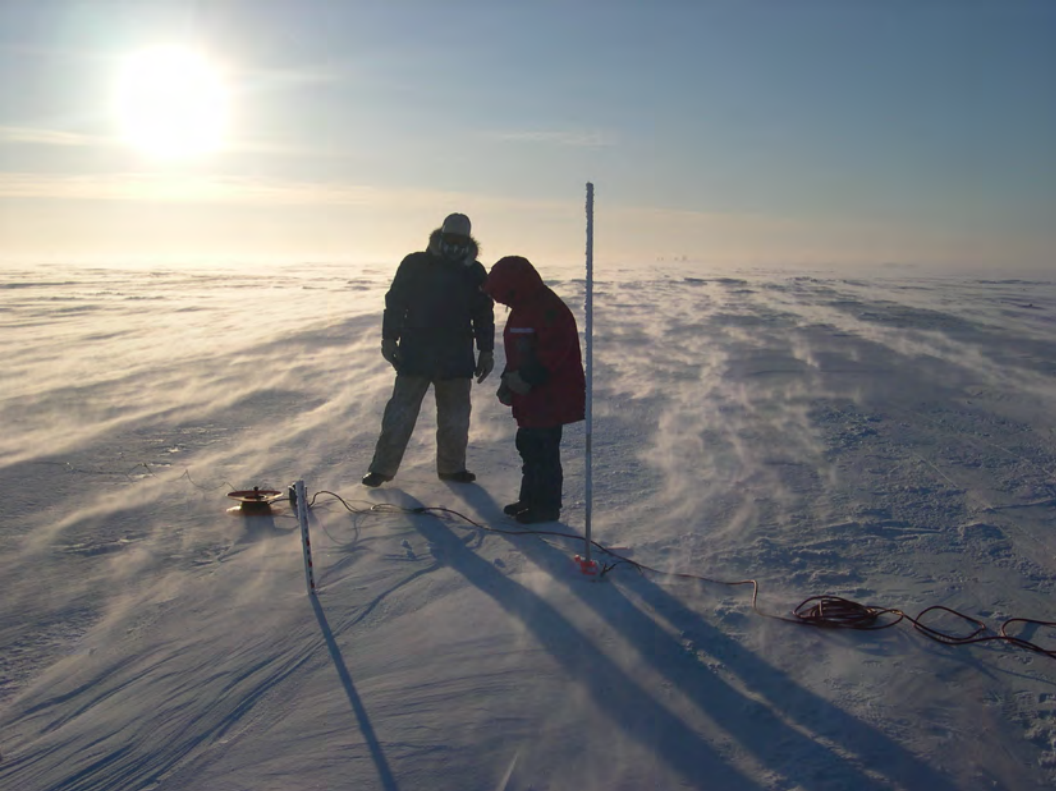


Stieltjes Functions and their Integral Representations in Sea Ice Modeling

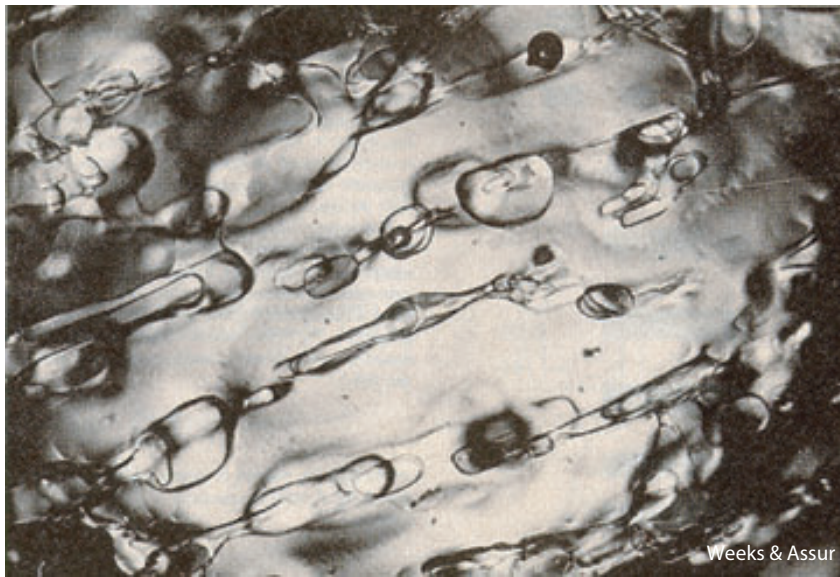
Kenneth M. Golden
Dept. of Mathematics, Univ. of Utah



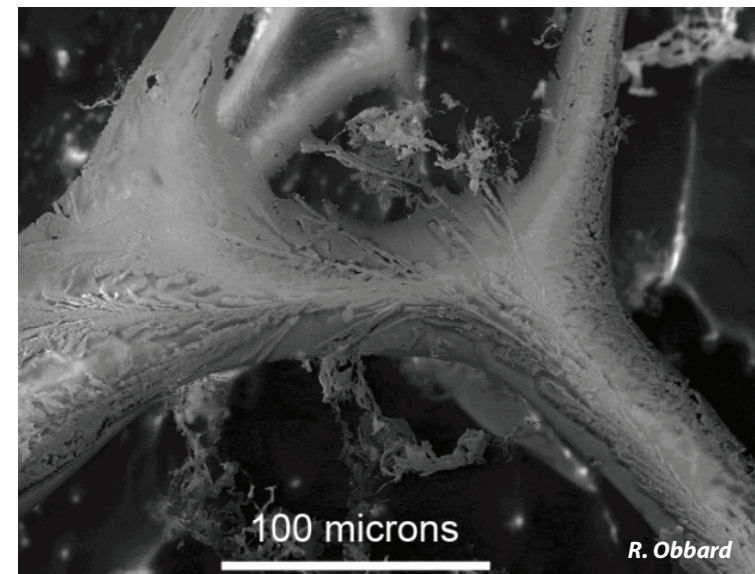
CIRM Conference on Herglotz-Nevanlinna functions and their
applications to dispersive systems and composite materials
26 May 2022



*sea ice may appear to be a
barren, impermeable cap ...*



brine inclusions in sea ice (mm)



micro - brine channel (SEM)

***sea ice is a
porous composite***

pure ice with brine, air, and salt inclusions

brine channels (cm)



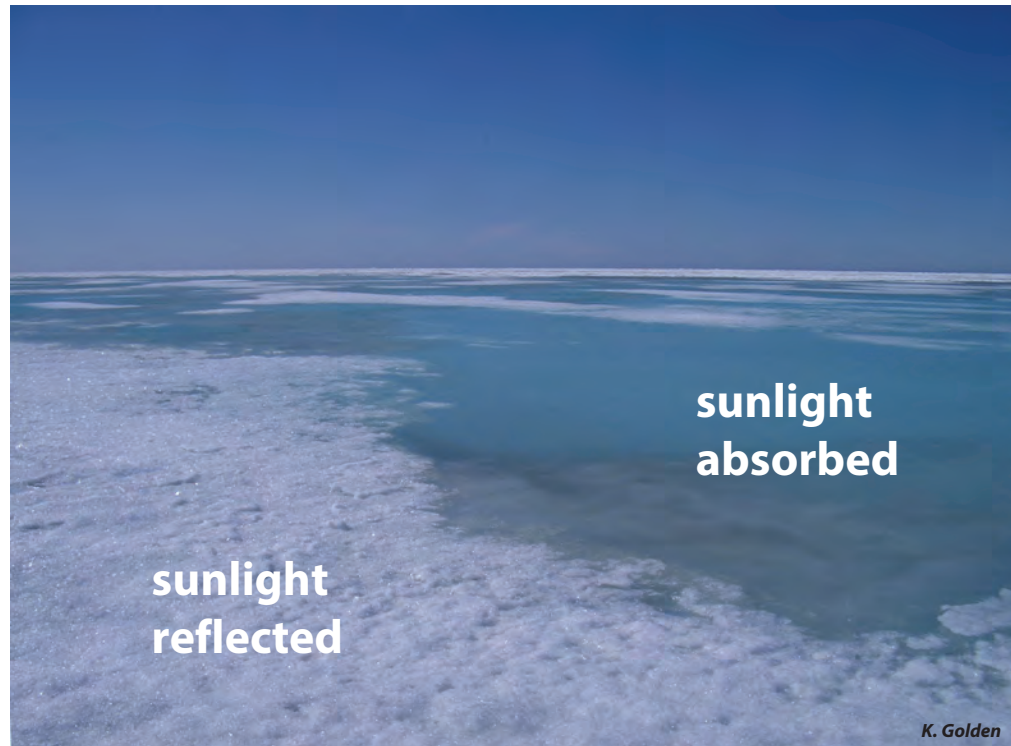
horizontal section



vertical section

fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

*evolution of Arctic melt ponds and sea ice **albedo***



nutrient flux for algal communities



***Antarctic surface flooding
and snow-ice formation***

September
snow-ice
estimates

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO₂*

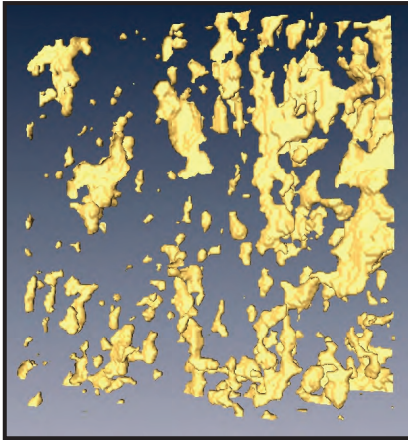
Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

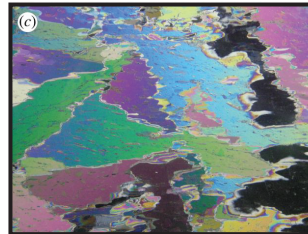
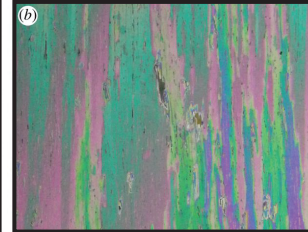
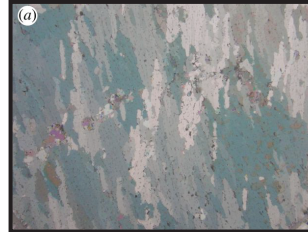


Weeks & Assur 1969



H. Eicken
Golden et al. GRL 2007

polycrystals

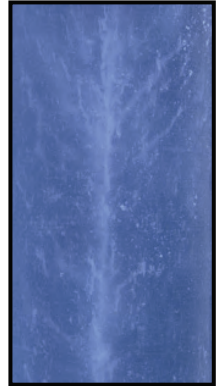


Gully et al. Proc. Roy. Soc. A 2015

brine channels



D. Cole



K. Golden

millimeters

centimeters

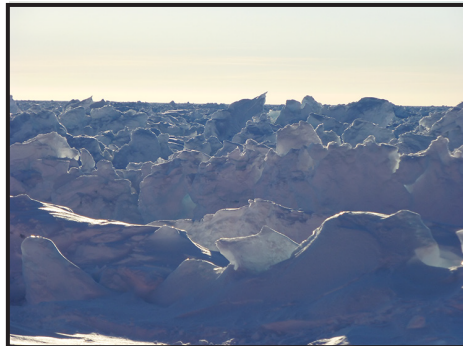
mesoscale

Arctic melt ponds



K. Frey

Antarctic pressure ridges



K. Golden

meters

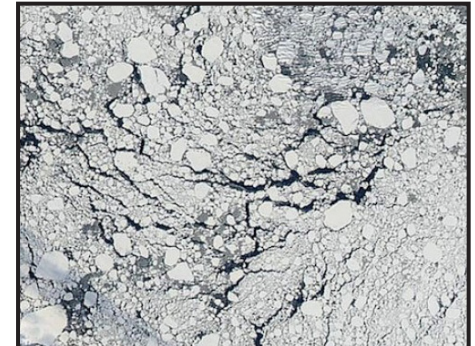
macroscale

sea ice floes



J. Weller

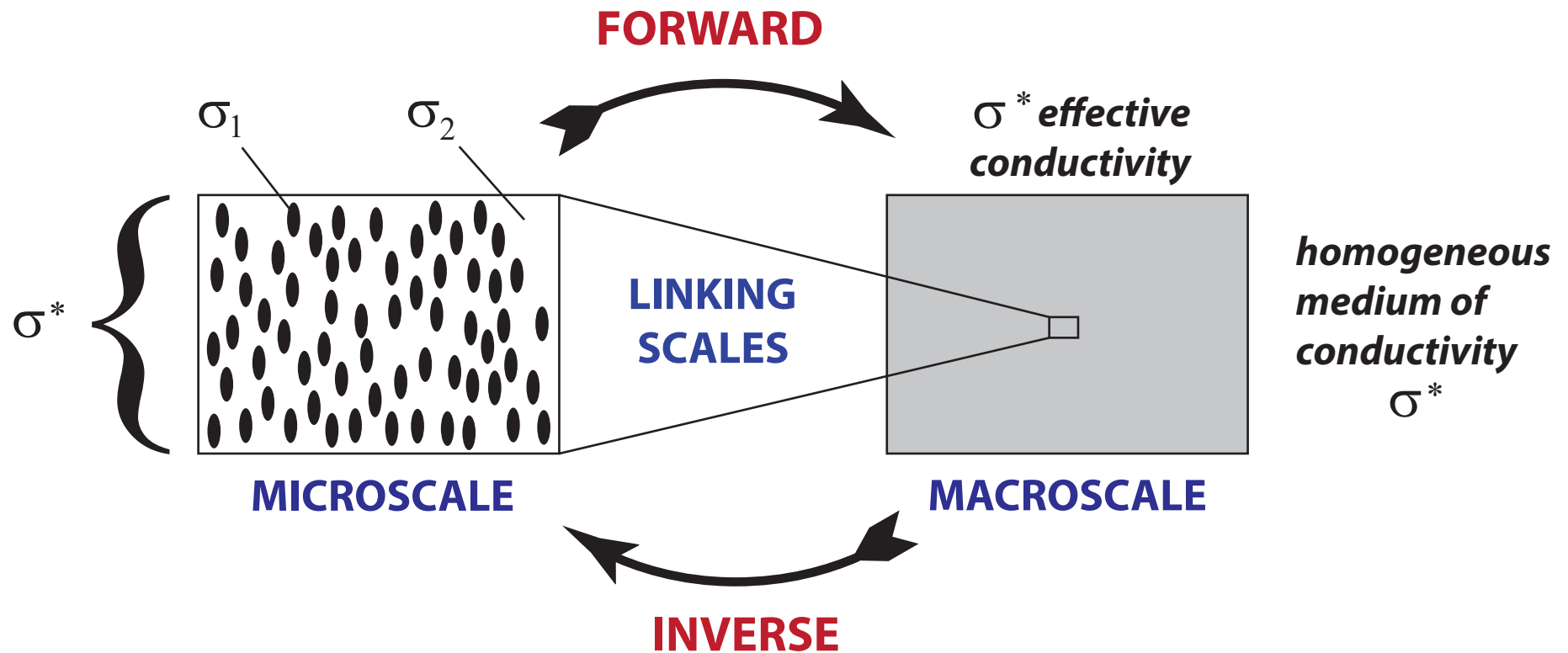
sea ice pack



NASA

kilometers

HOMOGENIZATION for Composite Materials



Maxwell 1873 : effective conductivity of a dilute suspension of spheres

Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

*Wiener 1912 : arithmetic and harmonic mean **bounds** on effective conductivity*

*Hashin and Shtrikman 1962 : variational **bounds** on effective conductivity*

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

What is this talk about?

the role of “microstructure” in determining sea ice effective properties

Using **homogenization and statistical physics** to compute effective behavior on scales relevant to coarse-grained sea ice and climate models, process studies, ...

MICROSCALE: brine + polycrystalline microstructure; EM, fluid transport

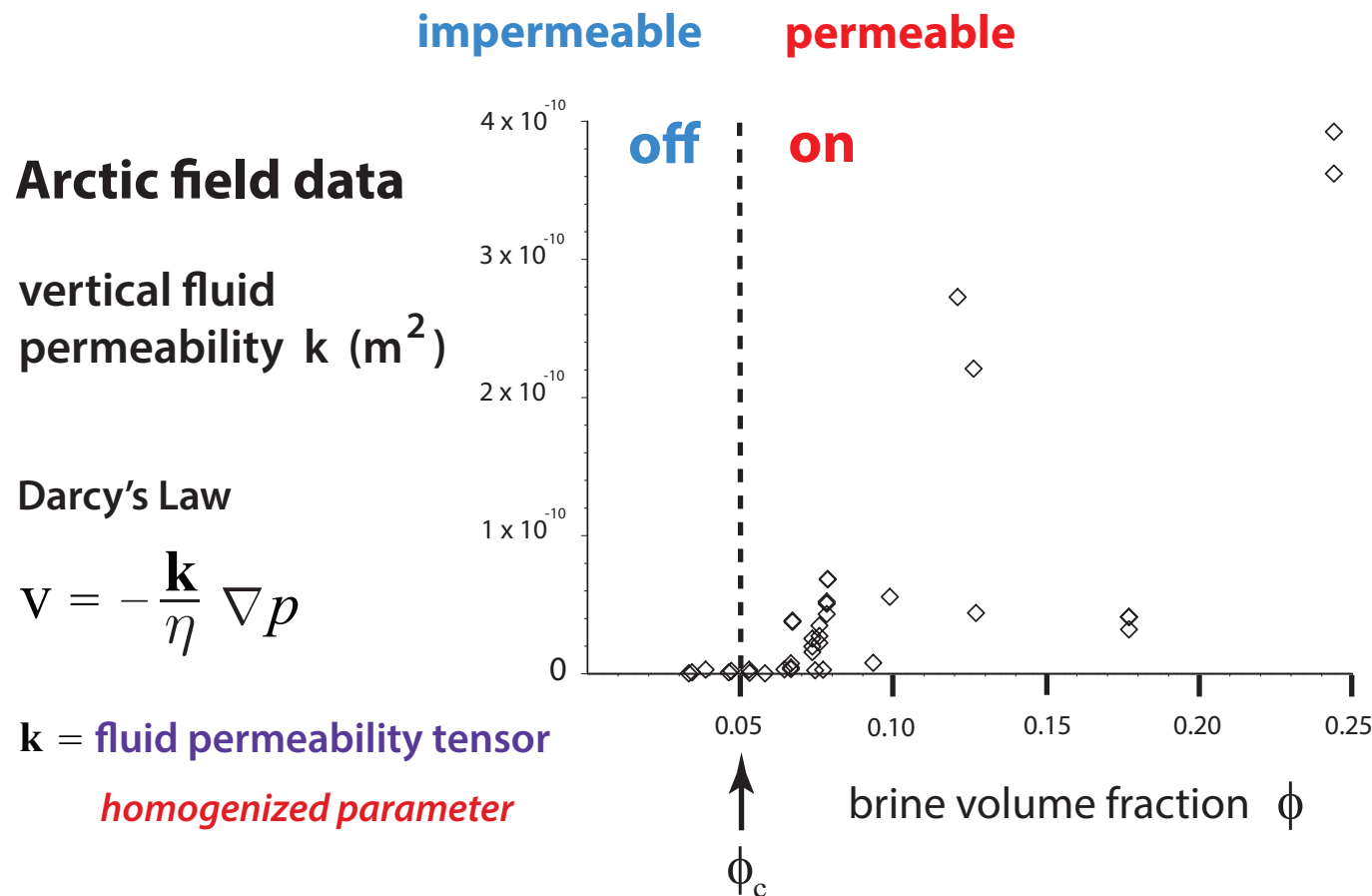
MESOSCALE: advection diffusion, thermal transport, ocean waves

A tour of Stieltjes functions in the study of sea ice and its role in climate.

Solving problems in the physics of sea ice drives advances in theory of composite materials.

microscale

Critical behavior of fluid transport in sea ice



***“on - off” switch
for fluid flow***

critical brine volume fraction $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ \text{C}, S \approx 5 \text{ ppt}$

RULE OF FIVES

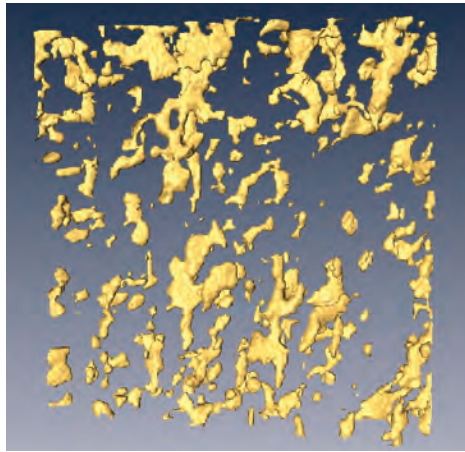
Golden, Ackley, Lytle Science 1998

Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007

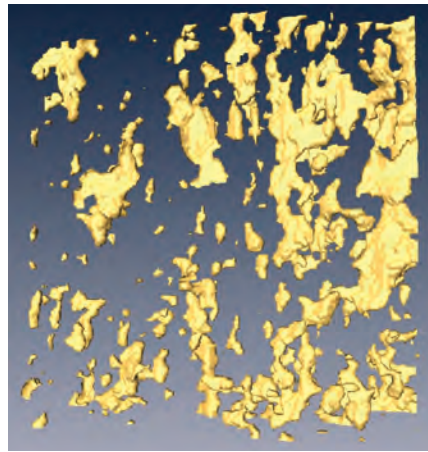
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

sea ice ~ compressed powder in stealthy composites

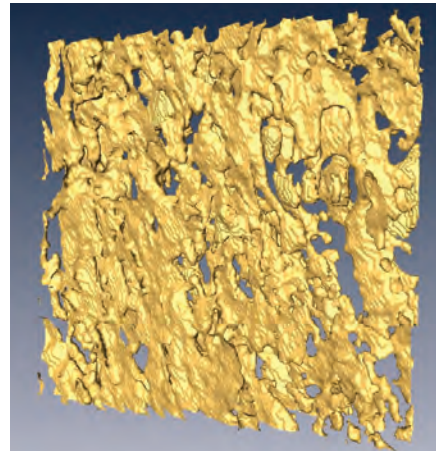
brine volume fraction and **connectivity** increase with temperature



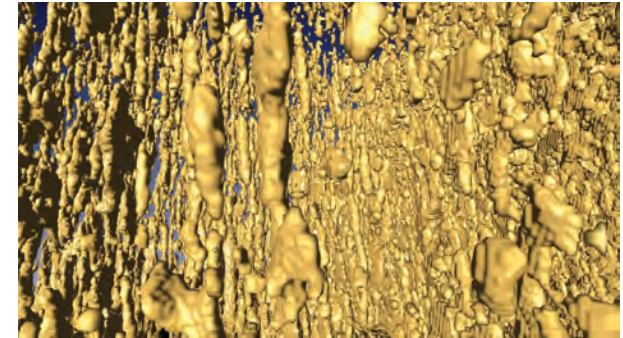
$T = -15\text{ }^{\circ}\text{C}$, $\phi = 0.033$



$T = -6\text{ }^{\circ}\text{C}$, $\phi = 0.075$



$T = -3\text{ }^{\circ}\text{C}$, $\phi = 0.143$



$T = -4\text{ }^{\circ}\text{C}$, $\phi = 0.113$

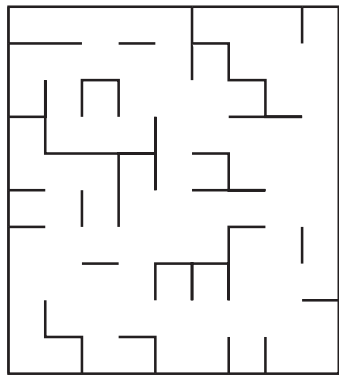
X-ray tomography for brine phase in sea ice

Golden, Eicken, *et al.*, *Geophysical Research Letters* 2007

PERCOLATION THRESHOLD $\phi_c \approx 5\%$

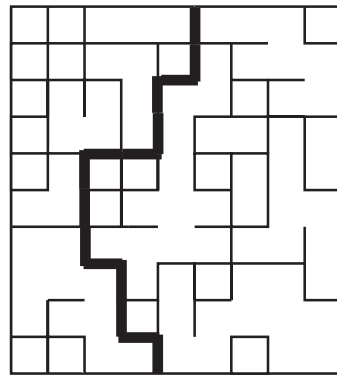
Golden, Ackley, Lytle, *Science* 1998

impermeable



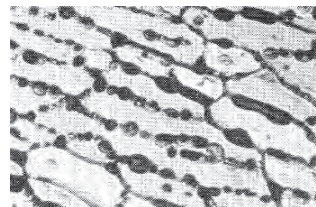
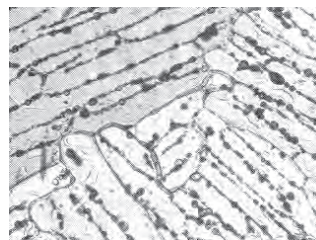
$p = 1/3$

permeable

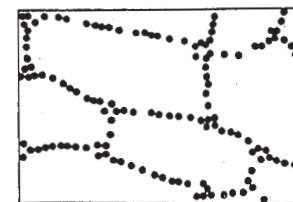
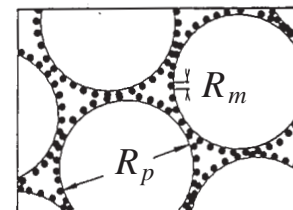


$p = 2/3$

lattice percolation



sea ice



compressed powder

Kusy, Turner
Nature 1971



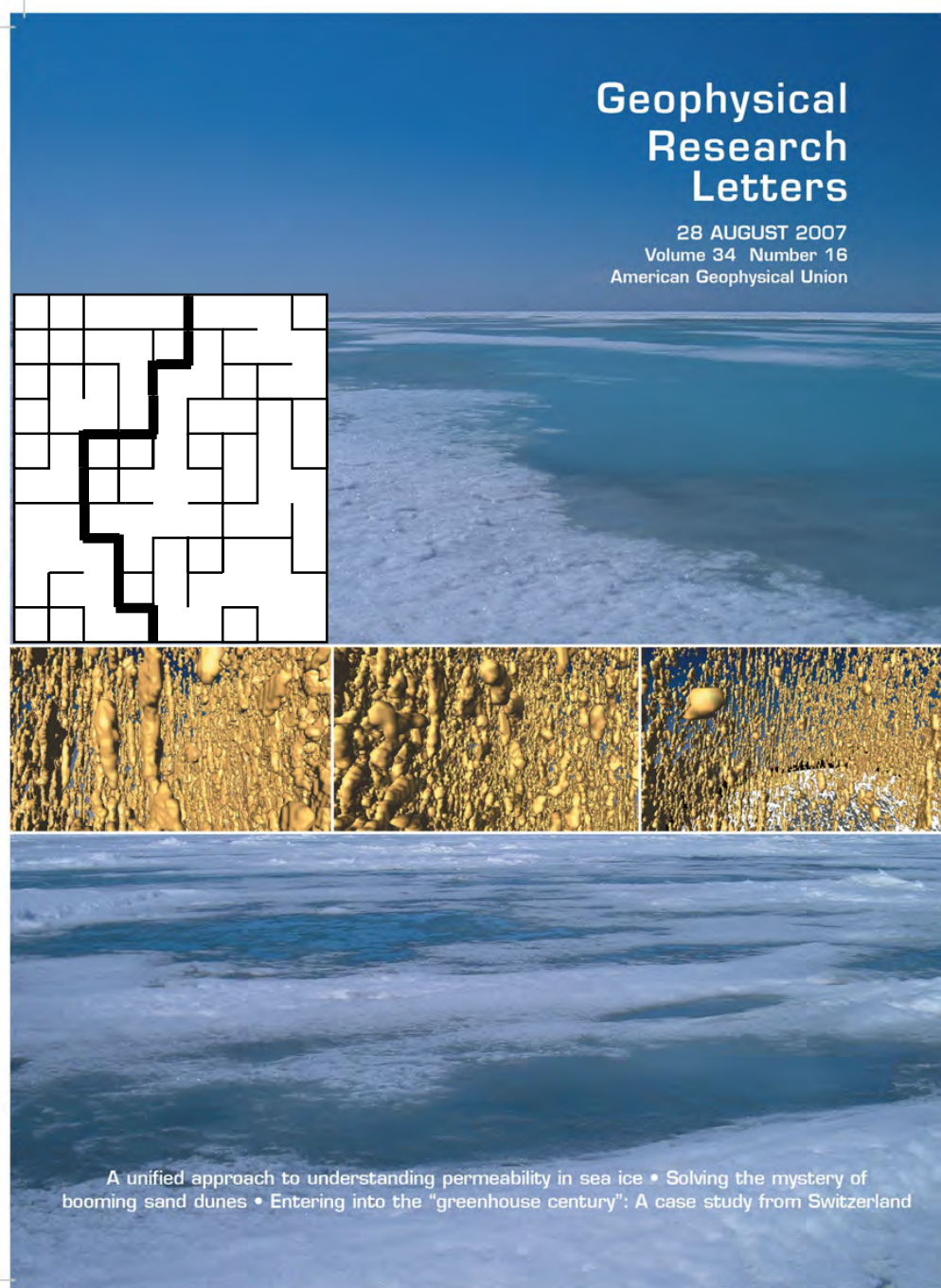
B-2 Stealth Bomber
F-117 Nighthawk
F-35

stealth

continuum percolation

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton*, Miner, Pringle, Zhu, *Geophysical Research Letters* 2007



**percolation theory
for fluid permeability**

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical
exponent
 t

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

from critical path analysis
in **hopping conduction**

hierarchical model

rock physics

network model

rigorous bounds

**X-ray tomography for
brine inclusions**

confirms rule of fives

*Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009*

**theories agree closely
with field data**

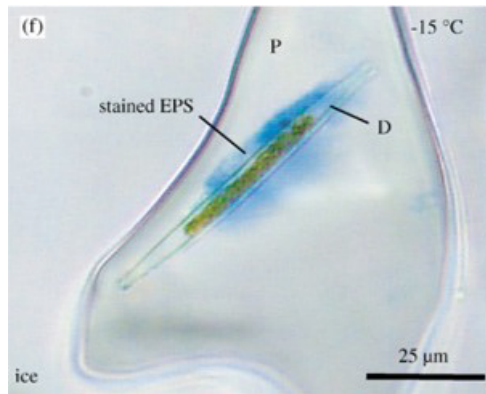
microscale
governs

mesoscale
processes

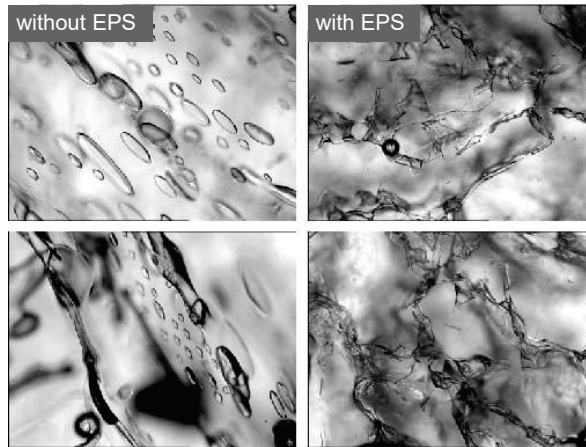
**melt pond
evolution**

Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

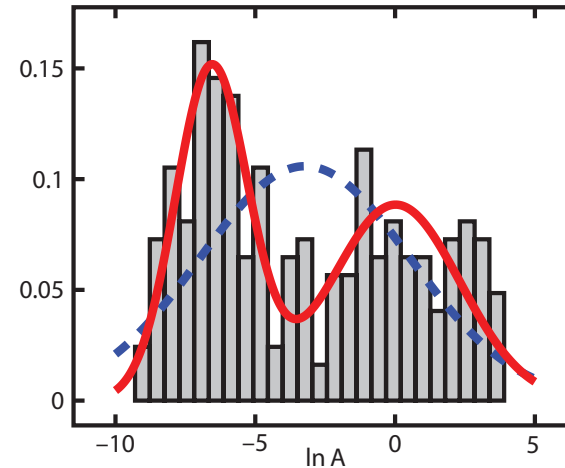
How does EPS affect fluid transport? How does the biology affect the physics?



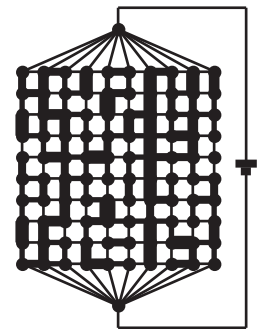
Krembs



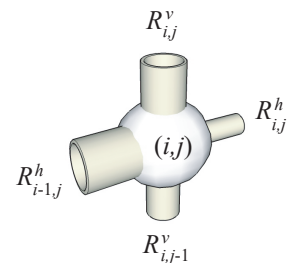
Krembs, Eicken, Deming, PNAS 2011



**RANDOM
PIPE
MODEL**



- 2D random pipe model with bimodal distribution of pipe radii
- Rigorous bound on permeability k ; results predict observed drop in k

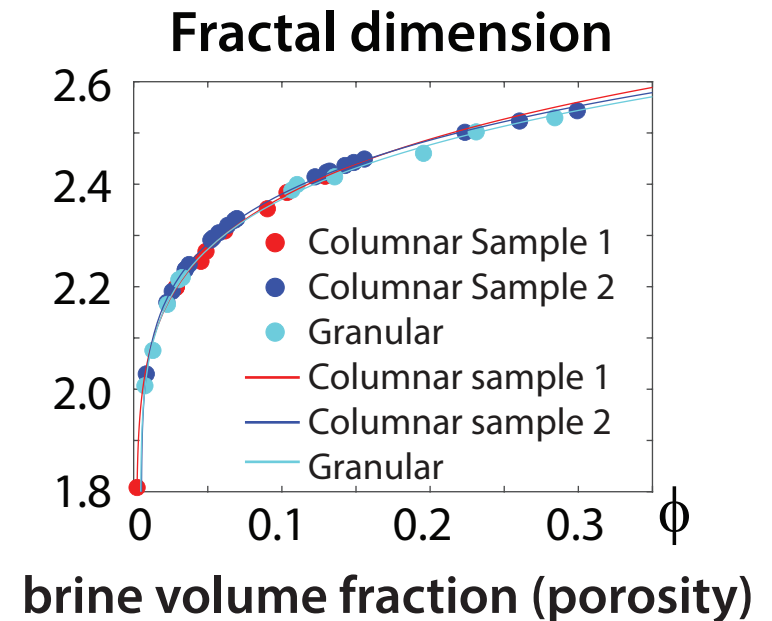
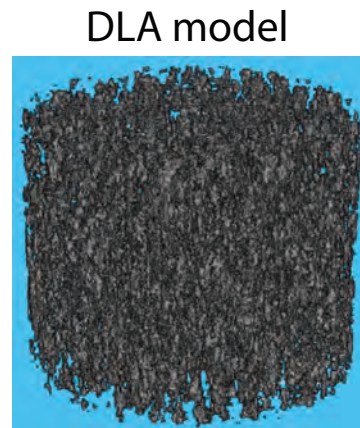
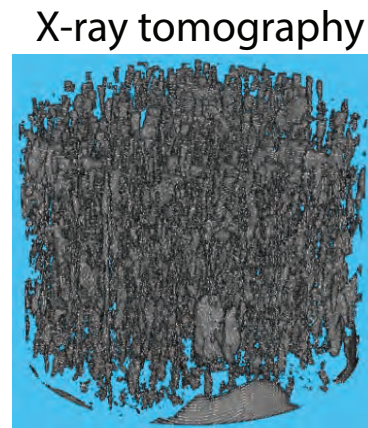
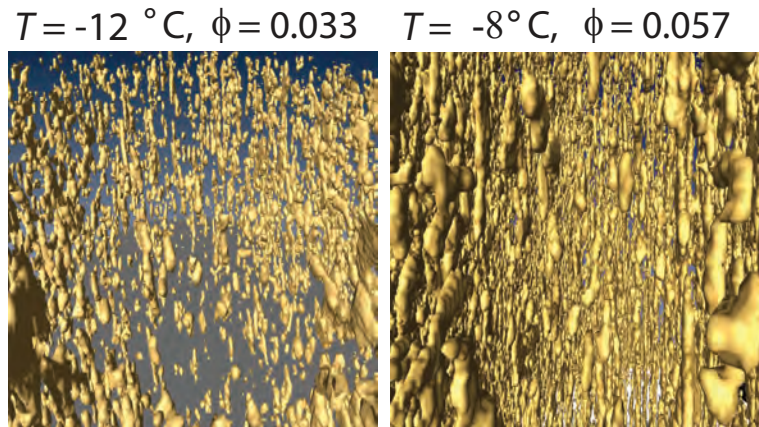


Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
Multiscale Modeling and Simulation, 2018

Zhu, Jabini, Golden,
Eicken, Morris
Ann. Glac. 2006

Thermal evolution of the fractal geometry of the brine microstructure in sea ice

N. Ward, D. Hallman, J. Reimer, H. Eicken, M. Oggier and K. M. Golden, 2022



theory of porosity as a
function of fractal dimension

invert

excellent correspondence with data

+ implications for brine phase as a habitat

Katz and Thompson, *PRL*, 1985

Arctic and Antarctic field experiments

*develop electromagnetic methods
of monitoring fluid transport and
microstructural transitions*

extensive measurements of fluid and
electrical transport properties of sea ice:

2007 Antarctic SIPEX

2010 Antarctic McMurdo Sound

2011 Arctic Barrow AK

2012 Arctic Barrow AK

2012 Antarctic SIPEX II

2013 Arctic Barrow AK

2014 Arctic Chukchi Sea



Notices

of the American Mathematical Society

May 2009

Volume 56, Number 5

Climate Change and
the Mathematics of
Transport in Sea Ice

page 562

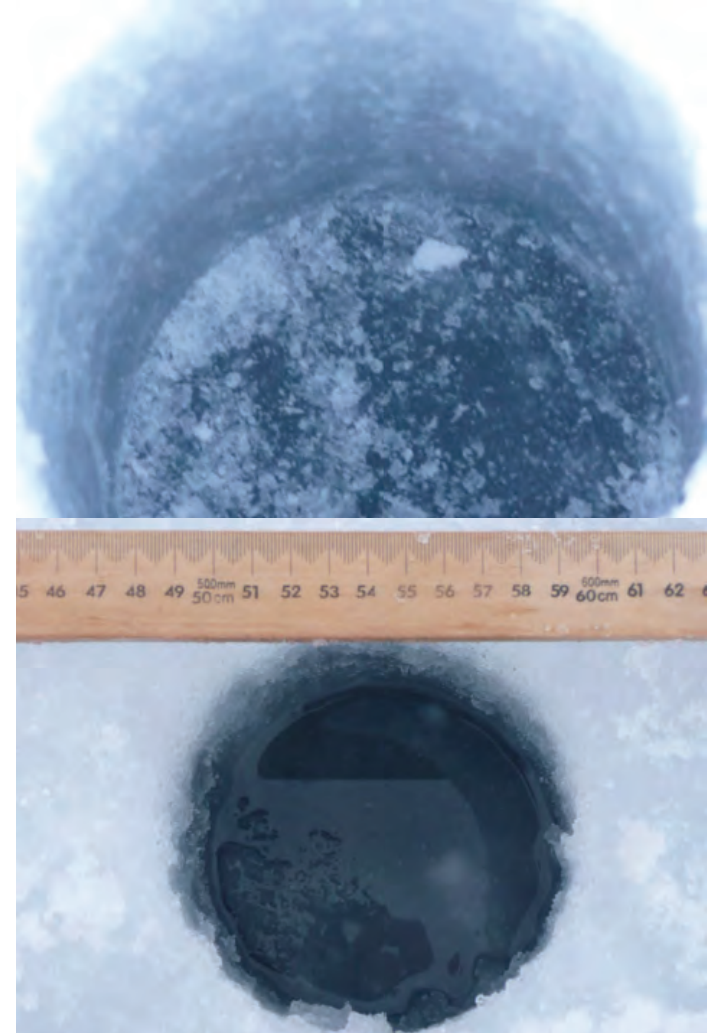
Mathematics and the
Internet: A Source of
Enormous Confusion
and Great Potential

page 586



photo by Jan Lieser

Real analysis in polar coordinates (see page 613)



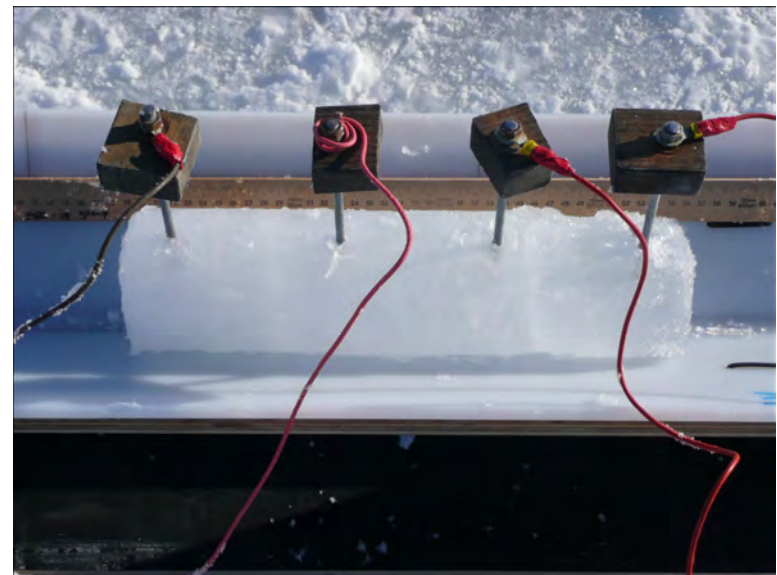
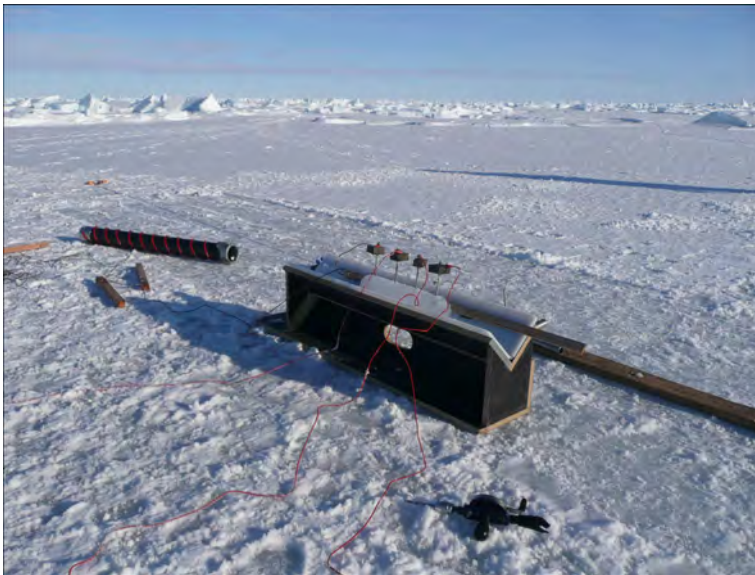
***measuring
fluid permeability
of Antarctic sea ice***

SIPEX 2007

electrical measurements



Wenner array



vertical conductivity

Zhu, Golden, Gully, Sampson *Physica B* 2010

Sampson, Golden, Gully, Worby *Deep Sea Research* 2011

cross borehole tomography



***Ingham, Jones, Buchanan
Victoria University, Wellington, NZ***

Measuring sea ice thickness





Remote sensing of sea ice



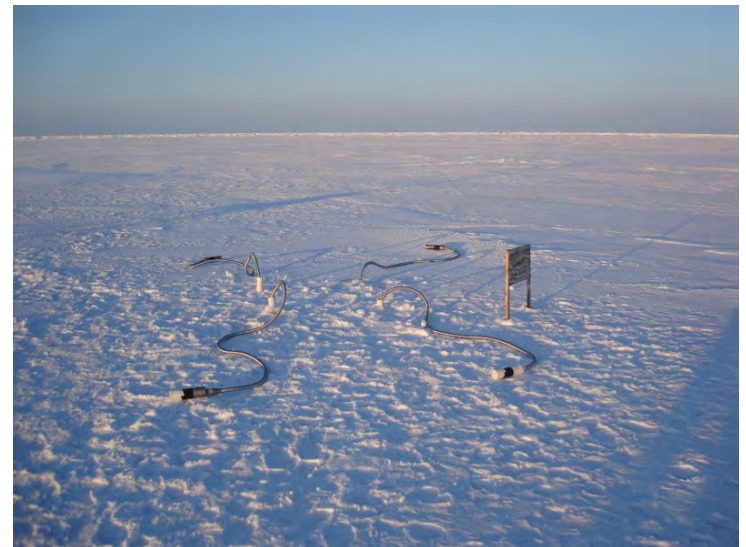
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice
properties from
electromagnetic
(EM) data

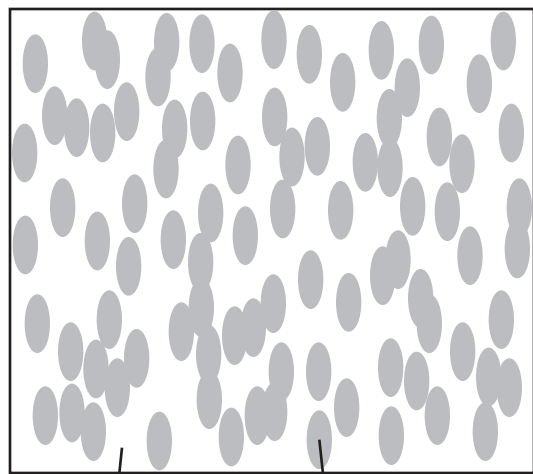
$$\epsilon^*$$

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



ϵ_1

ϵ_2

} ϵ^*

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1, p_2 = volume fractions of
the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2}, \text{ composite geometry} \right)$$

**What are the effective propagation characteristics
of an EM wave (radar, microwaves) in the medium?**

Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

separates geometry from parameters

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

← geometry

← material parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n -point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla.$$

χ = characteristic function of the brine phase

$$E = s (s + \Gamma\chi)^{-1} e_k$$

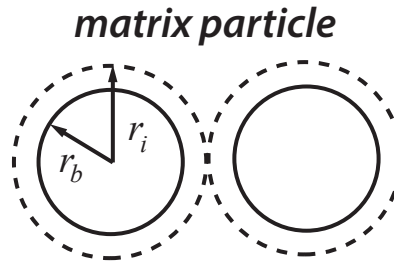
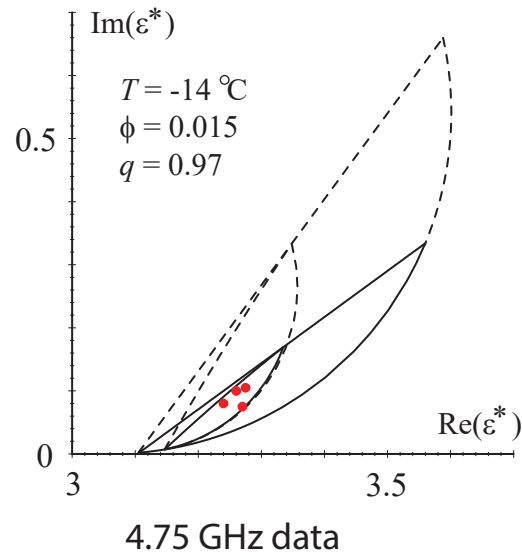
$\Gamma\chi$: microscale \rightarrow macroscale

$\Gamma\chi$ *links scales*

This representation distills the complexities of mixture geometry into the spectral properties of an operator like the Hamiltonian in physics.

forward and inverse bounds on the complex permittivity of sea ice

forward bounds

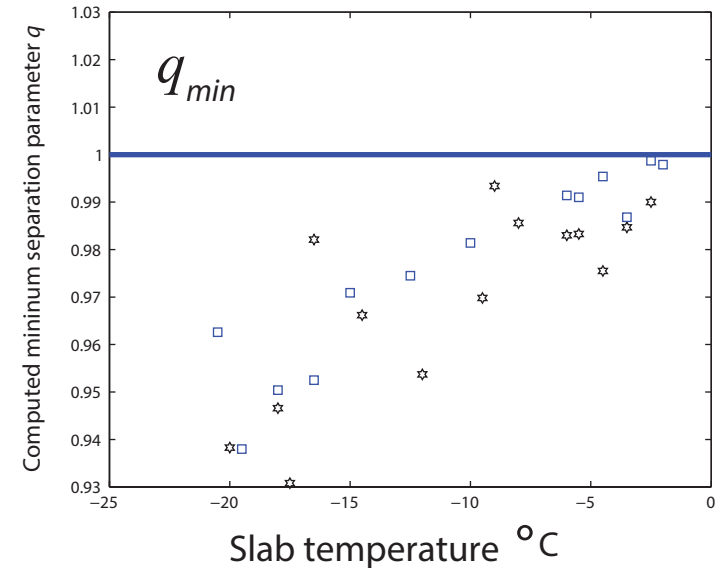


$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

inverse bounds



Inverse Homogenization

Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001), McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

ϵ^* \longrightarrow composite geometry
(spectral measure μ)

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

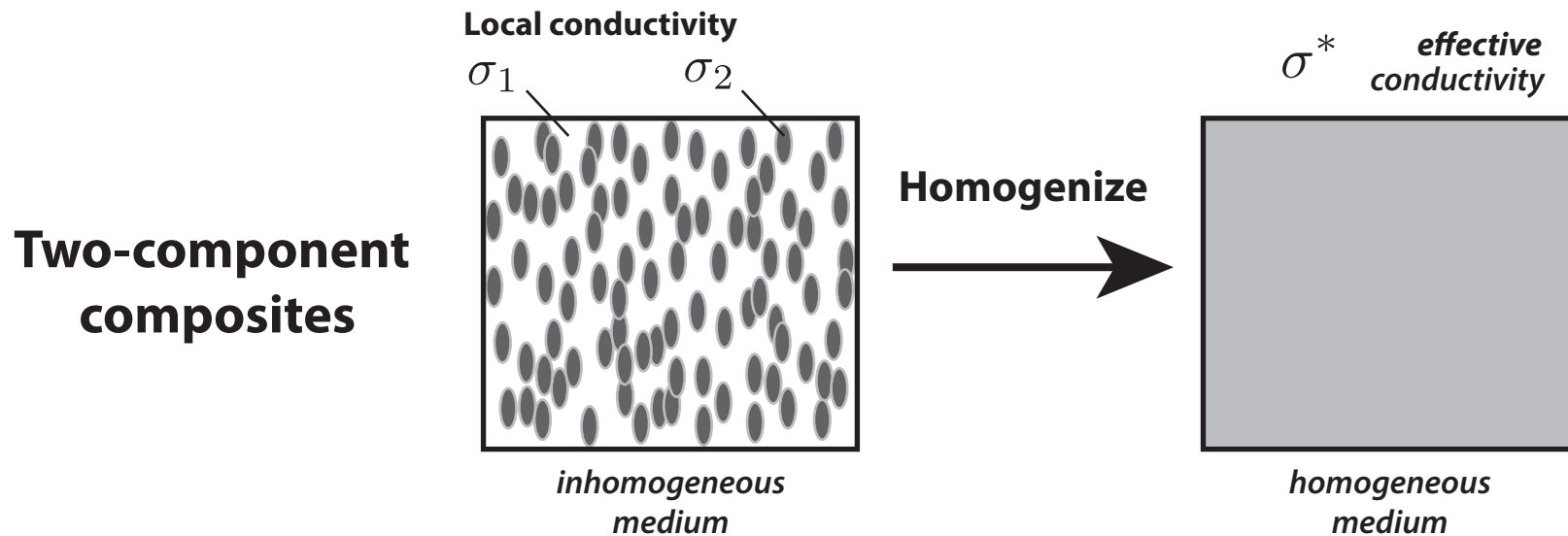
inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ϵ^*

rigorous inverse bound on spectral gap

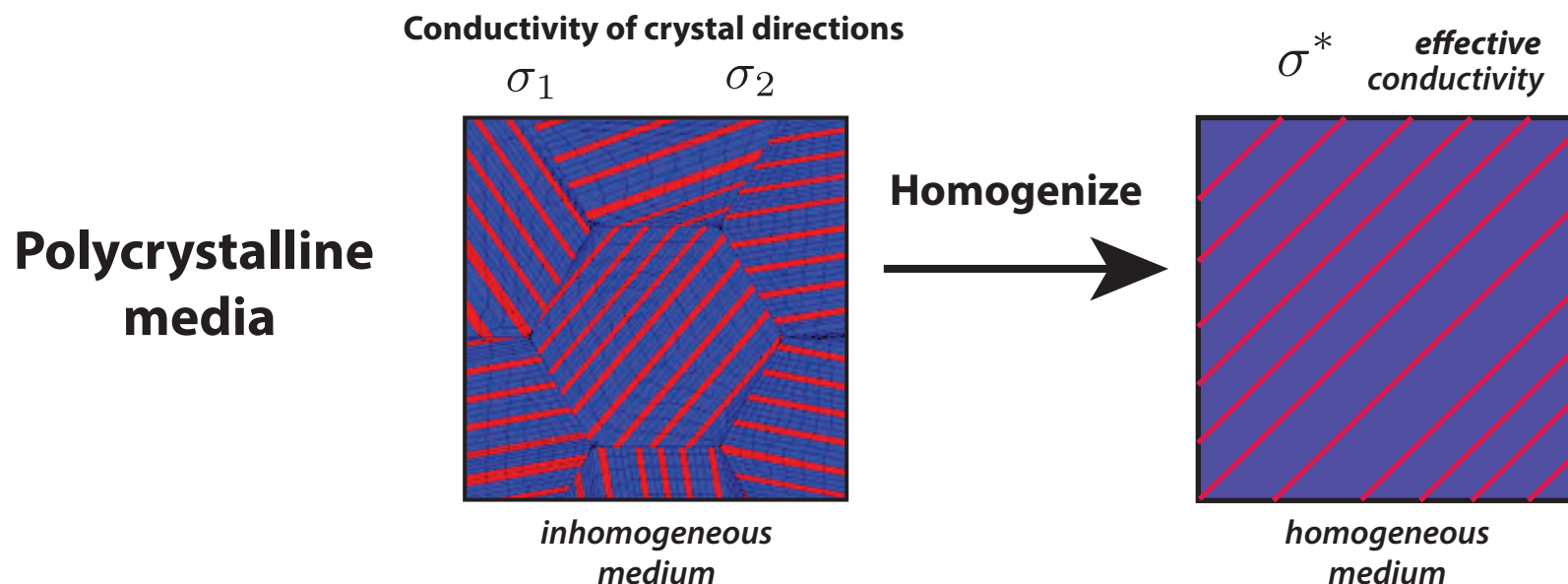
construct algebraic curves which bound admissible region in (p, q) -space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

Homogenization for polycrystalline materials



Find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium



Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**
Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**
orientation statistics
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



PROCEEDINGS A

350 YEARS
OF SCIENTIFIC
PUBLISHING

An invited review
commemorating 350 years
of scientific publishing at the
Royal Society

A method to distinguish
between different types
of sea ice using remote
sensing techniques

A computer model to
determine how a human
should walk so as to expend
the least energy



THE
ROYAL
SOCIETY
PUBLISHING

higher threshold for fluid flow in granular sea ice

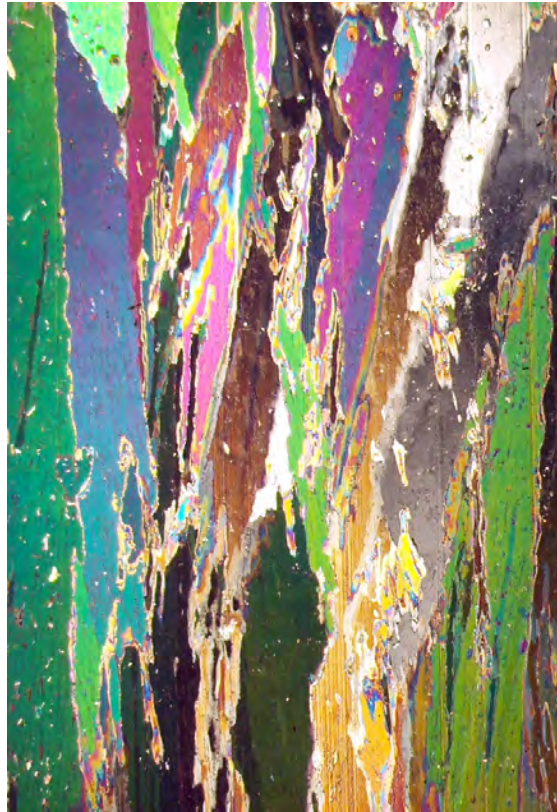
microscale details impact “mesoscale” processes

nutrient fluxes for microbes
melt pond drainage
snow-ice formation

columnar

granular

5%



10%



Golden, Sampson, Gully, Lubbers, Tison 2022

electromagnetically distinguishing ice types
Kitsel Lusted, Elena Cherkaev, Ken Golden

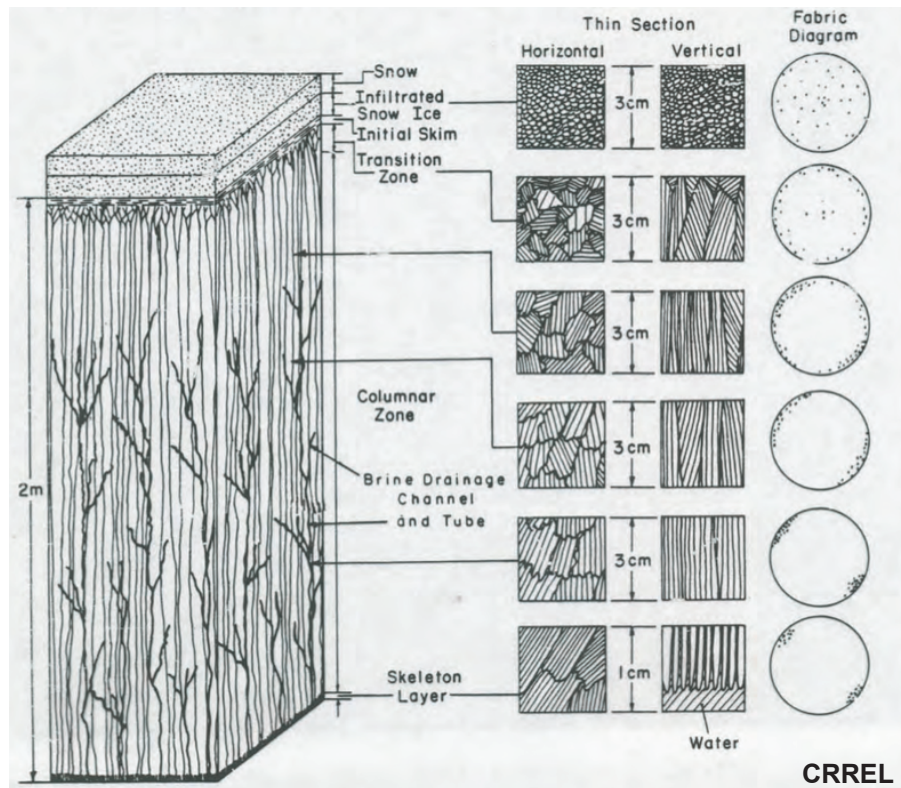
Rigorous bounds on the complex permittivity tensor of sea ice with polycrystalline anisotropy in the horizontal plane

Kenzie McLean, Elena Cherkaev, Ken Golden 2022

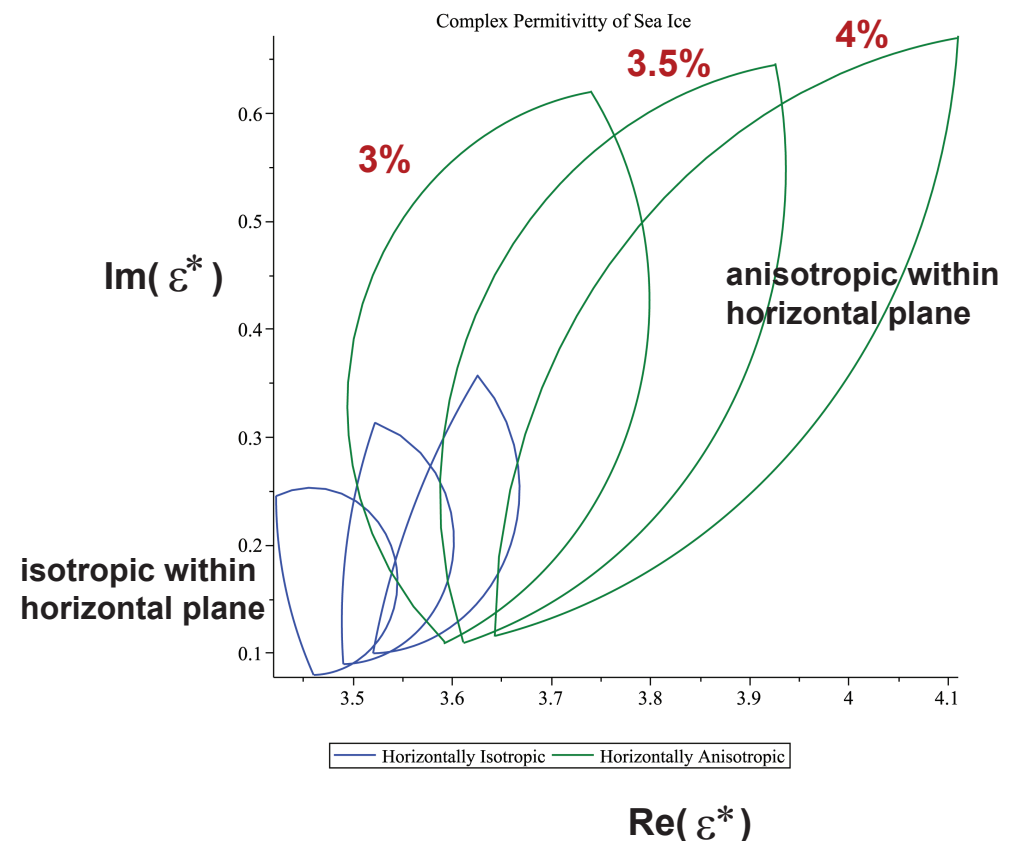
motivated by **Weeks and Gow, *JGR* 1979: c-axis alignment in Arctic fast ice off Barrow**

Golden and Ackley, *JGR* 1981: radar propagation model in aligned sea ice

input: orientation statistics



output: bounds



mesoscale

wave propagation in the marginal ice zone (MIZ)

Stieltjes integral representation and bounds for the complex viscoelasticity of the ice - ocean layer

Sampson, Murphy, Cherkaev, Golden 2022

first theory of key parameter in wave-ice interactions only fitted to wave data before

Keller, 1998

Mosig, Montiel, Squire, 2015

Wang, Shen, 2012

Analytic Continuation Method

Bergman (78) - Milton (79)
integral representation for ϵ^*

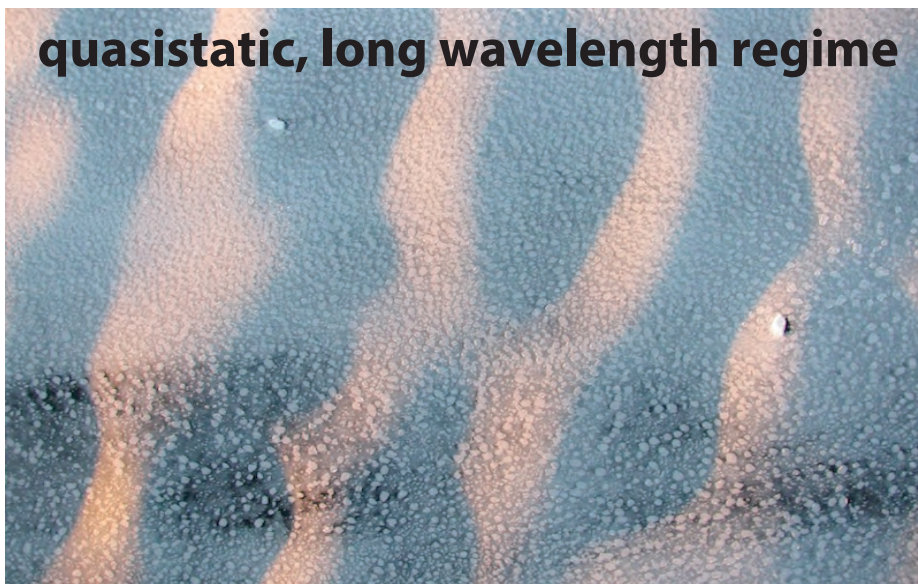
Golden and Papanicolaou (83)

Milton, *Theory of Composites* (02)

quasistatic, long wavelength regime

homogenized
parameter
depends on
sea ice
concentration
and ice floe
geometry

like EM waves

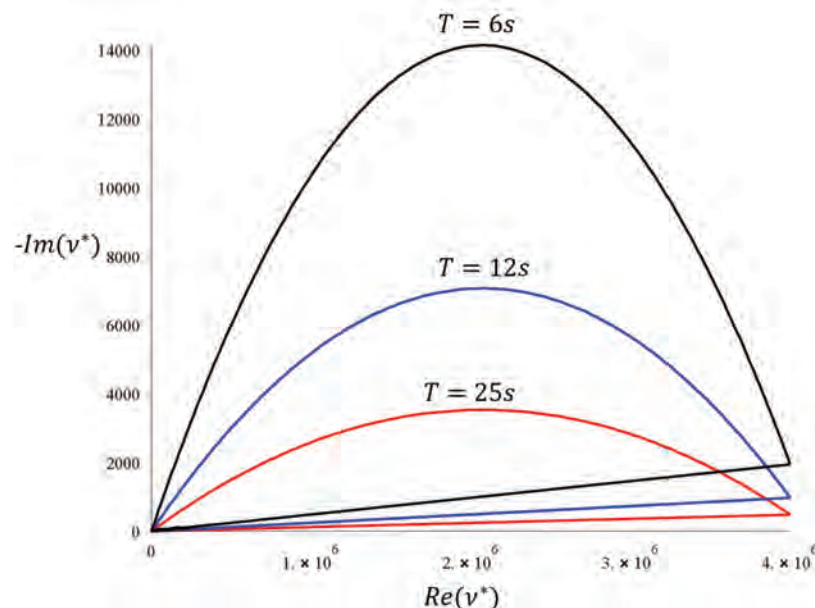


bounds on the effective complex viscoelasticity

$$V_1 = 10^7 + i 4875 \quad \text{pancake ice}$$

$$V_2 = 5 + i 0.0975 \quad \text{slush / frazil}$$

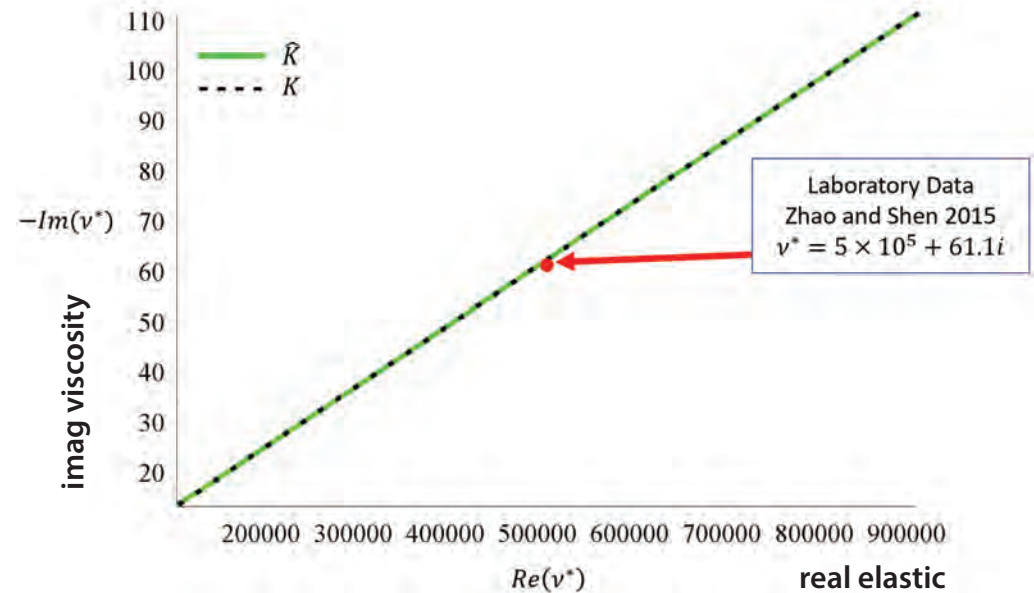
complex elementary bounds
(fixed area fraction of floes)



Elementary bounds for wave periods T .

high contrast

matrix-particle bounds



pancake ice

Golden

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere

advection diffusion equation with a velocity field \vec{u}

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$



homogenize

$$\frac{\partial \bar{T}}{\partial t} = \kappa^* \Delta \bar{T}$$

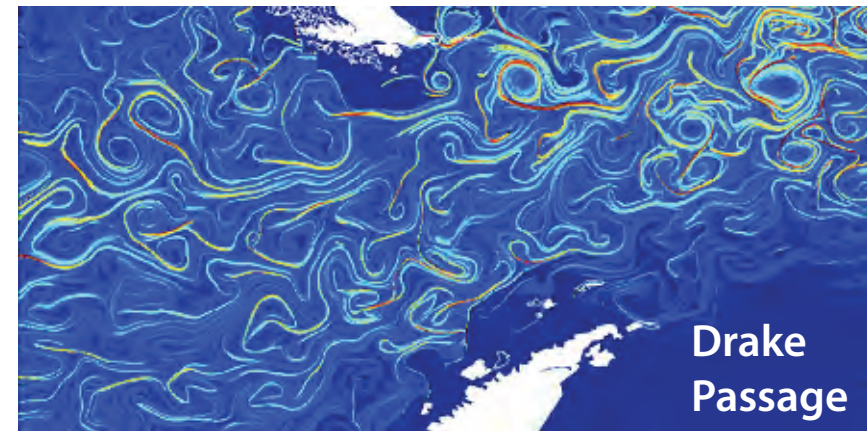
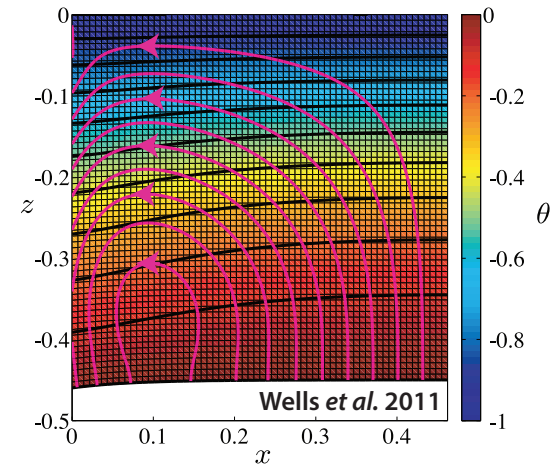
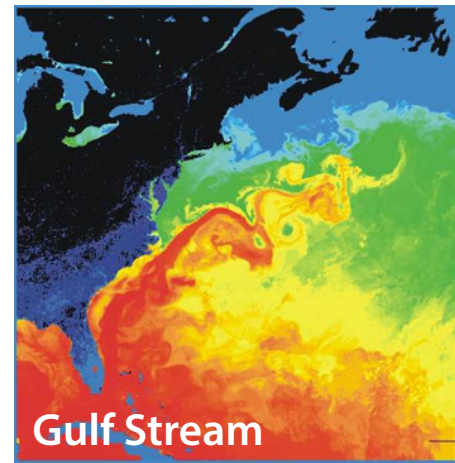
κ^* effective diffusivity

Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020



tracers flowing through inverted sea ice blocks



Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

$$\kappa^* = \kappa \left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2} \right), \quad F(\kappa) = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}$$

- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H\Gamma$
- H = stream matrix , κ = local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla$, Δ is the Laplace operator
- $i\Gamma H\Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ -plane

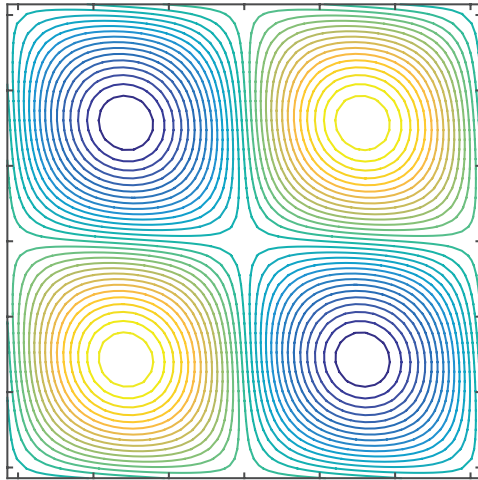
rigorous framework for numerical computations of spectral measures and effective diffusivity for model flows

new integral representations, theory of moment calculations

separation of material properties and flow field

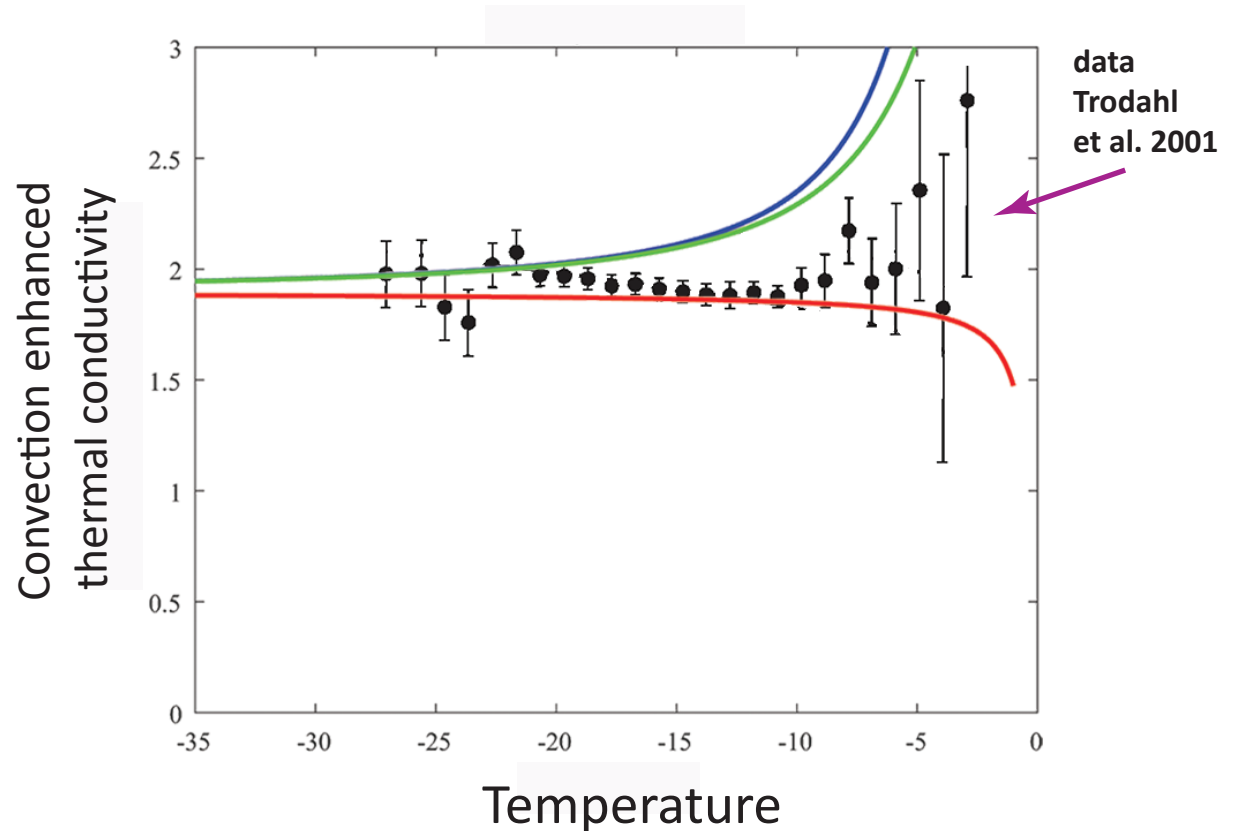
Rigorous bounds on convection enhanced thermal conductivity of sea ice

Kraitzman, Hardenbrook, Dinh, Murphy, Zhu, Cherkaev, Golden 2022



cat's eye flow model for
brine convection cells

similar bounds
for shear flows



rigorous Padé bounds from Stieltjes integral +
analytical calculations of moments of measure

direct calculation of spectral measures

Murphy, Hohenegger, Cherkaev, Golden, *Comm. Math. Sci.* 2015

- depends only on the composite geometry
- discretization of microstructural image gives binary network
- fundamental operator becomes a random matrix
- spectral measure computed from eigenvalues and eigenvectors

**once we have the spectral measure μ it can be used in
Stieltjes integrals for other transport coefficients:**

***electrical and thermal conductivity, complex permittivity,
magnetic permeability, diffusion, fluid flow properties***

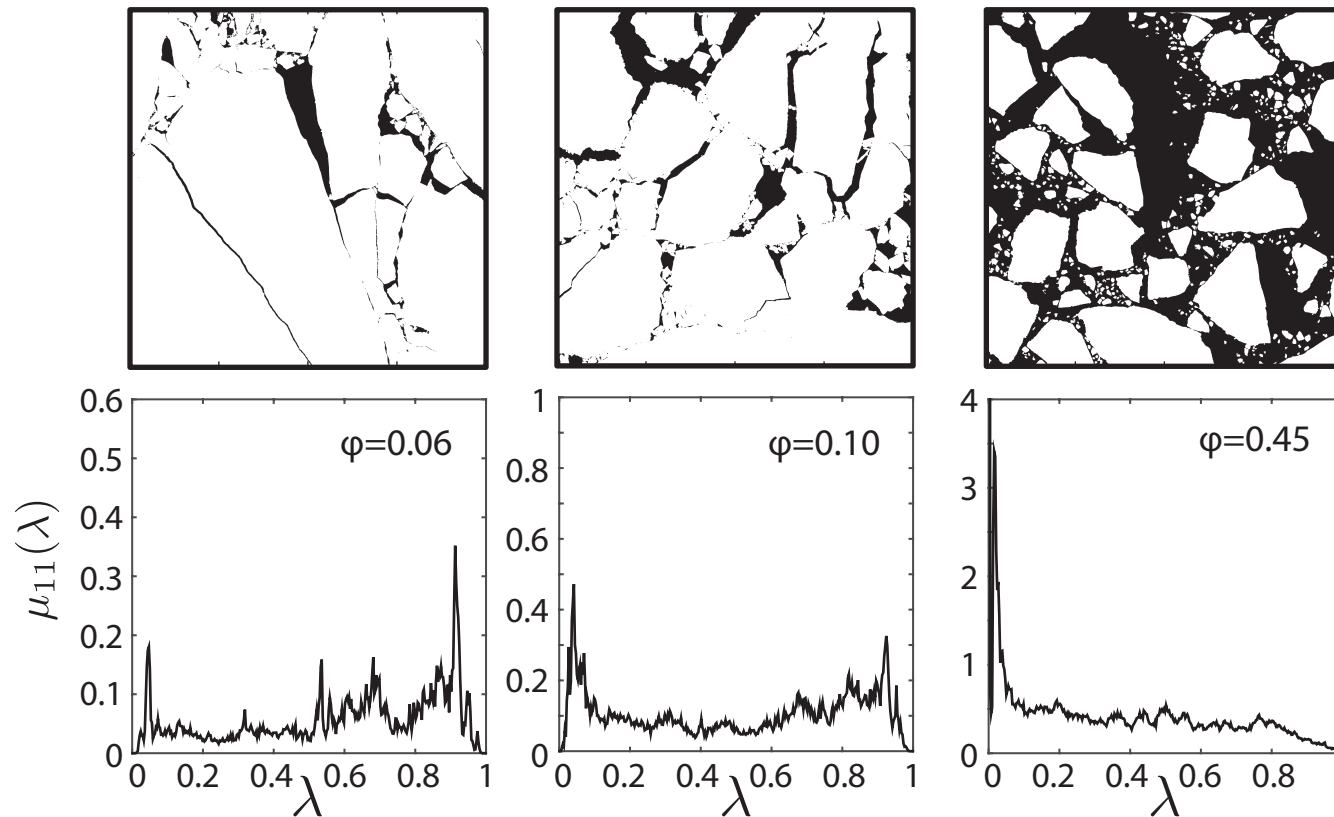
earlier studies of spectral measures

Day and Thorpe 1996

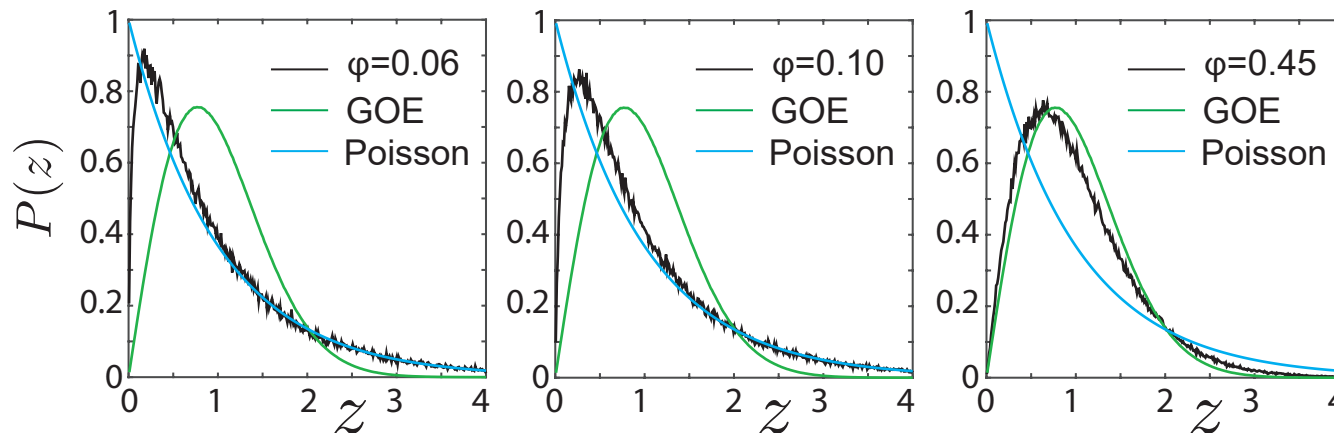
Helsing, McPhedran, Milton 2011

Spectral computations for sea ice floe configurations

spectral
measures



eigenvalue
spacing
distributions



uncorrelated



level repulsion

UNIVERSAL
Wigner-Dyson
distribution

Eigenvalue Statistics of Random Matrix Theory

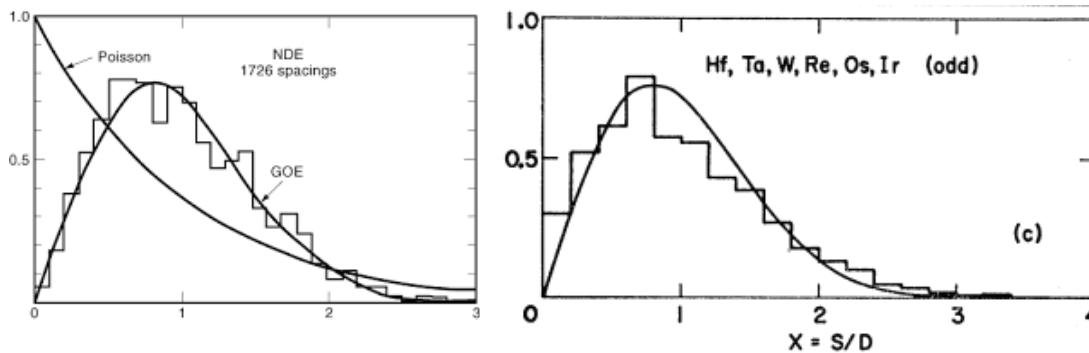
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[N]_{ij} \sim N(0,1), \quad A = (N + N^T)/2 \quad \text{Gaussian orthogonal ensemble (GOE)}$

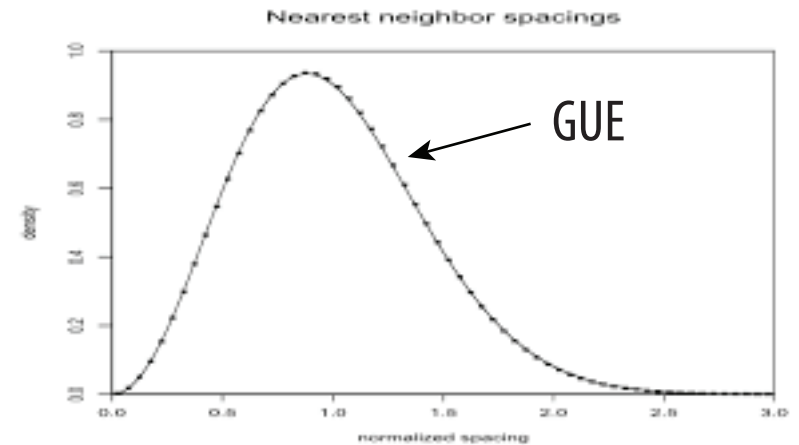
$[N]_{ij} \sim N(0,1) + iN(0,1), \quad A = (N + N^\dagger)/2 \quad \text{Gaussian unitary ensemble (GUE)}$

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics.

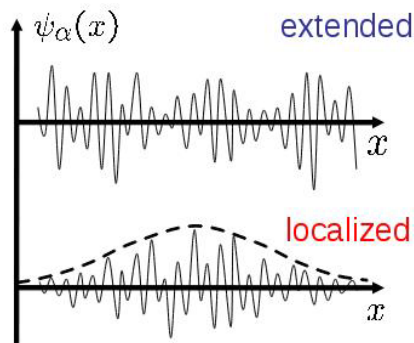
Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



Universal eigenvalue statistics arise in a broad range of “unrelated” problems!



electronic transport in semiconductors

metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

**Anderson transition in wave physics:
 quantum, optics, acoustics, water waves, ...**

from analysis of spectral measures for brine, melt ponds, ice floes

we find percolation-driven

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

**PERCOLATION
 TRANSITION**



**universal eigenvalue statistics (GOE)
 extended states, mobility edges**

-- but with NO wave interference or scattering effects ! --

local conductivity in 1D inhomogeneous material

$$\sigma(x) = 3 + \cos x + \cos kx$$

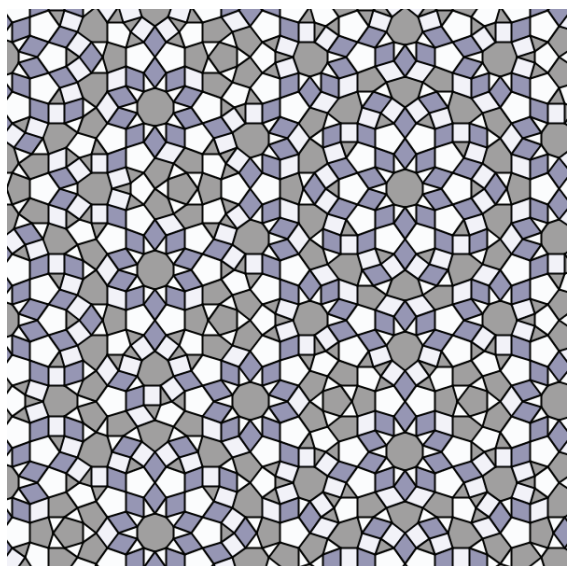
effective conductivity

$$\sigma^*(k) = \begin{cases} \text{constant} & k \text{ irrational} & \text{quasiperiodic} \\ f(k) & k \text{ rational} & \text{periodic} \end{cases}$$

Golden, Goldstein, Lebowitz, Phys. Rev. Lett. 1985

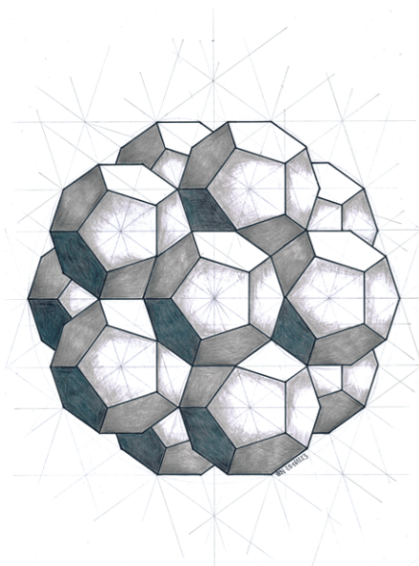
Order to Disorder in Quasiperiodic Composites

D. Morison (Physics), N. B. Murphy, E. Cherkaev, K. M. Golden, *Communications Physics* 2022



quasiperiodic checkerboard

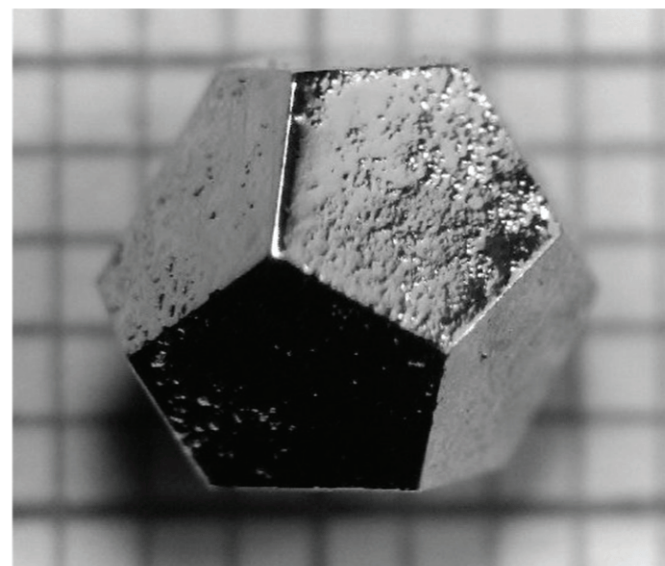
Stampfli, 2013



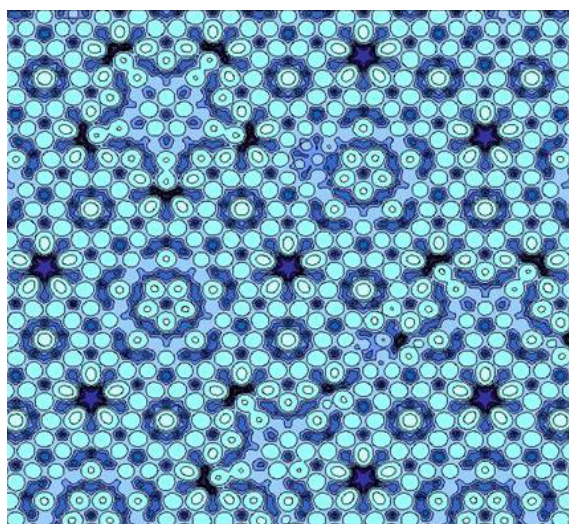
dense packing of dodecahedra

3D Penrose tiling

Tripkovic, 2019



Holmium-magnesium-zinc quasicrystal



energy surface Al-Pd-Mn quasicrystal

Unal et al., 2007

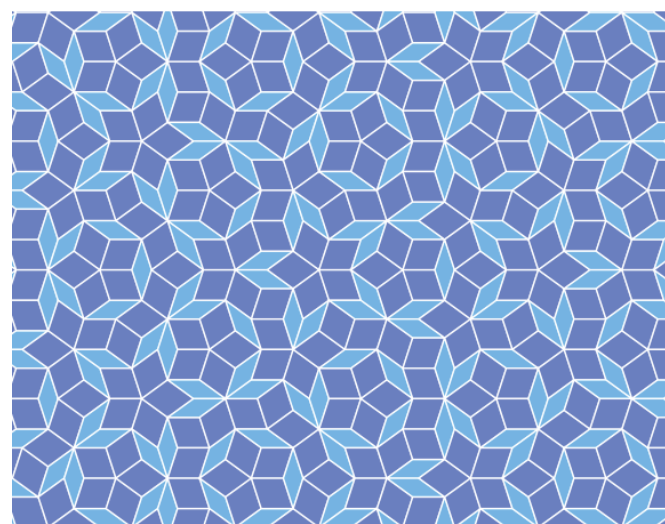
quasiperiodic crystal
quasicrystal

ordered but aperiodic

lacks translational symmetry

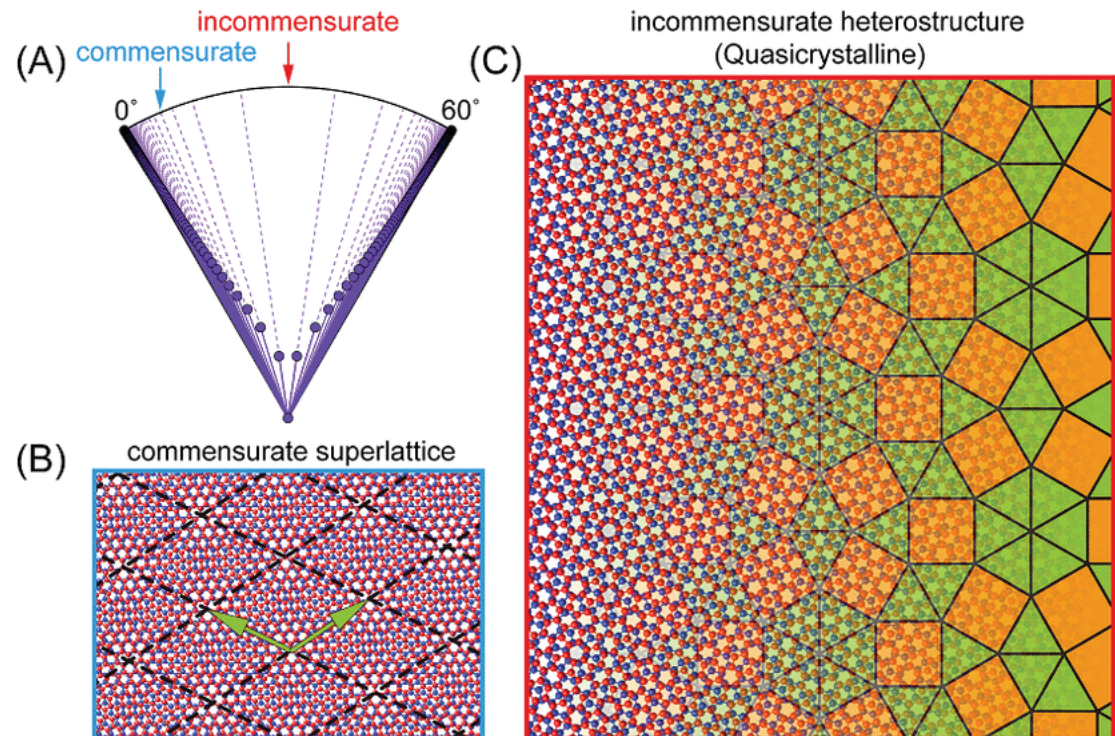
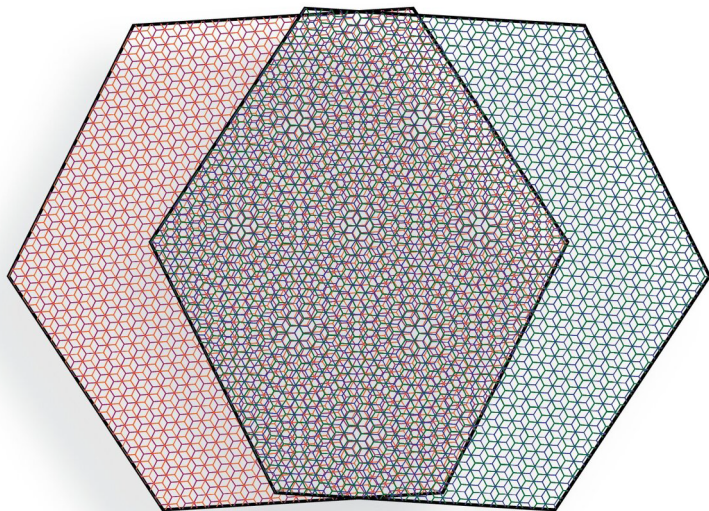
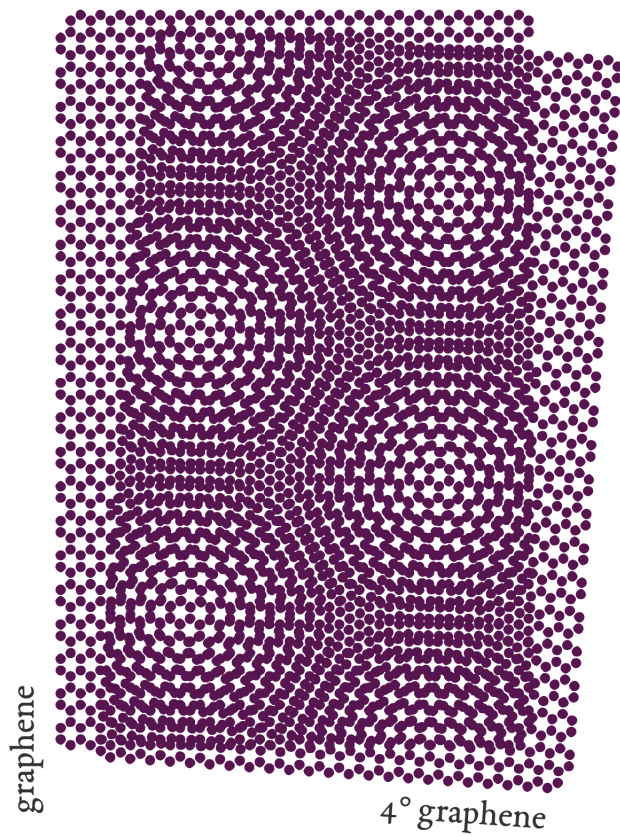
Schechtman et al., 1984

Levine & Steinhardt, 1984

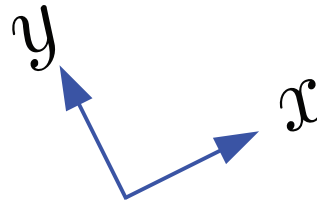
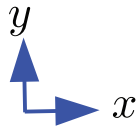


aperiodic tiling of the plane - R. Penrose 1970s

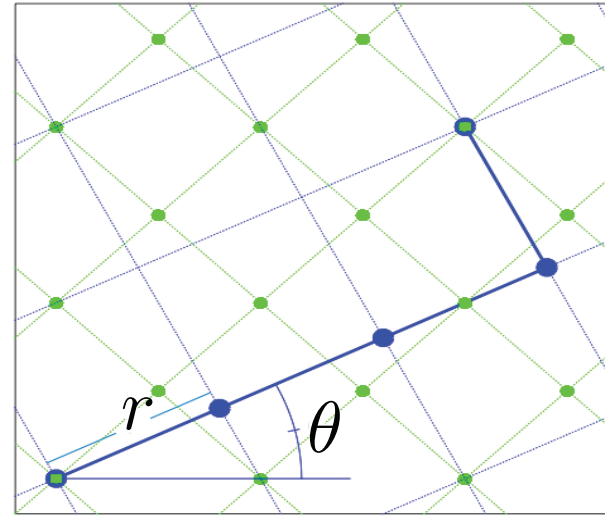
twisted bilayer graphene



Moiré patterns generate two component composites



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = r \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

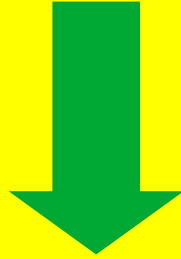


$$\psi(x', y') = \cos 2\pi x' \cos 2\pi y'$$

$$\chi = \begin{cases} 1, & \psi \geq 0 \\ 0, & \psi < 0 \end{cases}$$

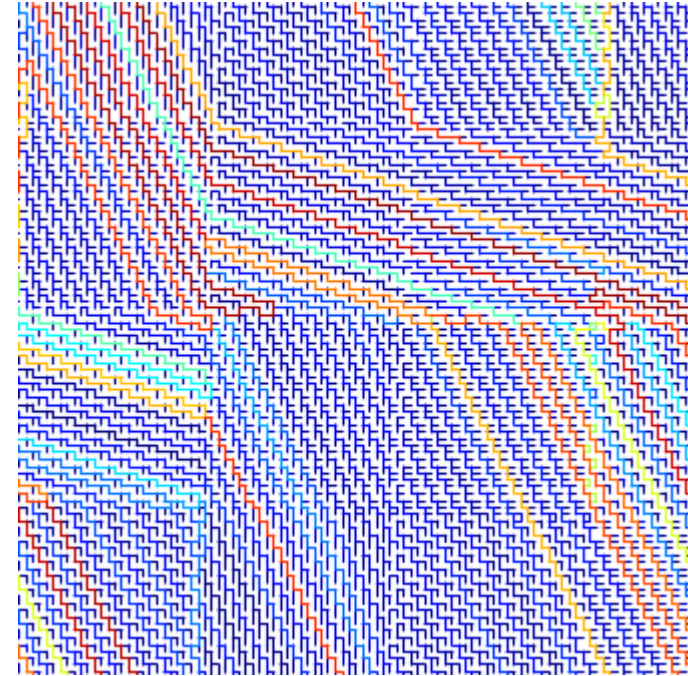
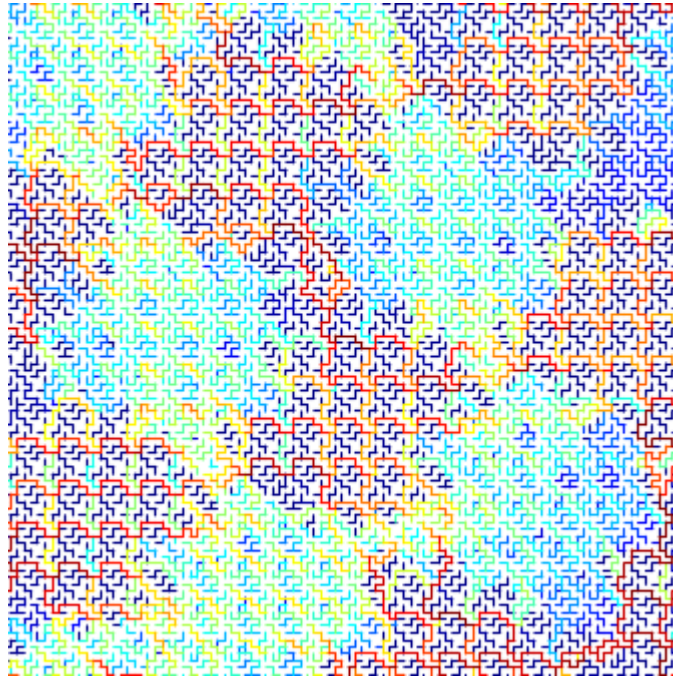
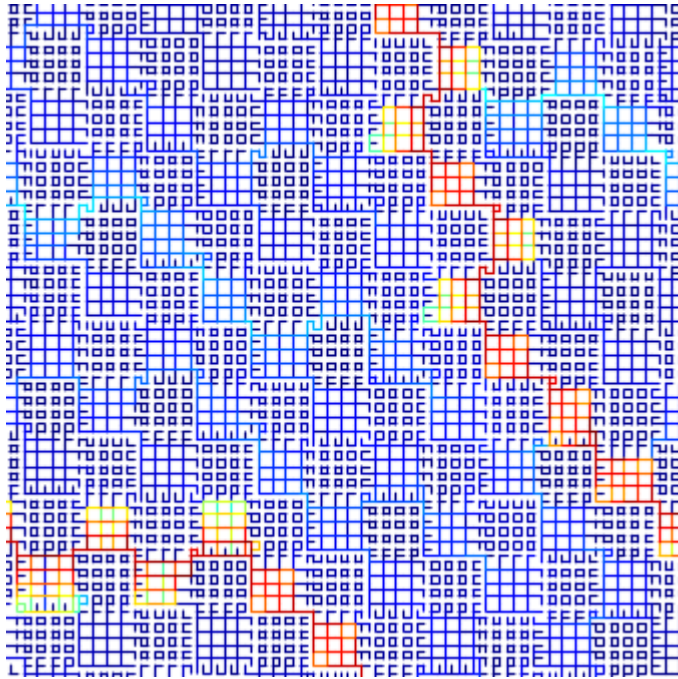
rotation and dilation

Small Difference in Moiré Parameters



Big Difference in Material Properties

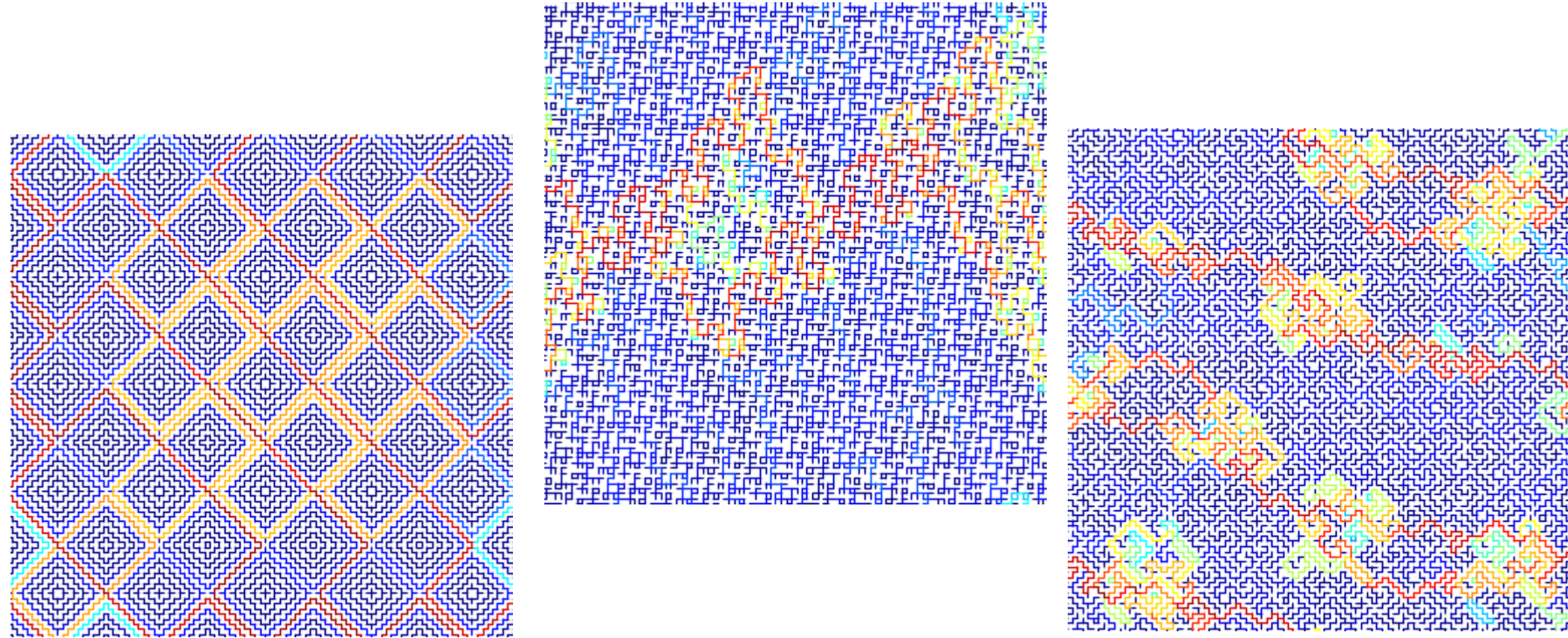
Wide Variety of Microgeometries



E



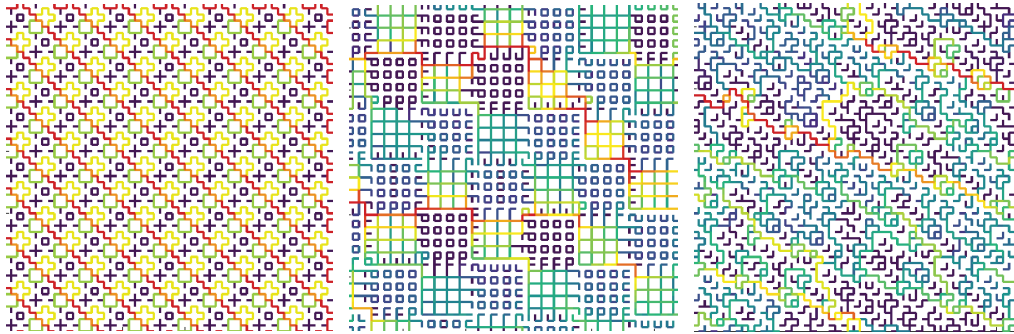
Wide Variety of Microgeometries



Order to disorder in quasiperiodic composites

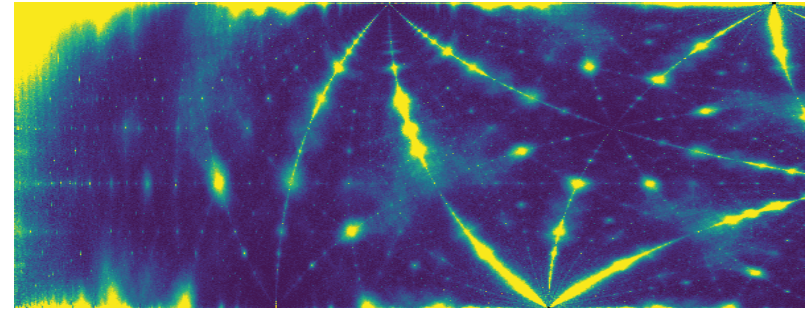
Morison, Murphy, Cherkaev, Golden, *Commun. Phys.* 2022

Parameterized Moiré Pattern Creates Tunable Microgeometry



constellation of periodic systems in a sea of randomness

Poisson
Wigner-Dyson



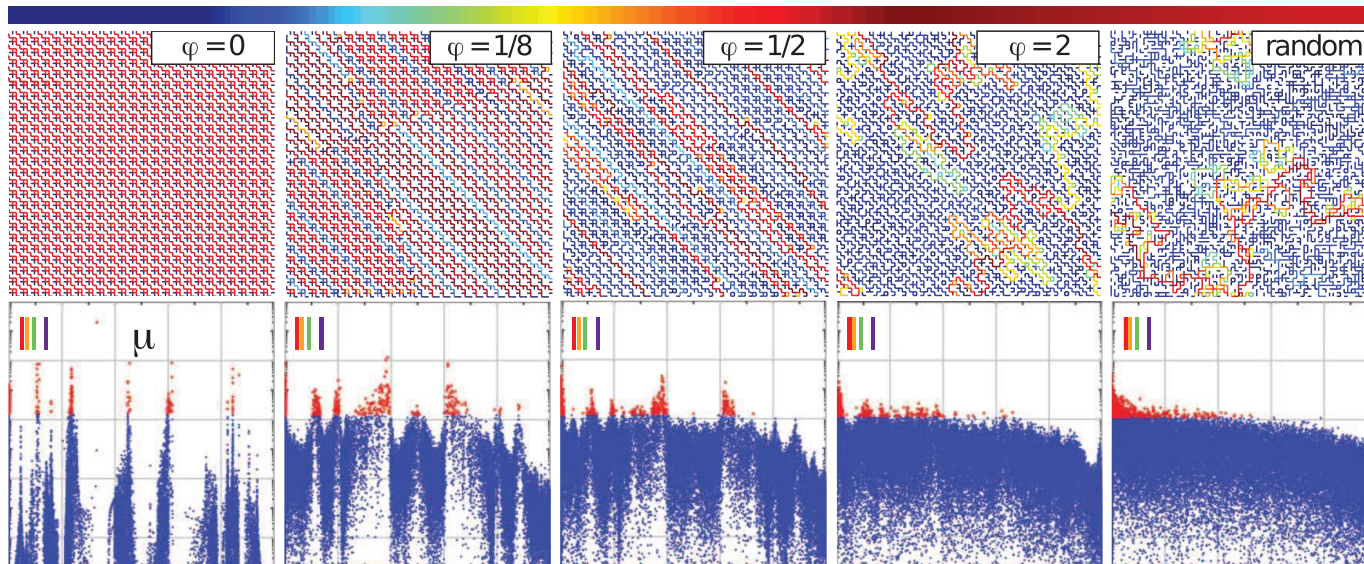
parameter space

periodic



quasiperiodic

electric field
strength

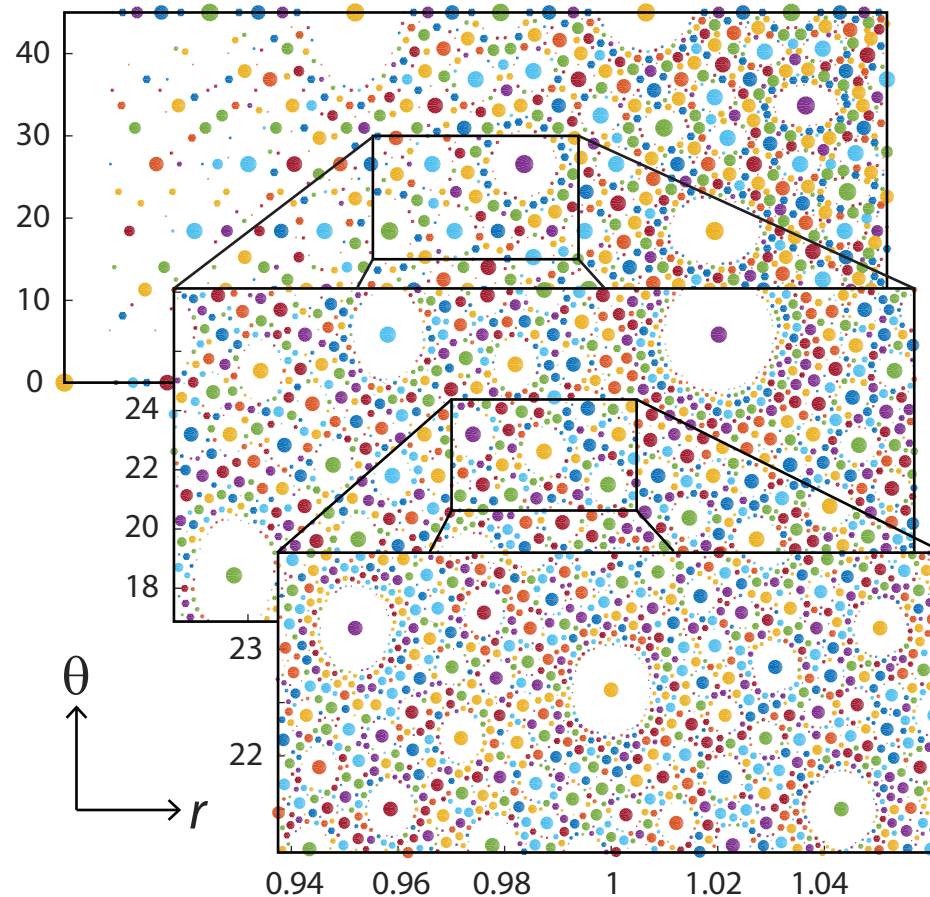


RRN at
percolation
threshold

we bring the framework of solid state physics of electronic transport and band gaps in semiconductors to classical transport in periodic and quasiperiodic composites

photonic crystals and quasicrystals

Fractal arrangement of periodic systems



Sequential insets zooming into smaller regions of parameter space.

size of the dots \sim length of period

(large dot \sim small period; small dot \sim large period; white space \sim "infinite" period)

Conclusions

1. Sea ice is a fascinating multiscale composite with structure similar to many other natural and man-made materials.
2. **Homogenization and statistical physics help *link scales in the sea ice system***; provide rigorous methods for finding effective behavior; advance sea ice representations in climate models.
3. **Stieltjes functions and their integral representations** provide powerful methods of homogenization for EM waves, advection diffusion, polycrystalline media, and ocean surface waves in the ice cover.
4. Mathematical methods developed for sea ice advance the theory of composites, and quasiperiodic media in particular.
5. Our research is helping to **improve projections of climate change**, the fate of Earth's sea ice packs, and the ecosystems they support.

University of Utah Sea Ice Modeling Group (2017-2021)

Senior Personnel: Ken Golden, Distinguished Professor of Mathematics
Elena Cherkaev, Professor of Mathematics
Court Strong, Associate Professor of Atmospheric Sciences
Ben Murphy, Adjunct Assistant Professor of Mathematics

Postdoctoral Researchers: Noa Kraitzman (now at ANU), Jody Reimer

Graduate Students: Kyle Steffen (now at UT Austin with Clint Dawson)
Christian Sampson (now at UNC Chapel Hill with Chris Jones)
Huy Dinh (now a sea ice MURI Postdoc at NYU/Courant)
Rebecca Hardenbrook
David Morison (Physics Department)
Ryleigh Moore
Delaney Mosier
Daniel Hallman

Undergraduate Students: Kenzie McLean, Jacqueline Cinella Rich,
Dane Gollero, Samir Suthar, Anna Hyde,
Kitsel Lusted, Ruby Bowers, Kimball Johnston,
Jerry Zhang, Nash Ward, David Gluckman

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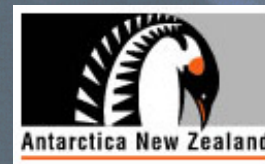
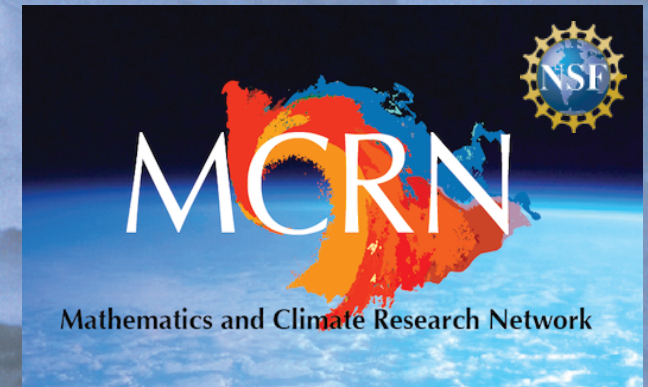
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National Science Foundation

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Division of Polar Programs



Buchanan Bay, Antarctica Mertz Glacier Polynya Experiment July 1999