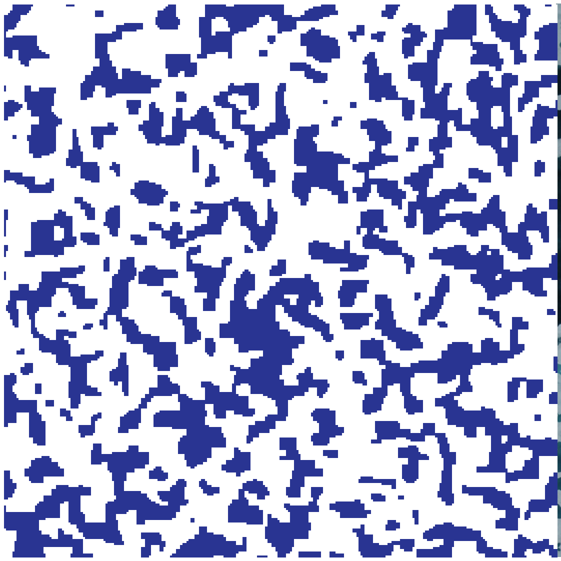
pond area fraction $F = \frac{(M+1)}{2}$

~ albedo

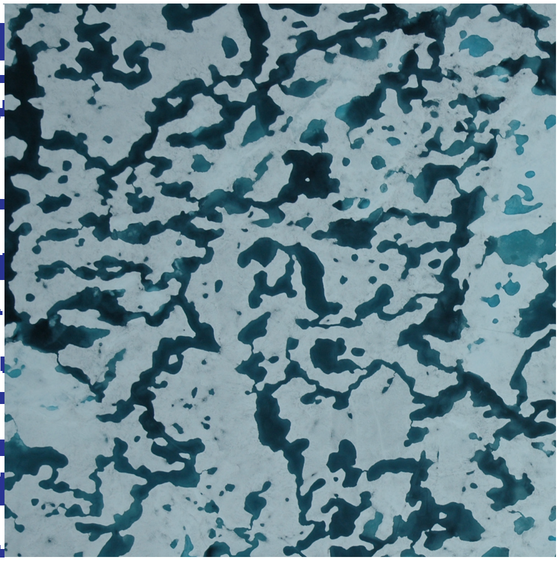
only nearest neighbor patches interact

Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system "flows" toward metastable equilibria.

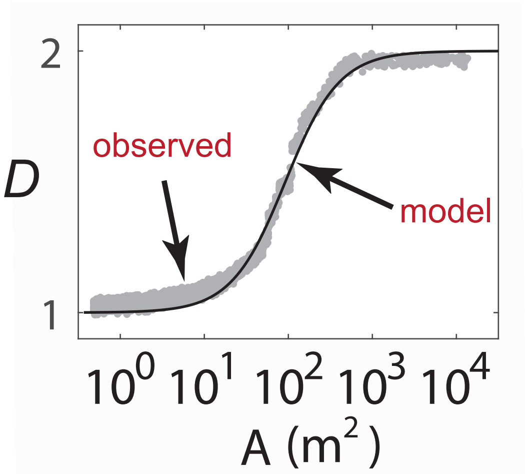
Order from Disorder



Ising model



melt pond photo (Perovich)



pond size distribution exponent

observed -1.5

(Perovich, *et al.* 2002)

model -1.58

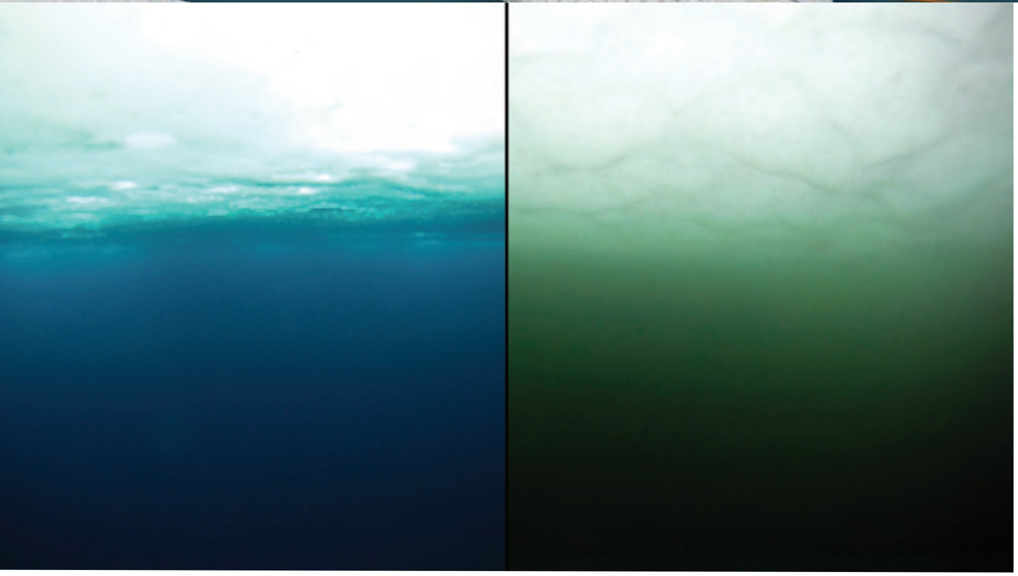
Scientific American
EOS, PhysicsWorld, ...

ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data

Perovich

Melt ponds control transmittance of solar energy through sea ice, impacting upper ocean ecology.

WINDOWS



no bloom bloom

massive under-ice algal bloom

Arrigo et al., *Science* 2012

Have we crossed into a new ecological regime?

The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

Horvat, Rees Jones, Iams, Schroeder,
Flocco, Feltham, ***Science Advances*** 2017

The effect of melt pond geometry on the distribution of solar energy under first year sea ice

Horvat, Flocco, Rees Jones, Roach, Golden
Geophys. Res. Lett. 2019

(2015 AMS MRC)

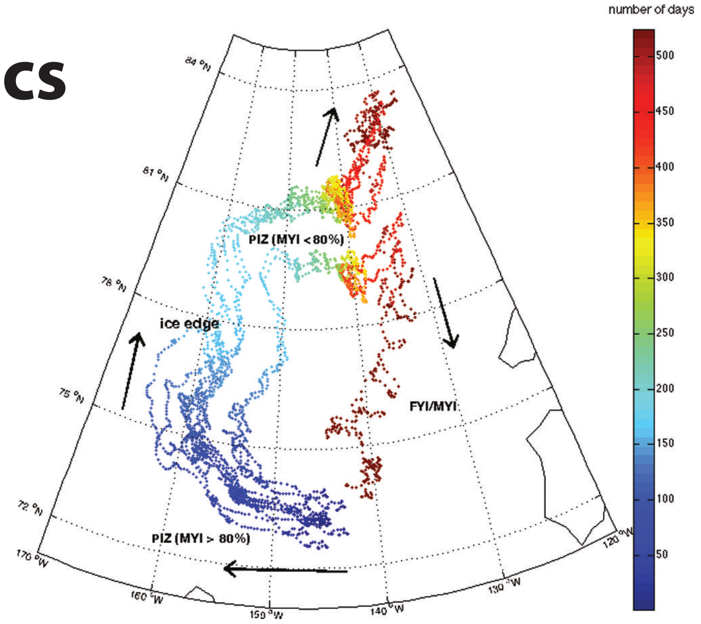
macroscale

Anomalous diffusion in sea ice dynamics

Ice floe diffusion in winds and currents

observations from GPS data

Lukovich, Hutchings, Barber, *Ann. Glac.* 2015



Floe scale model of advection diffusion

Huy Dinh, Tyler Evans, Kaeden George, Ben Murphy, Elena Cherkaev, Ken Golden 2025

$$\left\langle \left|\mathbf{x}(t)-\mathbf{x}(0)-\langle\mathbf{x}(t)-\mathbf{x}(0)\rangle\right|^2 \right\rangle \sim t^\alpha$$

α = Hurst exponent

diffusive $\alpha = 1$

sub-diffusive $\alpha < 1$

super-diffusive $\alpha > 1$

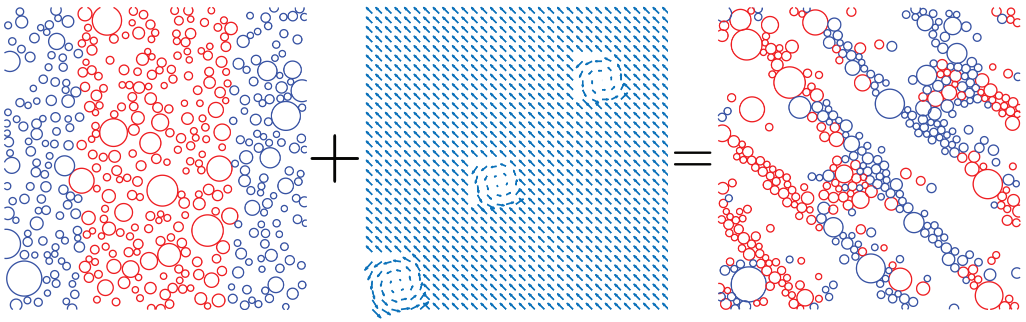
Model Approximations

Power Law Size Distribution: $N(D) \sim D^{-k}$

D. A. Rothrock and A. S. Thorndike Journal of Geophysical Research 1984

Floe-Floe Interactions: Linear Elastic Collisions

Advective Forcing: Passive, Linear Drag Law

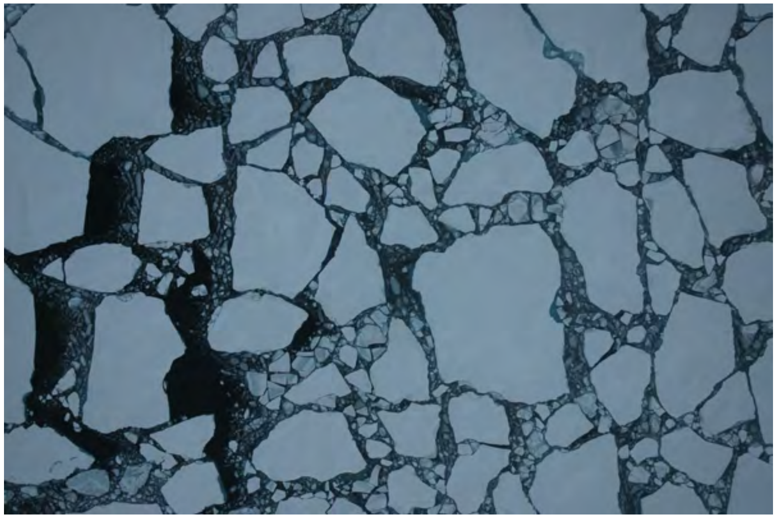


$\psi = 0.3$

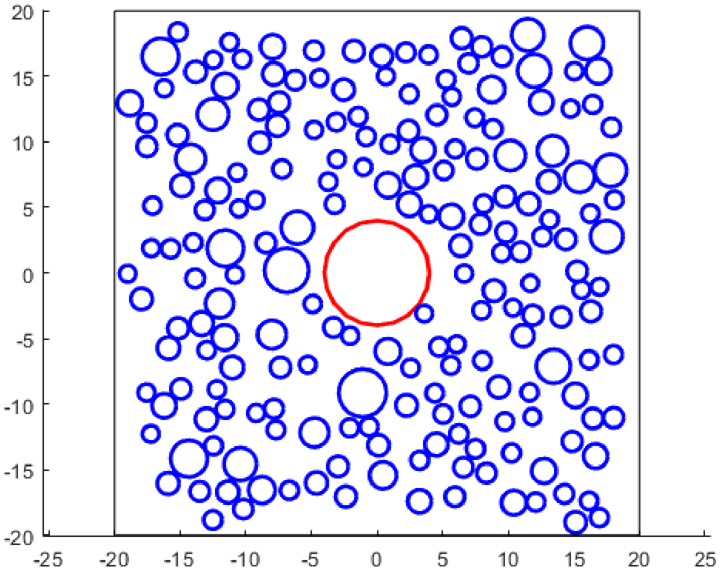
Vorticity Dominated Forcing

Fractional PDE

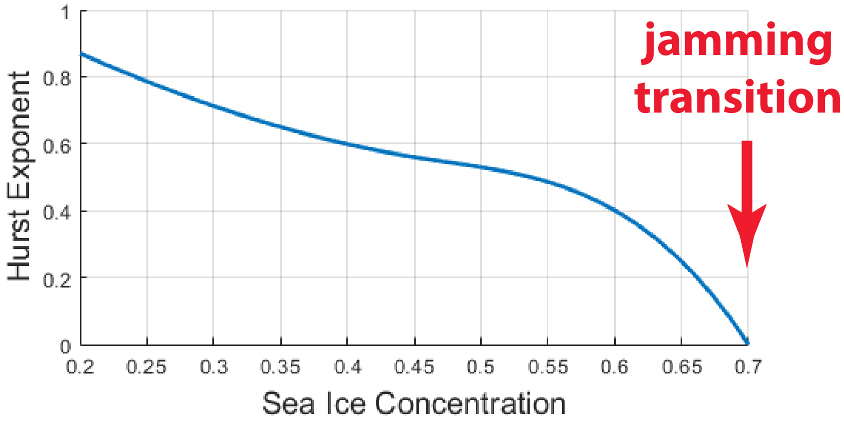
Arctic sea ice pack with tagged particle



Einstein's pollen grain

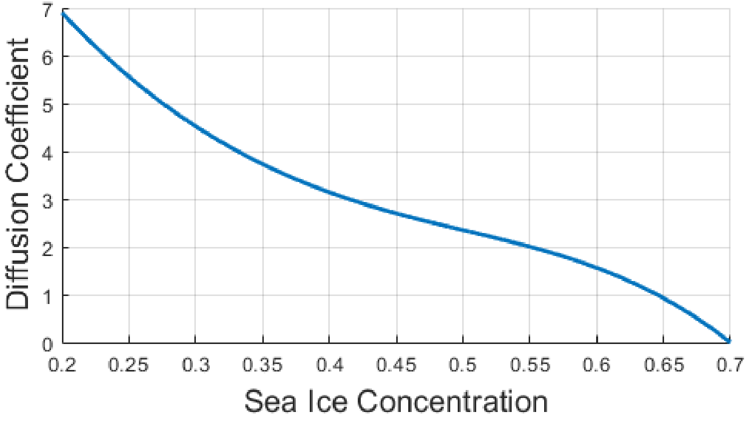


Hurst exponent



jamming transition

diffusion coefficient



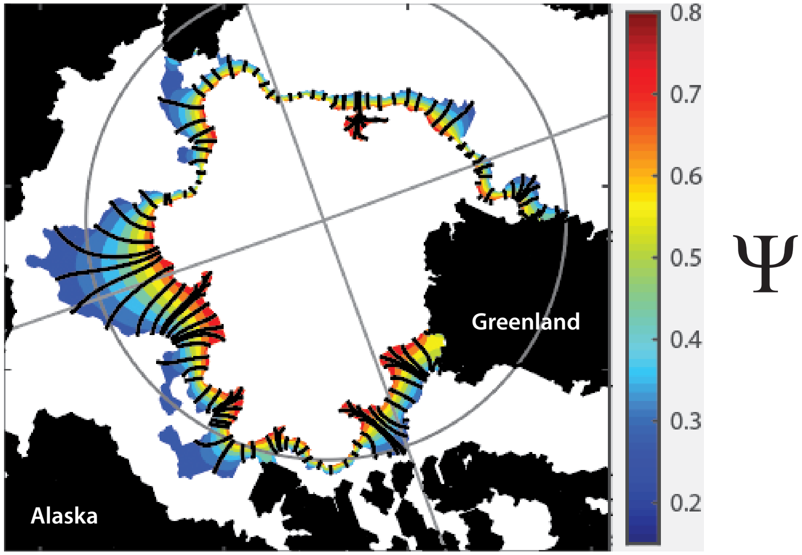
Marginal Ice Zone (MIZ)

transitional region between

dense interior pack $\Psi > 0.8$

sparse outer fringes $\Psi < 0.15$

- biologically active region
- intense wave-ice interactions
- strong air-ice-ocean exchanges



MIZ WIDTH

fundamental length scale of ecological and climate dynamics

streamlines of harmonic Ψ

MIZ width definition from medical imaging

Strong, *Climate Dynamics* 2012
Strong & Rigor, *GRL* 2013

39% widening
1979 - 2012

Strong, Foster, Cherkaev, Eisenman, Golden, *J. Atmos. Oceanic Tech.* 2017

Strong & Golden, *SIAM News* 2017

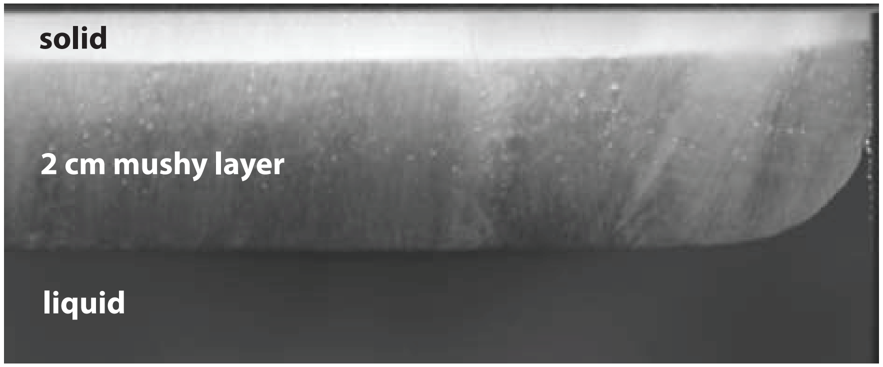
Multiscale mushy layer model for marginal ice zone dynamics

Strong, Cherkaev, Golden *Scientific Reports* 2024

MIZ - transitional region between dense pack ice and open ocean

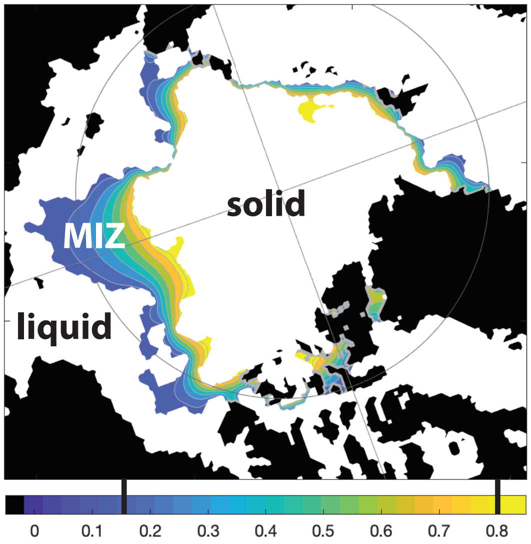
OBJECTIVE: model & predict dramatic annual cycle
impacts climate dynamics, polar ecology, human activities

mushy layer physics in the lab



NaCl-H2O in lab
(Peppin et al., 2007, J. Fluid Mech.)

Arctic MIZ as a mushy layer



sea ice concentration ψ

MIZ as a moving phase transition region

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + S$$

$$S = [\rho(c_l - c_s)T + \rho L] \frac{\partial \psi}{\partial t}$$

$$\psi = 1 - \left(\frac{T - T_s}{T_l - T_s} \right)^{\alpha}$$

$$k_x = \left(\frac{\psi}{k_s} + \frac{1 - \psi}{k_l} \right)^{-1}$$

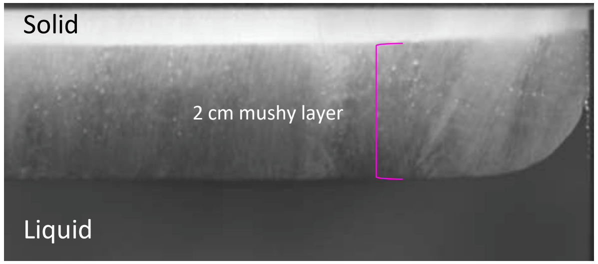
$$k_z = \psi k_s + (1 - \psi) k_l$$

homogenization

ρ effective density
T temperature
c specific heat
L latent heat of fusion

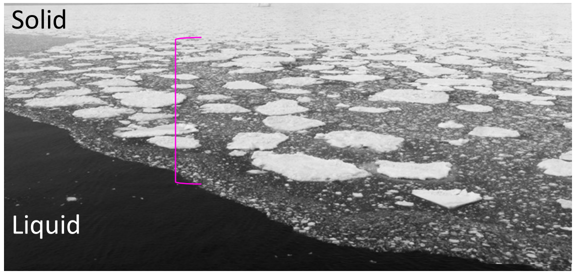
S models nonlinear phase change
ψ sea ice concentration
k effective diffusivity
l liquid, s solid

Classical small-scale application



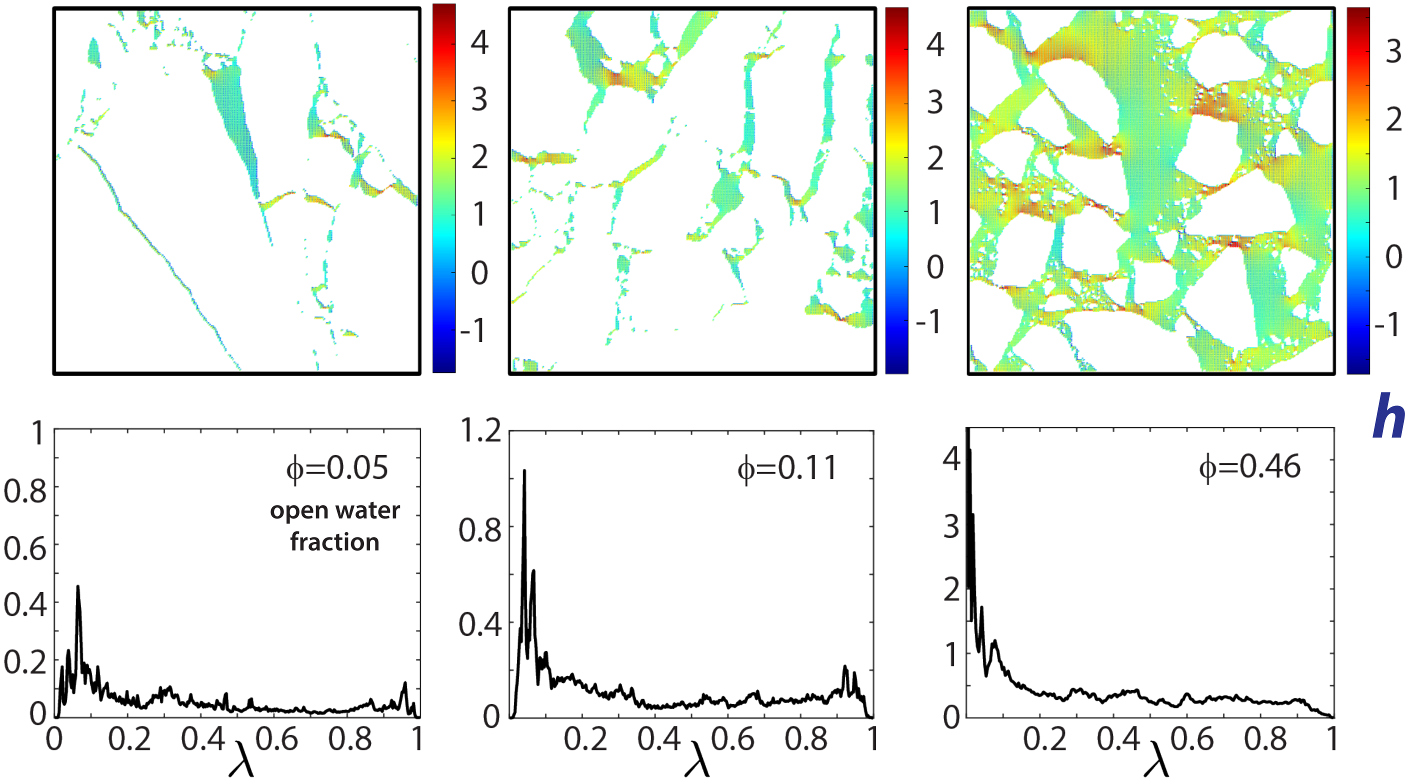
NaCl-H_2O in lab
(Peppin et al., 2007;, J. Fluid Mech.)

Macroscale application



- Develop multiscale PDE model for simulating phase transition fronts to predict MIZ seasonal cycles and decadal trends
- Model simulates MIZ as a large-scale mushy layer with effective thermal conductivity derived from physics of composite materials

thermal flow field through the ice cover: multiscale granular composite



homogenize

spectral measures for 2D horizontal thermal conductivity

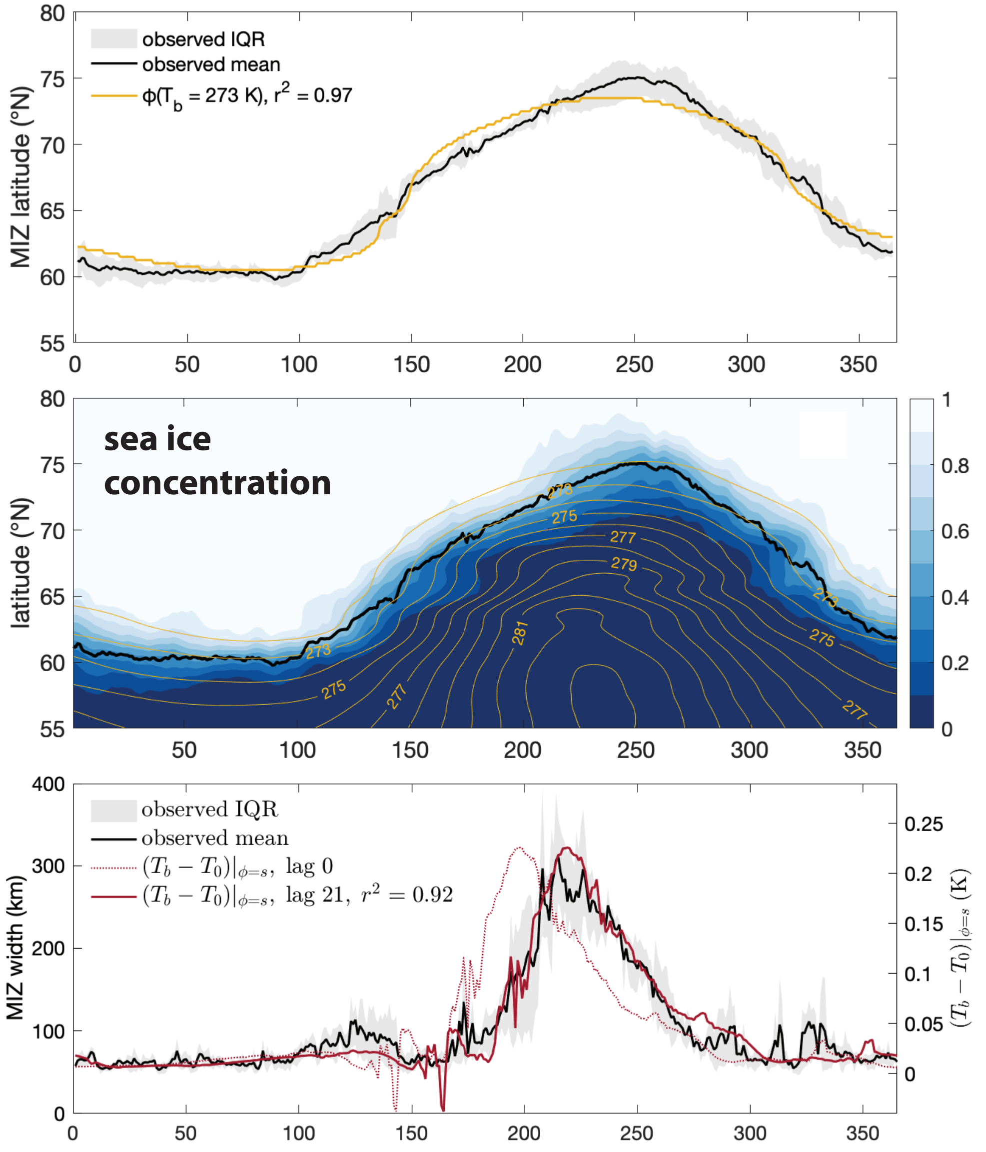
homogenized thermal conductivity is a key parameter in MIZ mushy layer model

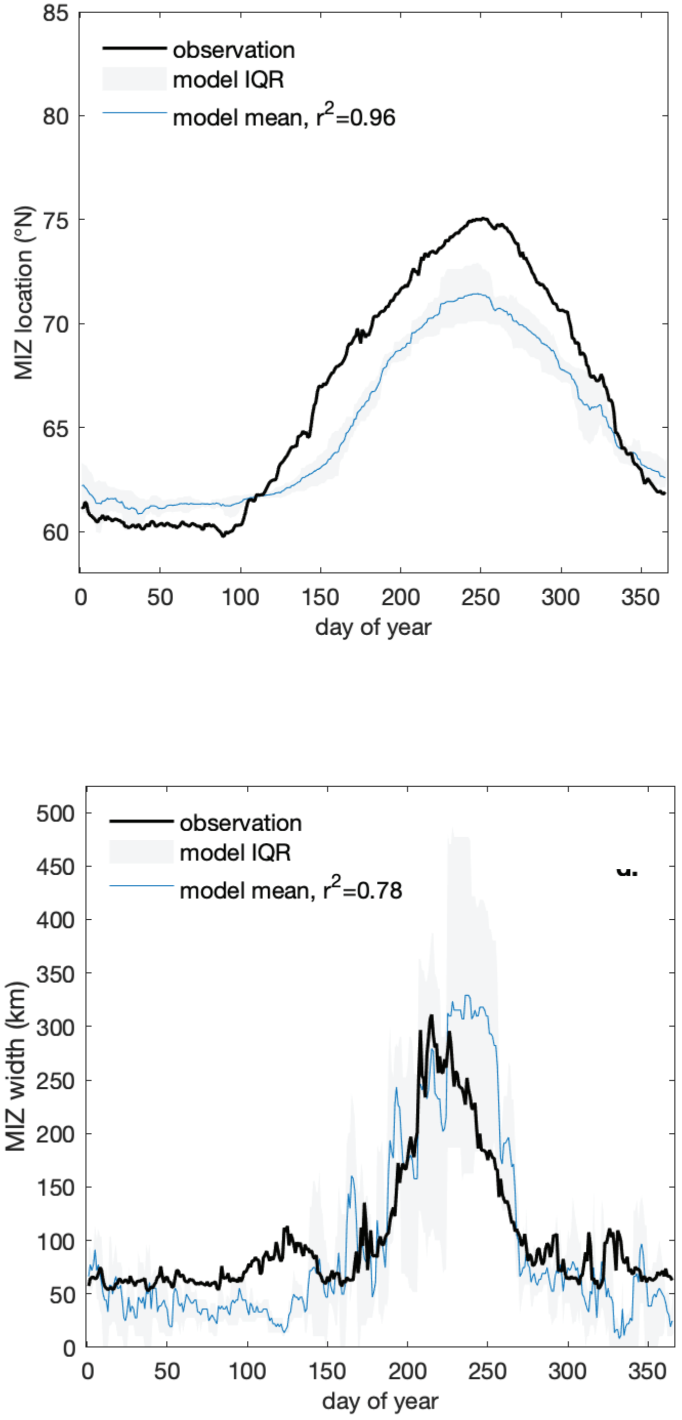
MIZ observations

MIZ model vs. observations

location

width



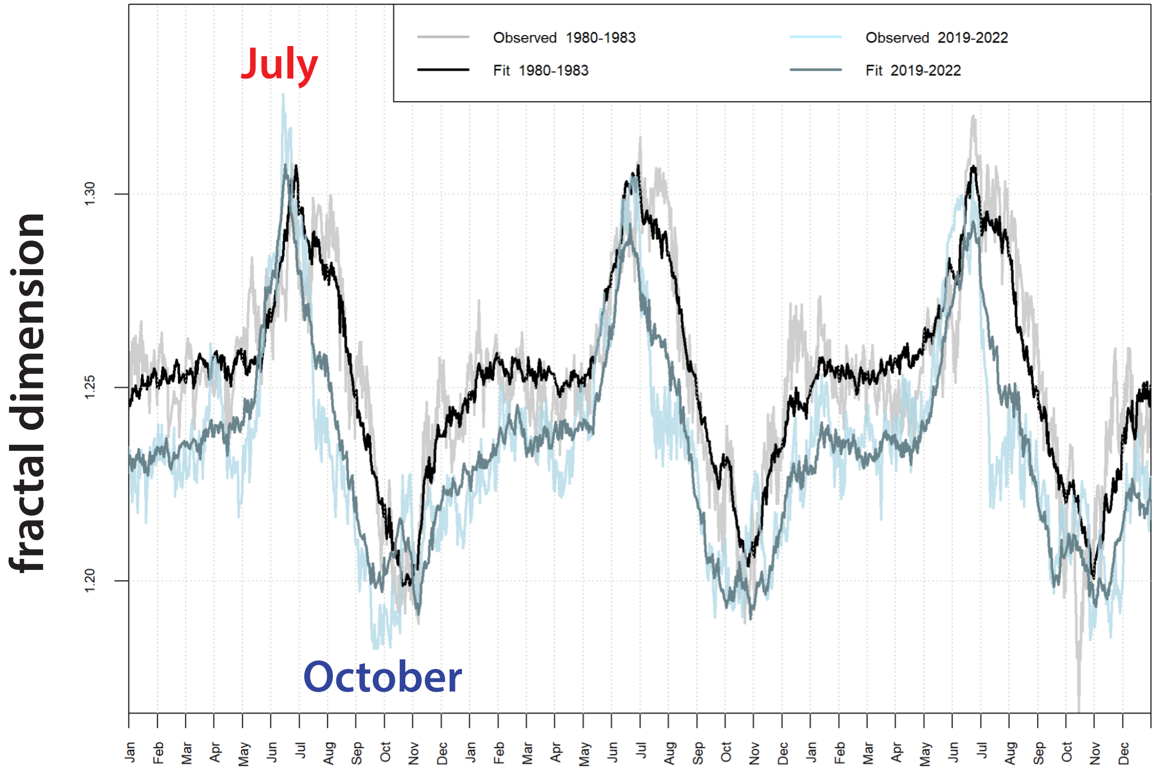


Model captures basic physics of MIZ dynamics.

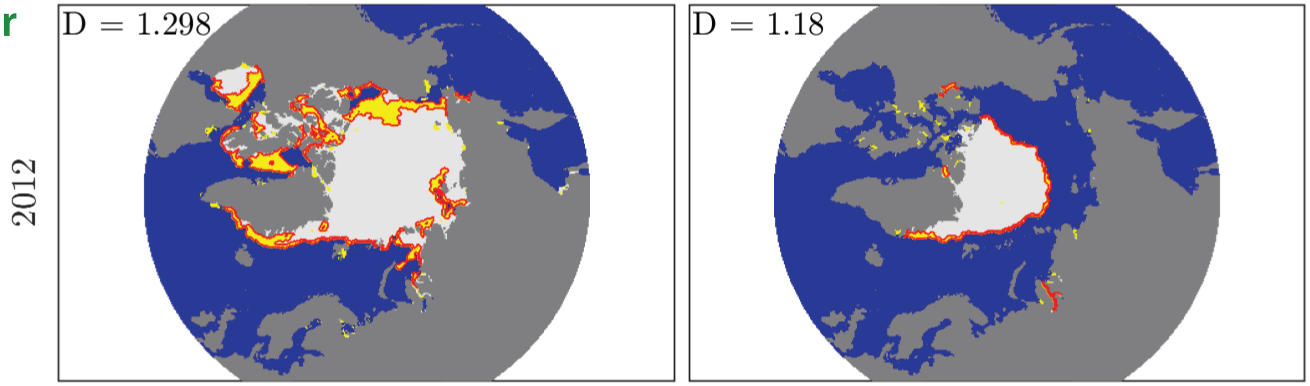
Identifying Fractal Geometry in Arctic Marginal Ice Zone Dynamics

Julie Sherman, Court Strong, Ken Golden 2024

Compute the fractal dimension of the boundary of the Arctic MIZ by boxcounting methods; analyze seasonal cycle and long term trends.



early summer



early autumn

wave and thermal interactions with fractal boundary

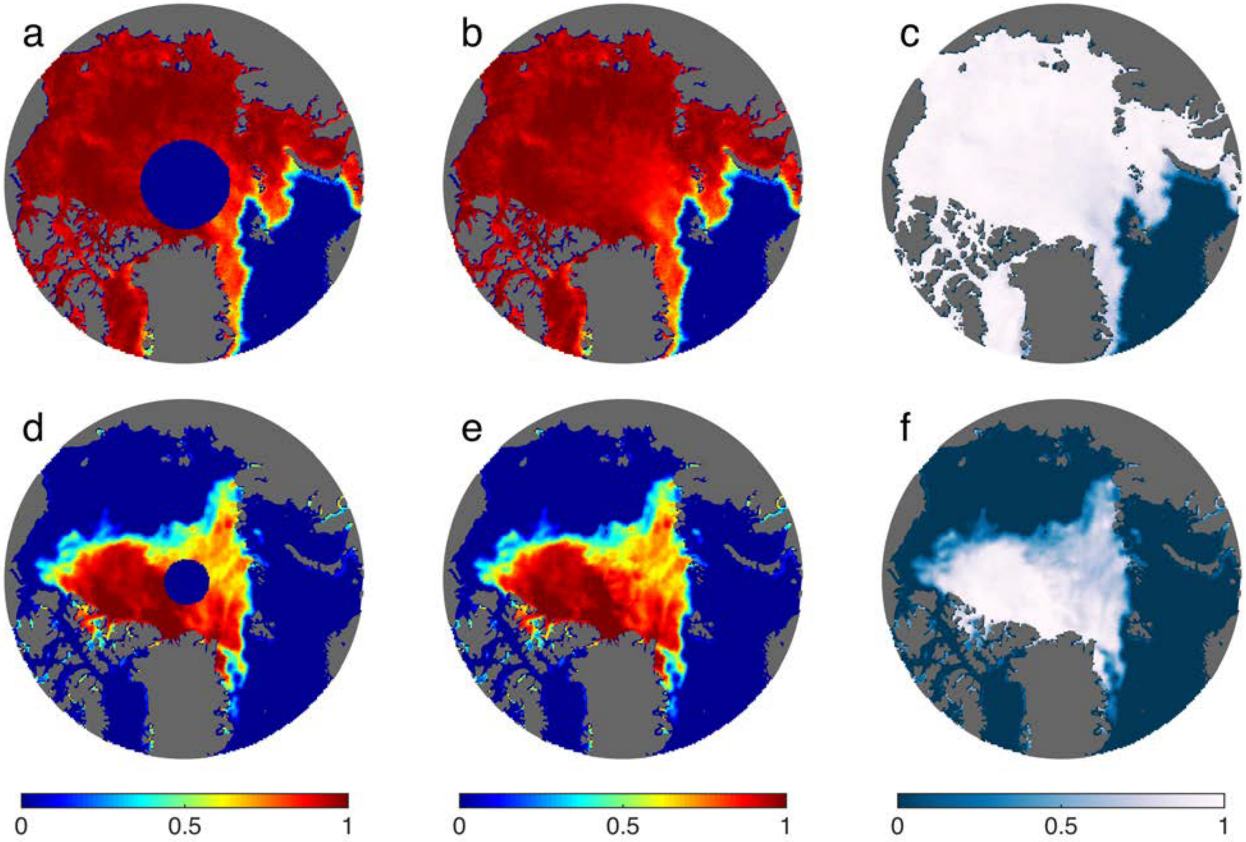
Filling the polar data gap with partial differential equations

hole in satellite coverage of sea ice concentration field

previously assumed ice covered

Gap radius: 611 km
06 January 1985

Gap radius: 311 km
30 August 2007



$$\Delta\psi=0$$

fill = harmonic function satisfying satellite BC's plus learned stochastic term

Strong and Golden, *Remote Sensing* 2016
Strong and Golden, *SIAM News* 2017

Global Sea Ice Concentration Climate Data Records, 2022
Lavergne, Sorensen, et al., Norwegian Met. Inst., ... OSI SAF

Conclusions

Our research is helping to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

Mathematics for sea ice advances the theory of composites, inverse problems, and other areas of science and engineering.

Modeling sea ice leads to unexpected areas of math and physics.

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Notices

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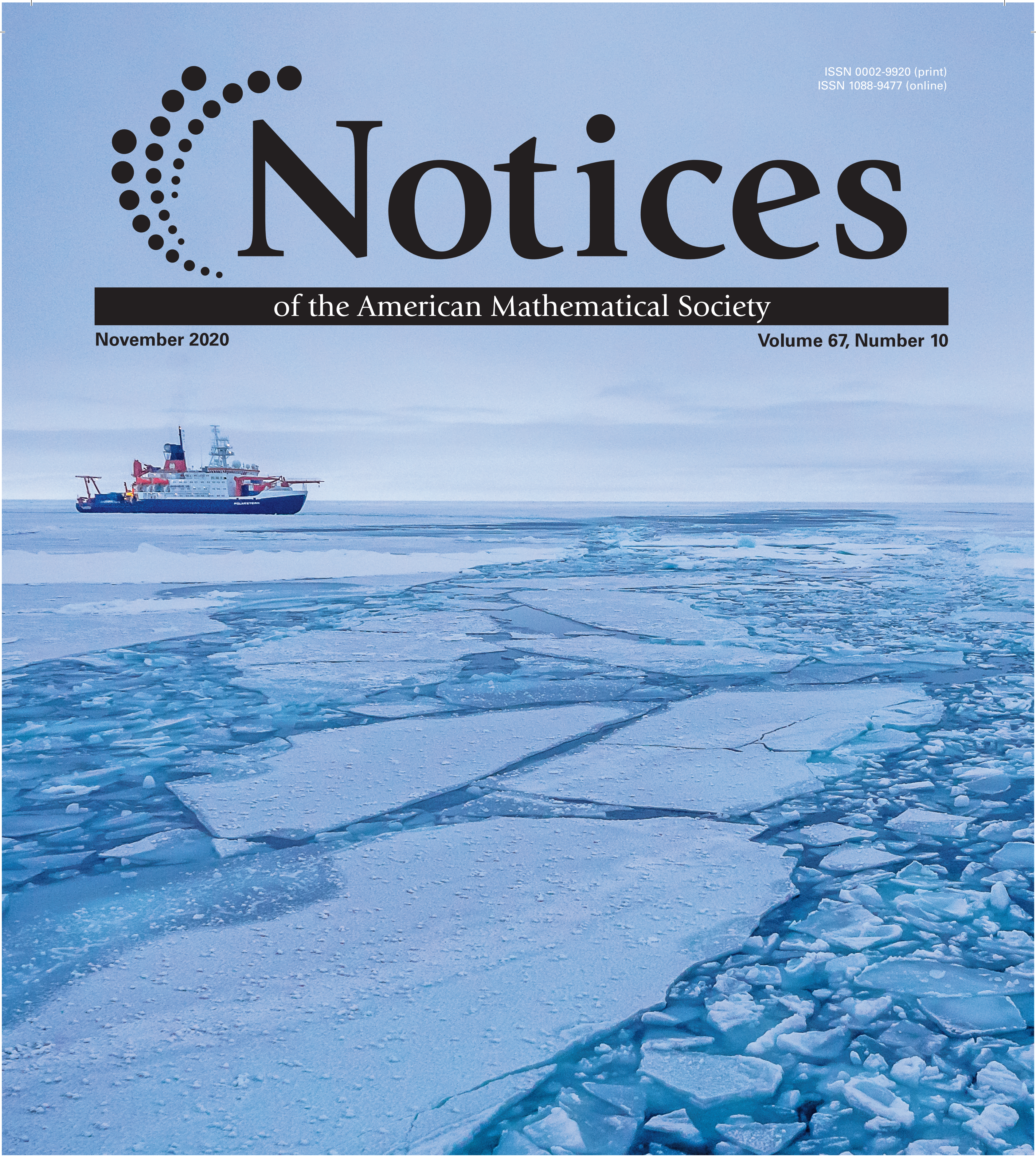


AMS
AMERICAN
MATHEMATICAL
SOCIETY
Advancing research. Creating connections.

The cover is based on "Modeling Sea Ice," page 1535.

NSF Research Training Grant (RTG) with 15 Applied Math faculty:

optimization and inverse problems

July 2022 - June 2027

Overall goal: Build an advanced, competitive U.S. STEM workforce.

- Strengthen our graduate and postdoctoral programs in applied math to attract top students in the nation, and place them in top jobs.
- Provide transformative experiences that draw students into math.

Arctic Mathpeditions - May 2024 & 2026

OPEN POSITIONS:
Postdoctoral, Ph.D., Undergraduate

NSF RTG Arctic Mathpedition, May 2024

on the frozen Arctic Ocean north of Utqiaġvik, AK

We took 7 math students working on sea ice models to the Arctic to do *experiments* on the physics and biology of sea ice.

Jody Reimer, Ken Golden

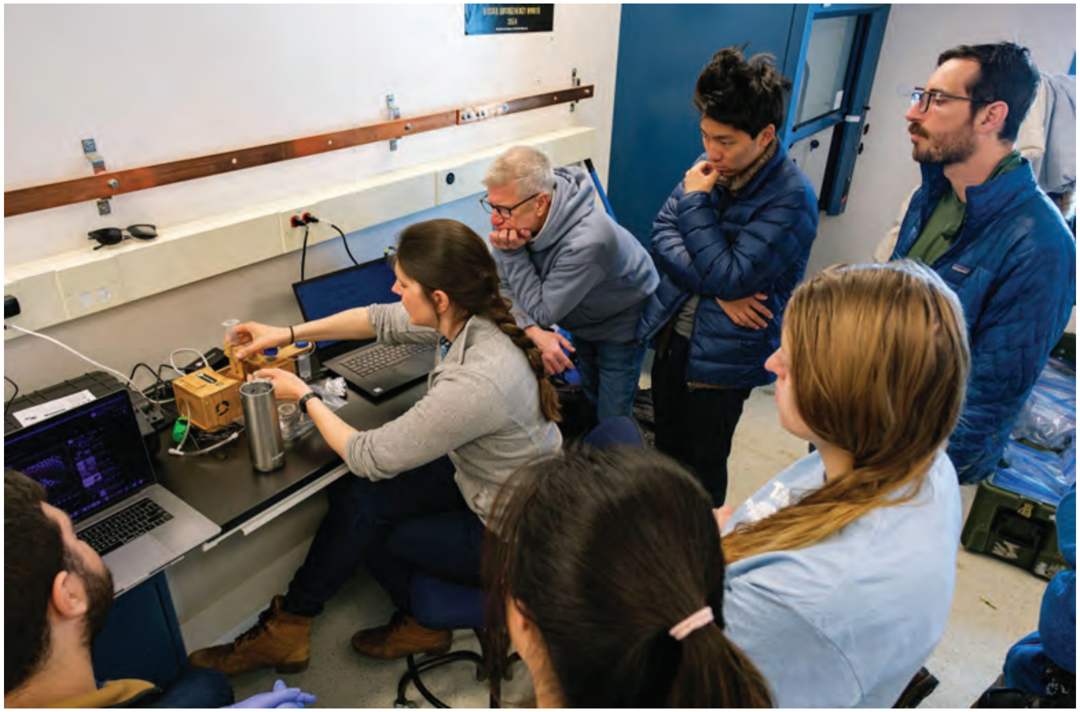
[Seth & Tarn]

Anthony Lee	High School
David Gluckman	Undergraduate
Kathy Lin	Undergraduate
Nash Ward	Undergraduate
Daniel Hallman	Graduate Student
Anthony Jajeh	Graduate Student
Delaney Mosier	Graduate Student
Marco Lozzi	Student Photojournalist

see what you're modeling; close the gap between theory and experiment; connect physics & bio; experience climate change first-hand; math outreach to locals

Math Dept Colloquium, Nov 21

NSF RTG Arctic Mathpedition 2, May 2026

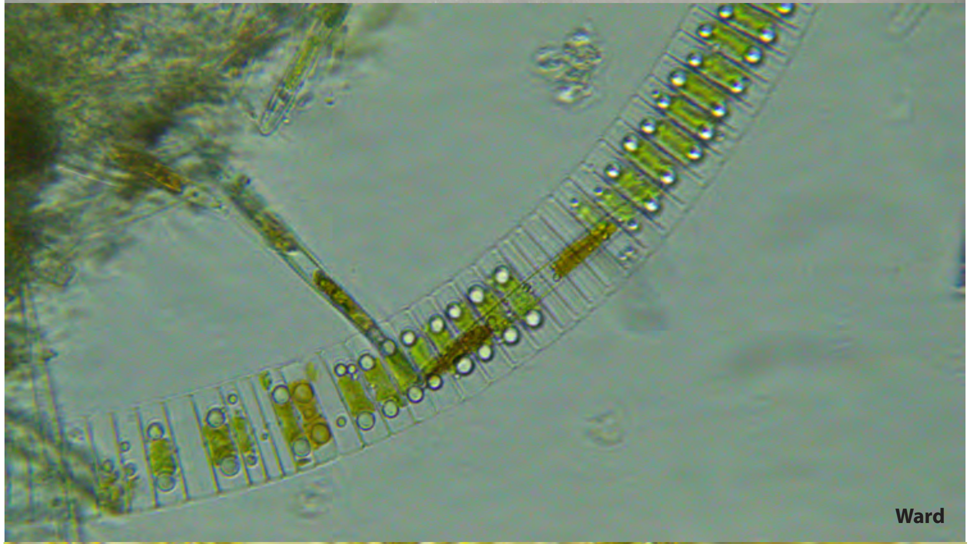



DEWALT
DEWALT

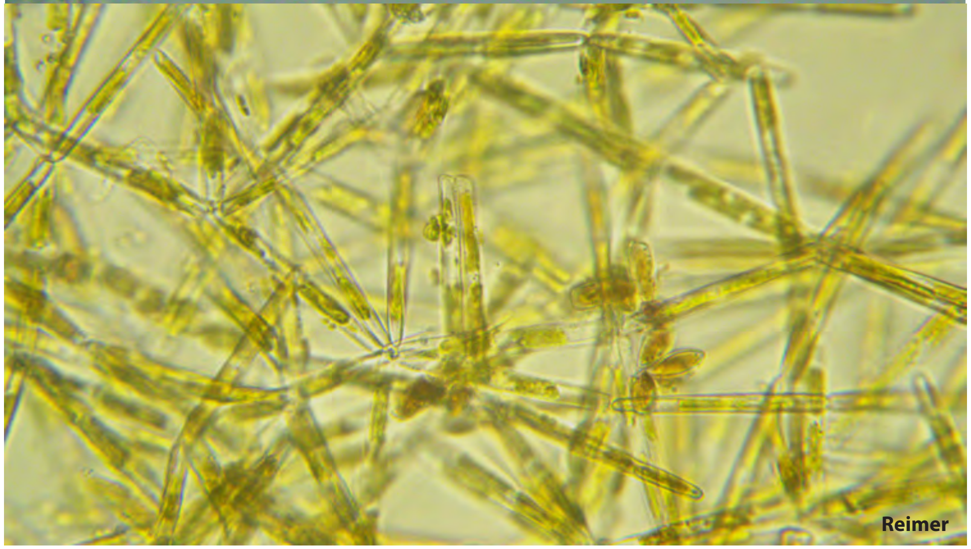
Reimer

bottom of a sea ice core

Golden

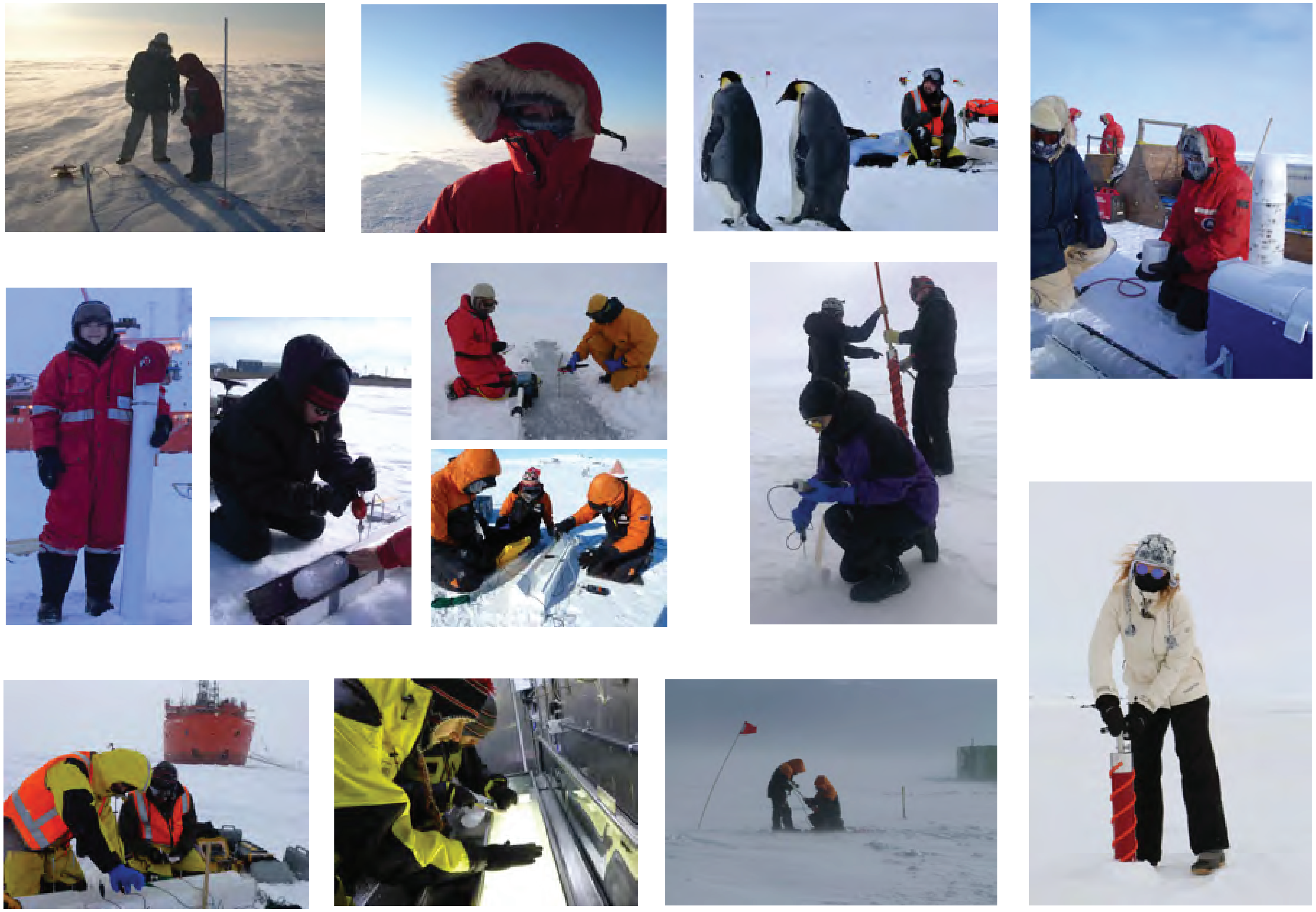


Ward



Reimer

Thank you to so many postdocs, graduate students, undergraduates, high school students and colleagues who contributed to this work!



U. of Utah students in the Arctic and Antarctic (2003-2022): closing the gap between theory and observation - making math models come alive and experiencing climate change firsthand.

THANK YOU

Office of Naval Research

Applied and Computational Analysis Program

Arctic and Global Prediction Program

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

UNITED STATES COAST GUARD

DEPARTMENT OF THE NAVY
ONR
Science & Technology

MCRN
Mathematics and Climate Research Network

Australian Government
Department of the Environment and Water Resources
Australian Antarctic Division

ace
CRC
ANTARCTIC CLIMATE & ECOSYSTEMS
COOPERATIVE RESEARCH CENTRE

NSF

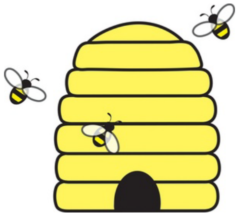
Antarctica New Zealand

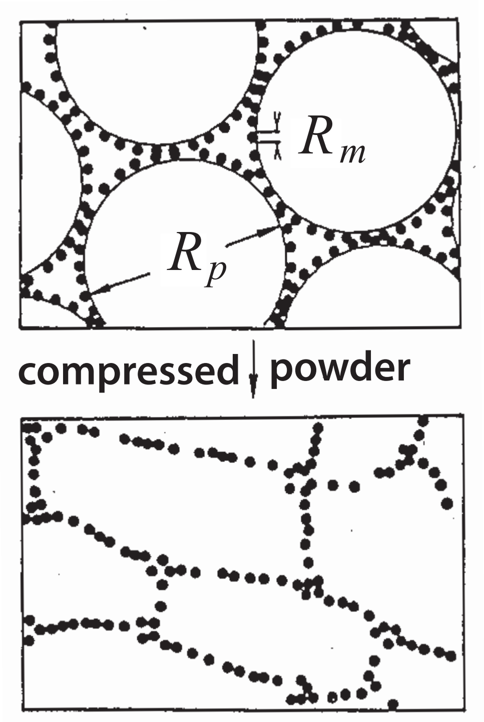
UAF
UNIVERSITY OF ALASKA FAIRBANKS

THE UNIVERSITY OF UTAH

Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999

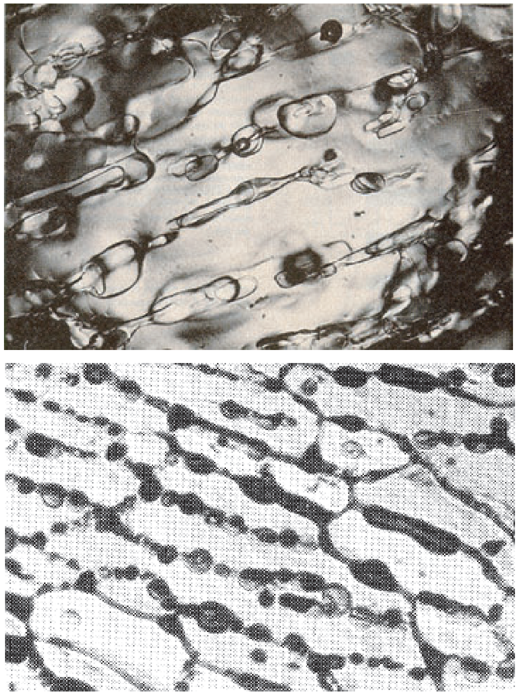
cross pollination





radar absorbing coating

Kusy & Turner
Nature 1971

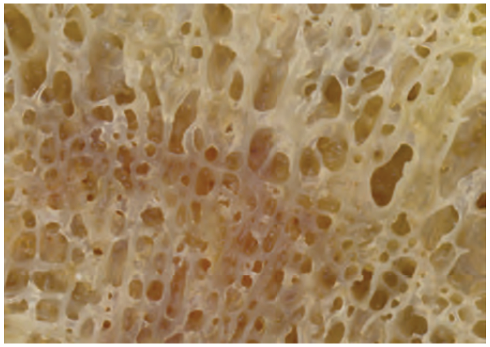


sea ice

Golden, Ackley, Lytle
Science 1998

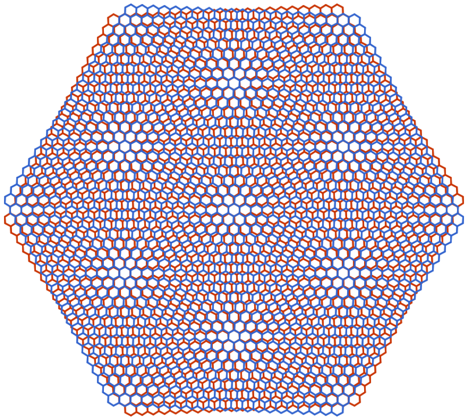
Rule of Fives
fluid flow

spectral analysis & RMT



human bone

Golden, Murphy, Cherkaev
J. Biomechanics 2011



twisted bilayer composites

Morison, Murphy, Cherkaev, Golden
Communications Physics 2022

stealth technology, climate science, medical imaging, twistronics

recent losses
in comparison to
the United States

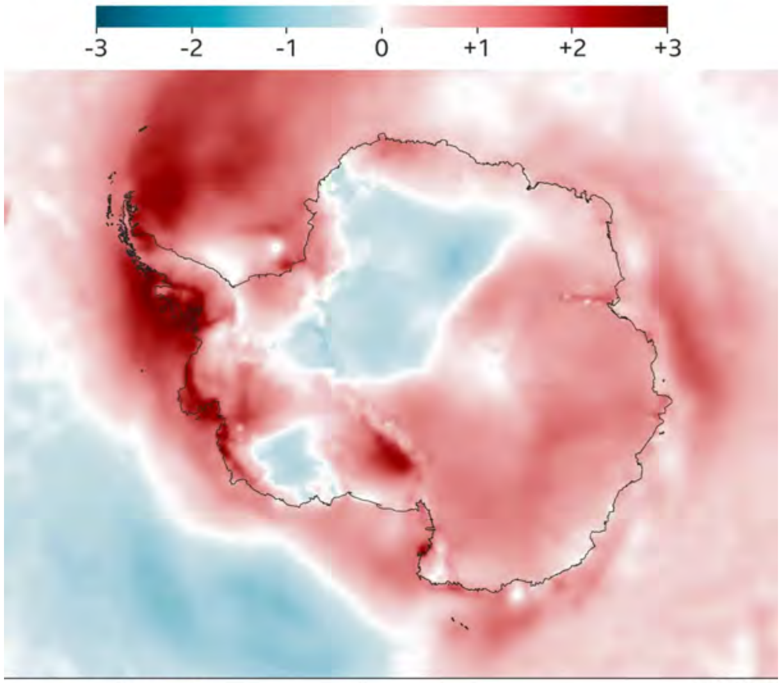
Perovich

New Record Low for Antarctic Sea Ice

February 13, 2023

Much of Antarctica warmer than average

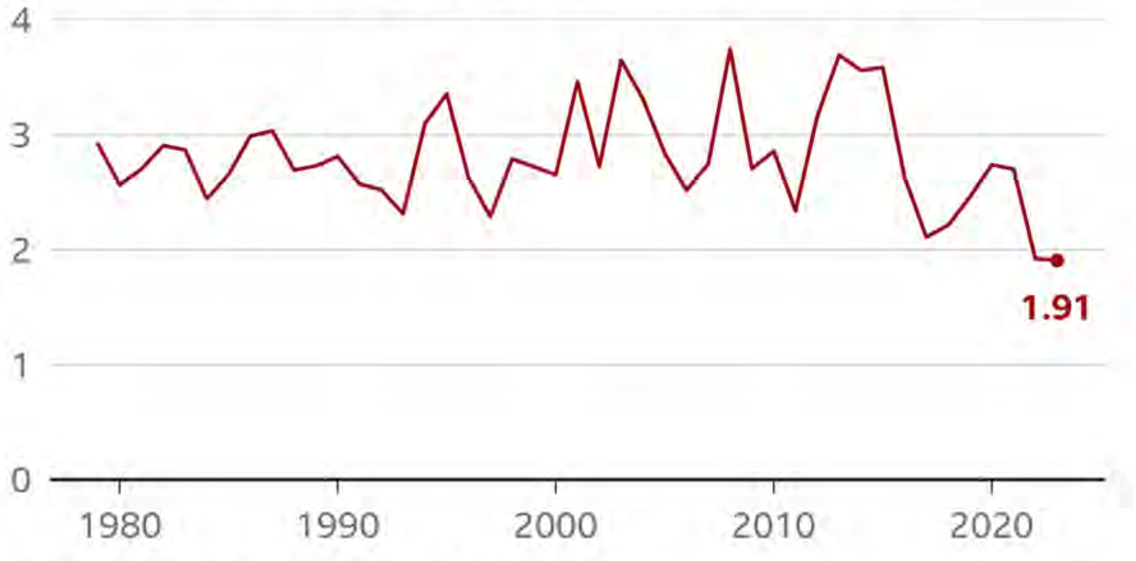
Mean 2022 surface air temp compared with 1991-2022 (°C)



Source: ECMWF ERA5

BBC

Minimum extent 1979-2023 (million sq km)



Five-day rolling average of sea-ice extent

Source: National Snow and Ice Data Center (NSIDC)

BBC

sea ice algal communities

D. Thomas 2004

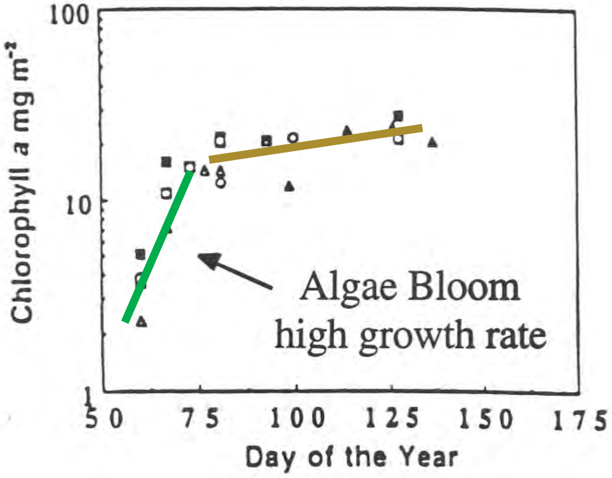
nutrient replenishment controlled by ice permeability

biological activity turns on or off according to *rule of fives*

Golden, Ackley, Lytle ***Science 1998***

Fritsen, Lytle, Ackley, Sullivan ***Science 1994***

critical behavior of microbial activity

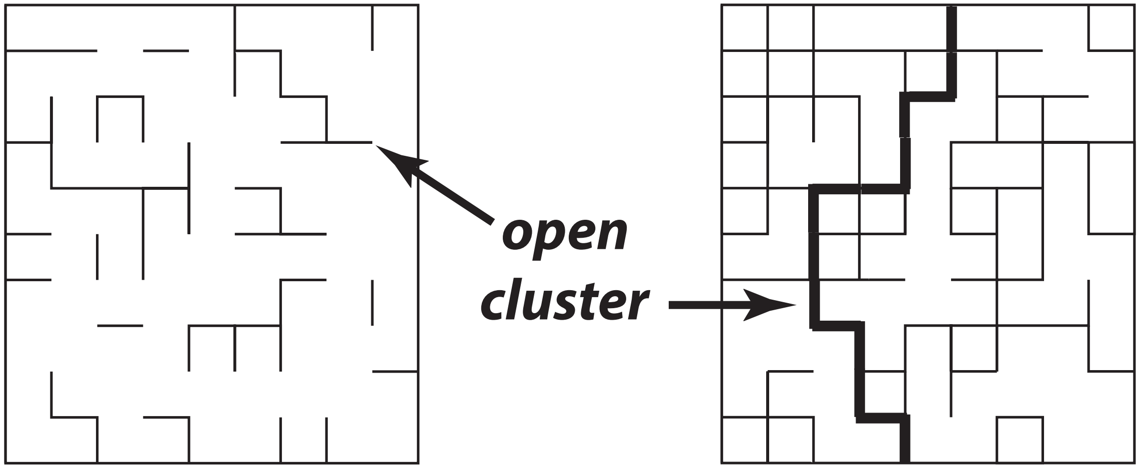


Convection-fueled algae bloom
Ice Station Weddell

percolation theory

probabilistic theory of connectedness

impermeable **permeable**



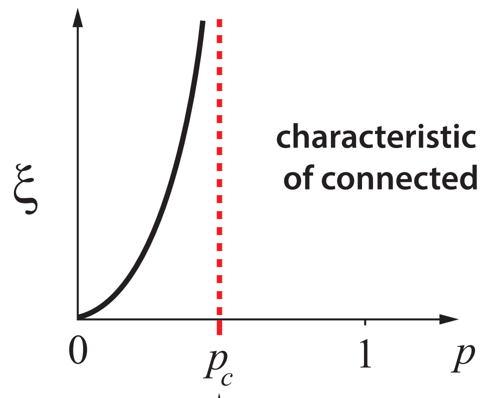
$p = 1/3$ $p = 2/3$

bond $\longrightarrow$ **open** *with probability* p

closed *with probability* $1-p$

correlation length

development of long range order



characteristic scale of connectedness

percolation threshold

$$\xi(p) \sim |p - p_c|^{-\nu} \qquad p \to p_c$$

ν **universal: depends only on *d***

p_c depends on type of lattice and *d*

percolation threshold

$$p_c = 1/2 \quad \text{for} \quad d = 2$$

smallest *p* for which there is an infinite open cluster

Critical behavior of fluid transport in sea ice

Arctic field data

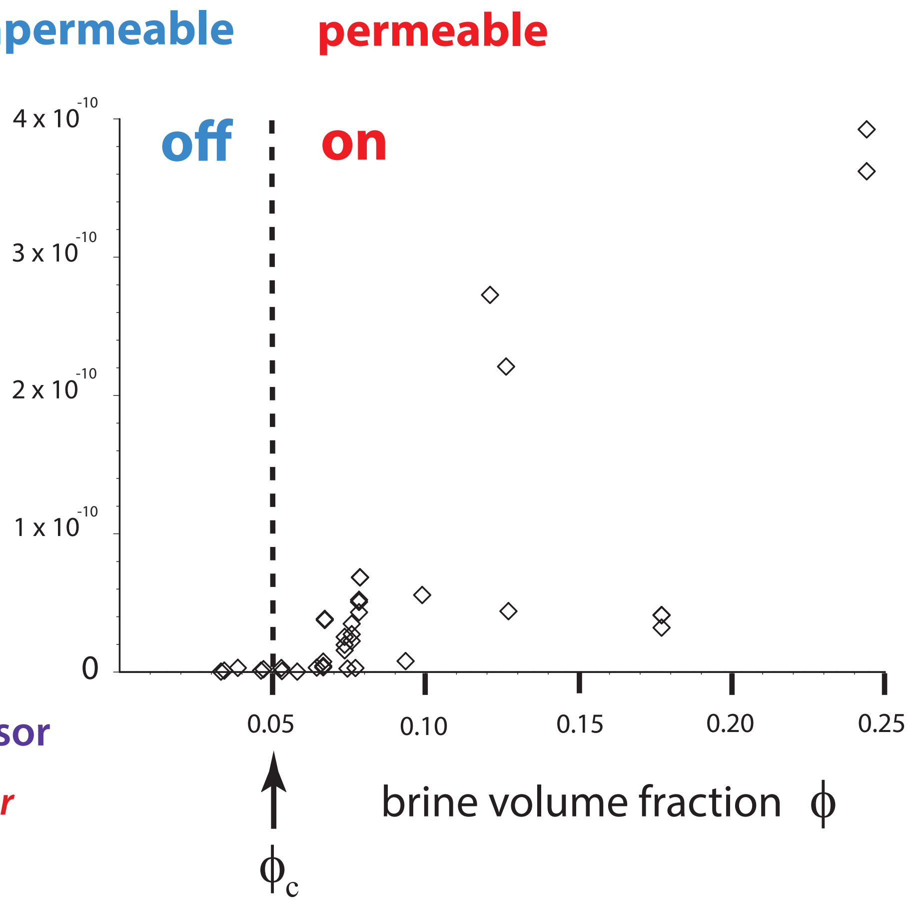
vertical fluid permeability k (m^2)

Darcy's Law

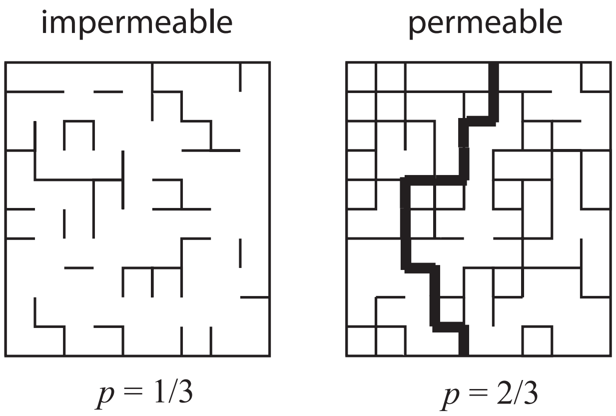
$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\nabla p$$

$\mathbf{k}$ = fluid permeability tensor

homogenized parameter



"on - off" switch for bulk fluid flow



$p = 1/3$ $p = 2/3$

lattice percolation

FRACTAL percolation clusters

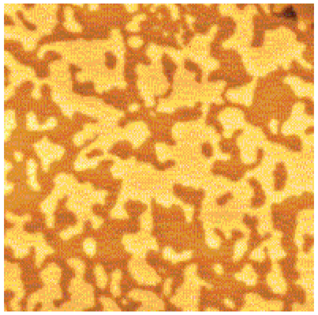
PERCOLATION THRESHOLD $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ C,\ S \approx 5$ ppt

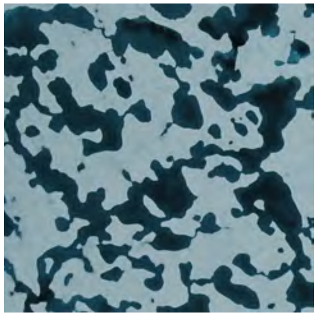
RULE OF FIVES

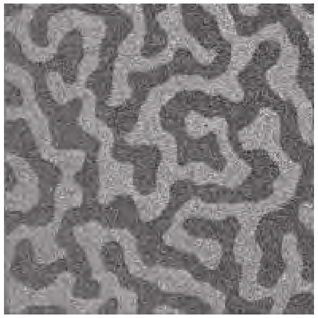
Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

From magnets to melt ponds

100 year old model for magnetic materials used to explain melt pond geometry

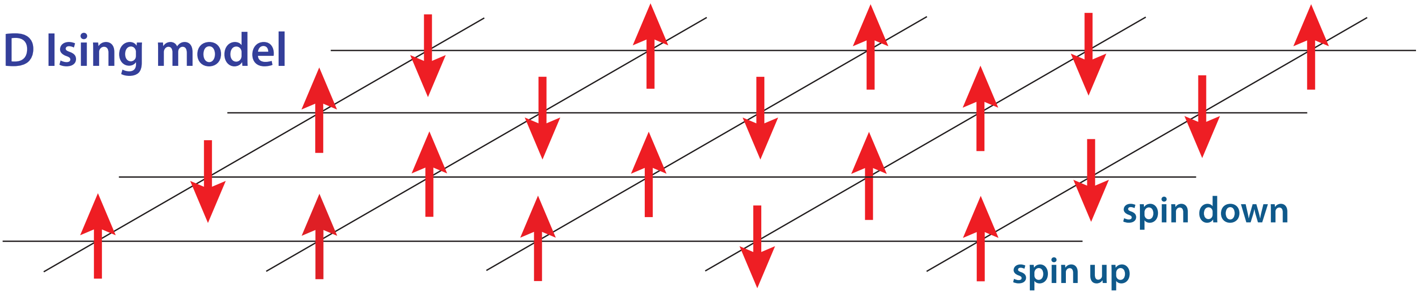

magnetic domains
in cobalt

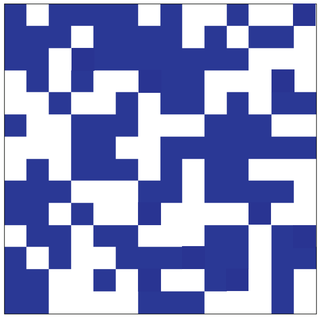

Arctic melt ponds


magnetic domains
in cobalt-iron-boron

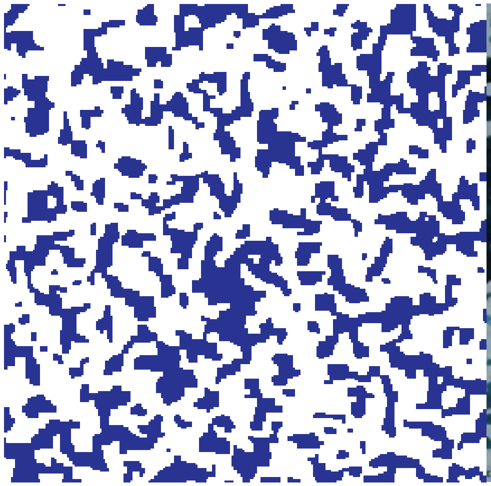

Arctic melt ponds

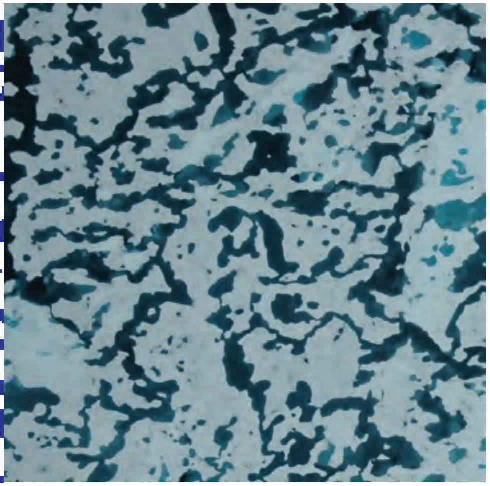
2D Ising model





model





real ponds (Perovich)

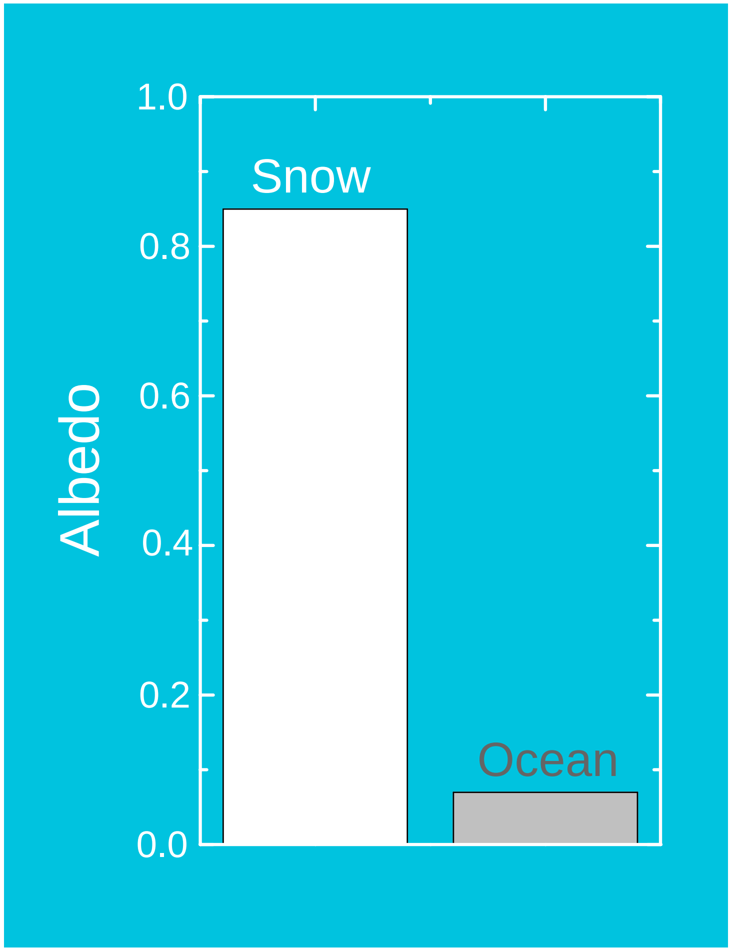
Ma, Sudakov, Strong, Golden, *New J. Phys.* 2019

Scientific American, EOS, PhysicsWorld, ...

Time evolution - William Harrison, Tyler Evans, Ken Golden 2024

polar ice caps critical to global climate in reflecting incoming solar radiation

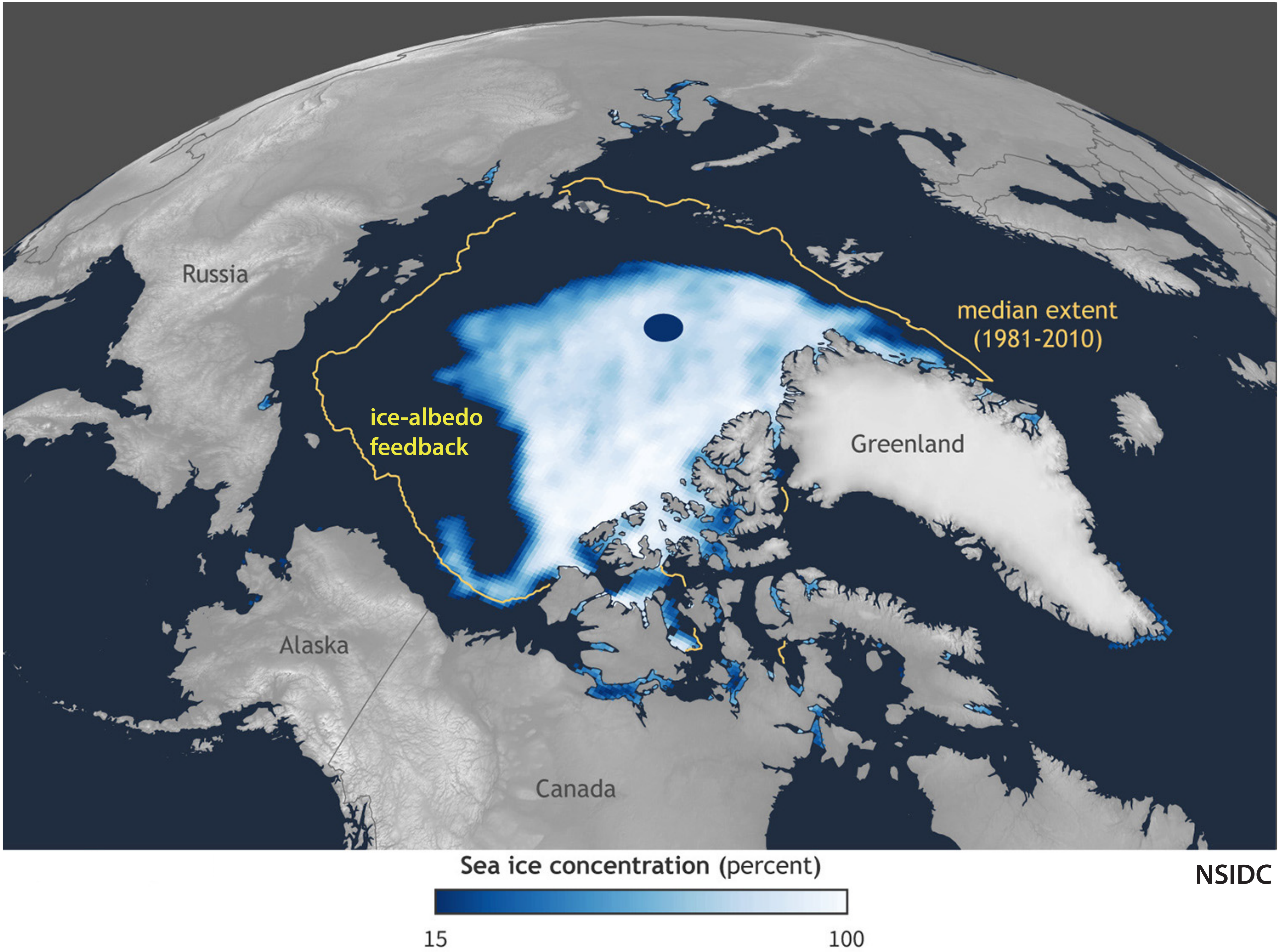
white snow and ice reflect



dark water and land absorb

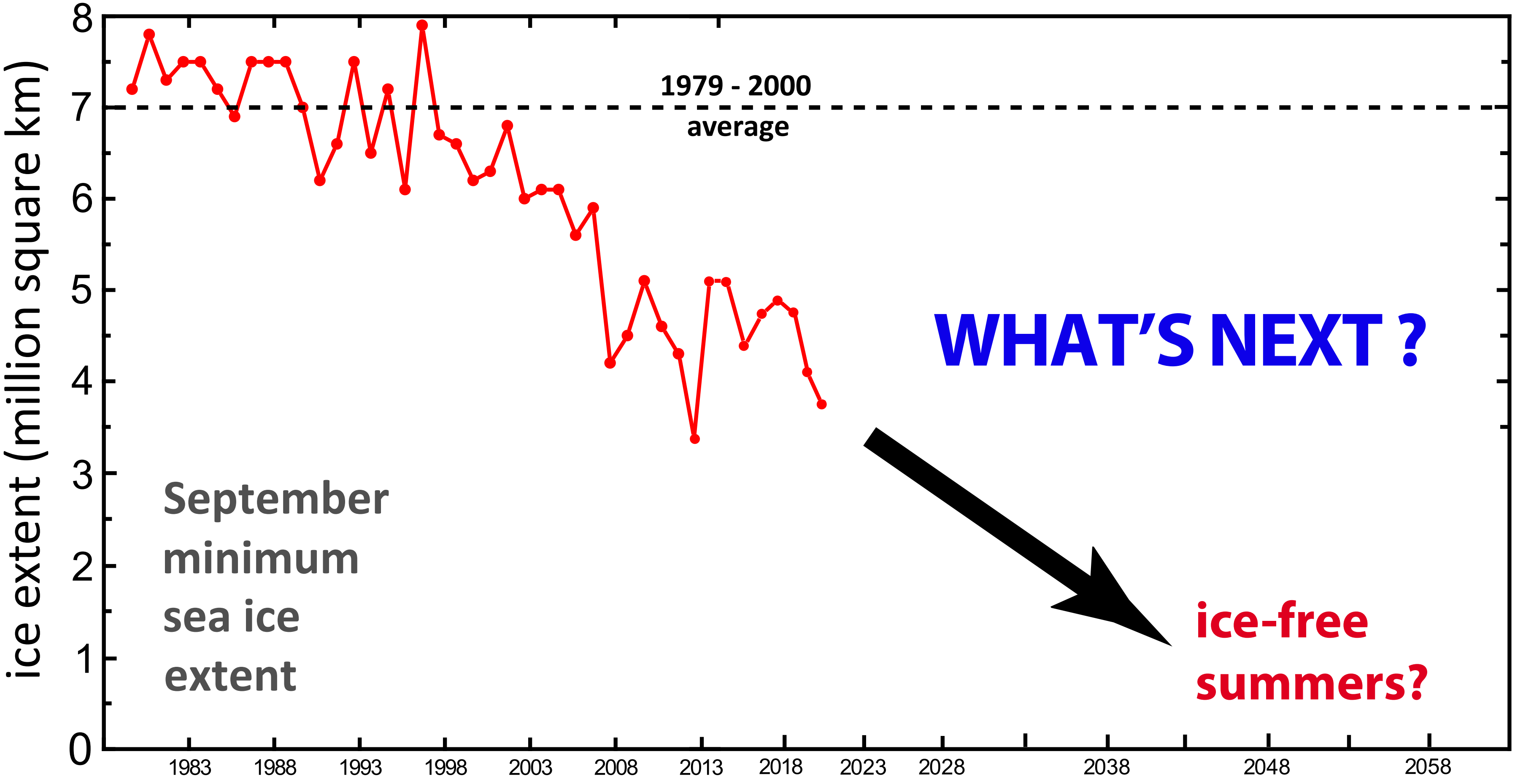
$$\textbf{\textit{albedo}}\ \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

Arctic sea ice extent September 15, 2020



NSIDC

ARCTIC summer sea ice loss



September minimum sea ice extent

WHAT'S NEXT ?

ice-free summers?

National Snow and Ice Data Center (NSIDC)

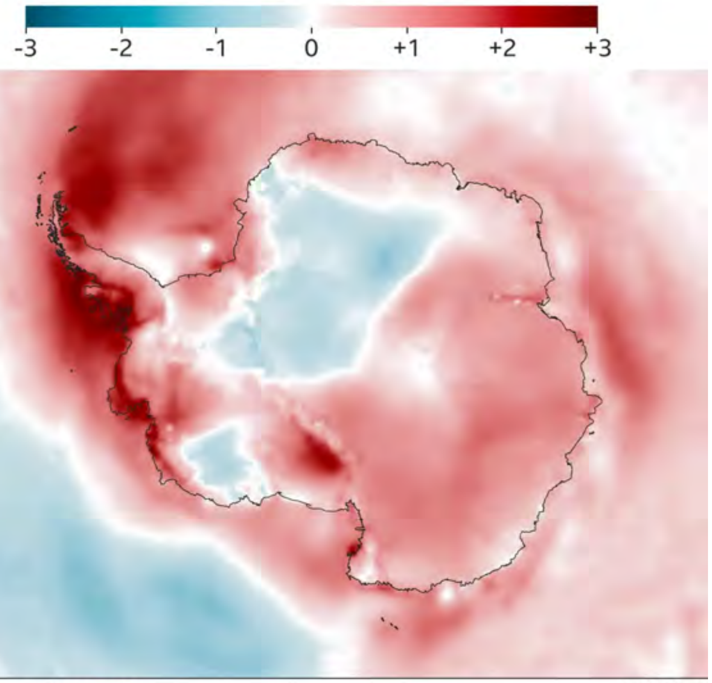
predictions require lots of math modeling

New Record Low for Antarctic Sea Ice

February 13, 2023

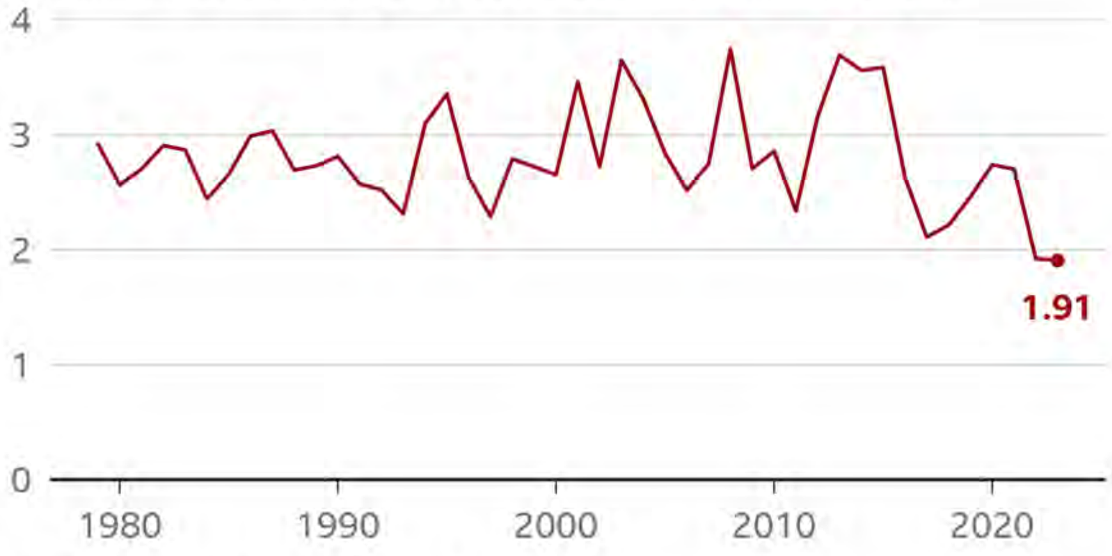
Much of Antarctica warmer than average

Mean 2022 surface air temp compared with 1991-2022 (°C)



Source: ECMWF ERA5

Minimum extent 1979-2023 (million sq km)



Five-day rolling average of sea-ice extent

Source: National Snow and Ice Data Center (NSIDC)