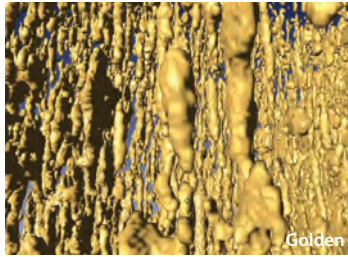
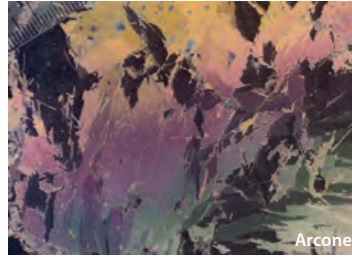


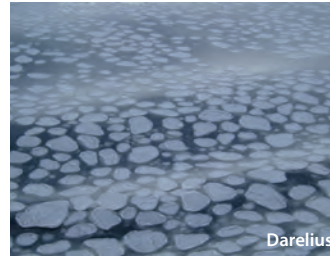
millimeters



centimeters



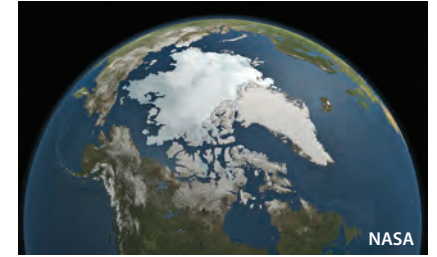
meters



kilometers



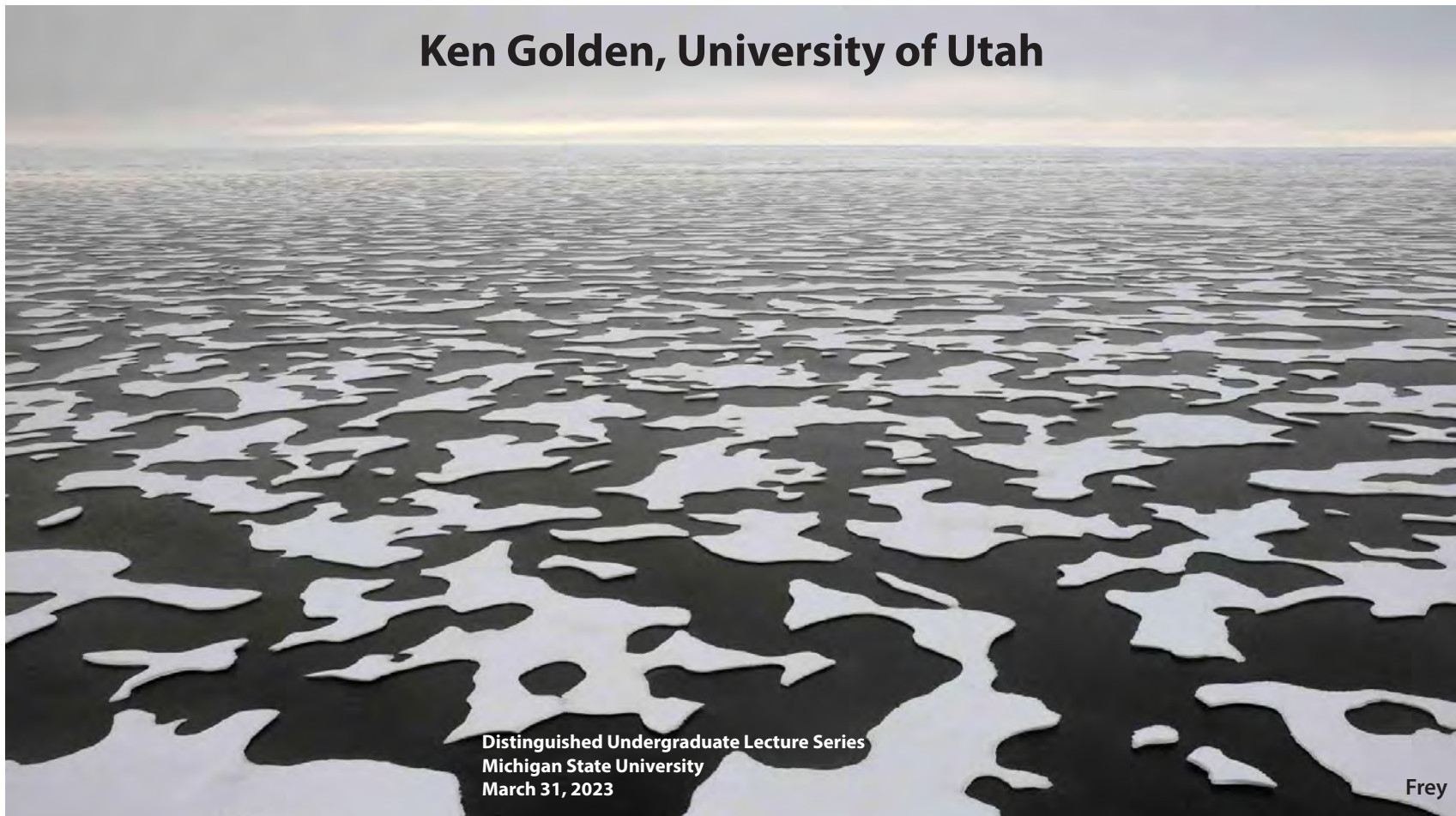
10^3 kilometers



Modeling the Melt

What math tells us about sea ice and polar ecosystems in a warming climate

Ken Golden, University of Utah



Distinguished Undergraduate Lecture Series
Michigan State University
March 31, 2023

Frey

SEA ICE covers ~12% of Earth's ocean surface

- boundary between ocean and atmosphere
- mediates exchange of heat, gases, momentum
- global ocean circulation
- hosts rich ecosystem
- indicator of **climate change**



polar ice caps critical to global climate in reflecting incoming solar radiation



white snow and ice
reflect

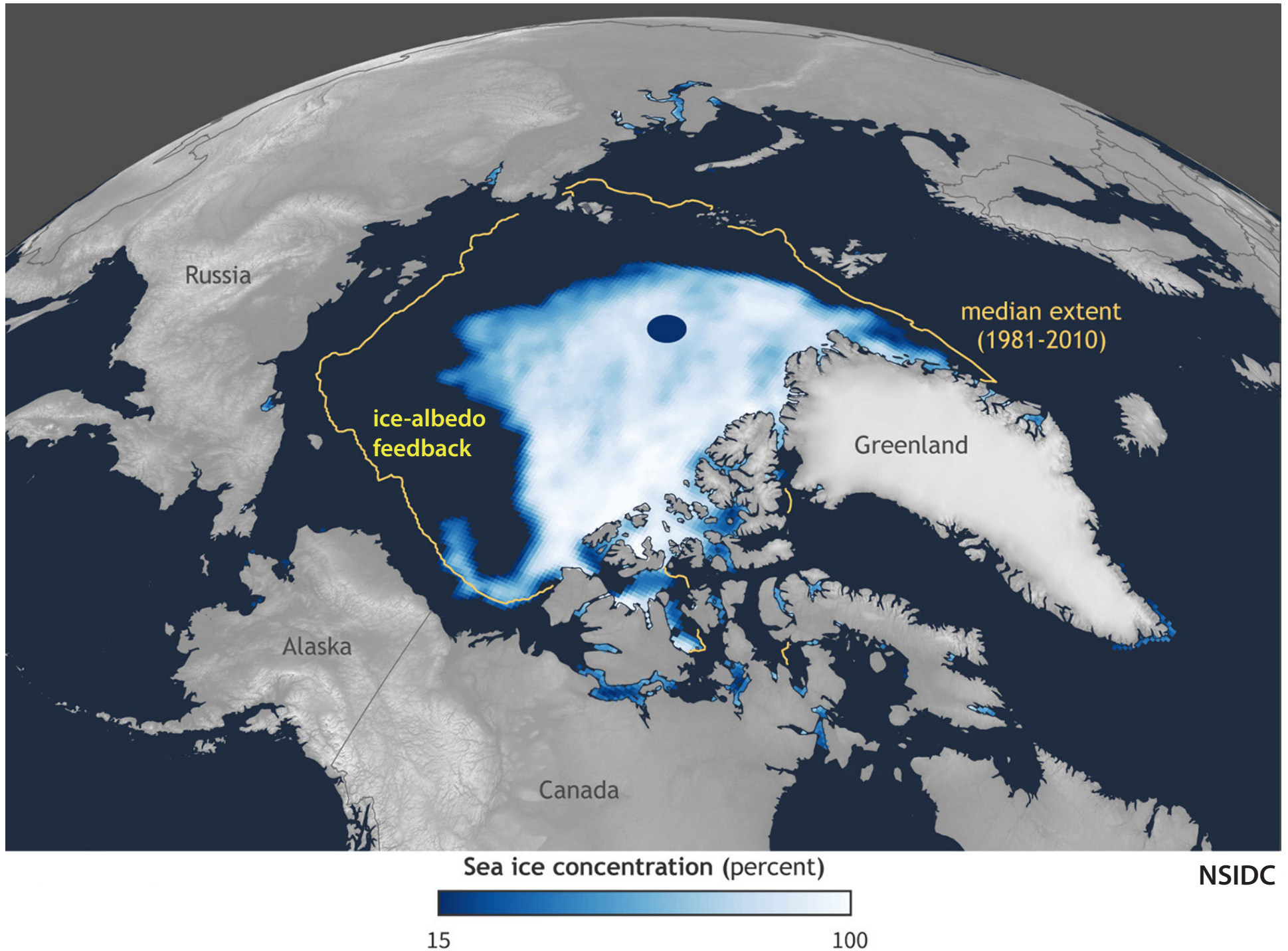


dark water and land
absorb

$$\text{albedo } \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

Arctic sea ice extent

September 15, 2020



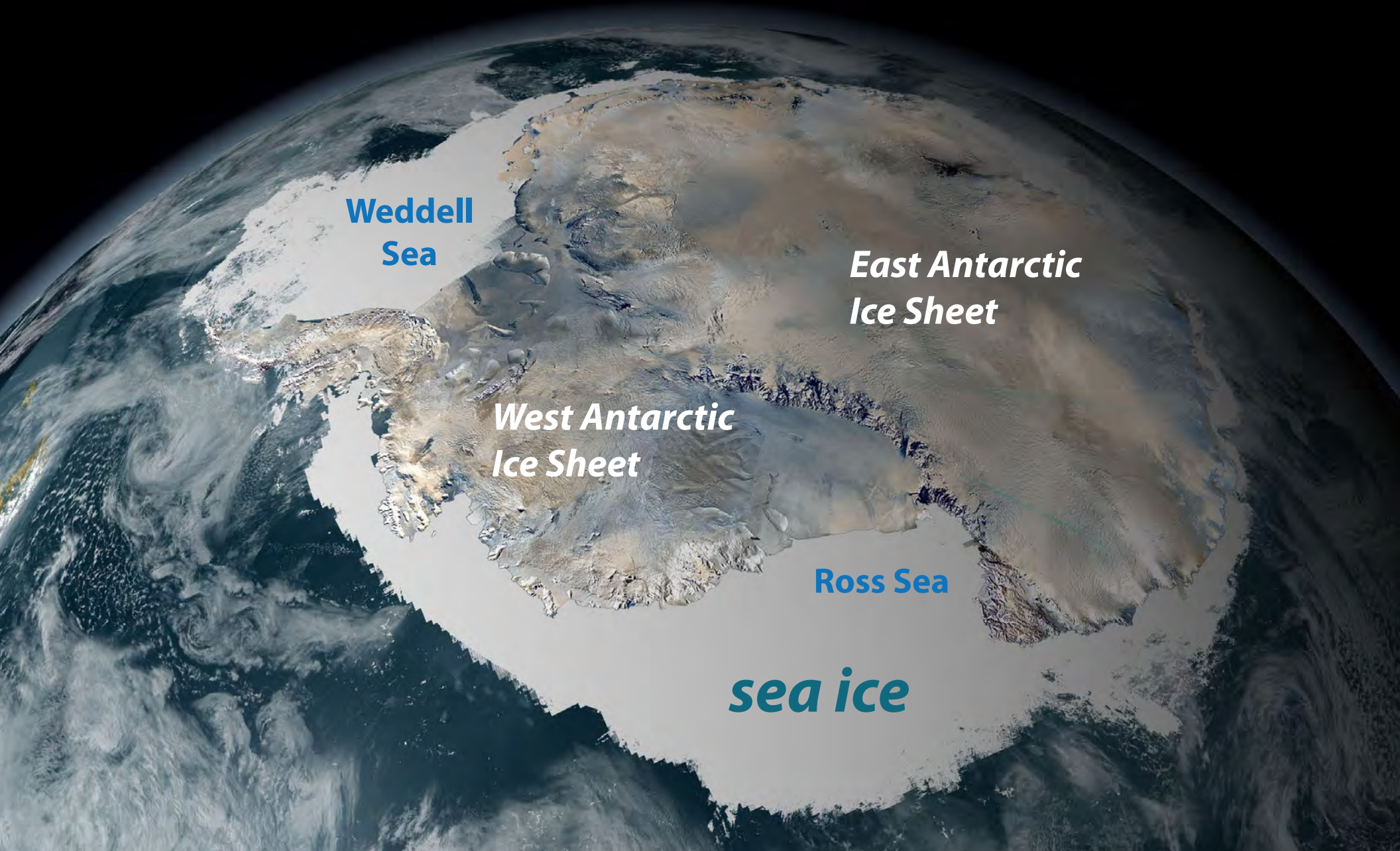


*recent losses
in comparison to
the United States*



ANTARCTICA

southern cryosphere



**Weddell
Sea**

***East Antarctic
Ice Sheet***

***West Antarctic
Ice Sheet***

Ross Sea

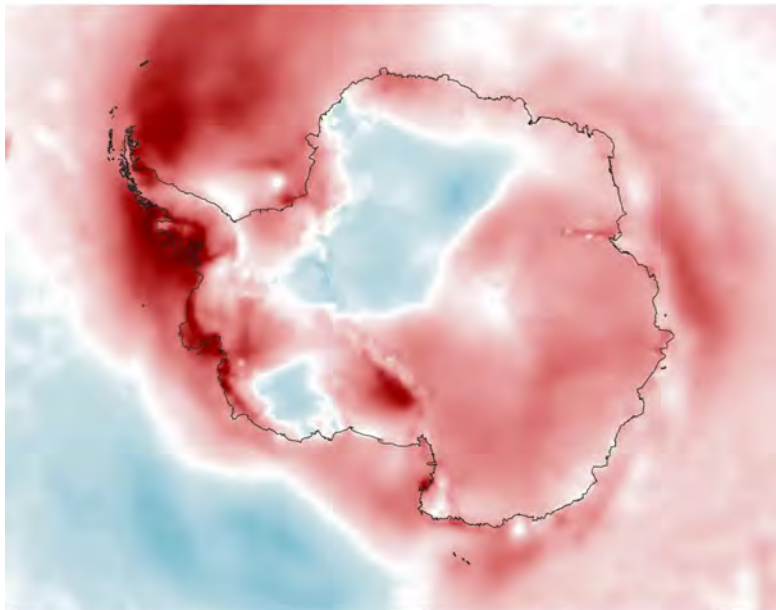
sea ice

New Record Low for Antarctic Sea Ice

February 13, 2023

**Much of Antarctica
warmer than average**

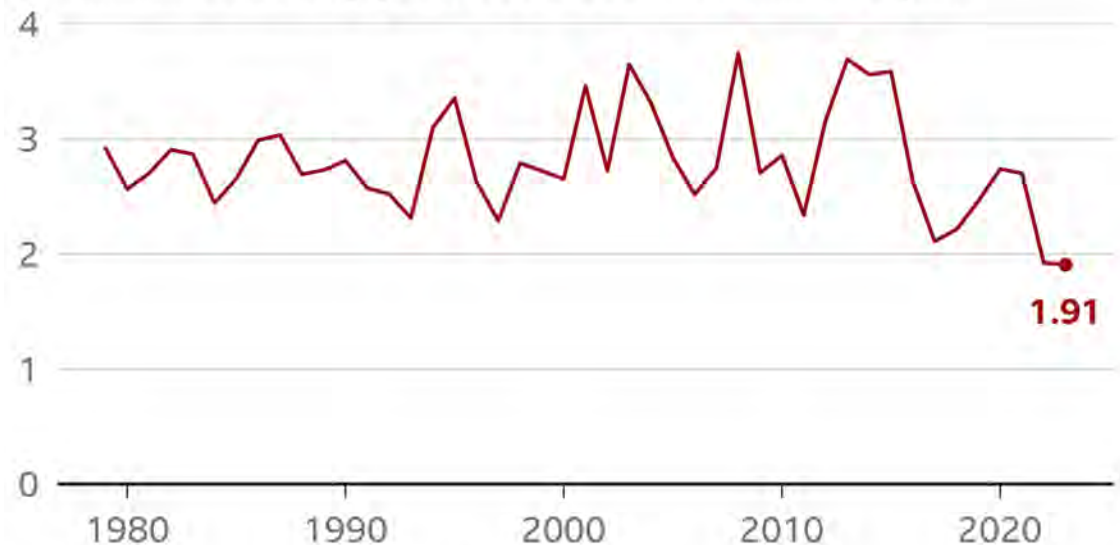
Mean 2022 surface air temp
compared with 1991-2022 ($^{\circ}\text{C}$)



Source: ECMWF ERA5

BBC

**Minimum extent 1979-2023
(million sq km)**

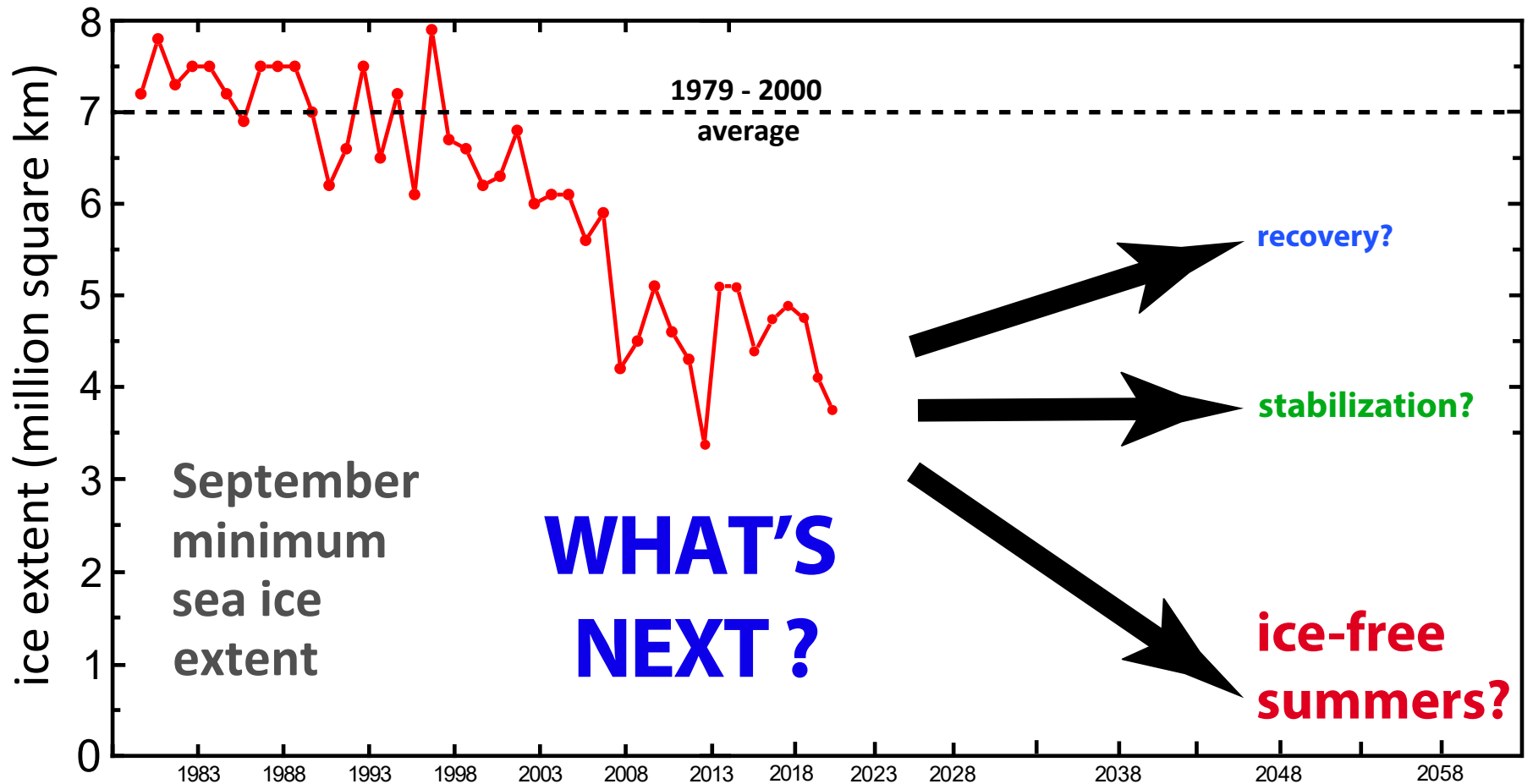


Five-day rolling average of sea-ice extent

Source: National Snow and Ice Data Center (NSIDC)

BBC

Predicting what may come next requires lots of math modeling.



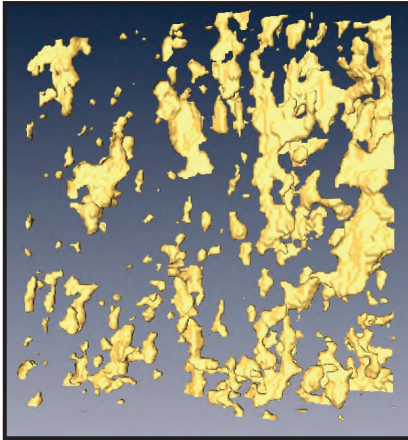
Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

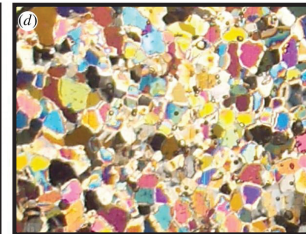
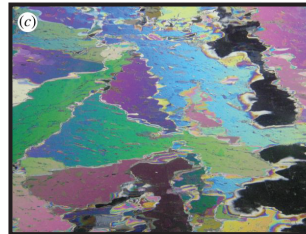
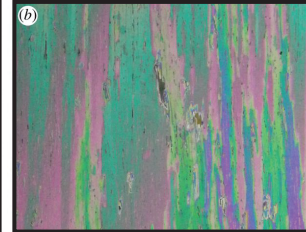
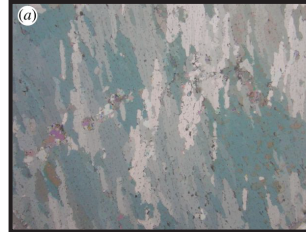


Weeks & Assur 1969



H. Eicken
Golden et al. GRL 2007

polycrystals

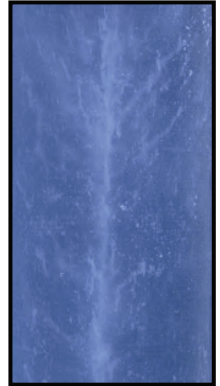


Gully et al. Proc. Roy. Soc. A 2015

brine channels



D. Cole



K. Golden

millimeters

centimeters

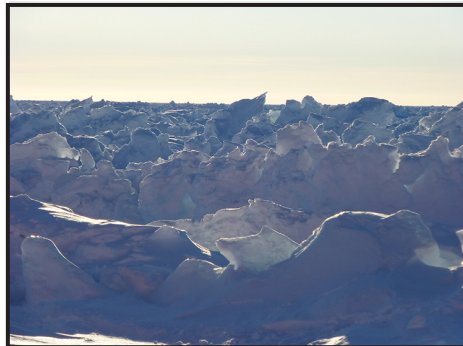
mesoscale

Arctic melt ponds



K. Frey

Antarctic pressure ridges



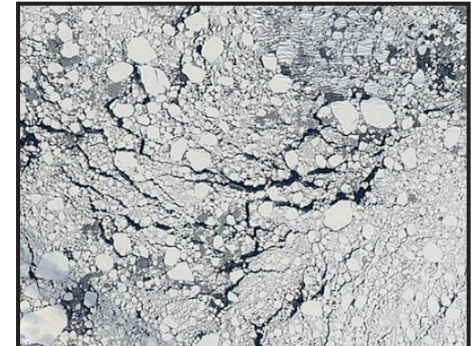
K. Golden

sea ice floes



J. Weller

sea ice pack



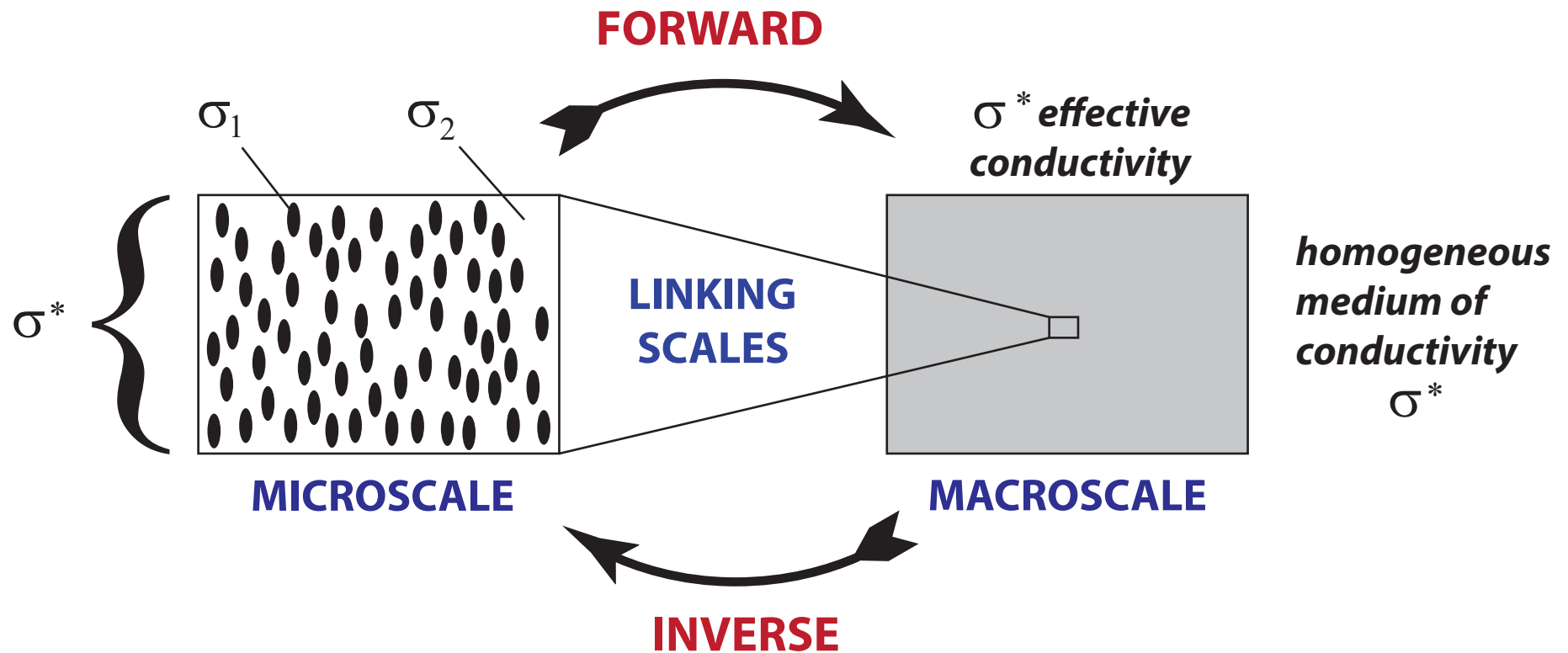
NASA

meters

kilometers

macroscale

HOMOGENIZATION for Composite Materials



Maxwell 1873 : effective conductivity of a dilute suspension of spheres

Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

*Wiener 1912 : arithmetic and harmonic mean **bounds** on effective conductivity*

*Hashin and Shtrikman 1962 : variational **bounds** on effective conductivity*

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

What is this talk about?

A tour of recent results on multiscale modeling of physical and ecological processes in the sea ice system, with a focus on novel mathematics.

microscale

mesoscale

macroscale

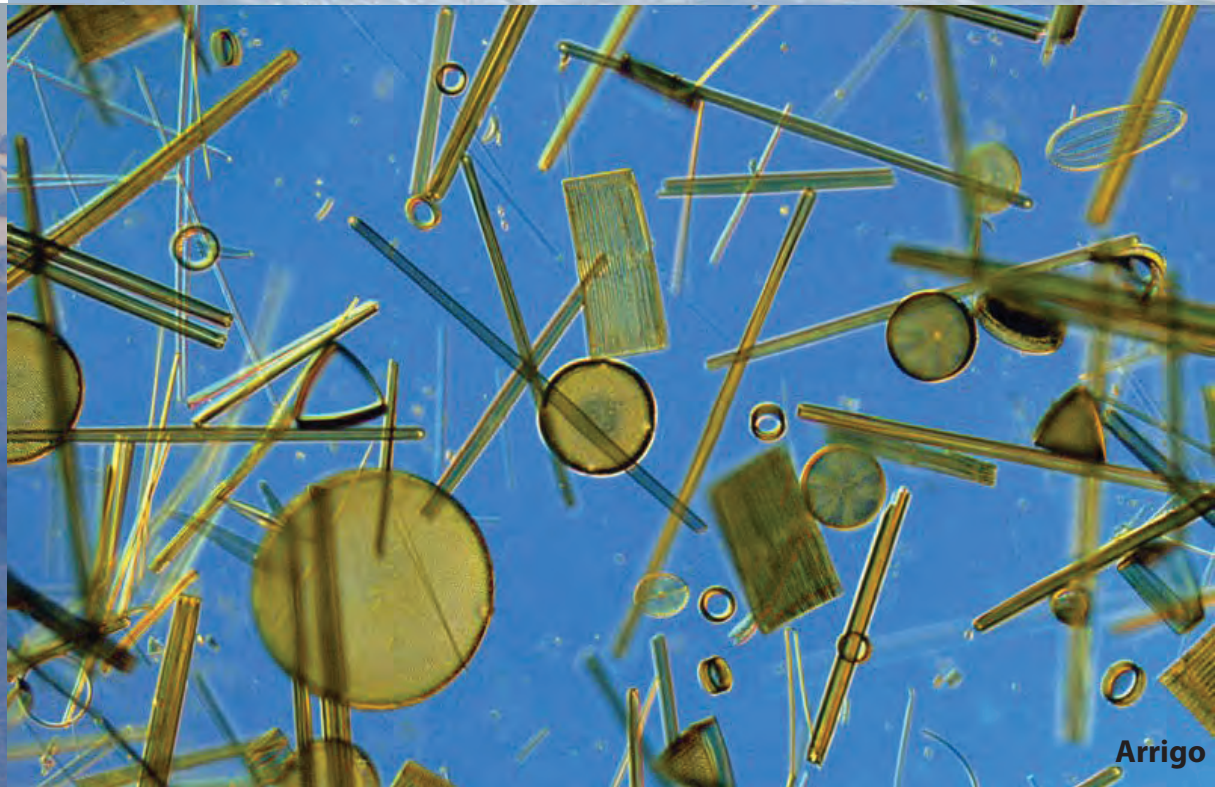
Microbial Ecology and the Physics of Sea Ice

How do the physical properties of sea ice affect microbial communities?

How does the presence of microbial life in sea ice affect its physical properties?

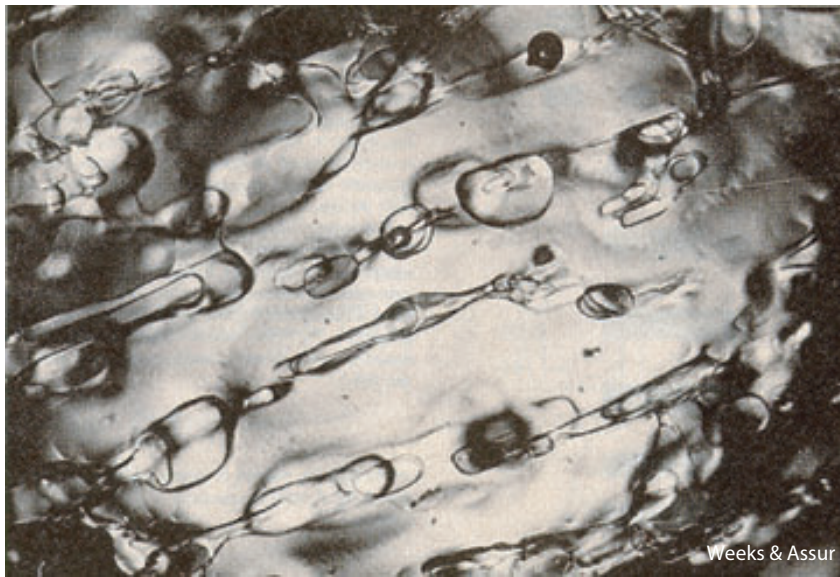


Golden

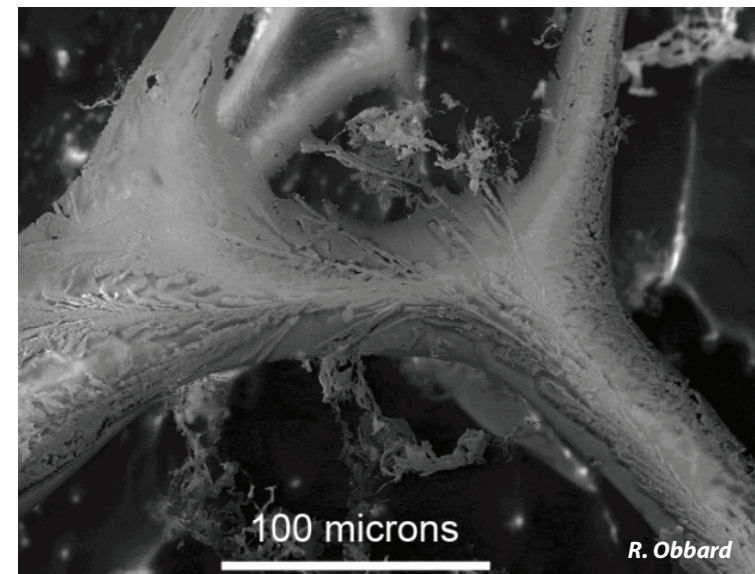


Arrigo

microscale



brine inclusions in sea ice (mm)



micro - brine channel (SEM)

***sea ice is a
porous composite***

pure ice with brine, air, and salt inclusions

brine channels (cm)



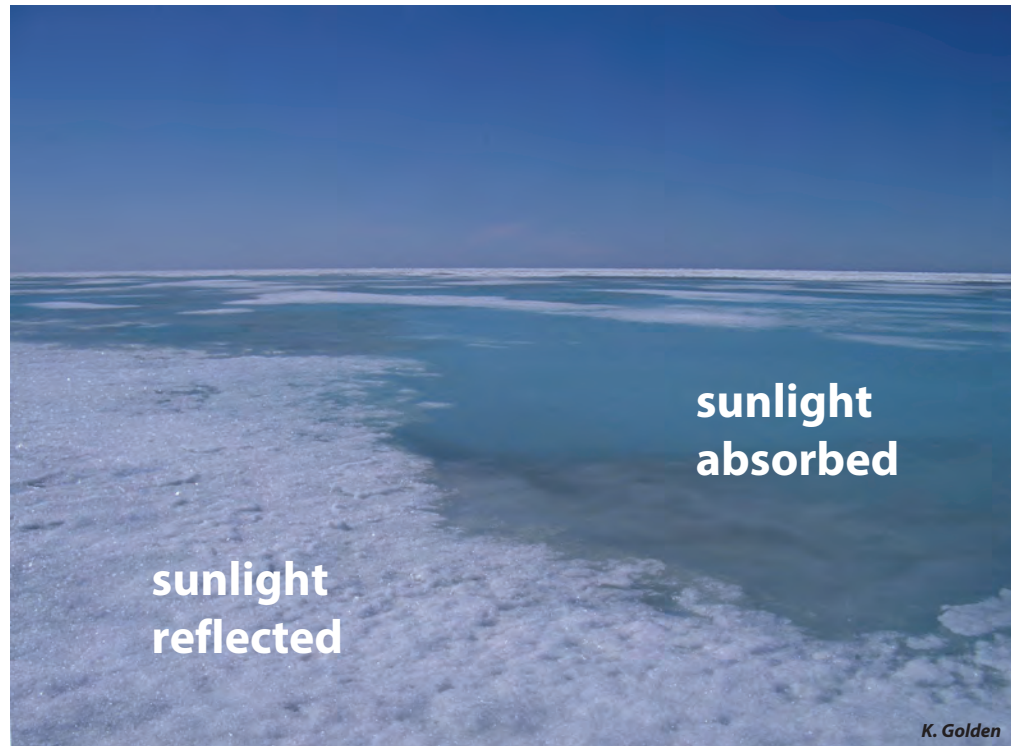
horizontal section



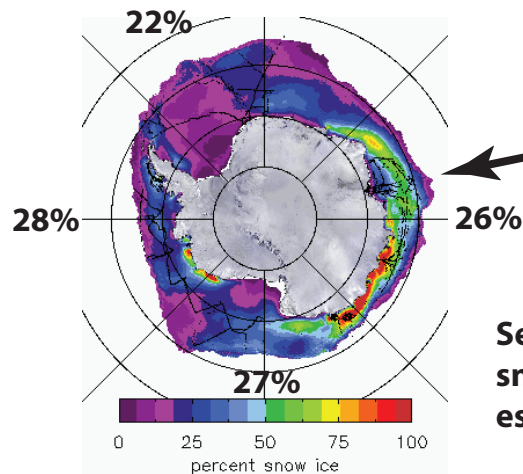
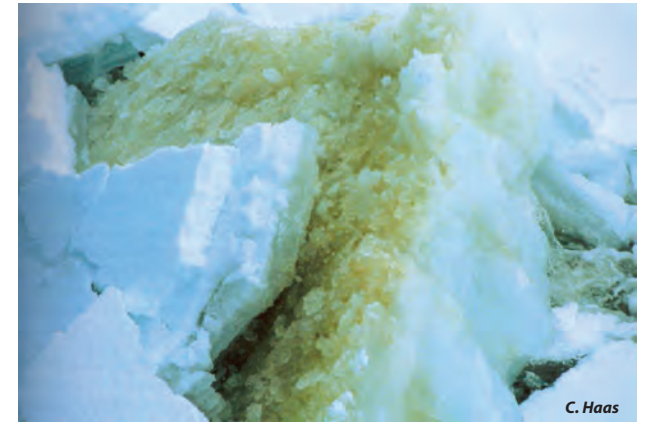
vertical section

fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

*evolution of Arctic melt ponds and sea ice **albedo***



nutrient flux for algal communities

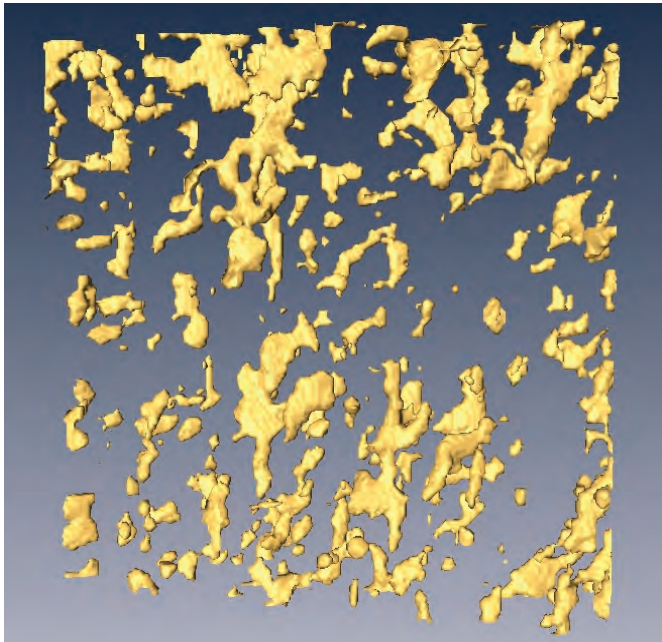


***Antarctic surface flooding
and snow-ice formation***

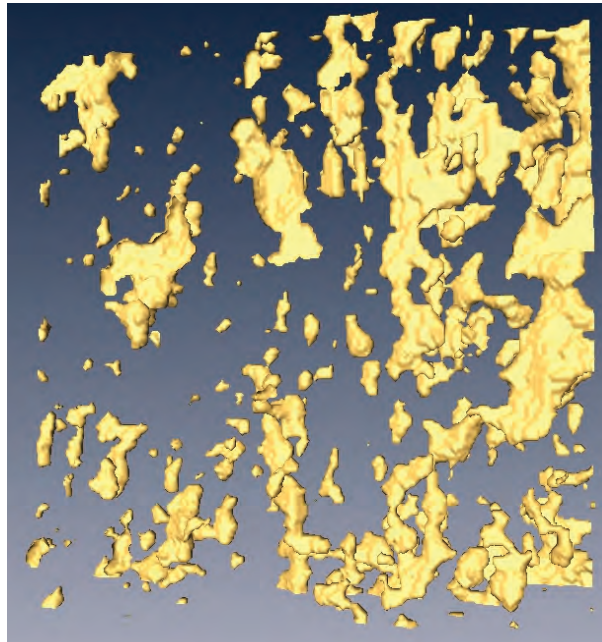
September
snow-ice
estimates

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO₂*

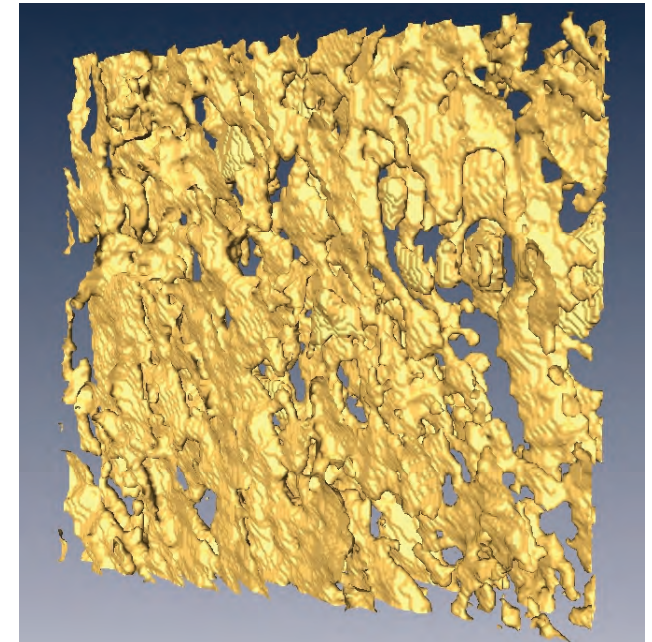
brine volume fraction and **connectivity** increase with temperature



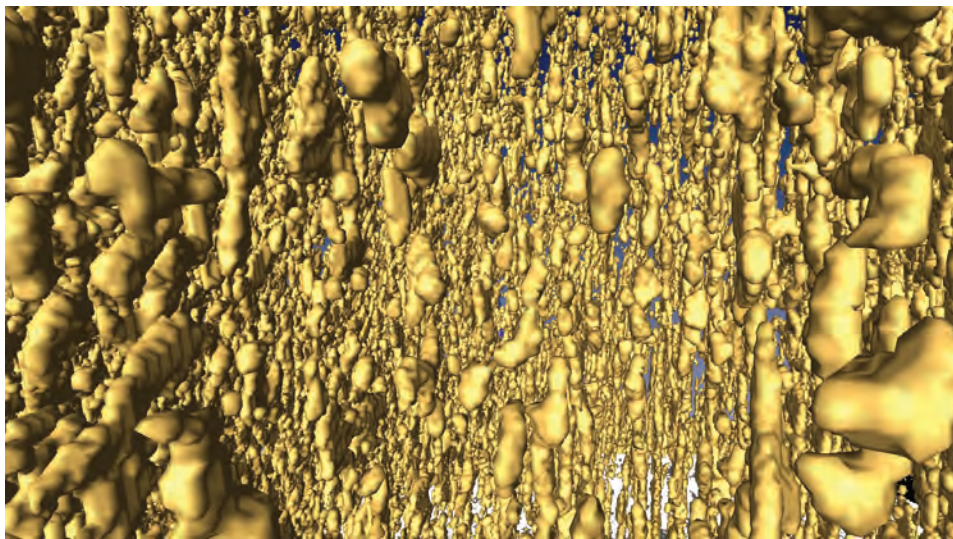
$T = -15\text{ }^{\circ}\text{C}$, $\phi = 0.033$



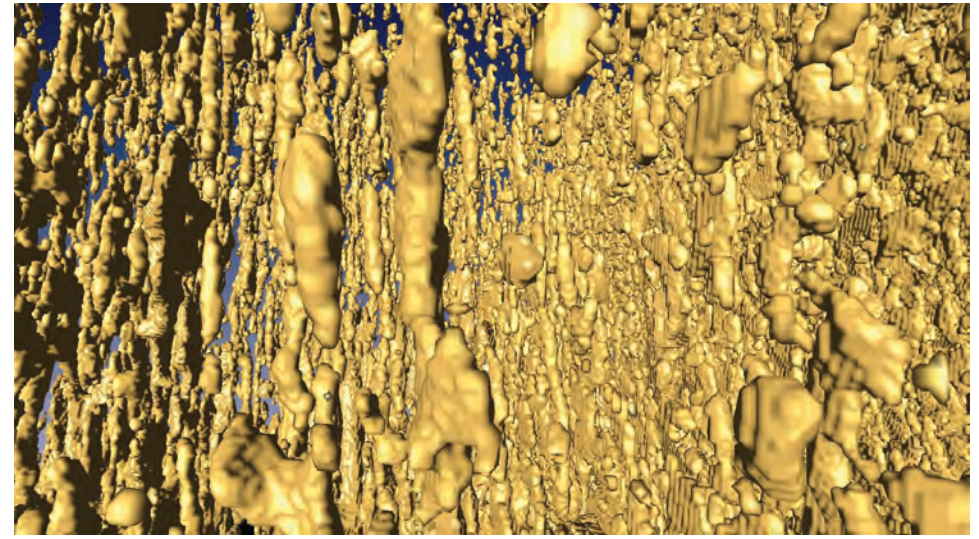
$T = -6\text{ }^{\circ}\text{C}$, $\phi = 0.075$



$T = -3\text{ }^{\circ}\text{C}$, $\phi = 0.143$



$T = -8\text{ }^{\circ}\text{C}$, $\phi = 0.057$



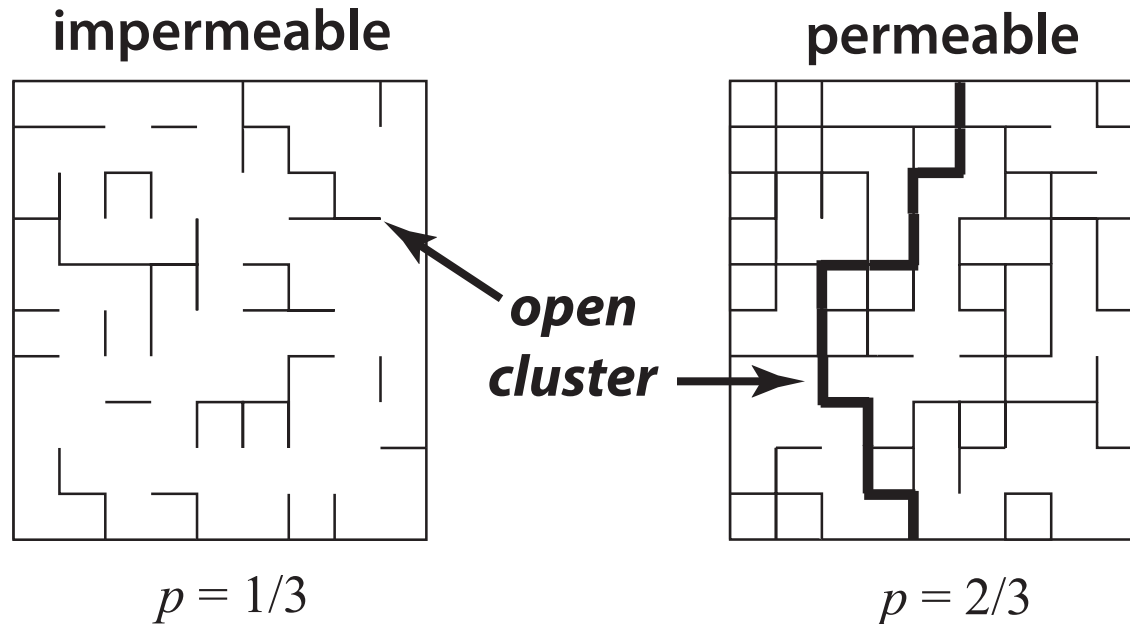
$T = -4\text{ }^{\circ}\text{C}$, $\phi = 0.113$

X-ray tomography for brine in sea ice

Golden et al., *Geophysical Research Letters*, 2007

percolation theory

probabilistic theory of connectedness



bond \longrightarrow open with probability p
closed with probability $1-p$

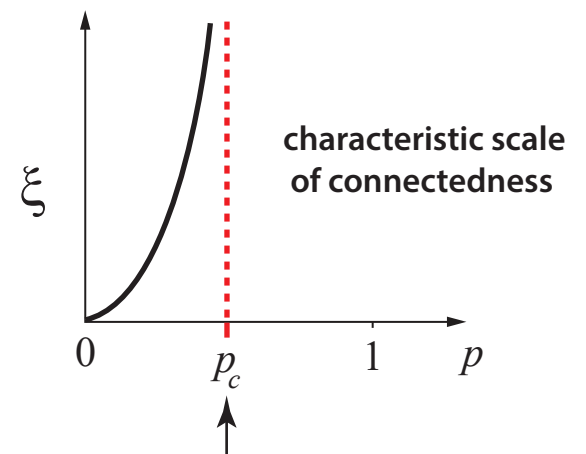
percolation threshold

$$p_c = 1/2 \quad \text{for } d = 2$$

smallest p for which there is an infinite open cluster

correlation length

development of long range order



percolation threshold

$$\xi(p) \sim |p - p_c|^{-\nu} \quad p \rightarrow p_c$$

ν universal: depends only on d

p_c depends on type of lattice and d

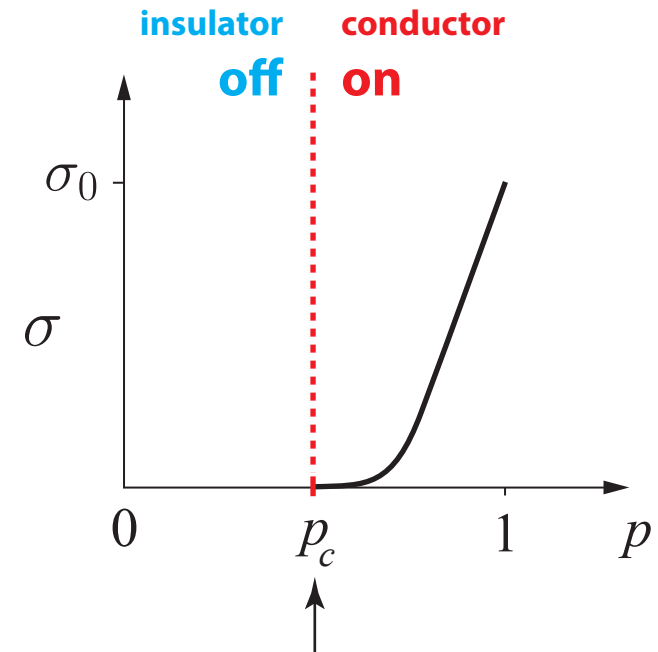
transport in percolation theory

MICRO $\xrightarrow{\text{lattice homogenization}}$ MACRO

local conductivity (electrical or fluid)

effective conductivity or fluid permeability

bond $\rightarrow \begin{cases} \sigma_0 & \text{probability } p \\ 0 & \text{probability } 1 - p \end{cases}$



percolation threshold

$$\sigma(p) \sim \sigma_0 (p - p_c)^t \quad p \rightarrow p_c^+$$

consider local conductivities

1 and $h > 0$

smooths, softens transition

UNIVERSAL critical exponents for lattices -- depend only on dimension

$1 \leq t \leq 2$ (for idealized model), Golden, *Phys. Rev. Lett.* 1990 ; *Comm. Math. Phys.* 1992

non-universal behavior in continuum

Critical behavior of fluid transport in sea ice

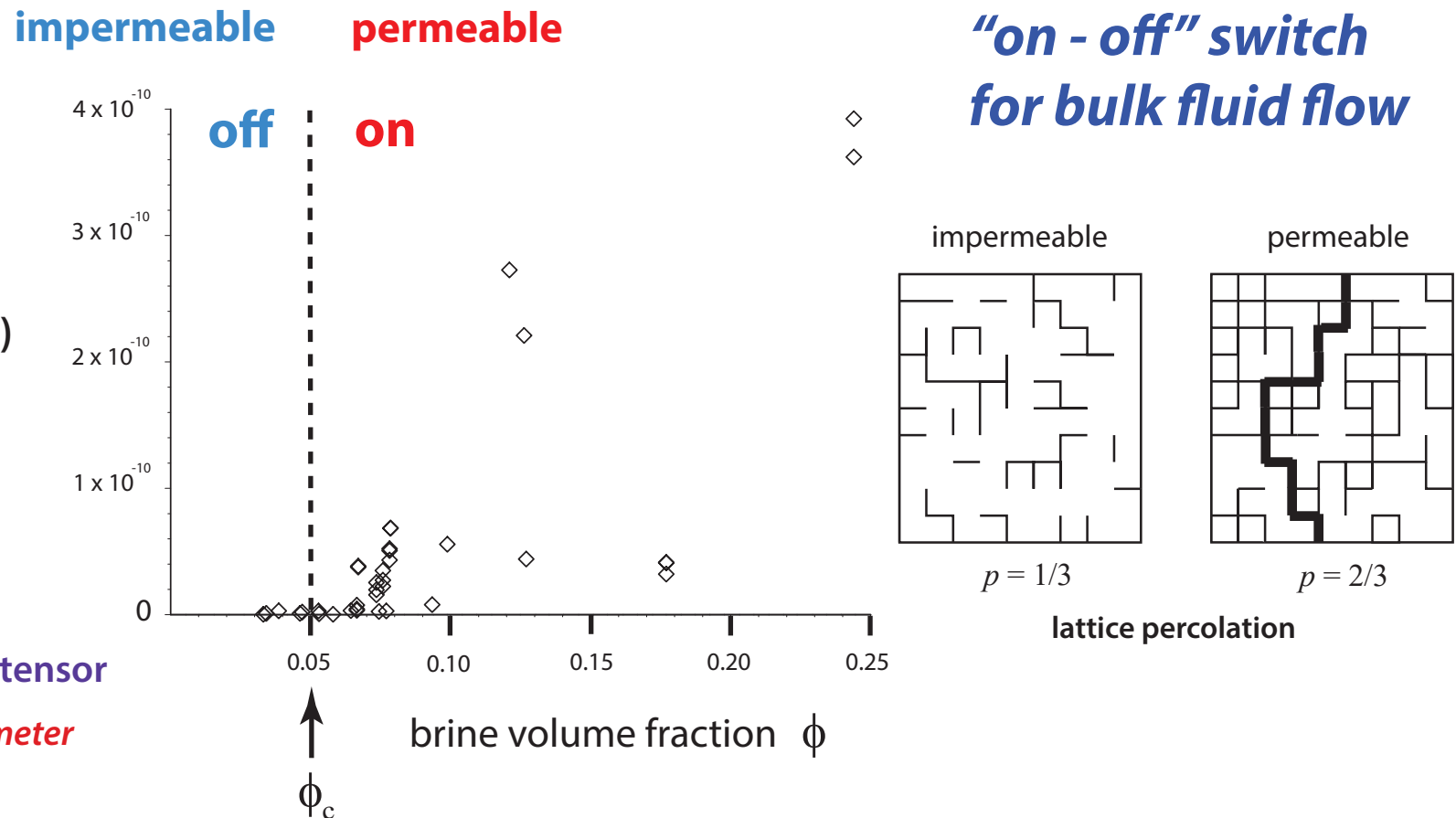
Arctic field data

vertical fluid permeability k (m^2)

Darcy's Law

$$\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$$

\mathbf{k} = fluid permeability tensor
homogenized parameter



PERCOLATION THRESHOLD $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ \text{C}, S \approx 5 \text{ ppt}$

RULE OF FIVES

Golden, Ackley, Lytle *Science* 1998

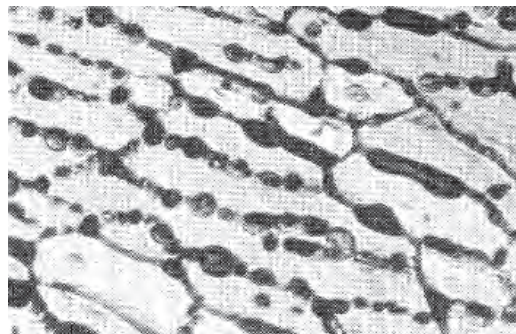
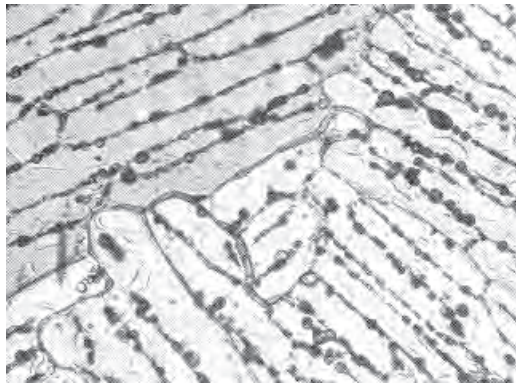
Golden, Eicken, Heaton, Miner, Pringle, Zhu *GRL* 2007

Pringle, Miner, Eicken, Golden *J. Geophys. Res.* 2009

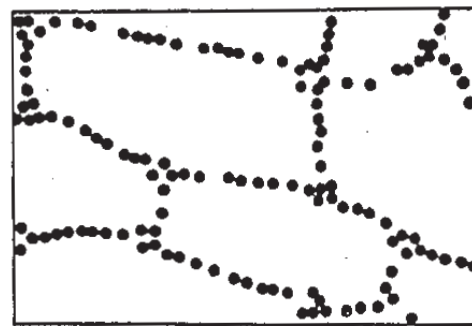
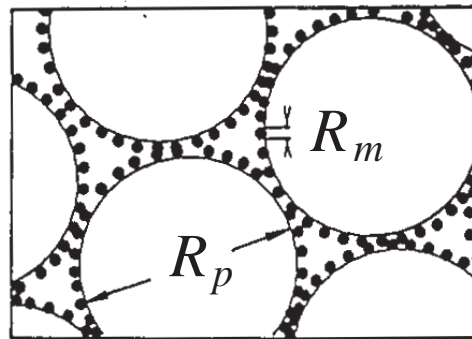
Continuum percolation model for **stealthy** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on **ice production** and **algal growth**

$$\phi_c \approx 5 \%$$

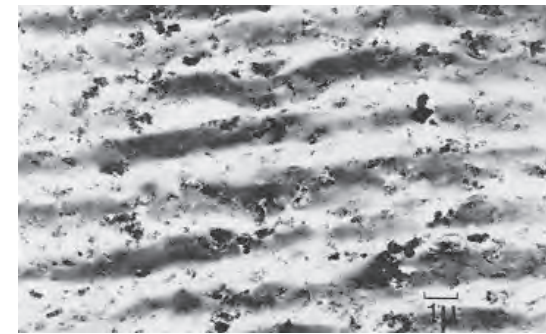
Golden, Ackley, Lytle, *Science*, 1998



sea ice



compressed
powder



radar absorbing
composite

sea ice is radar absorbing



sea ice algal communities

D. Thomas 2004

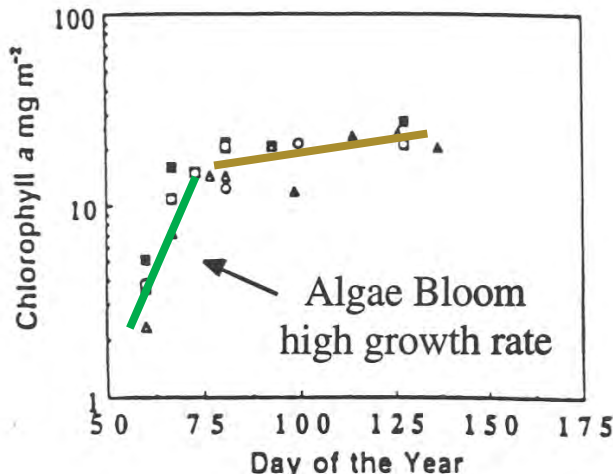
nutrient replenishment
controlled by ice permeability

biological activity turns on
or off according to
rule of fives

Golden, Ackley, Lytle Science 1998

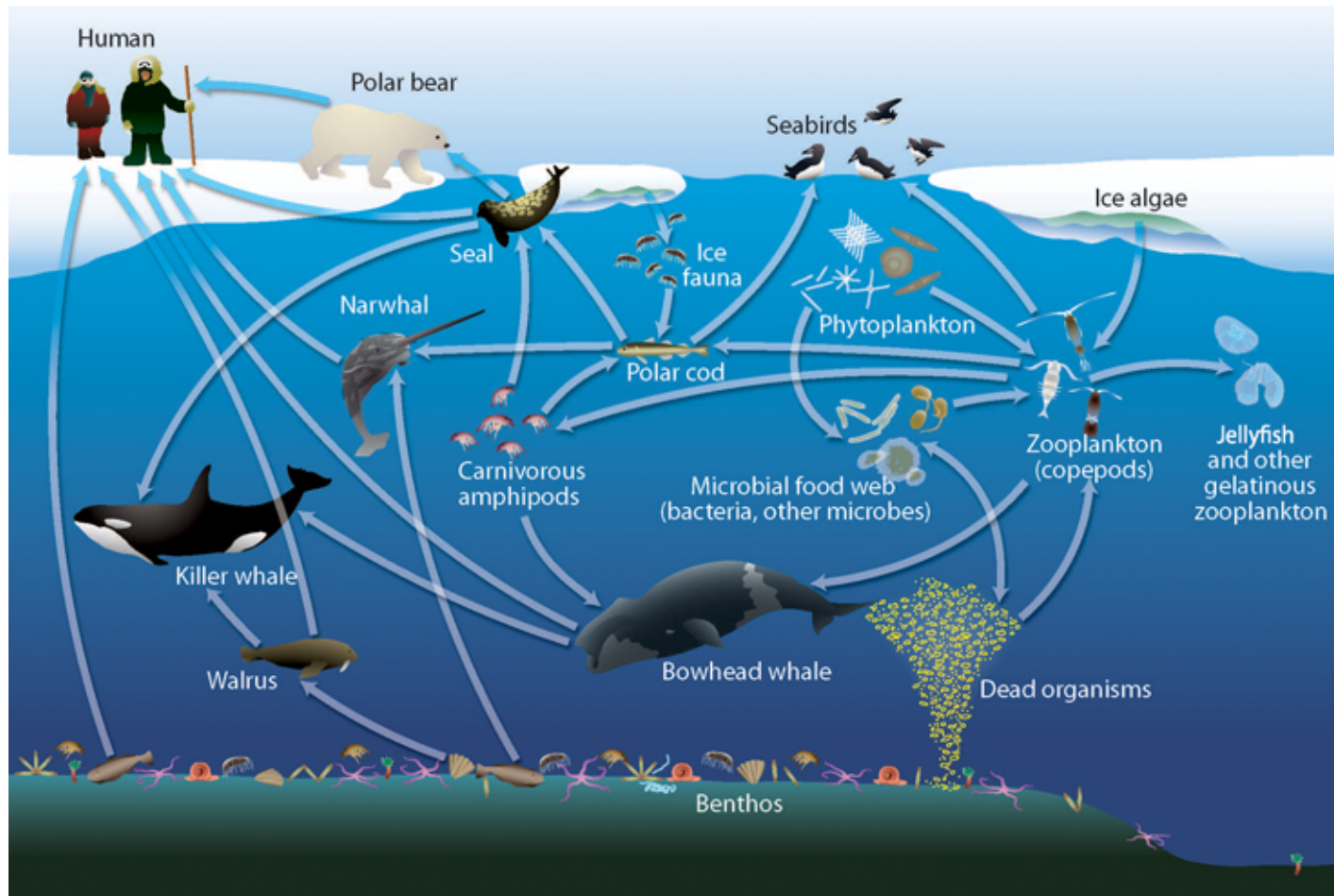
Fritsen, Lytle, Ackley, Sullivan Science 1994

critical behavior of microbial activity



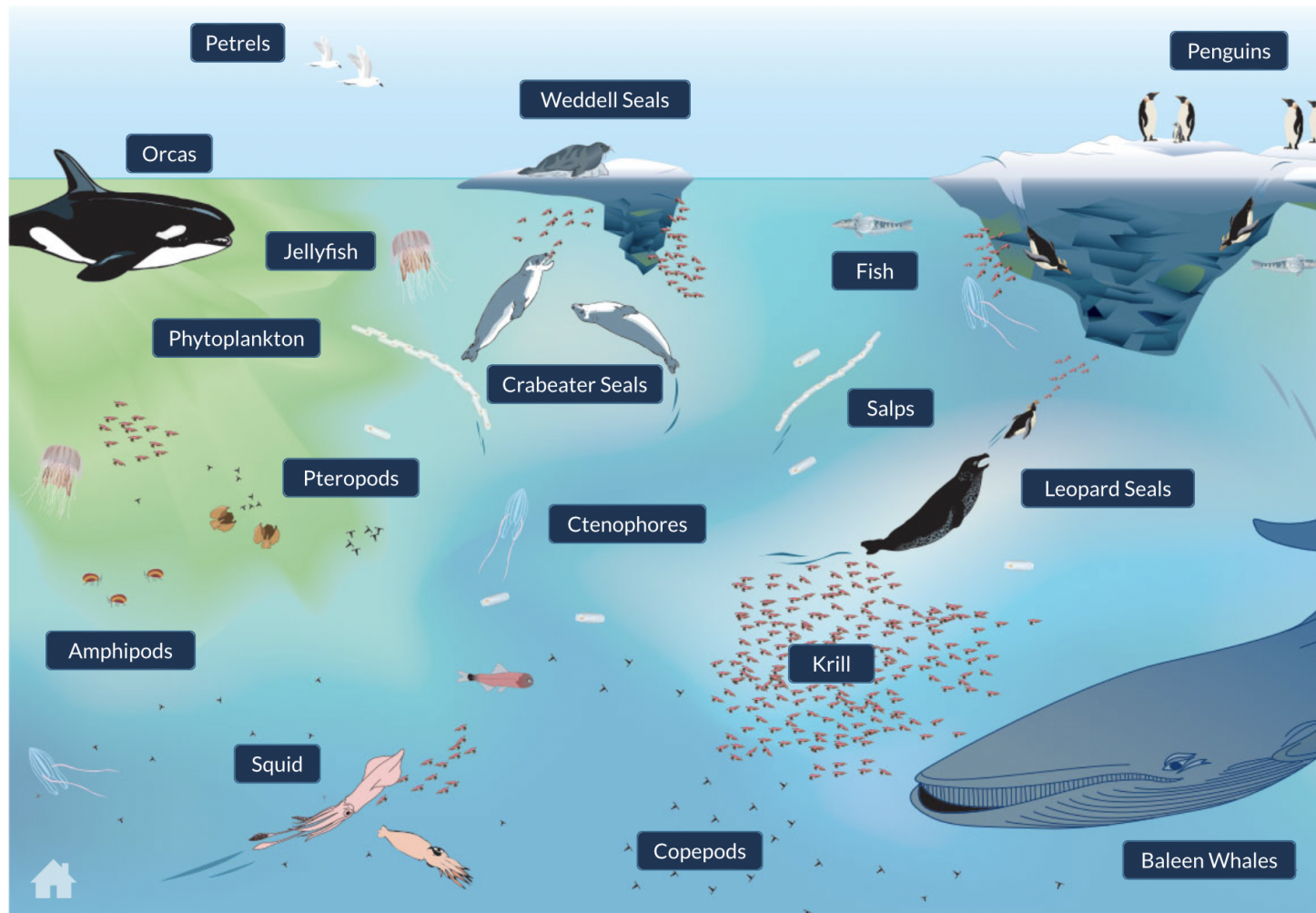
Convection-fueled algae bloom
Ice Station Weddell

Arctic marine ecosystem



Darnis et al.

Antarctic marine ecosystem



WHOI

sea ice ecosystem



sea ice algae
support life in the polar oceans

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, *Geophysical Research Letters* 2007



percolation theory
for fluid permeability

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical exponent t

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

from critical path analysis
in hopping conduction

hierarchical model
rock physics
network model
rigorous bounds

X-ray tomography for
brine inclusions

confirms rule of fives

brine percolation threshold
of $\phi = 5\%$ for bulk fluid flow

*Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009*

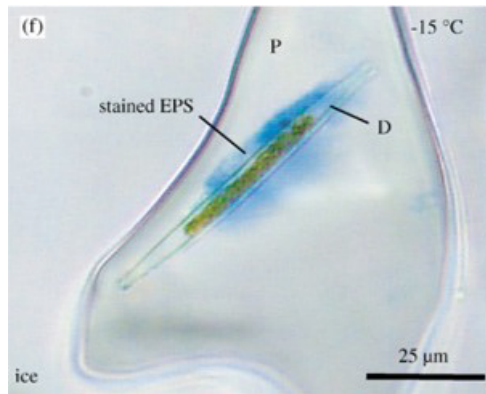
theories agree closely
with field data

microscale
governs
mesoscale
processes

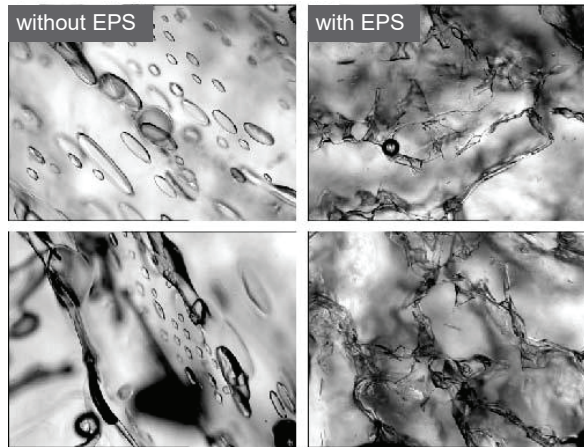
melt pond evolution

Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

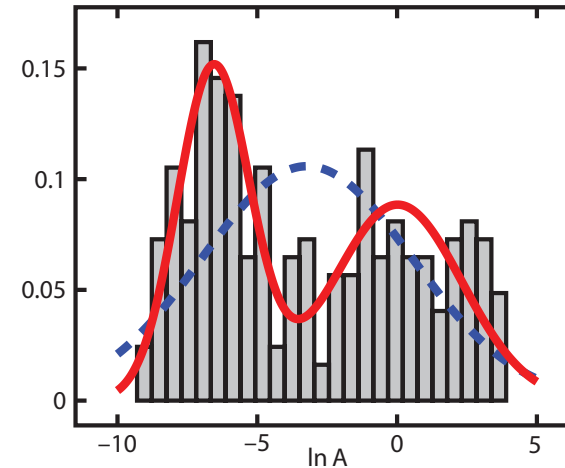
How does EPS affect fluid transport? How does the biology affect the physics?



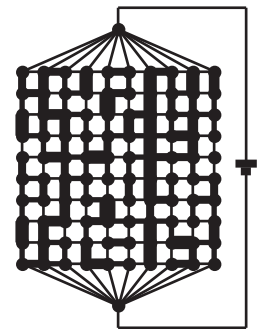
Krembs



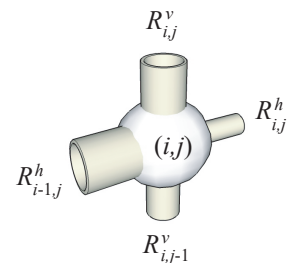
Krembs, Eicken, Deming, PNAS 2011



**RANDOM
PIPE
MODEL**

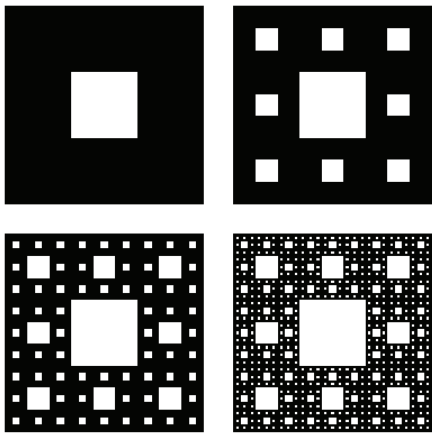


- 2D random pipe model with bimodal distribution of pipe radii
- Rigorous bound on permeability k ; results predict observed drop in k



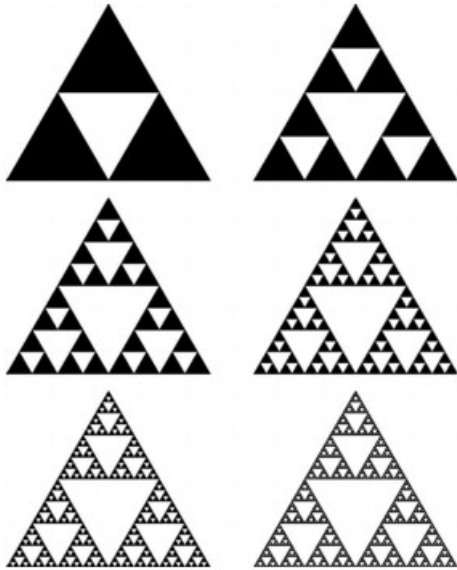
Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
Multiscale Modeling and Simulation, 2018

Zhu, Jabini, Golden,
Eicken, Morris
Ann. Glac. 2006

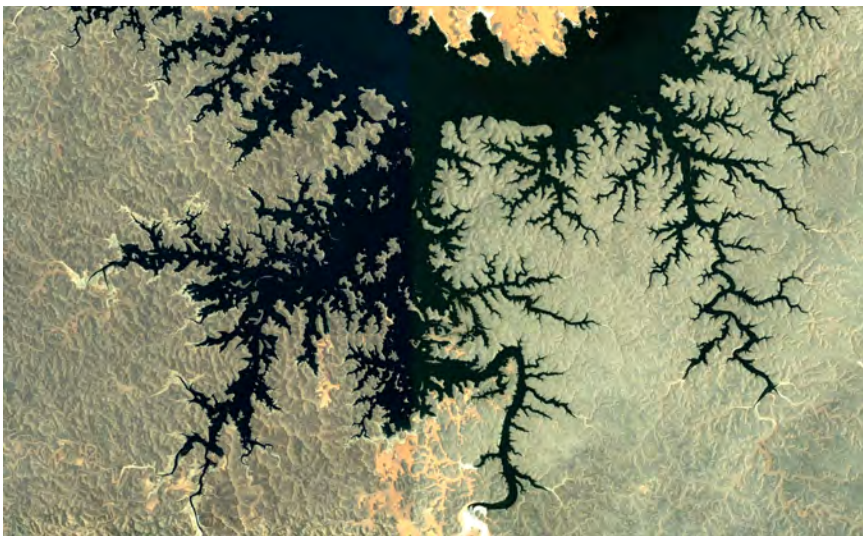


fractals

self-similar structure
non-integer dimension



$$D = \frac{\log 3}{\log 2} = 1.585...$$



polar bear foraging in a fractal icescape

Nicole Forrester
Jody Reimer
Ken Golden

It costs the polar bear
5 times the energy to
swim through water
than to walk on sea ice.

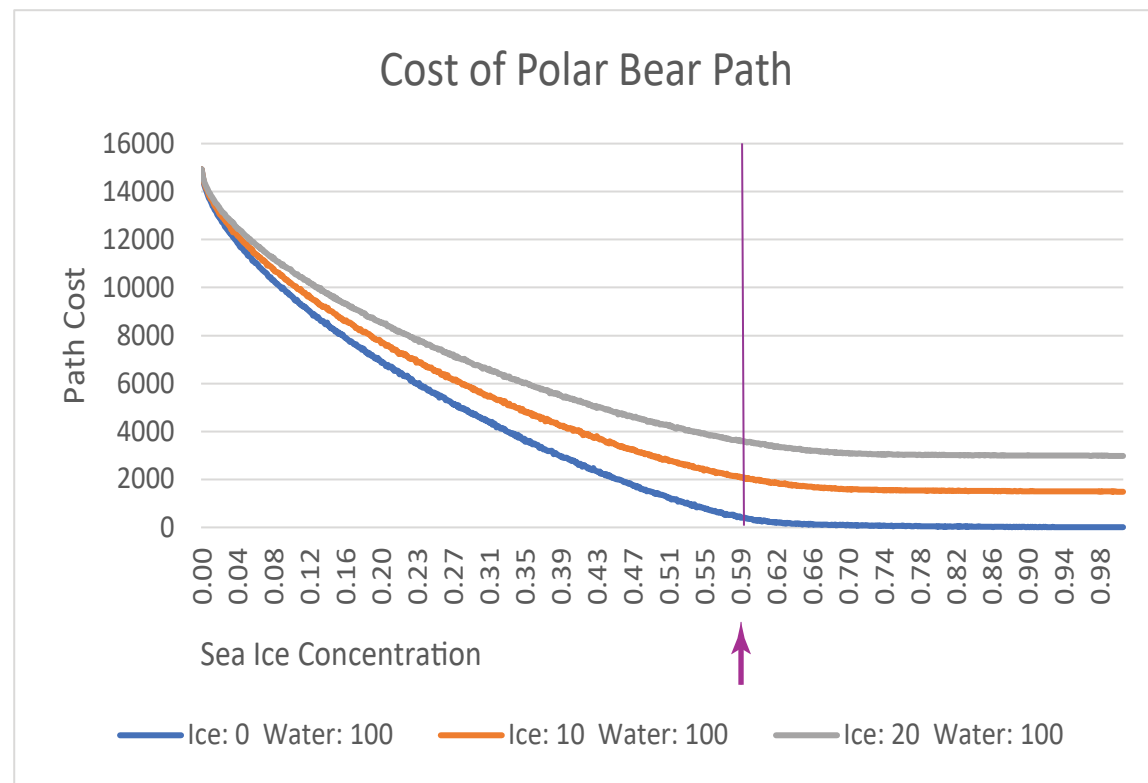


What pathway to a seal
minimizes energy spent?

Polar Bear Percolation

Optimal Movement of a Polar Bear in a Heterogenous Icescape

$C(p)$



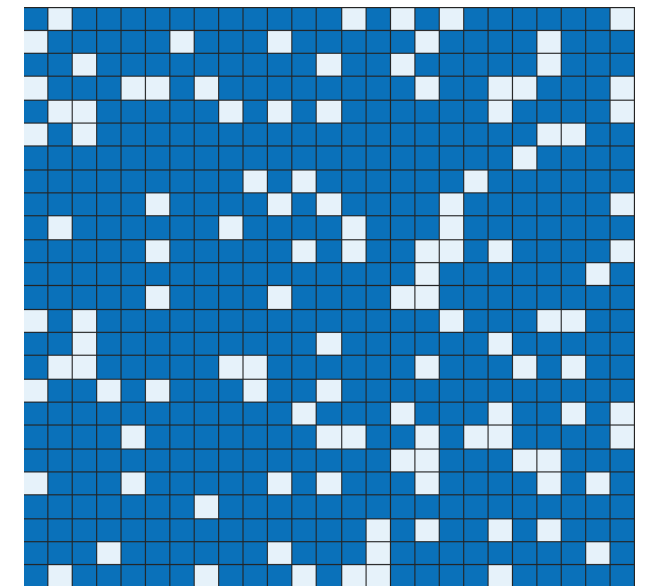
$$h = \frac{C_i}{C_w}$$

ratio of local
"conductivities"

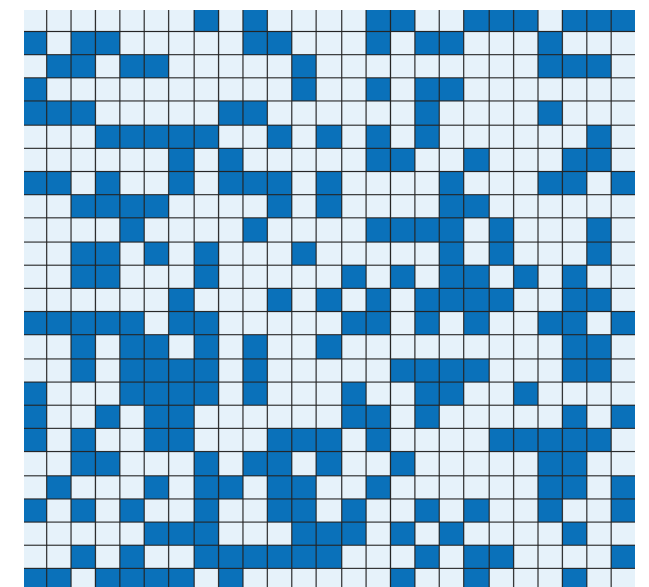
← $h = 0.2$
← $h = 0.1$
← $h = 0$

site percolation
threshold

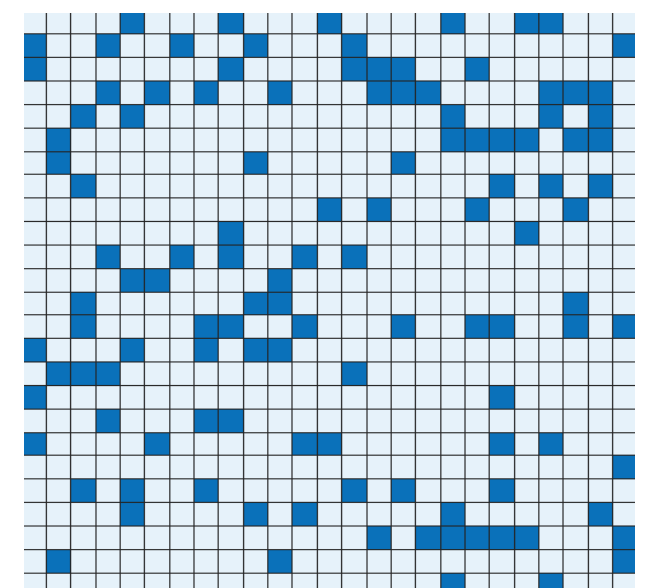
$p_c = 0.59$ for $d = 2$



20% Ice

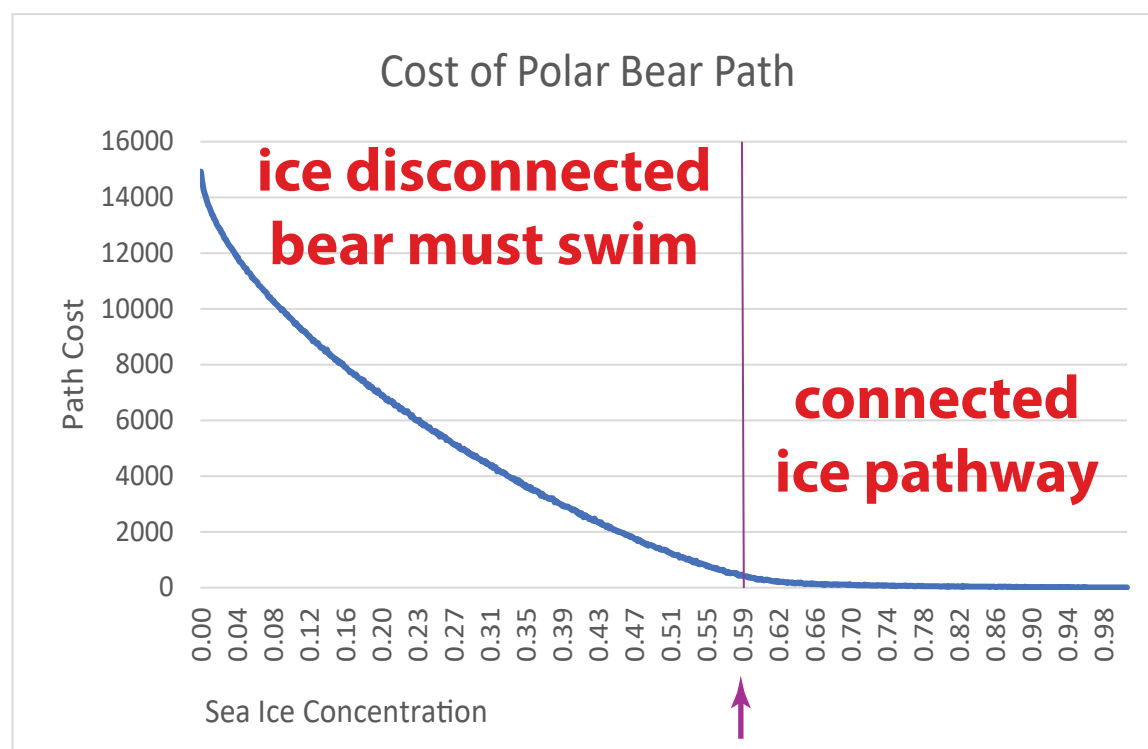


60% Ice



80% Ice

$C(p)$



← $h = 0$

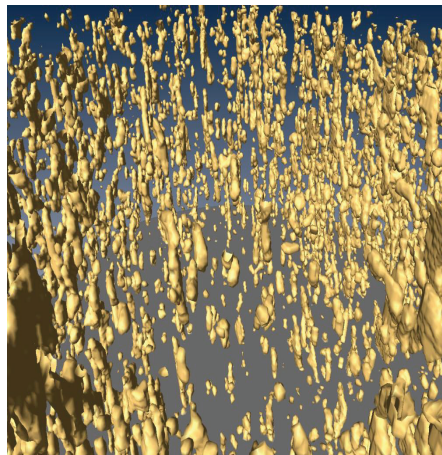
Thermal Evolution of Brine Fractal Geometry in Sea Ice

Nash Ward, Daniel Hallman, Benjamin Murphy, Jody Reimer,
Marc Oggier, Megan O'Sadnick, Elena Cherkaev and Kenneth Golden, 2023

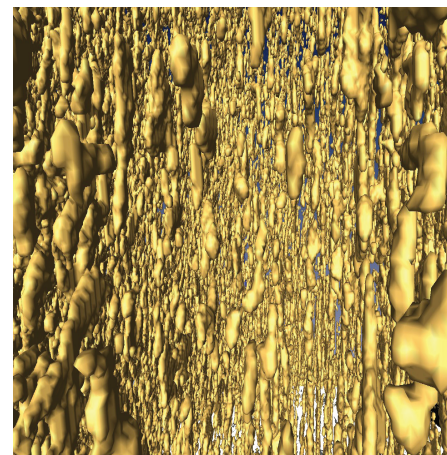


fractal dimension of the
British coastline by
box counting

$T = -12^{\circ} \text{C}$, $\phi = 0.033$



$T = -8^{\circ} \text{C}$, $\phi = 0.057$



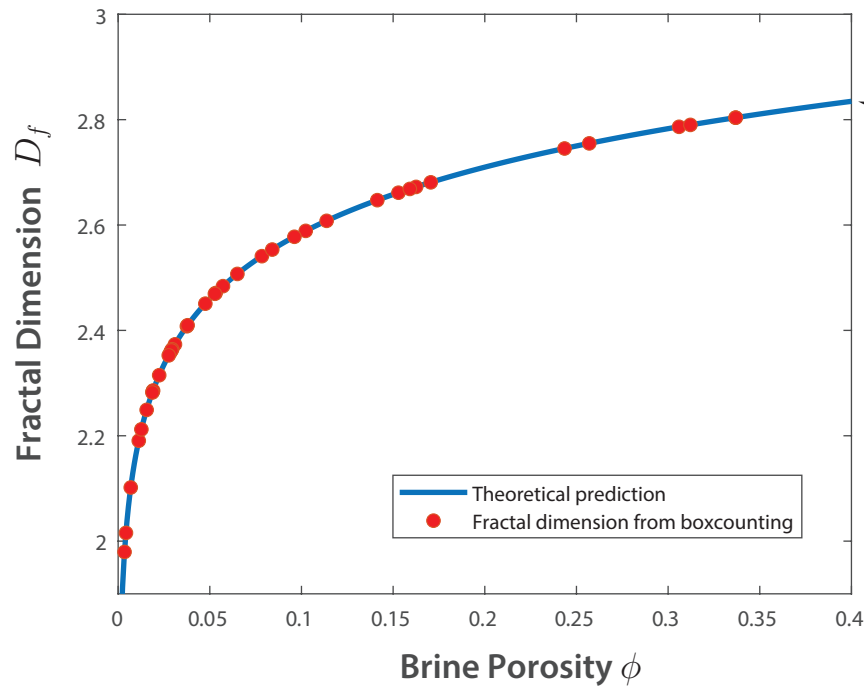
brine channels and
inclusions “look”
like fractals
(from 30 yrs ago)

X-ray computed
tomography of
brine in sea ice

columnar and granular

Golden, Eicken, et al. *GRL*, 2007

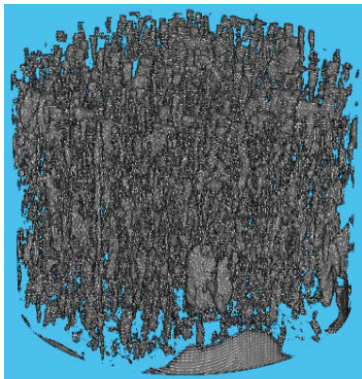
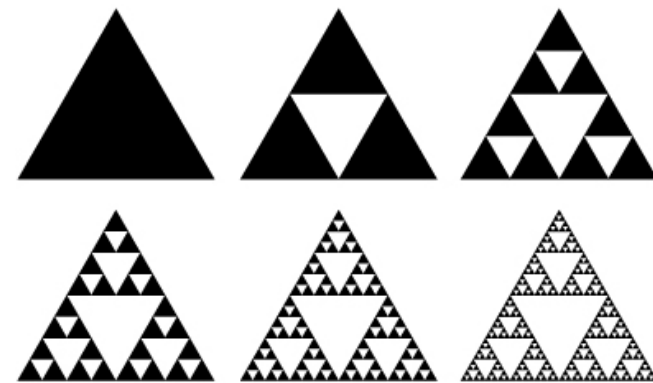
The first comprehensive, quantitative study of the fractal dimension of brine in sea ice and its strong dependence on temperature and porosity.



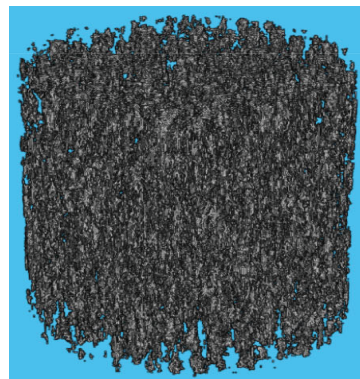
$$D_f = 3 - \frac{\ln \phi}{\ln(\lambda_{min}/\lambda_{max})}$$

The blue curve is exact for the Sierpinski gasket (an exactly self-similar geometry); discovered for sandstones - statistically self-similar porous media.

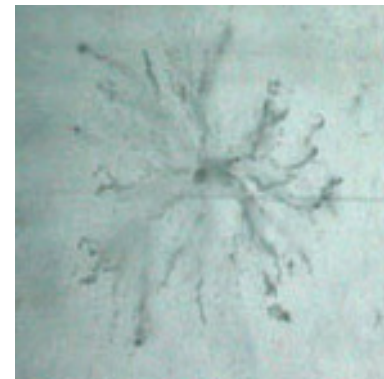
Katz and Thompson, 1985
Yu and Li, 2001



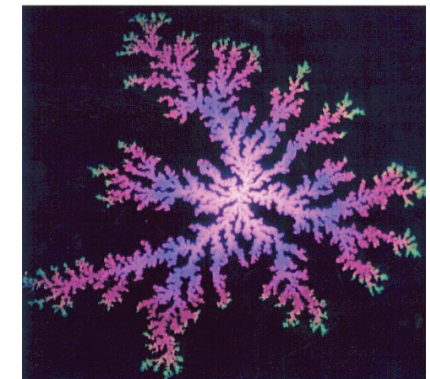
X-ray tomography



DLA model

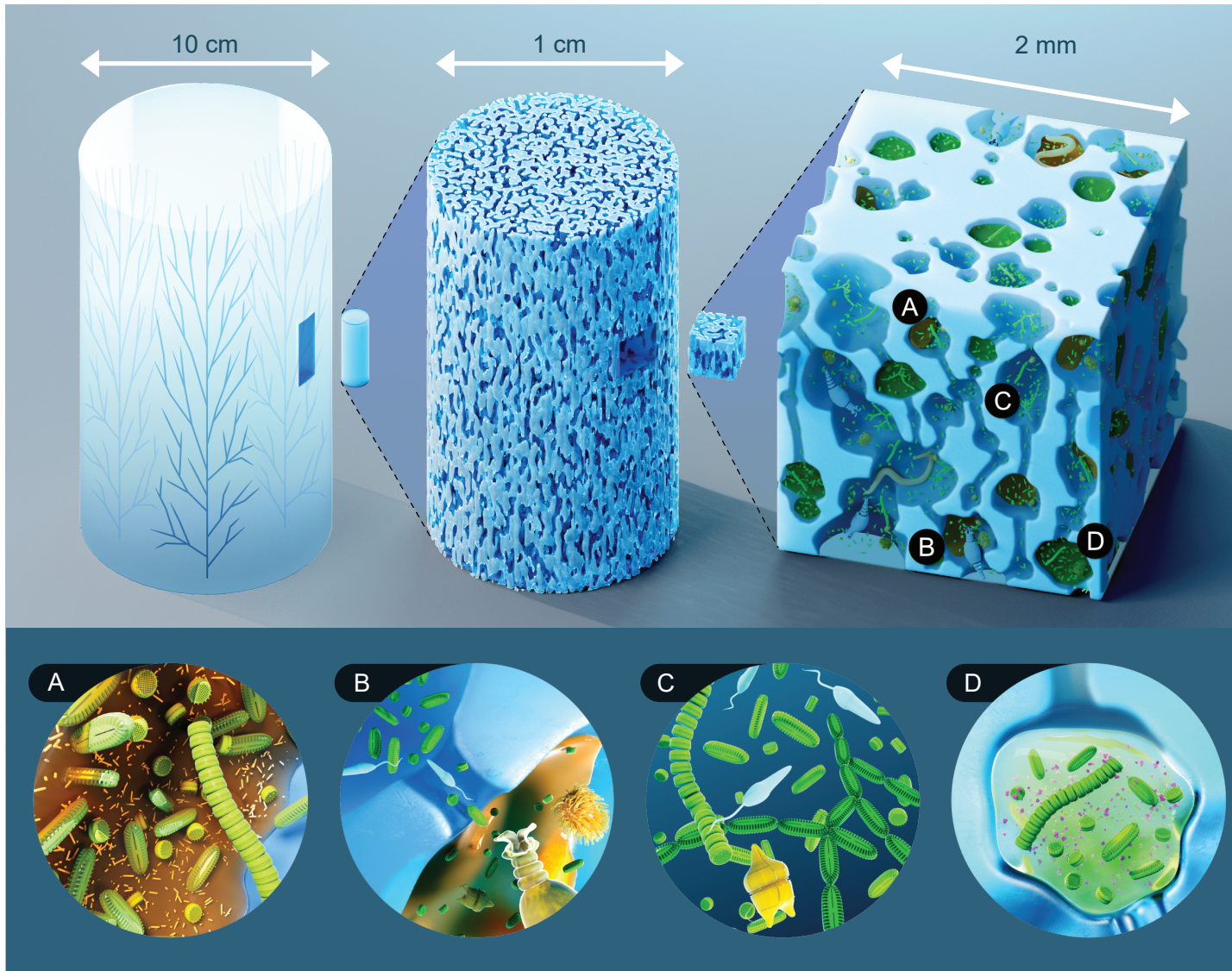


brine channel
in sea ice



diffusion limited
aggregation

Implications of brine fractal geometry on sea ice ecology and biogeochemistry



Brine inclusions are home to ice endemic organisms, e.g., bacteria, diatoms, flagellates, rotifers, nematodes.

The habitability of sea ice for these organisms is inextricably linked to its complex brine geometry.

- (A) Many sea ice organisms attach themselves to inclusion walls; inclusions with a higher fractal dimension have greater surface area for colonization.
- (B) Narrow channels prevent the passage of larger organisms, leading to refuges where smaller organisms can multiply without being grazed, as in (C).
- (D) Ice algae secrete extracellular polymeric substances (EPS) which alter inclusion geometry and may further increase the fractal dimension.

Arctic and Antarctic field experiments

*develop electromagnetic methods
of monitoring fluid transport and
microstructural transitions*

extensive measurements of fluid and
electrical transport properties of sea ice:

2007 Antarctic SIPEX

2010 Antarctic McMurdo Sound

2011 Arctic Barrow AK

2012 Arctic Barrow AK

2012 Antarctic SIPEX II

2013 Arctic Barrow AK

2014 Arctic Chukchi Sea



Notices

of the American Mathematical Society

May 2009

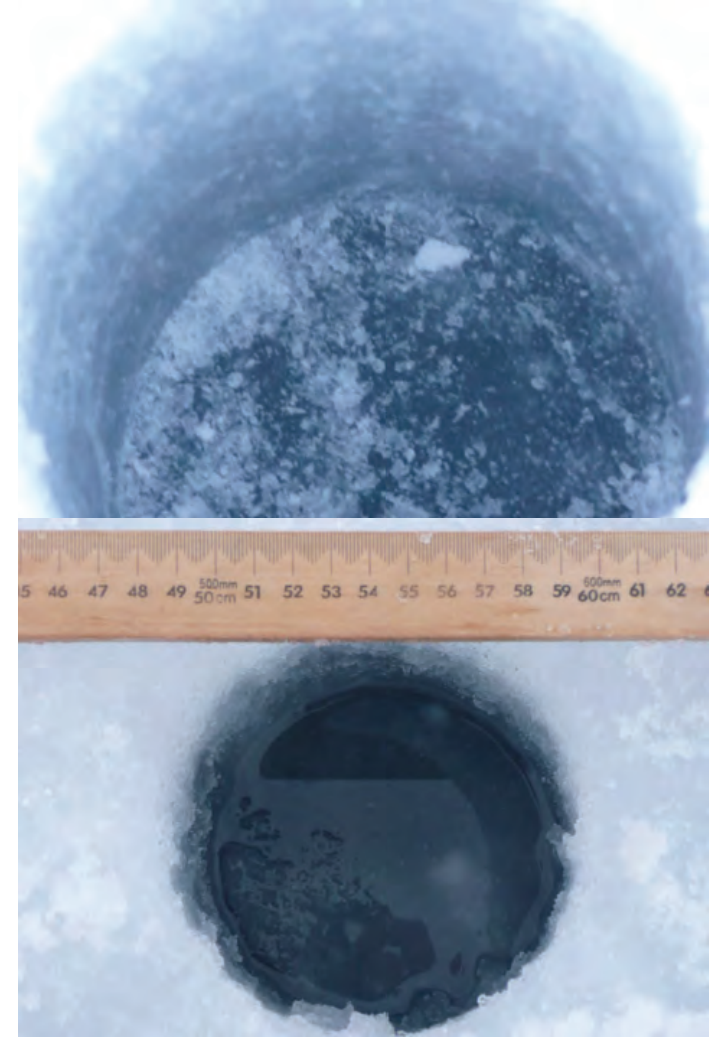
Volume 56, Number 5

Climate Change and
the Mathematics of
Transport in Sea Ice
page 562

Mathematics and the
Internet: A Source of
Enormous Confusion
and Great Potential
page 586

photo by Jan Lieser

Real analysis in polar coordinates (see page 613)



***measuring
fluid permeability
of Antarctic sea ice***

SIPEX 2007



Remote sensing of sea ice



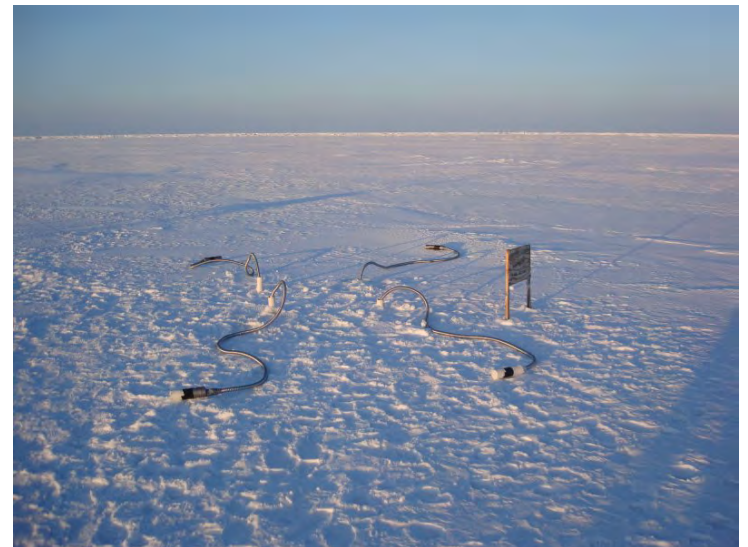
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice
properties from
electromagnetic
(EM) data

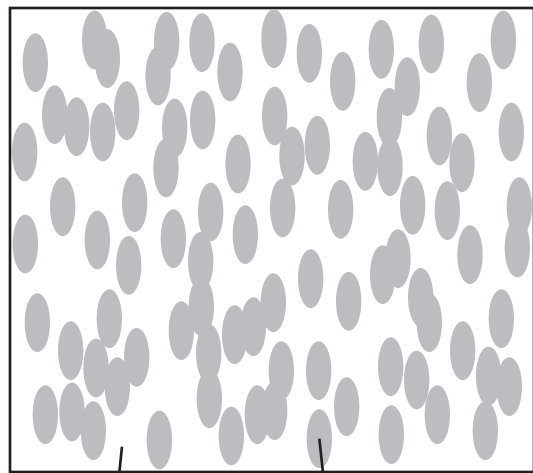
$$\epsilon^*$$

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



ϵ_1

ϵ_2



ϵ^*

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

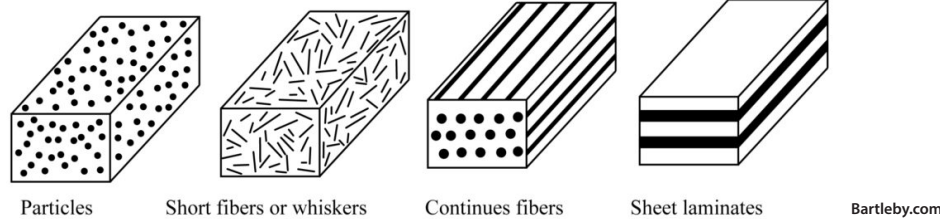
$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1, p_2 = volume fractions of
the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2}, \text{ composite geometry} \right)$$

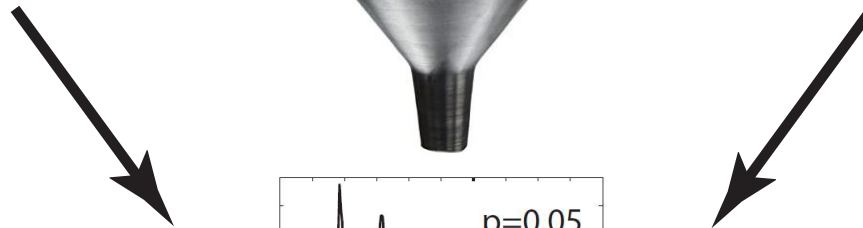
**What are the effective propagation characteristics
of an EM wave (radar, microwaves) in the medium?**

complexities of mixture geometry



distilled

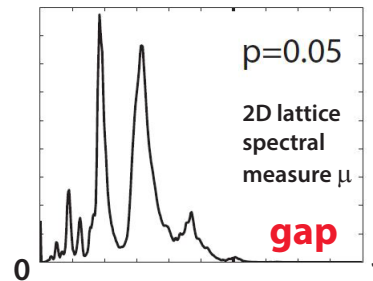
distilled



Analytic Continuation Method

Stieltjes Integral Representations
for Homogenized Parameters

Bergman 1978, Milton 1979
Golden & Papanicolaou 1983



spectral properties of operator (matrix)
~ quantum states, energy levels for atoms

eigenvectors

eigenvalues

EXTEND to: polycrystals, advection diffusion, waves through ice pack

Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

separates geometry from parameters

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

← geometry

← material parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n -point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla.$$

χ = characteristic function of the brine phase

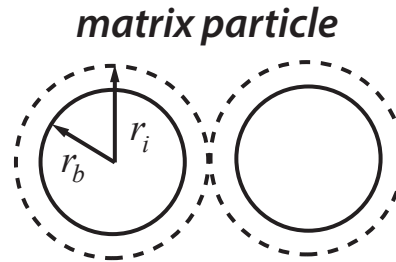
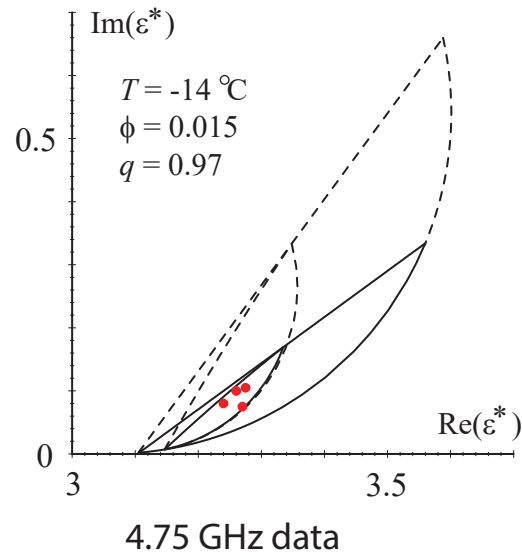
$$E = s (s + \Gamma\chi)^{-1} e_k$$

$\Gamma\chi$: microscale \rightarrow macroscale

$\Gamma\chi$ *links scales*

forward and inverse bounds on the complex permittivity of sea ice

forward bounds

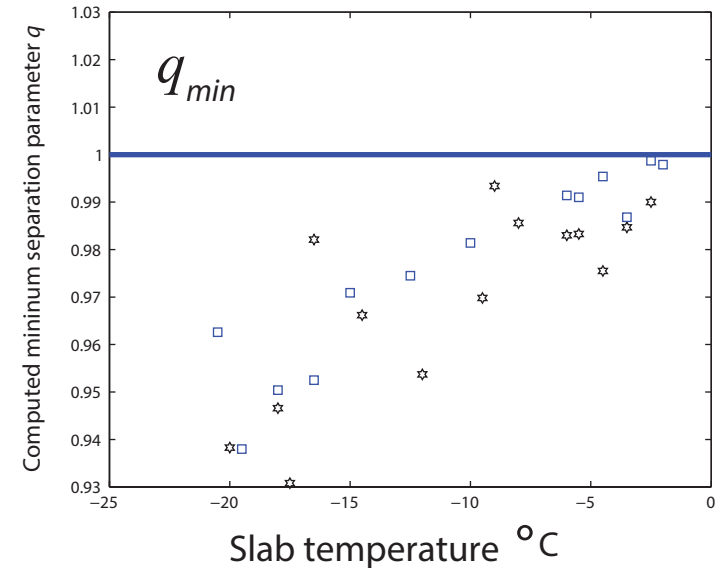


$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

inverse bounds



Inverse Homogenization

Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001), McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

ϵ^* \longrightarrow composite geometry
(spectral measure μ)

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ϵ^*

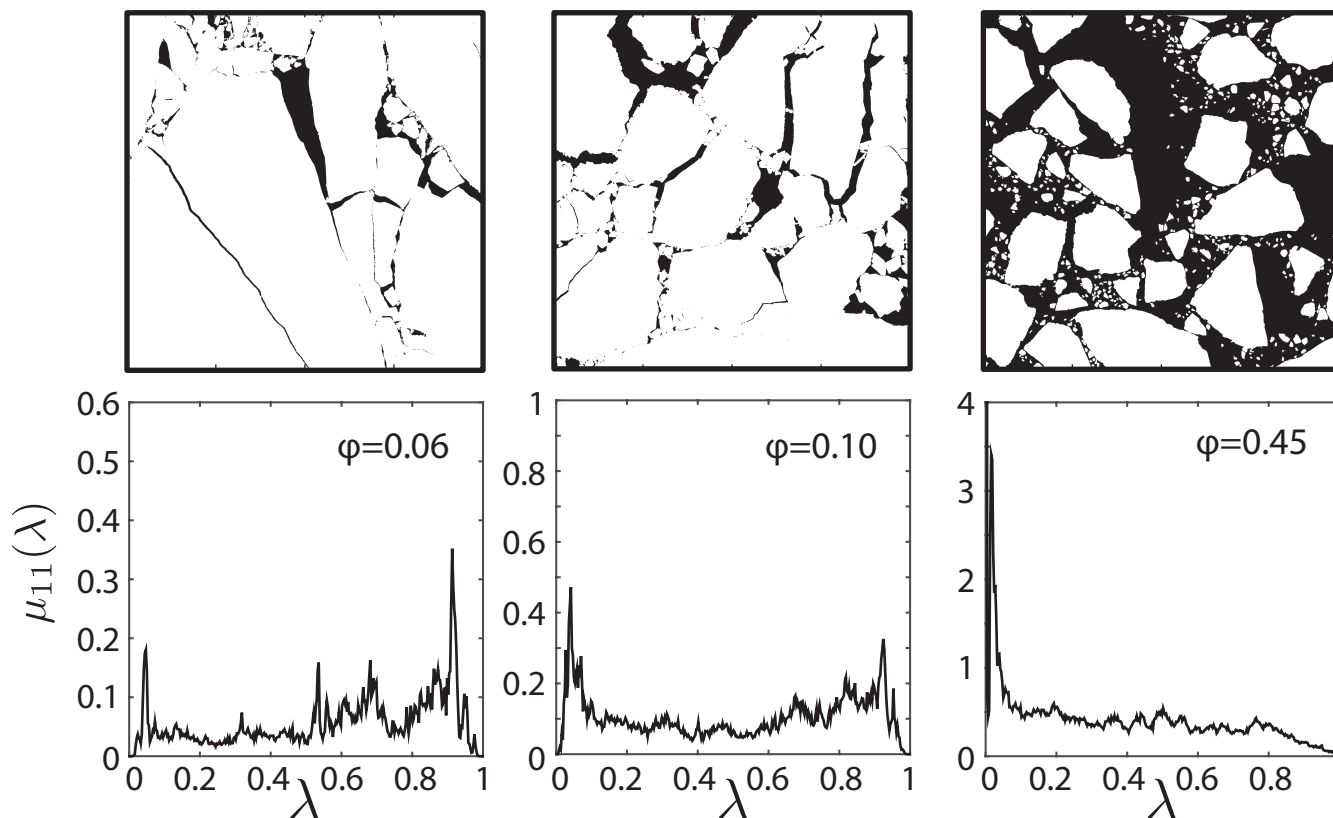
rigorous inverse bound on spectral gap

construct algebraic curves which bound admissible region in (p, q) -space

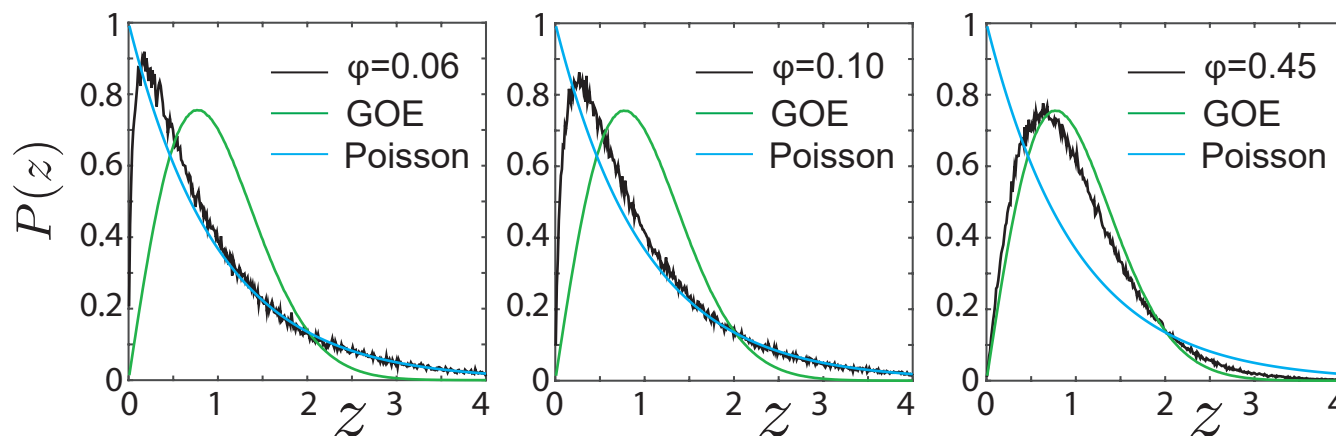
Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

Spectral computations for sea ice floe configurations

spectral
measures



eigenvalue
spacing
distributions



uncorrelated



level repulsion

**UNIVERSAL
Wigner-Dyson
distribution**

Eigenvalue Statistics of Random Matrix Theory

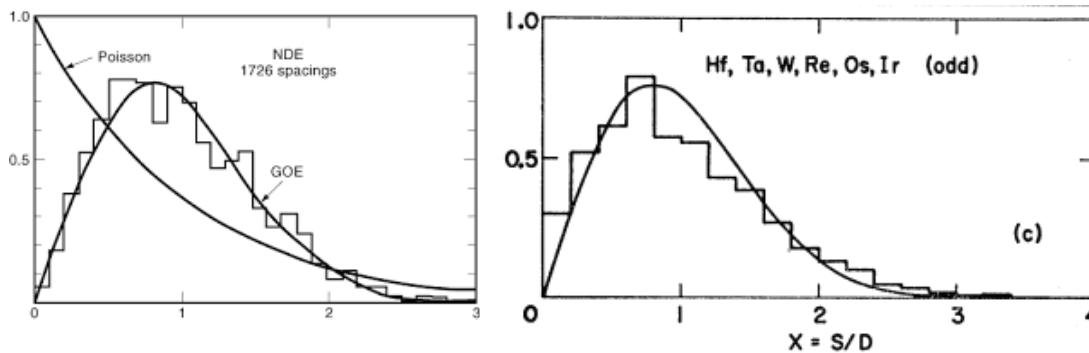
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[N]_{ij} \sim N(0,1), \quad A = (N + N^T)/2 \quad \text{Gaussian orthogonal ensemble (GOE)}$

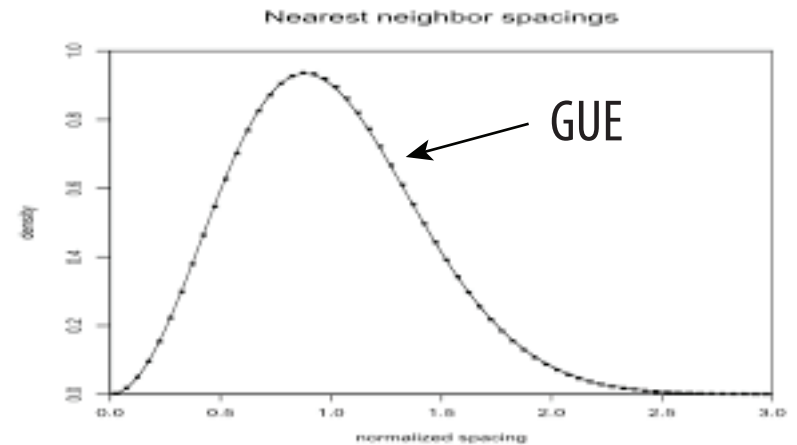
$[N]_{ij} \sim N(0,1) + iN(0,1), \quad A = (N + N^\dagger)/2 \quad \text{Gaussian unitary ensemble (GUE)}$

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics.

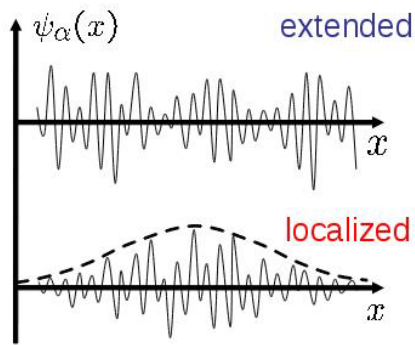
Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



Universal eigenvalue statistics arise in a broad range of “unrelated” problems!



Anderson localization

disorder-driven

metal / insulator transition

Anderson 1958
Mott 1949
Evangelou 1992
Shklovshii et al 1993

propagation vs. localization in wave physics:
quantum, optics, acoustics, water waves

Wave equations

Laplace + Diffusion
equations

we find percolation-driven

Anderson transition for classical transport in composites

mobility edges, localization, universal spectral statistics

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

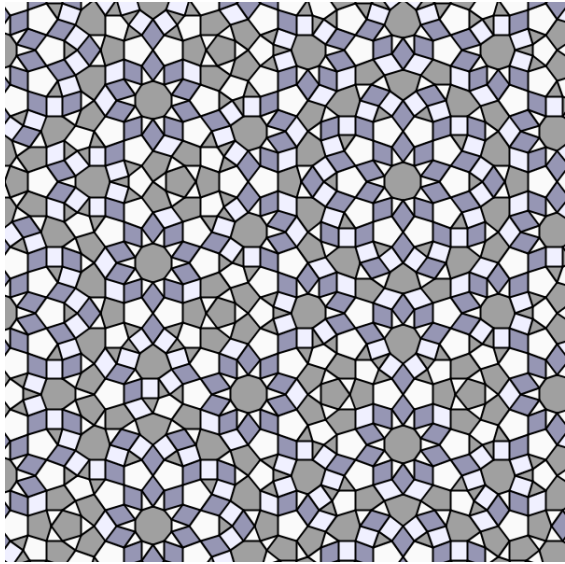
but no wave interference or scattering effects at play!

Order to Disorder in Quasiperiodic Composites

D. Morison (Physics), N. B. Murphy, E. Cherkaev, K. M. Golden, *Communications Physics* 2022

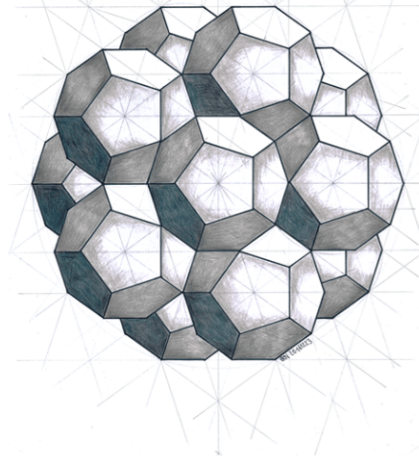
quasiperiodic crystal

quasicrystal



quasiperiodic checkerboard

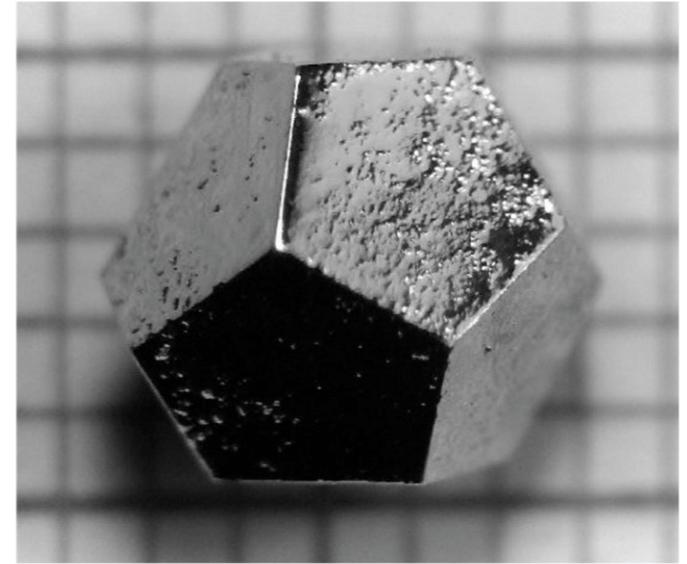
Stampfli, 2013



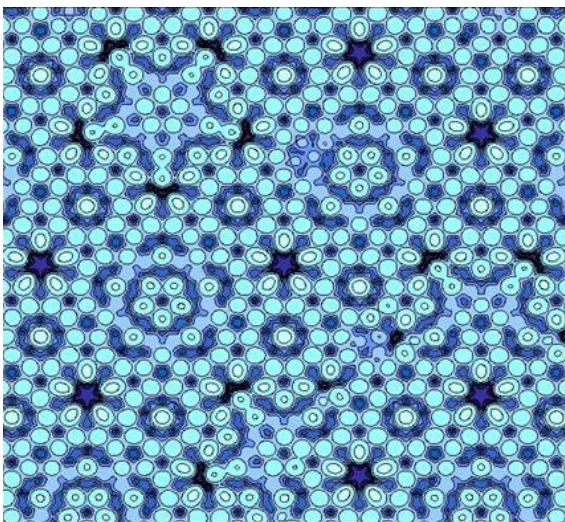
dense packing of dodecahedra

3D Penrose tiling

Tripkovic, 2019



Holmium-magnesium-zinc quasicrystal



energy surface Al-Pd-Mn quasicrystal

Unal et al., 2007

ordered but aperiodic

lacks translational symmetry

Shechtman et al., *Phys. Rev. Lett.*, 1984

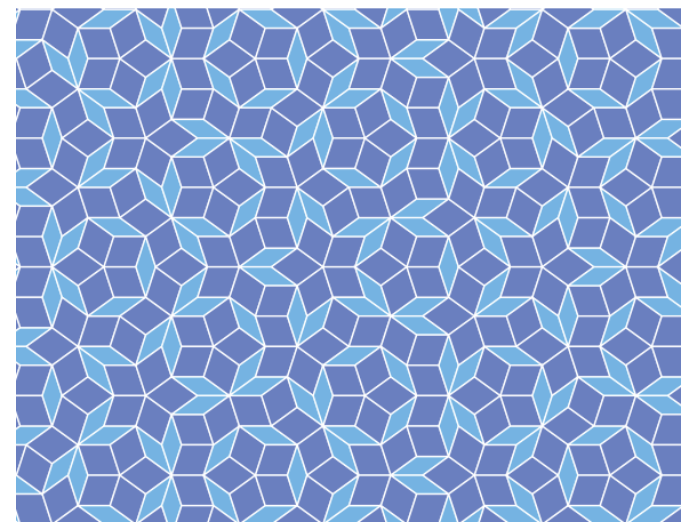
Levine & Steinhardt, *Phys. Rev. Lett.*, 1984

**classical transport in
quasiperiodic media**

Golden, Goldstein & Lebowitz, *Phys. Rev. Lett.*, 1985

Golden, Goldstein & Lebowitz, *J. Stat. Phys.*, 1990

⋮



aperiodic tiling of the plane - R. Penrose 1970s

Order to disorder in quasiperiodic composites

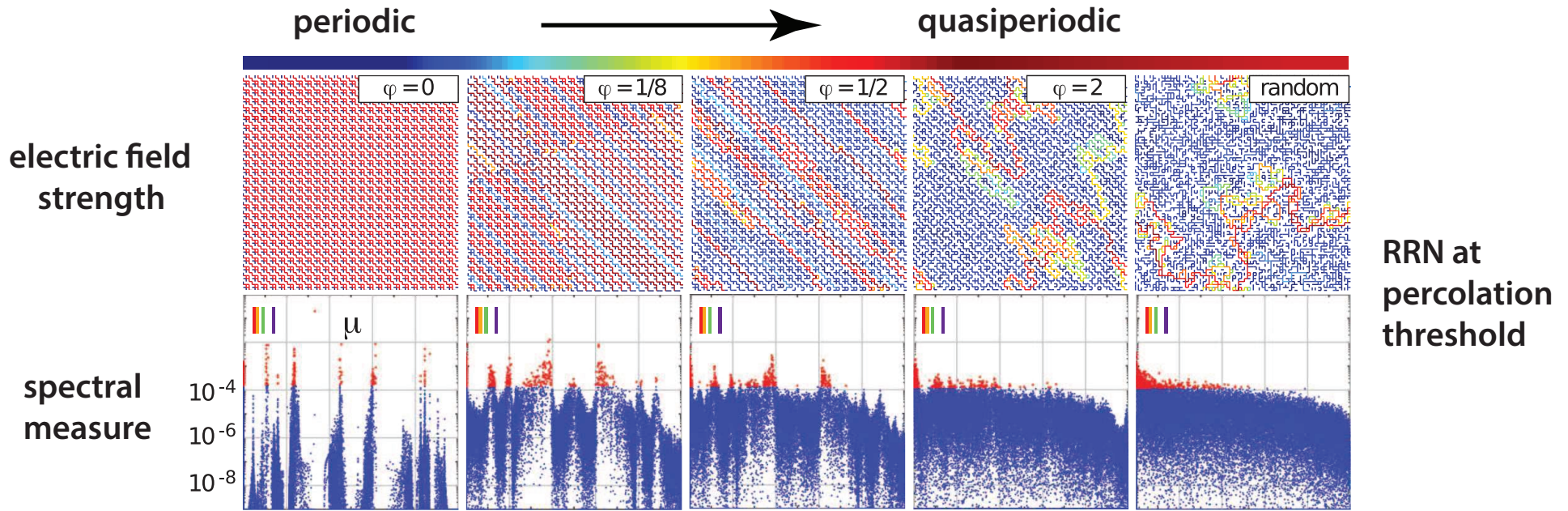
Morison, Murphy, Cherkhev, Golden, Comm. Phys. 2022

sea ice inspired - high tech spin off

tunable quasiperiodic composites with exotic properties

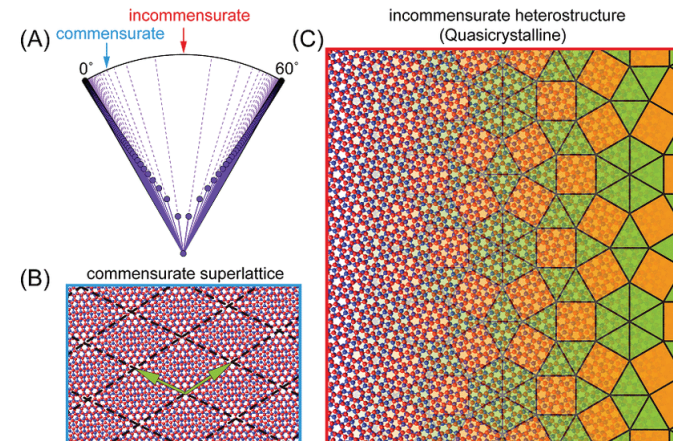
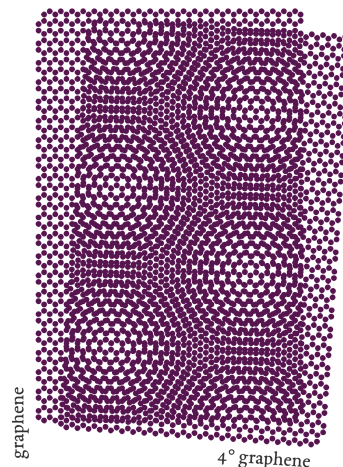
(optical, electrical, thermal, ...), Anderson localization; our Moiré patterned geometries are similar to **twisted bilayer graphene**

increasing twist angle between two lattices



twisted bilayer graphene

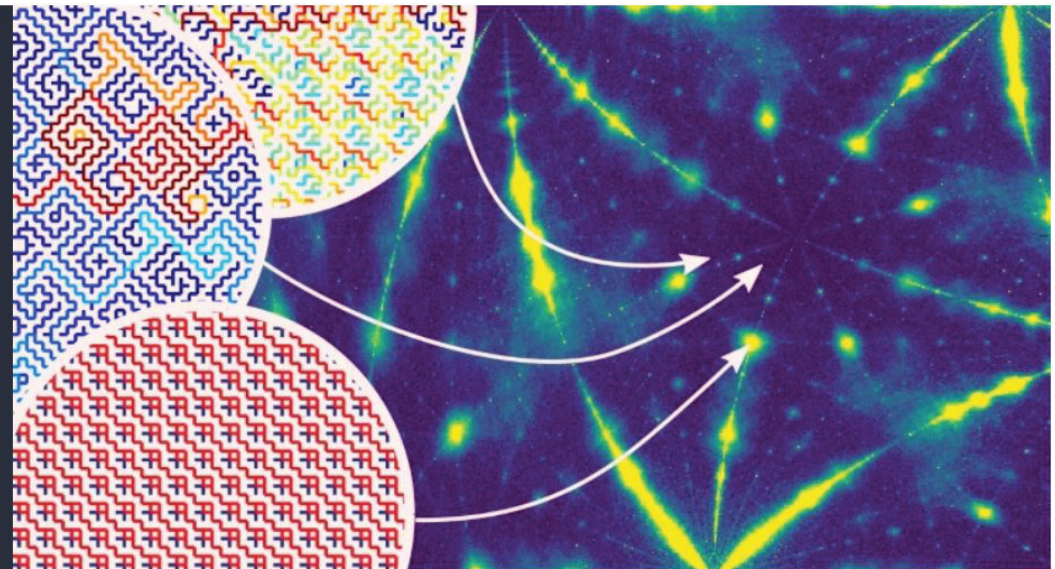
*superconducting
magic twist angle*



Yao et al., 2018

Order to disorder in quasiperiodic composites

David Morison, N. Benjamin Murphy ... Kenneth M. Golden
Article | 14 June 2022



Featured

Article

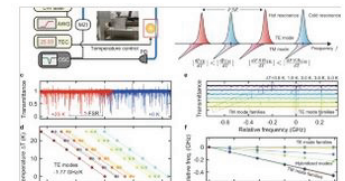
Open Access

10 Jan 2023

Versatile tuning of Kerr soliton microcombs in crystalline microresonators

High-repetition rate microresonator-based frequency combs offer powerful and compact optical frequency comb sources that are of great importance to various applications. Here, the authors extend the tunability of the Kerr soliton frequency combs by exploiting thermal effects and frequency stabilization techniques.

Shun Fujii, Koshiro Wada ... Takasumi Tanabe



Article

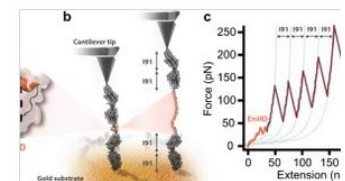
Open Access

12 Jan 2023

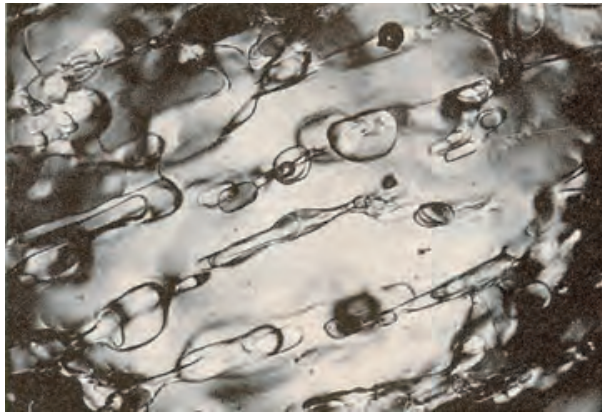
Compliant mechanical response of the ultrafast folding protein EnHD under force

Exhibiting low-energy (un)folding barriers and fast kinetics, ultrafast folding proteins are enticing models to study protein dynamics. The authors use single molecule force spectroscopy AFM to capture the compliant behaviour hallmarking the dynamics of ultrafast folding proteins under force.

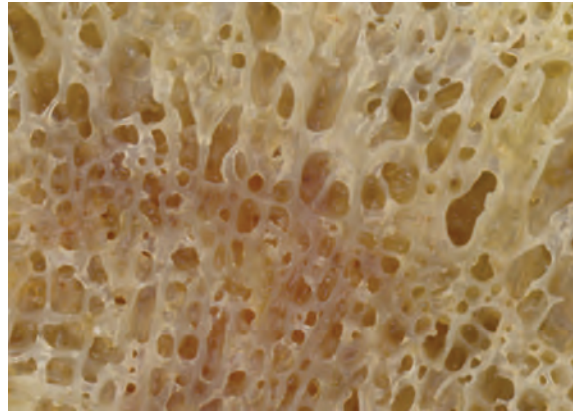
Antonio Reifs, Irene Ruiz Ortiz ... Raul Perez-Jimenez



SEA ICE

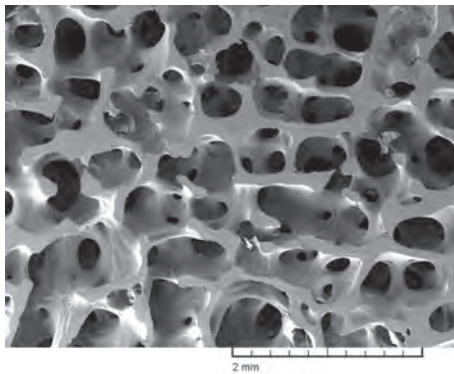


HUMAN BONE

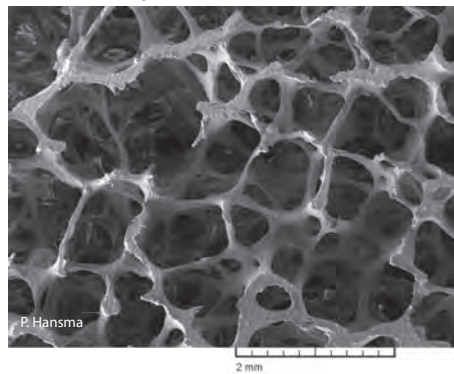


*spectral characterization
of porous microstructures
in human bone*

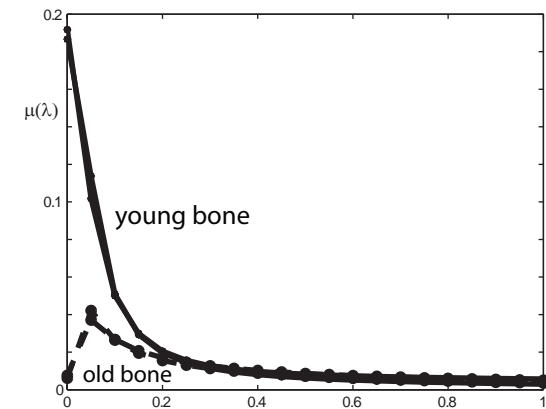
young healthy trabecular bone



old osteoporotic trabecular bone



reconstruct spectral measures
from complex permittivity data



use regularized inversion scheme

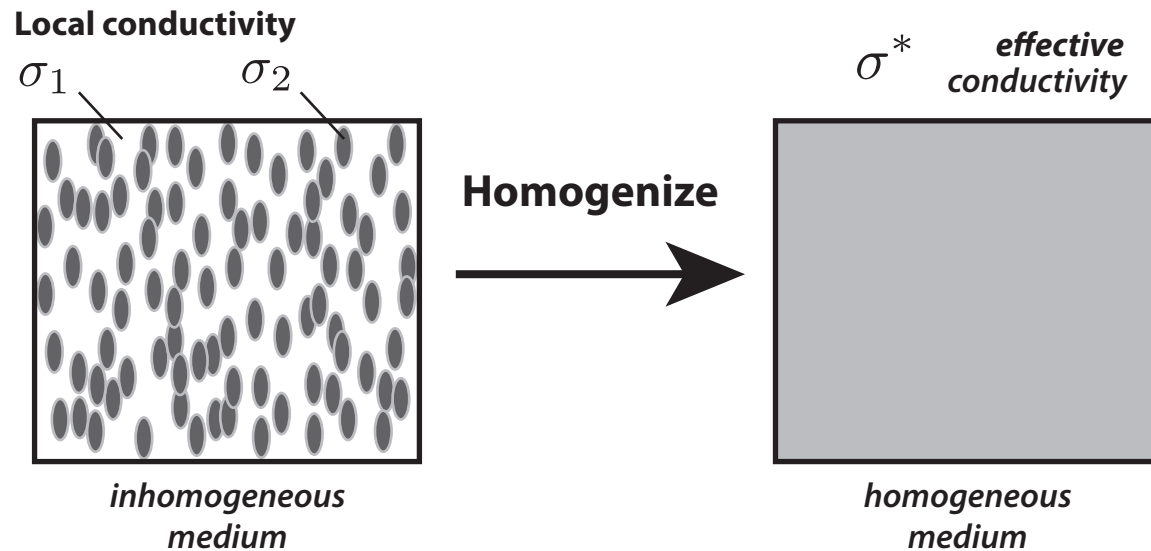
*apply spectral measure analysis of brine connectivity and
spectral inversion to electromagnetic monitoring of osteoporosis*

Golden, Murphy, Cherkaev, J. Biomechanics 2011

the math doesn't care if it's sea ice or bone!

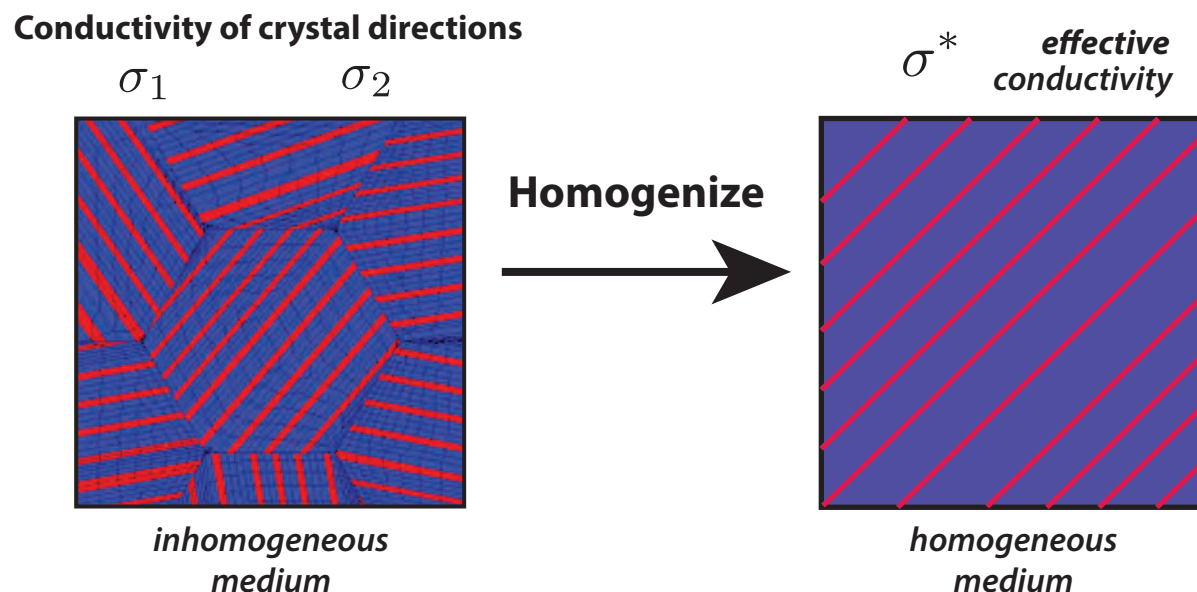
The math doesn't care if it's two phase composites or polycrystals!

**Two-component
composites**



Find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

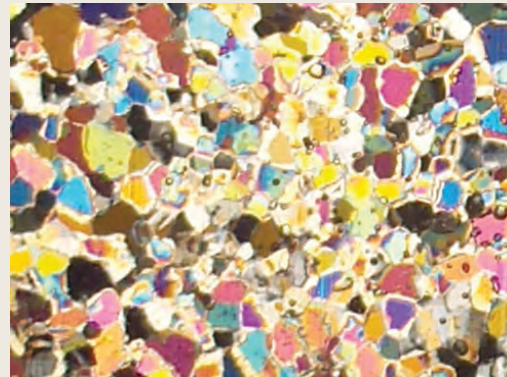
**Polycrystalline
media**



Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**
Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**
orientation statistics
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



PROCEEDINGS A

350 YEARS
OF SCIENTIFIC
PUBLISHING

An invited review
commemorating 350 years
of scientific publishing at the
Royal Society

A method to distinguish
between different types
of sea ice using remote
sensing techniques

A computer model to
determine how a human
should walk so as to expend
the least energy



THE
ROYAL
SOCIETY
PUBLISHING

higher threshold for fluid flow in granular sea ice

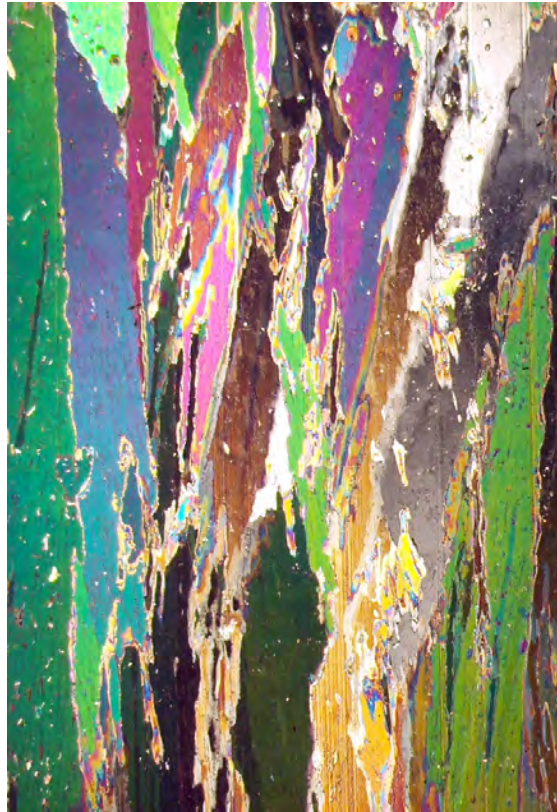
microscale details impact “mesoscale” processes

nutrient fluxes for microbes
melt pond drainage
snow-ice formation

columnar

granular

5%



10%



Golden, Sampson, Gully, Lubbers, Tison 2023

electromagnetically distinguishing ice types
Kitsel Lusted, Elena Cherkaev, Ken Golden

mesoscale

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere

advection diffusion equation with a velocity field \vec{u}

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$



homogenize

$$\frac{\partial \bar{T}}{\partial t} = \kappa^* \Delta \bar{T}$$

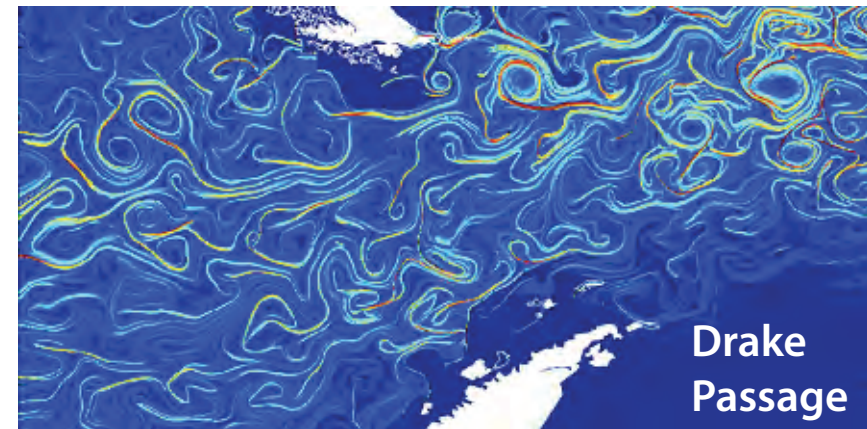
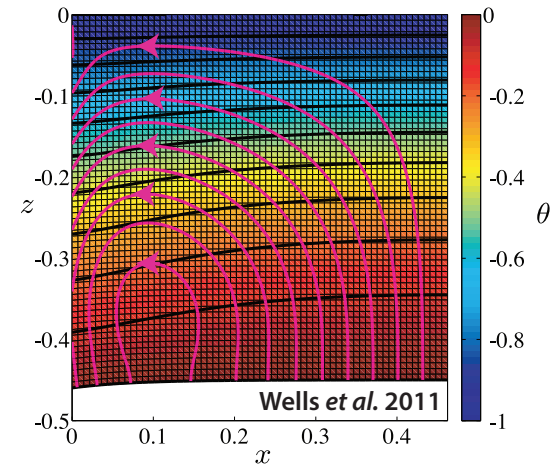
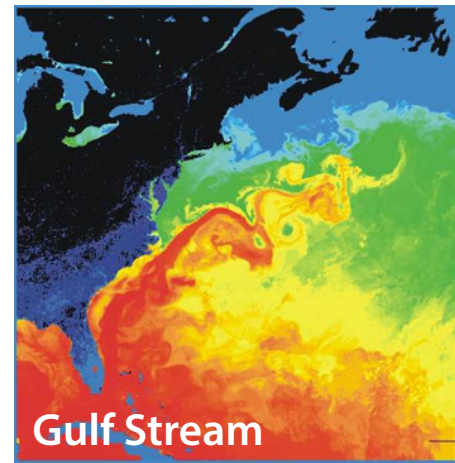
κ^* effective diffusivity

Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

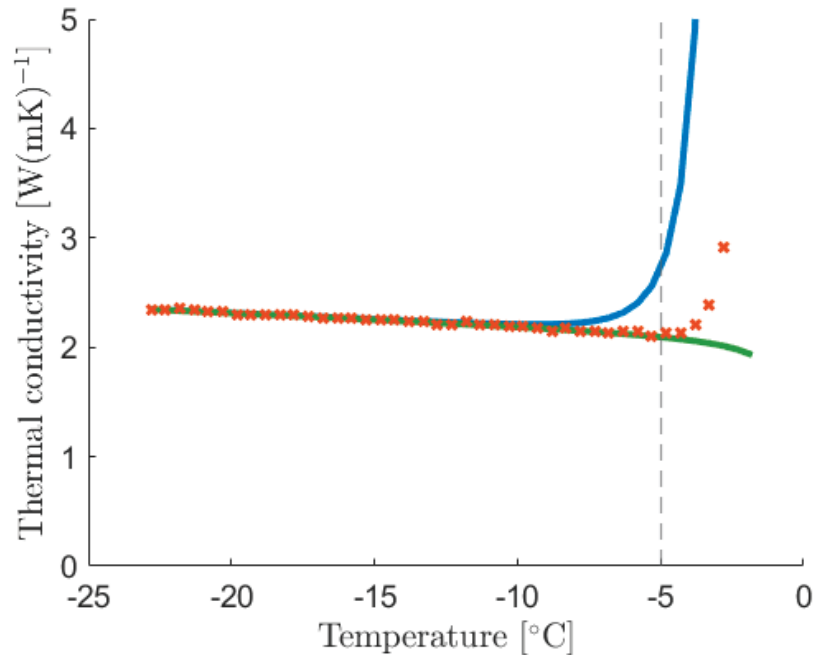


tracers flowing through inverted sea ice blocks



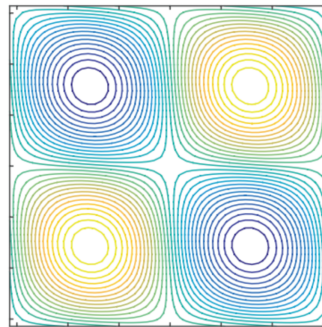
Bounds on Convection Enhanced Thermal Transport

simulations



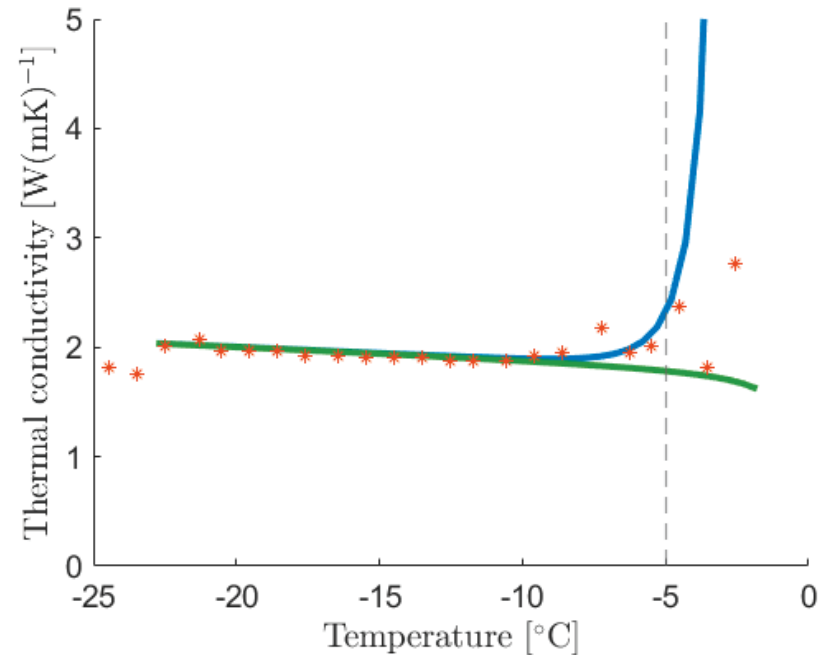
Monte-Carlo simulations of SDE with temperature dependent Péclet number P

strength of advection $B = \kappa P / 2\pi$
Euler-Maruyama and subsampling
methods for SDE



**cat's eye flow model for
brine convective flow**

data [Trodahl et al., 2001]



Rigorous Padé approximant bounds in terms of P using Stieltjes integral + analytic continuation method for the measure

Darcy velocity $v = 0.5$ [m/s]

wave propagation in the marginal ice zone (MIZ)

Stieltjes integral representation and bounds for the complex viscoelasticity of the ice - ocean layer

Sampson, Murphy, Cherkaev, Golden 2023

first theory of key parameter in wave-ice interactions only fitted to wave data before

Keller, 1998

Mosig, Montiel, Squire, 2015

Wang, Shen, 2012

Analytic Continuation Method

Bergman (78) - Milton (79)
integral representation for ϵ^*

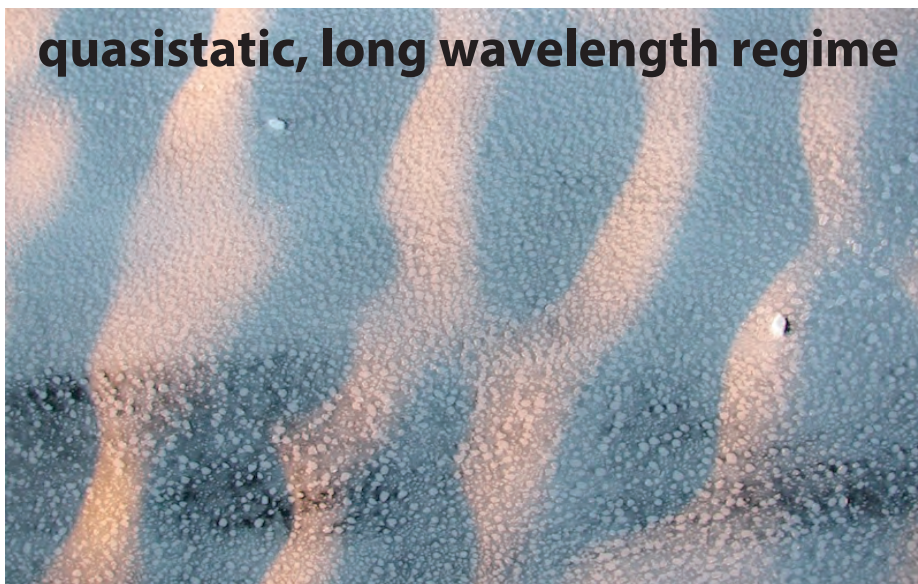
Golden and Papanicolaou (83)

Milton, *Theory of Composites* (02)

quasistatic, long wavelength regime

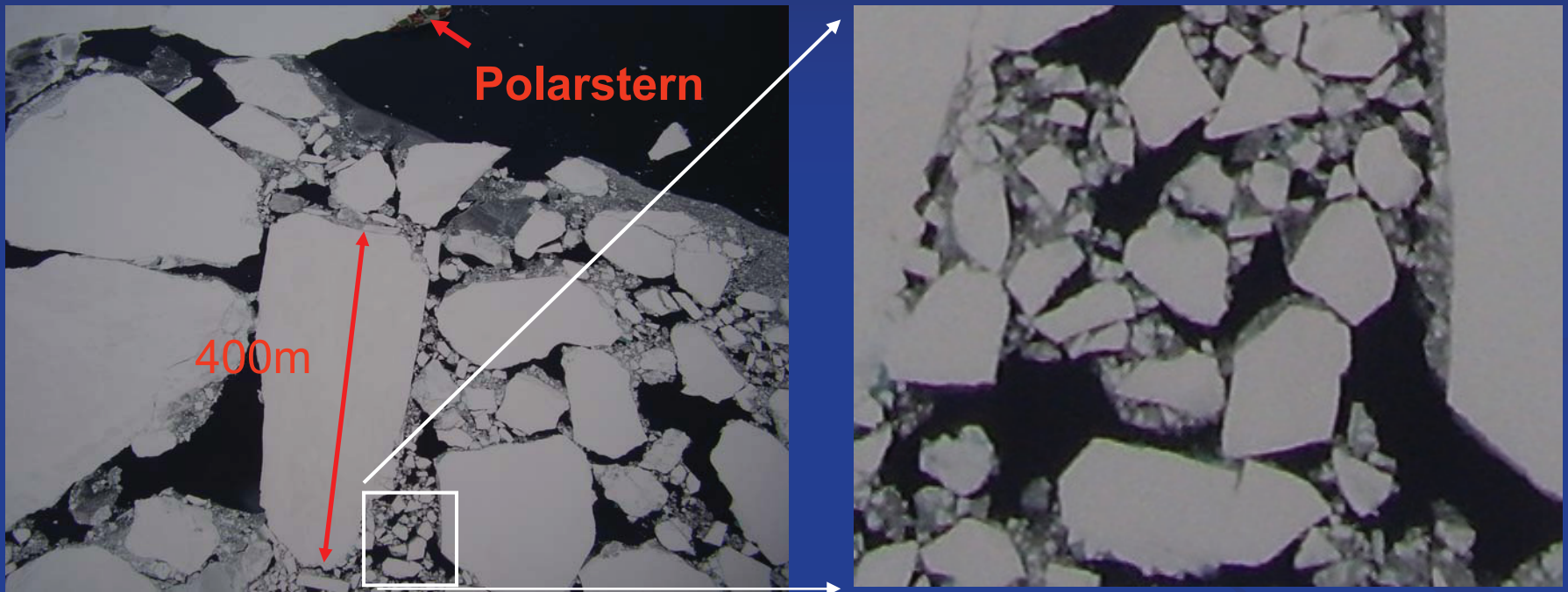
homogenized parameter depends on sea ice concentration and ice floe geometry

like EM waves



fractal sea ice cover

Self-similarity of sea ice pack



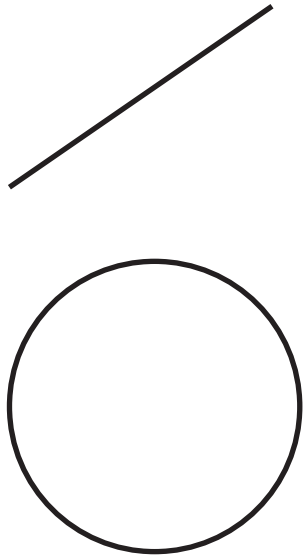
Weddell Sea, Antarctica
2006

spectral measures!

Takenobu Toyota

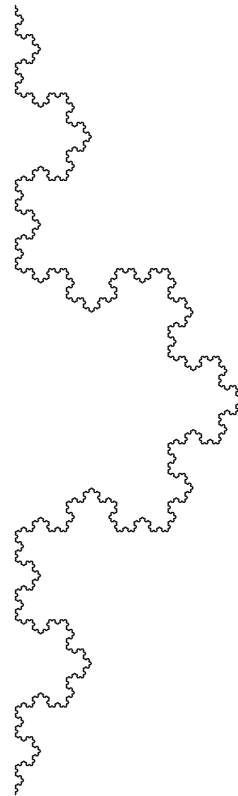
fractal curves in the plane

they wiggle so much that their dimension is >1



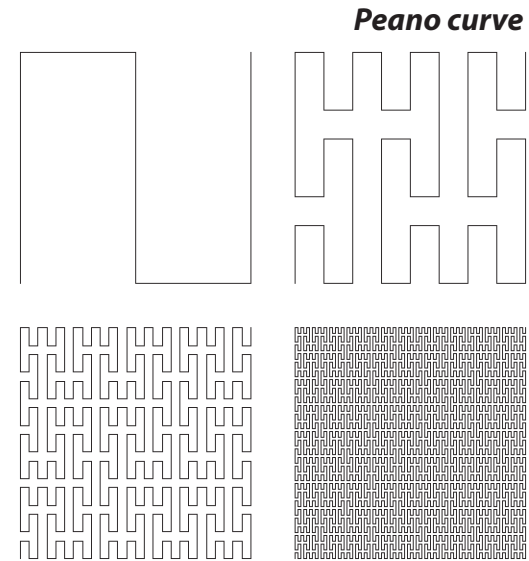
simple curves

$D = 1$

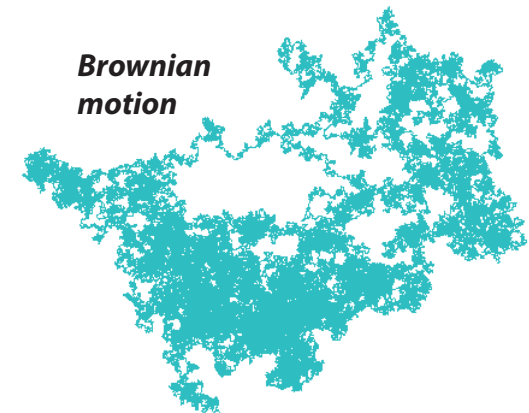


Koch snowflake

$D = 1.26$



Peano curve



Brownian motion

space filling curves

$D = 2$

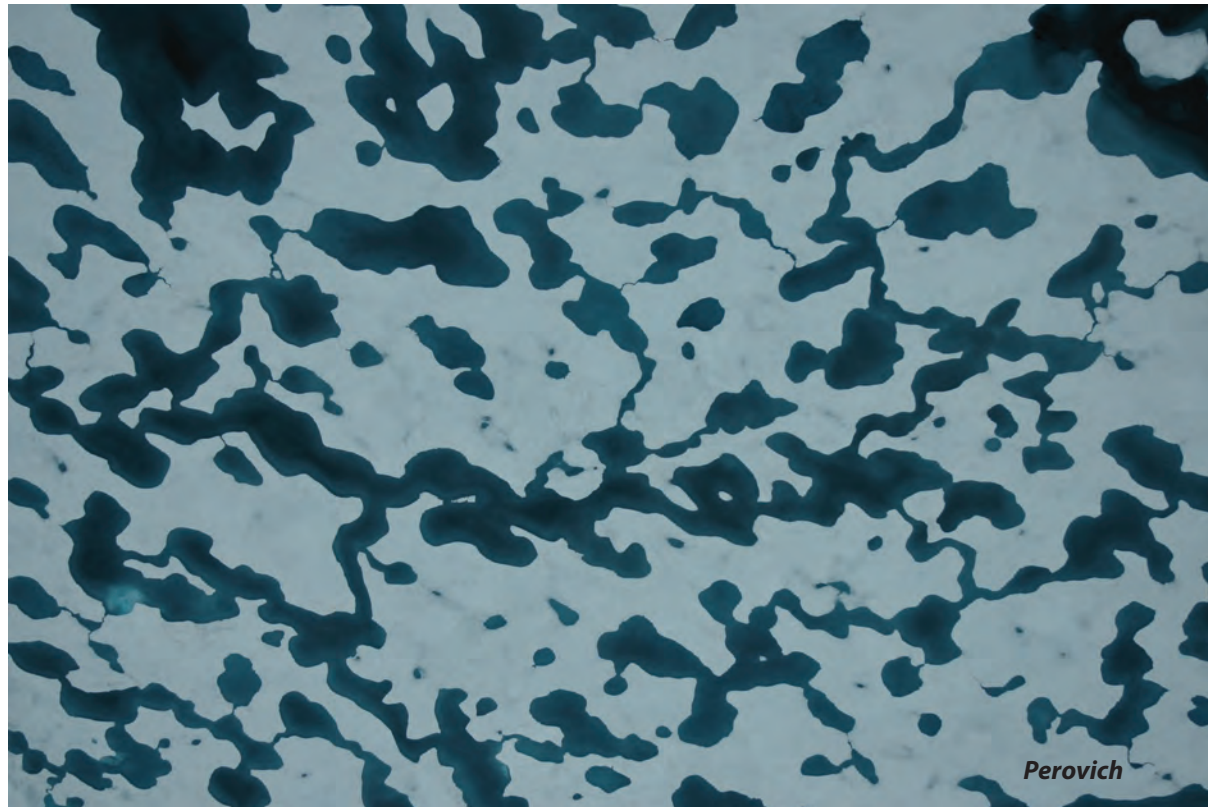
melt pond formation and albedo evolution:

- *major drivers in polar climate*
- *key challenge for global climate models*

numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham,
Taylor, Worster 2006
Flocco, Feltham 2007

Skyllingstad, Paulson,
Perovich 2009
Flocco, Feltham,
Hunke 2012



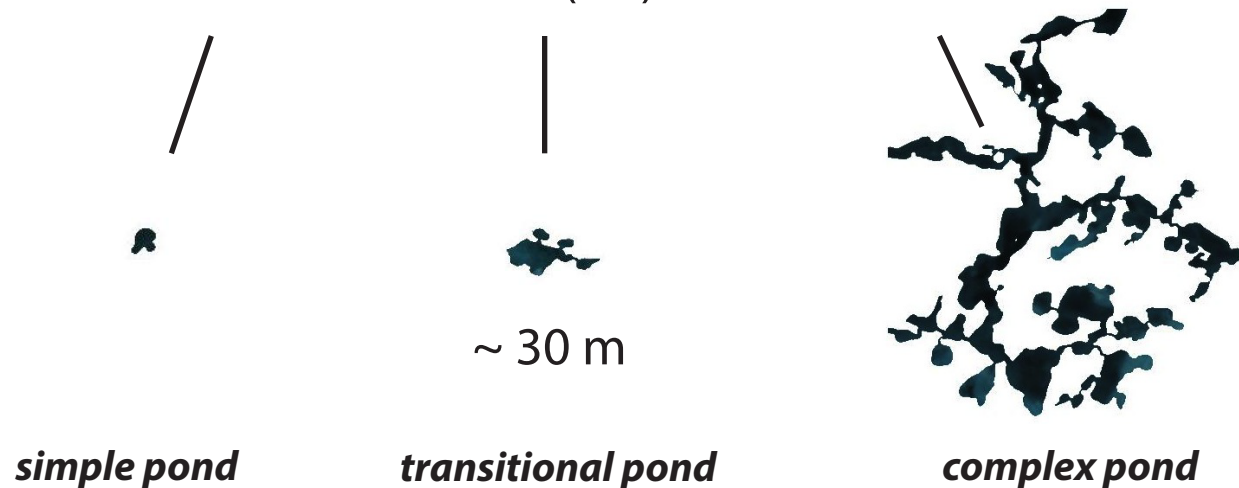
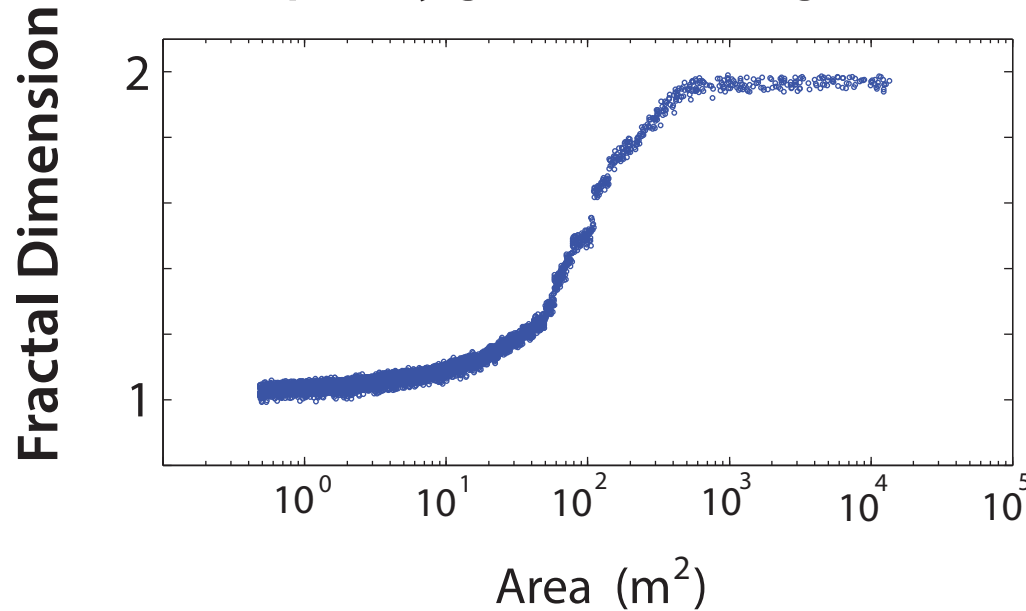
Are there universal features of the evolution similar to phase transitions in statistical physics?

Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

The Cryosphere, 2012

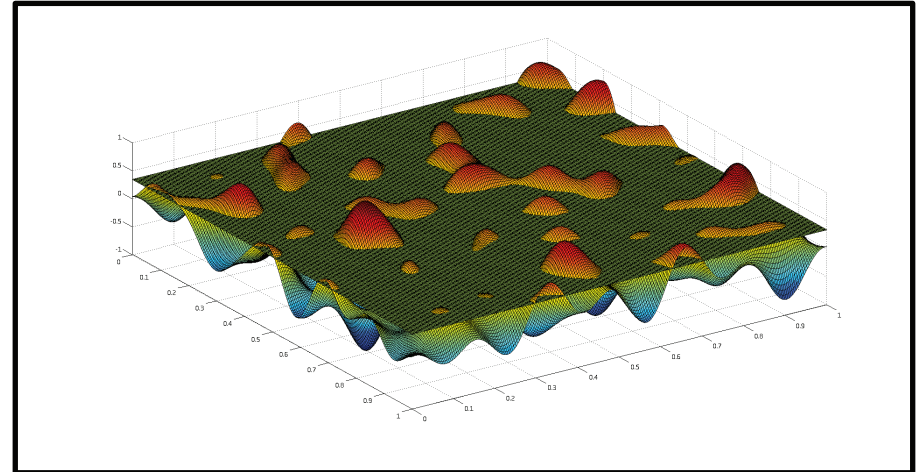
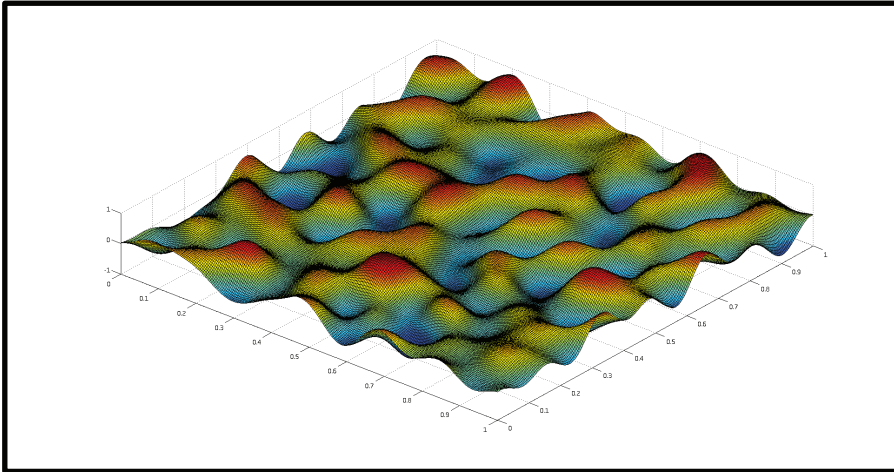
complexity grows with length scale



Continuum percolation model for melt pond evolution

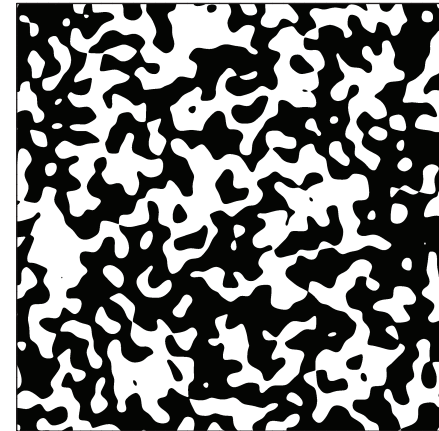
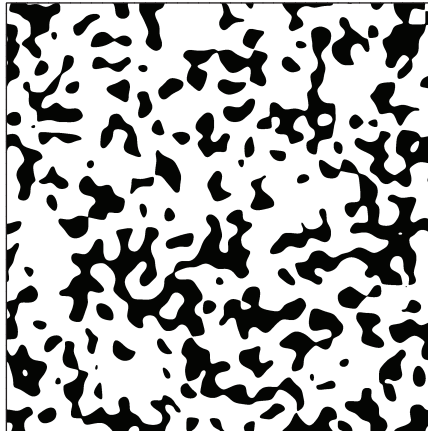
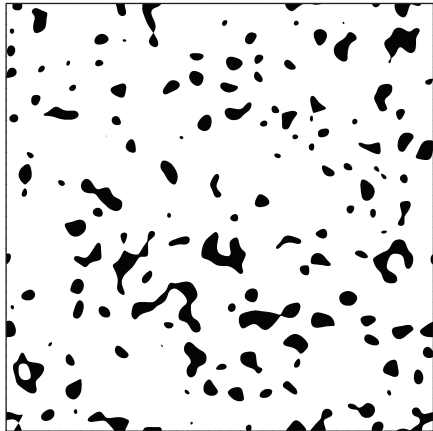
level sets of random surfaces

Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018



random Fourier series representation of surface topography

intersections of a plane with the surface define melt ponds



electronic transport in disordered media

diffusion in turbulent plasmas

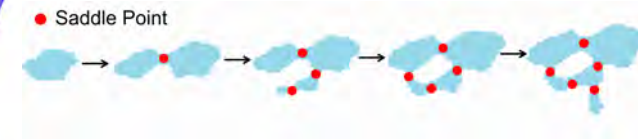
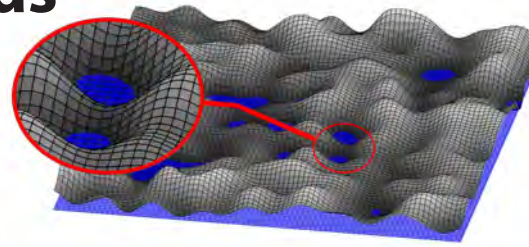
Isichenko, Rev. Mod. Phys., 1992

Saddle points of the sea ice surface and the fractal geometry of Arctic melt ponds

Physical Review Research (invited, in revision), Ryleigh Moore, Jacob Jones, Dane Gollero, Court Strong, Ken Golden

Morse Theory for Melt Ponds

ponds coalesce - change topology and complexify at saddle points; they drive the fractal transition

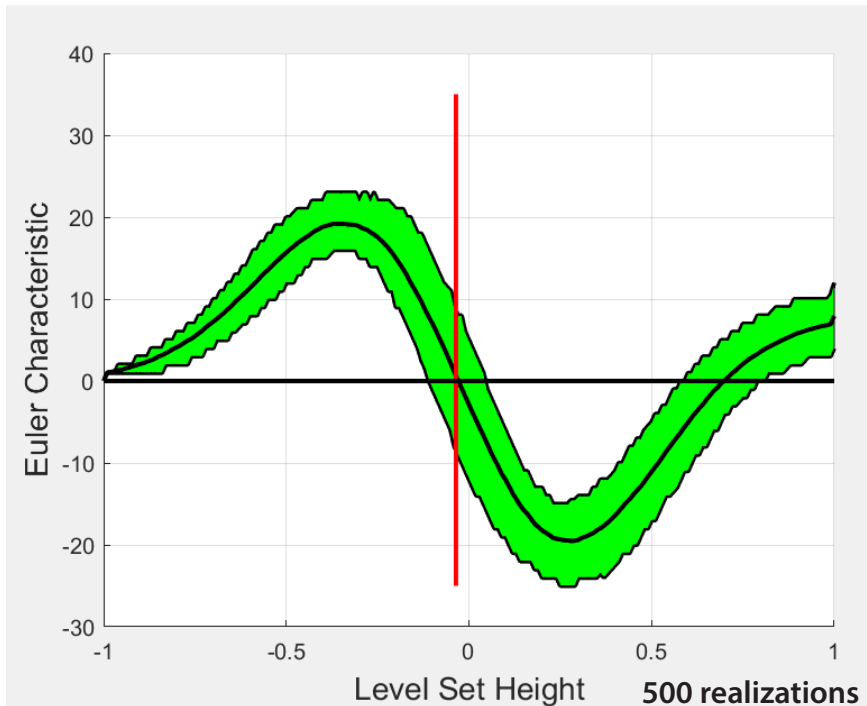


Topological Data Analysis (TDA) for Melt Ponds

persistent homology Euler characteristic = # maxima + # minima - # saddles

filtration - sequence of nested topological spaces, indexed by water level

Expected Euler Characteristic Curve (ECC)



tracks the evolution of the EC of the flooded surface as water rises

zero of ECC ~ percolation

percolation on a torus creates a giant cycle

Bobrowski & Skraba, 2020

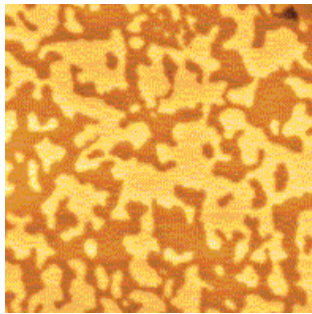
Carlsson, 2009

Vogel, 2002 GRF

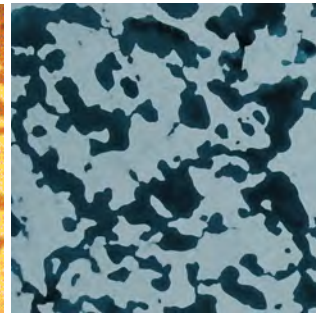
image analysis
porous media
cosmology
brain activity

From magnets to melt ponds

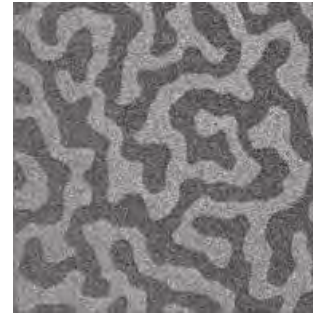
100 year old model for magnetic materials
used to explain melt pond geometry



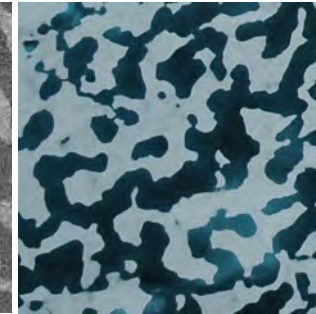
magnetic domains
in cobalt



Arctic melt ponds

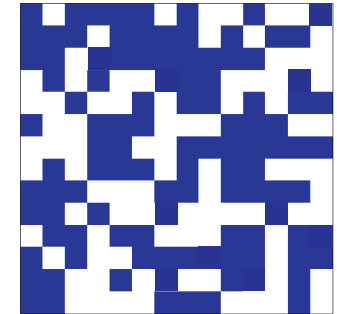
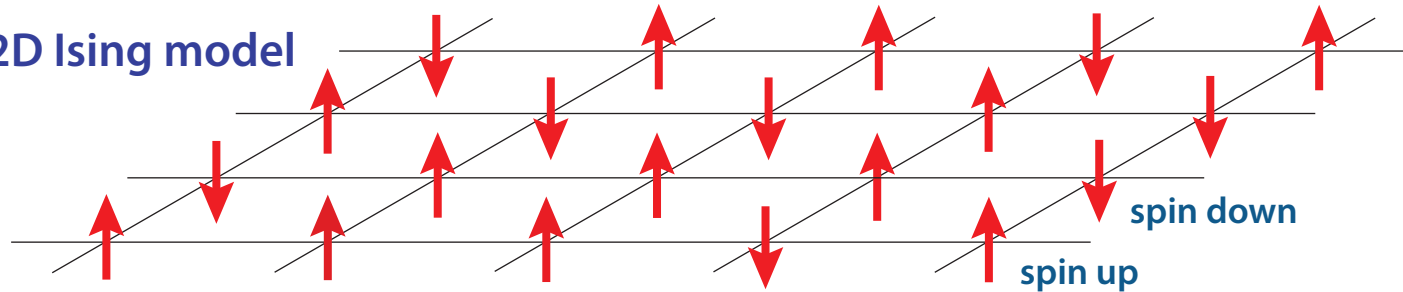


magnetic domains
in cobalt-iron-boron

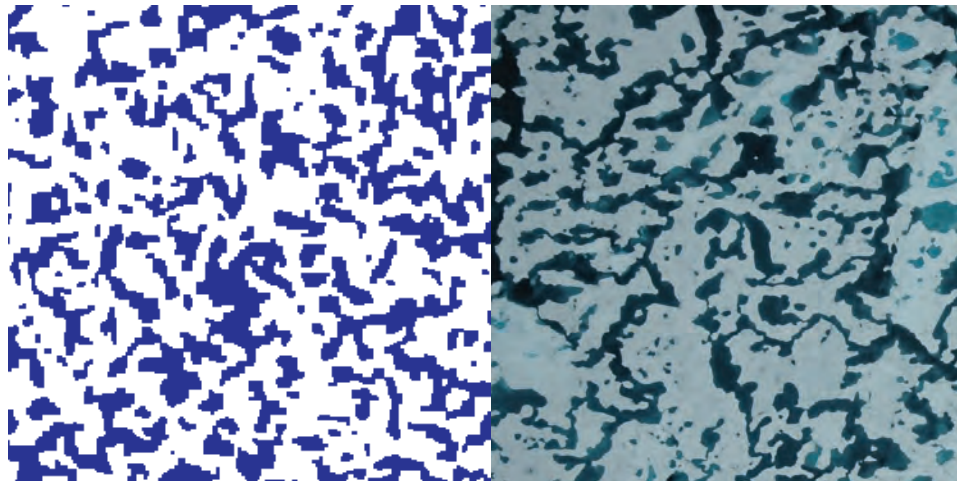


Arctic melt ponds

2D Ising model



model



real ponds
(Perovich)

Ma, Sudakov, Strong,
Golden, *New J. Phys.* 2019

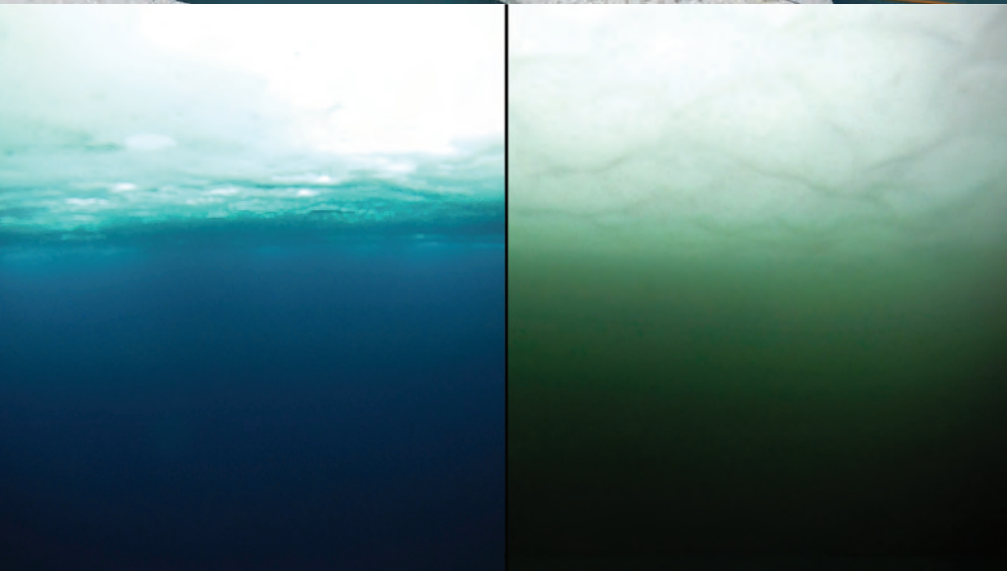
Scientific American,
EOS, PhysicsWorld, ...



Perovich

Melt ponds control transmittance of solar energy through sea ice, impacting upper ocean ecology.

WINDOWS



no bloom

bloom

massive under-ice **algal bloom**

Arrigo et al., *Science* 2012

Have we crossed into a new ecological regime?

The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

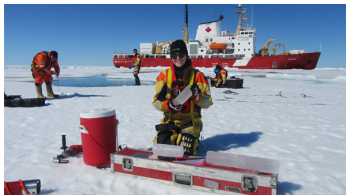
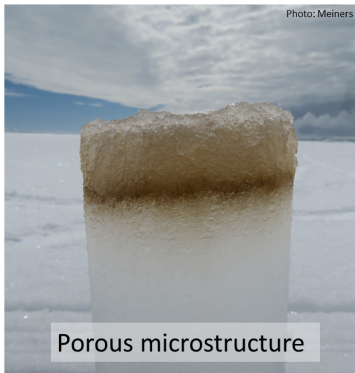
Horvat, Rees Jones, Iams, Schroeder, Flocco, Feltham, *Science Advances* 2017

The effect of melt pond geometry on the distribution of solar energy under first year sea ice

Horvat, Flocco, Rees Jones, Roach, Golden
Geophys. Res. Lett. 2019

(2015 AMS MRC)

SEA ICE ALGAE



Can we improve agreement between algae models and data?

80% of polar bear diet can be traced to ice algae*.

* Brown TA, et al. (2018). *PloS one*, 13(1), e0191631

METHOD

Uncertainty quantification for ecological models with random parameters

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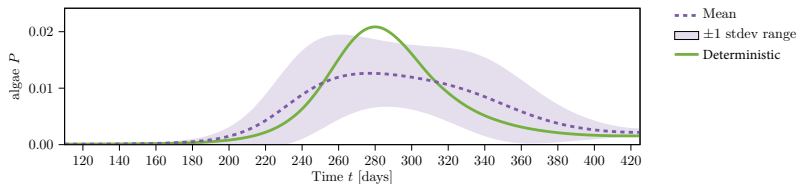
Email: reimer@math.utah.edu

Abstract

There is often considerable uncertainty in parameters in ecological models. This uncertainty can be incorporated into models by treating parameters as random variables with distributions, rather than fixed quantities. Recent advances in uncertainty quantification methods, such as polynomial chaos approaches, allow for the analysis of models with random parameters. We introduce these methods with a motivating case study of sea ice algal blooms in heterogeneous environments. We compare Monte Carlo methods with polynomial chaos techniques to help understand the dynamics of an algal bloom model with random parameters.

Introduce polynomial chaos approach to widely used ecological ODE models, but with random parameters.

ECOLOGICAL INSIGHTS



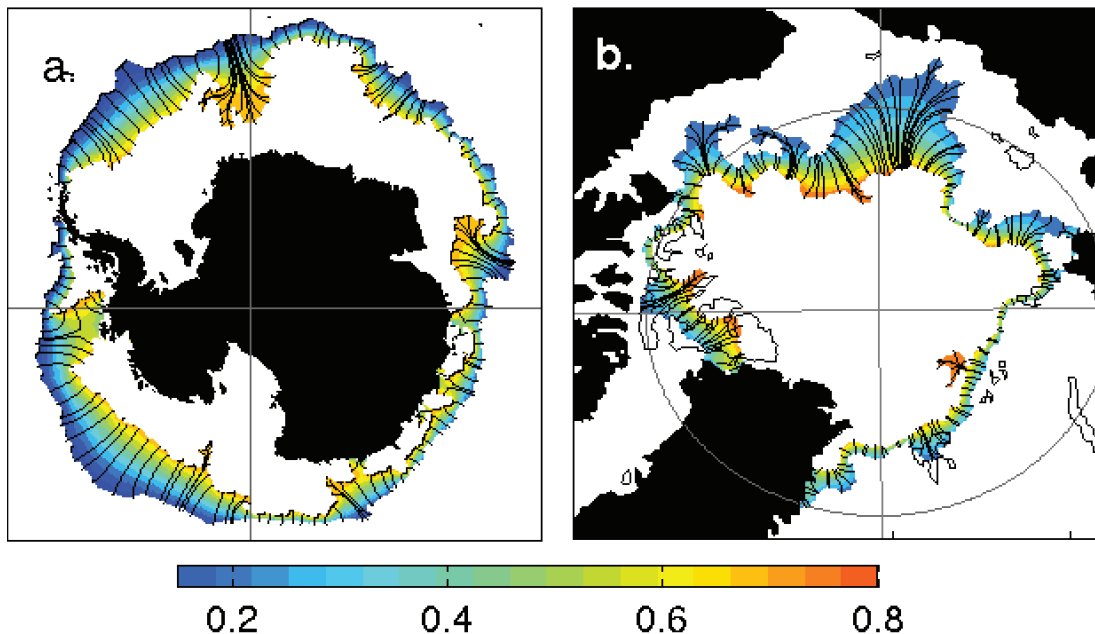
- lower peak bloom intensity
- longer bloom duration
- able to compare variance to data

macroscale

Marginal Ice Zone

MIZ

- biologically active region
- intense ocean-sea ice-atmosphere interactions
- region of significant wave-ice interactions



MIZ WIDTH

fundamental length scale of
ecological and climate dynamics

Strong, *Climate Dynamics* 2012

Strong and Rigor, *GRL* 2013

transitional region between
dense interior pack ($c > 80\%$)
sparse outer fringes ($c < 15\%$)

**How to objectively
measure the “width”
of this complex,
non-convex region?**

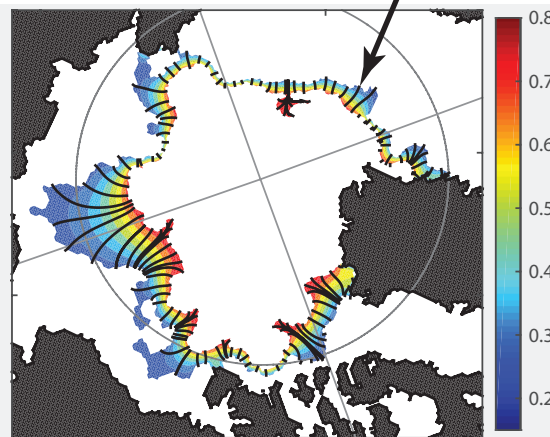
Objective method for measuring MIZ width motivated by medical imaging and diagnostics

Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

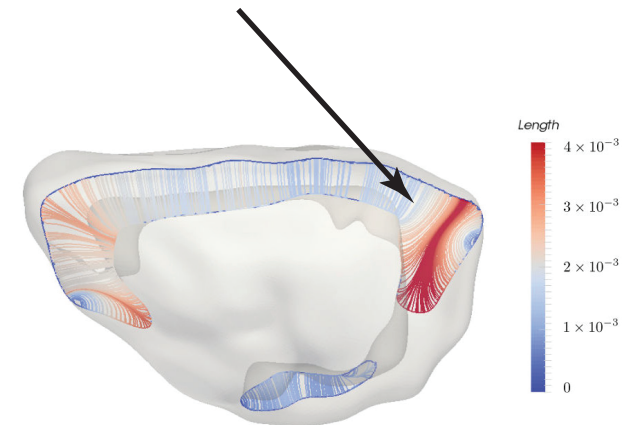
39% widening
1979 - 2012

“average” lengths of streamlines

streamlines of a solution
to Laplace’s equation



Arctic Marginal Ice Zone



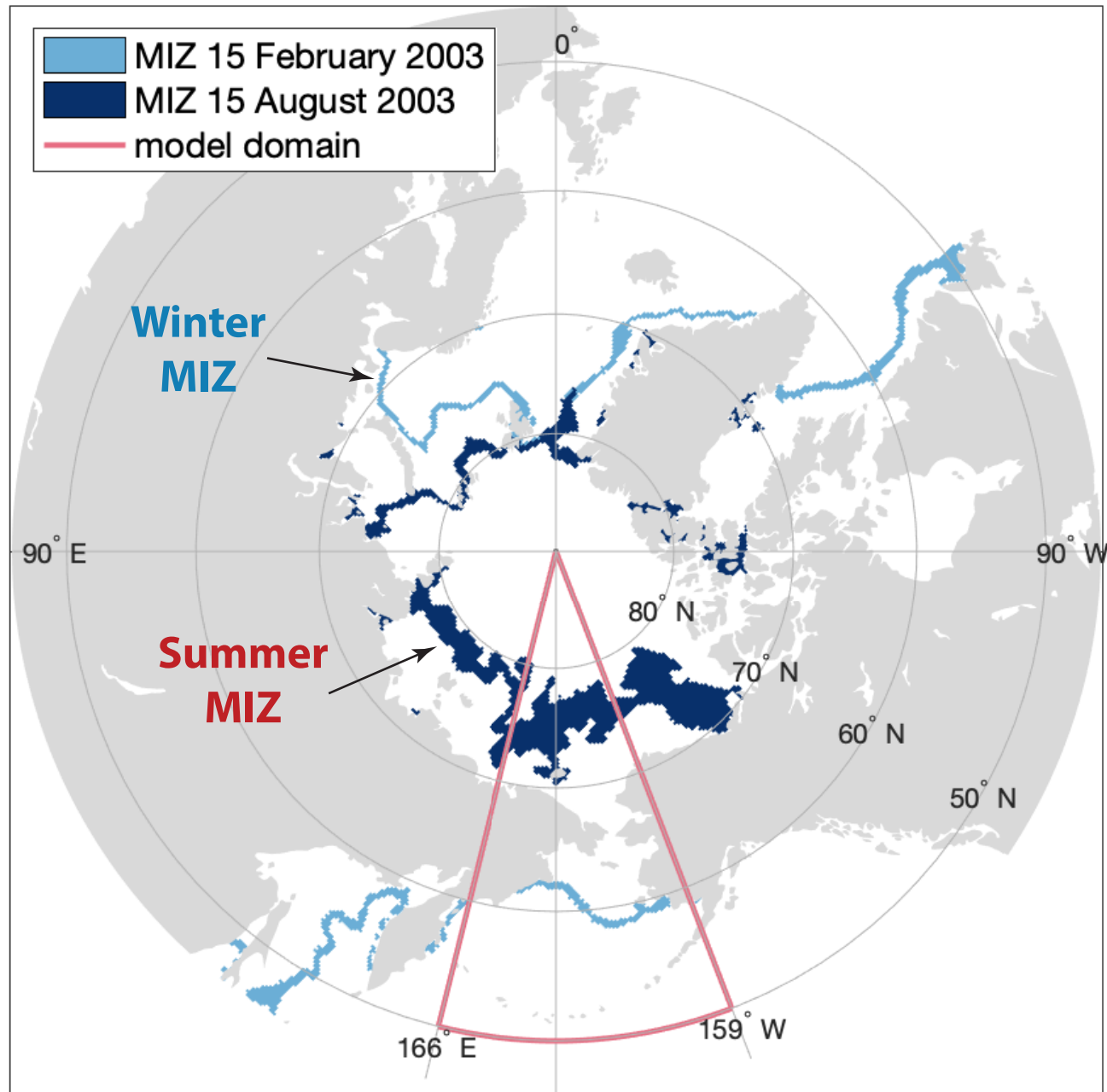
**crosssection of the
cerebral cortex of a rodent brain**

analysis of different MIZ WIDTH definitions

Strong, Foster, Cherkaev, Eisenman, Golden
J. Atmos. Oceanic Tech. 2017

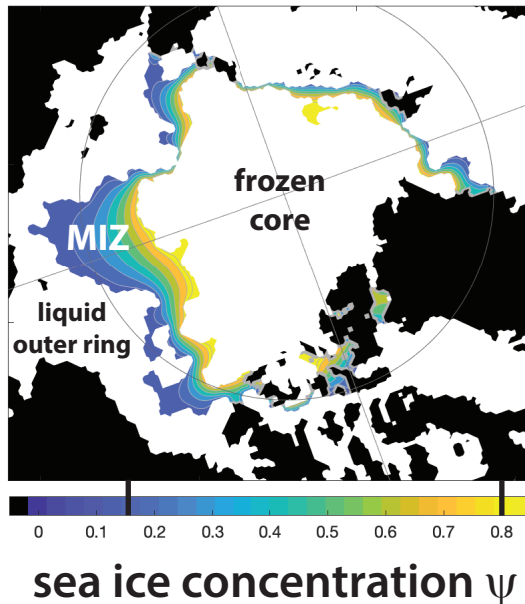
Strong and Golden
Society for Industrial and Applied Mathematics News, April 2017

Observed Arctic MIZ



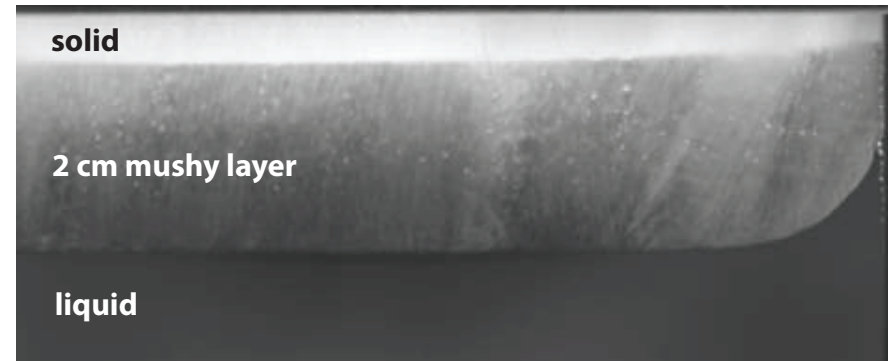
Model larger scale effective behavior
with partial differential equations that
homogenize complex local structure and dynamics.

Arctic MIZ



Predict MIZ width and location with
basin-scale phase change model.

seasonal and long term trends



NaCl-H₂O in lab
(Peppin et al., 2007;; J. Fluid Mech.)

Partial differential equation models
and deep learning for the sea ice
concentration field, 2023

Delaney Mosier, Court Strong,
Jingyi Zhu, Bao Wang, Ken Golden

Annual cycle of Arctic marginal ice zone location and
width explained by macroscale mushy layer model, 2023

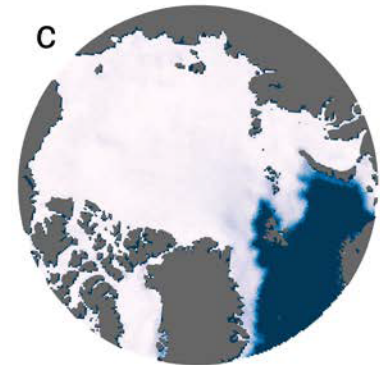
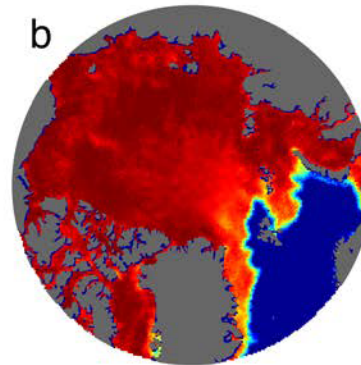
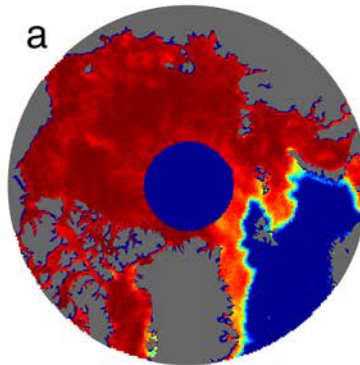
C. Strong, E. Cherkaev, and K. M. Golden

Filling the polar data gap with partial differential equations

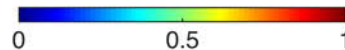
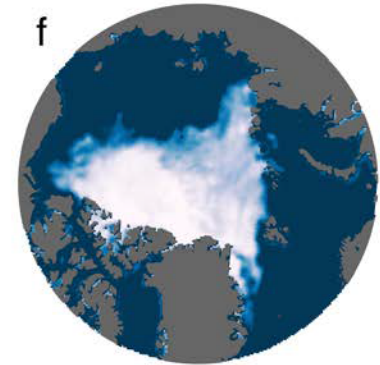
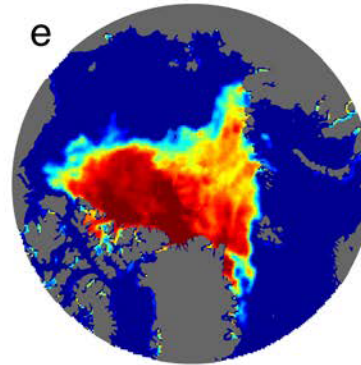
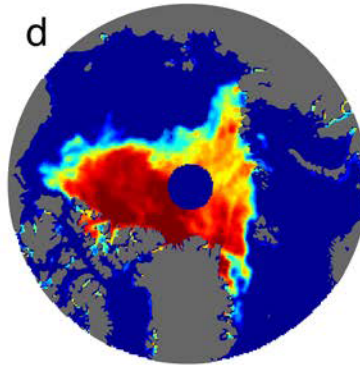
hole in satellite coverage
of sea ice concentration field

previously assumed
ice covered

Gap radius: 611 km
06 January 1985



Gap radius: 311 km
30 August 2007



$$\Delta\psi=0$$

fill = harmonic function with
learned stochastic term

Strong and Golden, *Remote Sensing* 2016
Strong and Golden, *SIAM News* 2017

NOAA/NSIDC Sea Ice Concentration CDR
product update will use our PDE method.

Conclusions

Our research is helping to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

Mathematics for sea ice advances the theory of composites, inverse problems, and other areas of science and engineering.

Sea ice is really cool! Modeling it leads to unexpected areas of math and physics.



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Notices

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University of Utah Sea Ice Modeling Group (2017-2023)

Senior Personnel: Ken Golden, Distinguished Professor of Mathematics
Elena Cherkaev, Professor of Mathematics
Court Strong, Associate Professor of Atmospheric Sciences
Ben Murphy, Adjunct Assistant Professor of Mathematics

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Graduate Students: Kyle Steffen (now at UT Austin)
Christian Sampson (now at NCAR)
Huy Dinh (MURI sea ice Postdoc at NYU/Courant)
Rebecca Hardenbrook (-> Dartmouth Postdoc)
David Morison (Physics Department)
Ryleigh Moore
Delaney Mosier, Daniel Hallman, Julie Sherman

Undergraduate Students: Kenzie McLean, Jacqueline Cinella Rich,
Dane Gollero, Samir Suthar, Anna Hyde,
Kitsel Lusted, Ruby Bowers, Kimball Johnston,
Jerry Zhang, Nash Ward, David Gluckman,
Kayla Stewart, Nicole Forrester, Megan Long

High School Students: J. Chapman, T. Quah, D. Webb, A. Lee, A. Dorsky

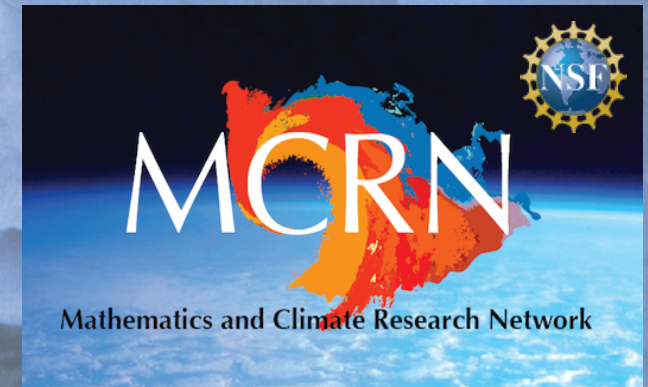
THANK YOU

Office of Naval Research

Applied and Computational Analysis Program
Arctic and Global Prediction Program

National Science Foundation

Division of Mathematical Sciences
Division of Polar Programs



Buchanan Bay, Antarctica Mertz Glacier Polynya Experiment July 1999

Fire endangers Hobart's ice ship

By DAVID CARRIGG

AN engine-room fire has left the Hobart-based Antarctic research ship *Aurora Australis* without power in dangerous sea ice off the Antarctic coast.

None of the 79 people on board was injured in the blaze, which broke out early yesterday morning while the ship was in deep water 185km off the coast.

The extent of the damage is not known.

Australian Antarctic Division director Rex Moncur said the fire was extinguished by flooding the engine room with an inert gas.

The gas had to be cleared before crew wearing breathing apparatus could enter and assess the situation.

He said it could be some time before the extent of damage was known.

The 25 crew and 54 expeditioners, mostly from Hobart, would wear thermal clothing and stay below decks to keep warm.

"There is always a risk of becoming ice-bound in these waters at this time of the year but at this stage we don't expect to launch a rescue mission from Hobart," Mr Moncur said.

The ship was in regular radio contact with the Antarctic Div-



A file photo of the *Aurora Australis* in Antarctica.

ision's Hobart office.

He expected the expeditioners and crew to abandon the pioneering winter voyage and return the ship to Hobart for repairs in about a week.

The Antarctic Division, which hires the ship from P&O Australia, would not be hiring another vessel for the expedition.

"It's a pretty specialist vessel so you couldn't get the sort of research capability that this ship has got readily available," Mr Moncur said.

"We hope the next voyage can still proceed on schedule, which is early September."

The *Aurora Australis* is owned by P&O Australia and chartered by the Antarctic Div-

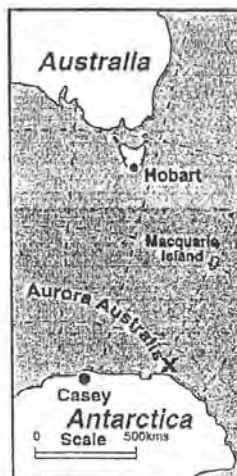
ision for about \$11 million a year.

P&O Australia managing director Richard Hein said yesterday the company was assessing the situation and a number of rescue options were being considered.

It was too early to say whether P&O would be liable for the cost of the aborted mission.

The vessel left Hobart last Wednesday for a seven-week voyage mainly to study a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

The ship was nearing the polynya when the fire broke out.



Oceanographers believe a closer study of the phenomenon will lead to a better understanding of climate change.

CSIRO Marine Research oceanographer Steve Rintoul said the dense bottom water, created only in a few places in Antarctica and to a lesser extent in the North Atlantic, was critical to the chemistry and biology of the world's oceans.

Fire strands Antarctic ship in sea ice

AN engine room fire has disabled the icebreaker *Aurora Australis* in sea ice, deep in Antarctic waters.

There were no injuries and the ship was not in danger after Tuesday night's fire.

Australian Antarctic Division director Mr Rex Moncur said. But Mr Moncur said he expected it would have to abandon its pioneering mid-winter voyage to the edge of the Ant-

arctic continent and return to Hobart for repairs.

The cause of the fire was not known but the engines have been turned off, with the ship 100 nautical miles from the Antarctic coast.

THE CANBERRA TIMES

Thursday 23 July 1998

Page 4

Antarctic voyage stopped by fire

HOBART: An engine room fire has disabled the Australian icebreaker *Aurora Australis* in sea ice, deep in Antarctic waters.

Australian Antarctic Division director Rex Moncur said there were no injuries and the ship was not in danger after Tuesday night's fire.

But Mr Moncur said he expected *Aurora Australis* would have to abandon its pioneering mid-winter voyage to the edge of the Antarctic continent to return to Hobart for repairs.

The fire had been extinguished and the engines were turned off, leaving the ship in sea ice about 100 nautical miles from the Antarctic coast, he said. The weather was good.

Crew had to wear breathing apparatus to enter the engine room and it was likely to be 24 hours before the damage could be fully assessed.

The *Aurora*, with 54 expeditioners and 25 crew, left Hobart last Wednesday for a seven-week voyage which was to have focused on a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

Mr Moncur said, the cause of the fire was not yet known.

2:45 am July 22, 1998

"Please don't be alarmed but we have an uncontrolled fire in the engine room"

about 10 minutes later ...

"Please don't be alarmed but we're lowering the lifeboats"

Sydney Morning Herald
23 July, 1998

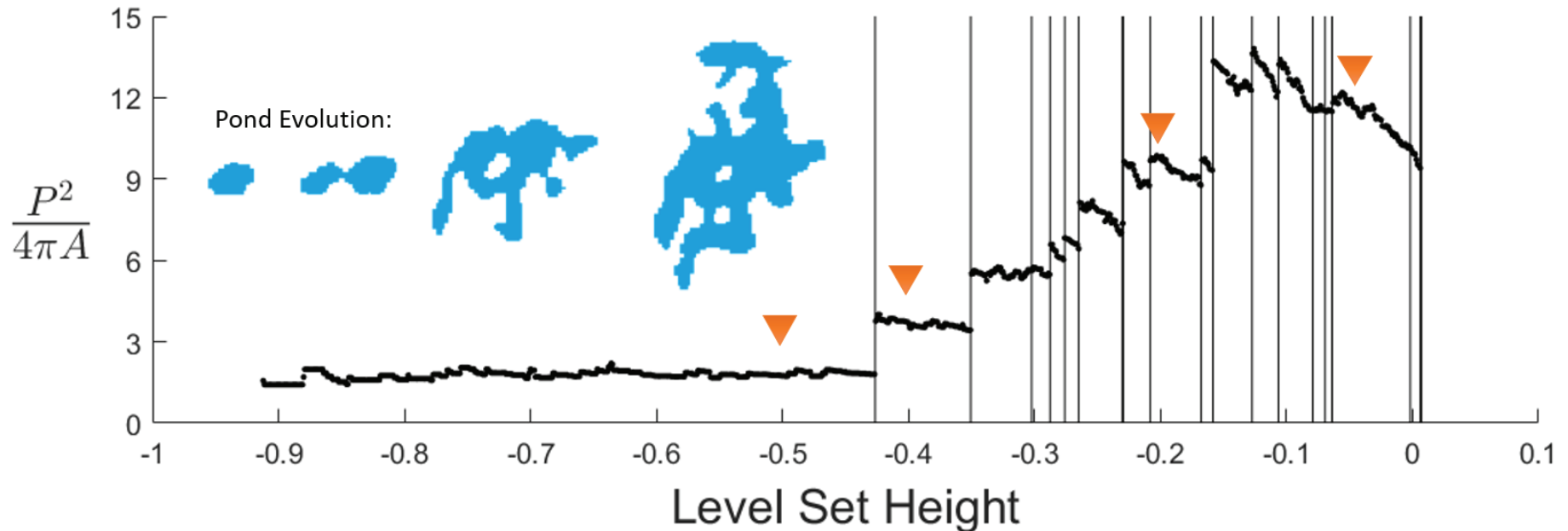
ICEBREAKER BURNS

A pioneering \$2-million Australian scientific voyage to the mid-winter Antarctic polynya is expected to be scrapped following an engine room fire on the *Aurora Australis* yesterday. The 54 people on board were forced on deck in the



Main results

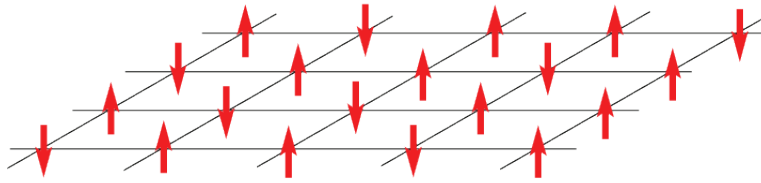
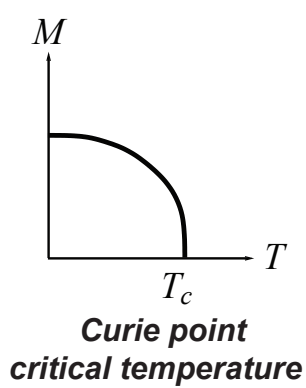
Isoperimetric quotient - as a proxy for fractal dimension - increases in discrete jumps when ponds coalesce at saddle points.



Horizontal fluid permeability "controlled" by saddles ~ electronic transport in 2D random potential.

drainage processes, seal holes

Ising Model for a Ferromagnet



$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases} \quad \begin{matrix} \text{blue} \\ \text{white} \end{matrix}$$

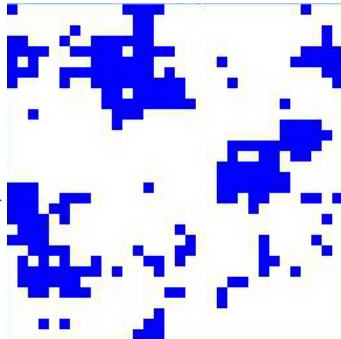
$$\mathcal{H} = -H \sum_i s_i - J \sum_{\langle i,j \rangle} s_i s_j$$

nearest neighbor Ising Hamiltonian

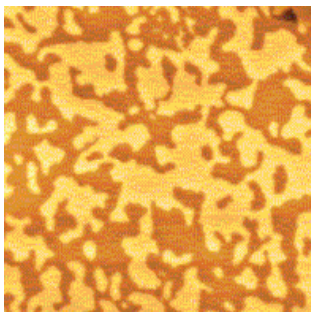
$$M(T, H) = \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$$

effective magnetization

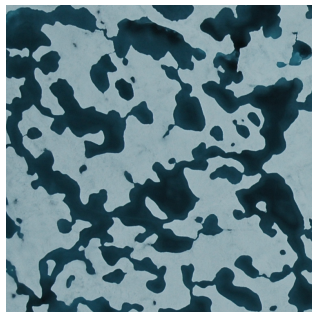
islands of like spins



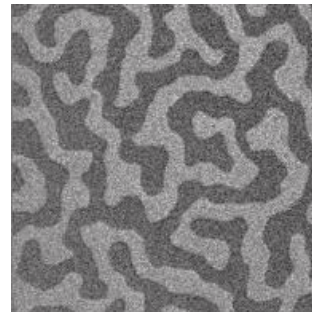
energy is lowered when nearby spins align with each other, forming **magnetic domains**



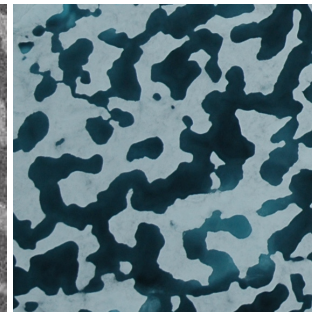
magnetic domains in cobalt



melt ponds (Perovich)



magnetic domains in cobalt-iron-boron



melt ponds (Perovich)

Ising model for ferromagnets \longrightarrow Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.*, 2019

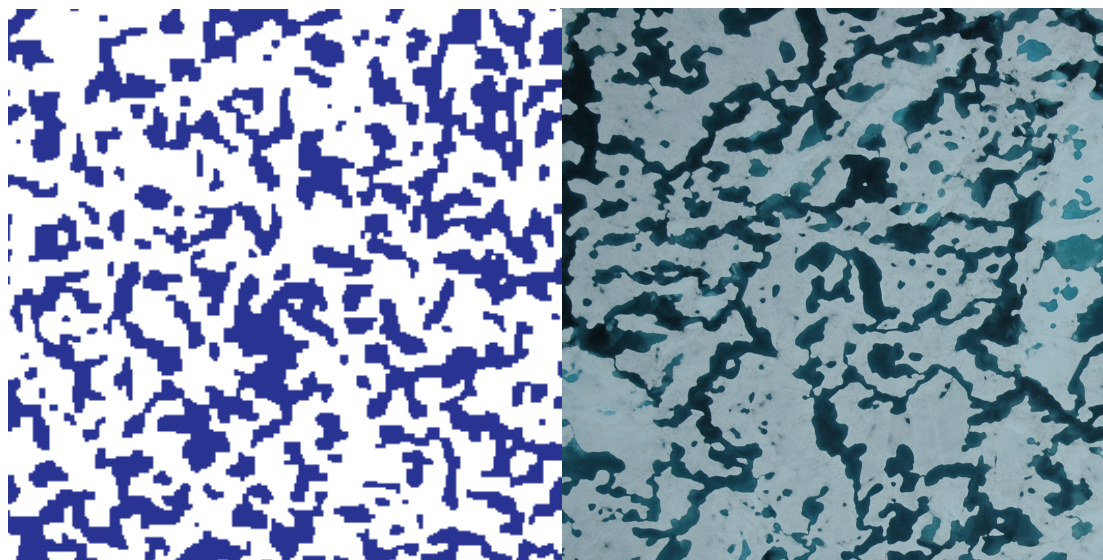
$$\mathcal{H} = - \sum_i^N H_i s_i - J \sum_{\langle i,j \rangle}^N s_i s_j \quad s_i = \begin{cases} \uparrow & +1 \text{ water (spin up)} \\ \downarrow & -1 \text{ ice (spin down)} \end{cases}$$

random magnetic field
represents snow topography

magnetization M pond area fraction $F = \frac{(M+1)}{2}$ only nearest neighbor patches interact
 \sim albedo

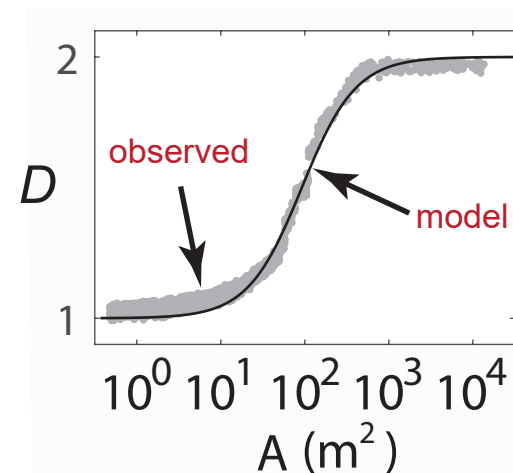
Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system “flows” toward metastable equilibria.

Order from Disorder



Ising
model

melt pond
photo (Perovich)



pond size
distribution exponent

observed -1.5

(Perovich, et al. 2002)

model -1.58

*Scientific American
EOS, PhysicsWorld, ...*

ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data

HETEROGENEITY IN INITIAL CONDITIONS

At each location within a larger region, we could consider

$$\text{Nutrients} \quad \frac{dN}{dt} = \alpha - \textcolor{brown}{B}NP - \eta N$$

$$\text{Algae} \quad \frac{dP}{dt} = \gamma \textcolor{brown}{B}NP - \delta P$$

$$N(0) = \textcolor{brown}{N}_0, \quad P(0) = \textcolor{brown}{P}_0$$

