### **Mathematics of Frozen Seas**

Ken Golden, University of Utah

sea ice microstructure and percolation

SLMath Summer School UAF June 16, 2025





sea ice may appear to be a barren, impermeable cap ...



brine inclusions in sea ice (mm)



micro - brine channel (SEM)

#### brine channels (cm)

# sea ice is a porous composite

pure ice with brine, air, and salt inclusions





horizontal section

vertical section

### cross-sections of sea ice structure

$$T_{freeze} = -1.8^{\circ} \mathrm{C}$$

### crystallographic texture



vertical thin section



















TUCKER ET AL.: FRAM STRAIT SUMMER SEA ICE

7205



Plate 2b



Plate 2c



### Sea Ice is a Multiscale Composite Material *microscale*

#### brine inclusions



H. Eicken

Golden et al. GRL 2007

Weeks & Assur 1969

### millimeters

polycrystals



Gully et al. Proc. Roy. Soc. A 2015

### centimeters

brine channels



D. Cole

K. Golden

### mesoscale

macroscale

Arctic melt ponds



Antarctic pressure ridges





sea ice floes

sea ice pack





K. Golden

J. Weller

kilometers

NASA

meters

### **Central theme:**

### How do we use "small scale" information to find effective behavior on larger scales relevant to climate and ecological models?

**OBJECTIVE:** advance how sea ice is represented in climate models improve projections of fate of SEA ICE and its ECOSYSTEMS

### **HOMOGENIZATION for Composite Materials**



Maxwell 1873, Einstein 1906 Wiener 1912, Hashin and Shtrikman 1962

### **STATISTICAL PHYSICS** percolation, phase transitions solid state, semiconductors

#### How do microscopic laws determine macroscopic behavior?

Banwell, Burton, Cenedese, Golden, Astrom, Physics of the Cryosphere, Nature Reviews Physics 2023



### sea ice microphysics

fluid transport

### fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

#### evolution of Arctic melt ponds and sea ice albedo



#### nutrient flux for algal communities







#### Antarctic surface flooding and snow-ice formation

- evolution of salinity profiles - ocean-ice-air exchanges of heat, CO<sub>2</sub>



 $\phi = 3.3 \%$  T = -18



 $\varphi = 7.8 \% \qquad T = -6$ 

### brine volume fraction and *connectivity* increase with temperature



### $T = -15 \,^{\circ}\text{C}, \ \phi = 0.033$ $T = -6 \,^{\circ}\text{C}, \ \phi = 0.075$ $T = -3 \,^{\circ}\text{C}, \ \phi = 0.143$



 $T = -8^{\circ} C, \phi = 0.057$ 

X-ray tomography for brine in sea ice



 $T = -4^{\circ} C, \phi = 0.113$ 

Golden et al., Geophysical Research Letters, 2007

### fluid permeability of a porous medium

how much fluid gets through the sample per unit time?

**Stokes equations** for fluid velocity  $\mathbf{v}^{\epsilon}$ , pressure  $p^{\epsilon}$ , force **f**:



 $\nabla p^{\epsilon} - \epsilon^2 \eta \Delta \mathbf{v}^{\epsilon} = \mathbf{f}, \quad x \in \mathcal{P}_{\epsilon}$  $\nabla \cdot \mathbf{v}^{\epsilon} = 0, \quad x \in \mathcal{P}_{\epsilon}$  $\mathbf{v}^{\epsilon} = 0, \quad x \in \partial \mathcal{P}_{\epsilon}$  $\eta = \text{fluid viscosity}$ 

HOMOGENIZE

via two-scale expansion

MACROSCOPIC EQUATIONS $\mathbf{v}^{\epsilon} \rightarrow \mathbf{v}$ ,  $p^{\epsilon} \rightarrow p$  as  $\epsilon \rightarrow 0$ Darcy's law $\mathbf{v} = -\frac{1}{\eta} \mathbf{k} \nabla p$ , $x \in \Omega$  $\mathbf{k}(x) =$  effective fluid $(\mathbf{f} = \mathbf{0})$  $\nabla \cdot \mathbf{v} = 0$ , $x \in \Omega$  $\mathbf{k}(x) =$  tensor

[Keller '80, Tartar '80, Sanchez-Palencia '80, J. L. Lions '81, Allaire '89, '91,'97]

### fluid permeability of a porous medium



Darcy's Law

for slow viscous flow in a porous medium



how much water gets through the sample per unit time?

**k** = fluid permeability tensor

### HOMOGENIZATION

mathematics for analyzing effective behavior of heterogeneous systems

### **Critical behavior of fluid transport in sea ice**



**PERCOLATION THRESHOLD**  $\phi_c \approx 5\%$   $\checkmark$   $T_c \approx -5^{\circ}C, S \approx 5$  ppt

### **RULE OF FIVES**

Golden, Ackley, Lytle Science 1998 Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007 Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009



# sea ice algal communities

D. Thomas 2004

nutrient replenishment controlled by ice permeability

biological activity turns on or off according to *rule of fives* 

Golden, Ackley, Lytle

Science 1998

Fritsen, Lytle, Ackley, Sullivan Science 1994

#### critical behavior of microbial activity



### percolation theory

### probabilistic theory of connectedness



 $p_c$  depends on type of lattice and d

smallest p for which there is an infinite open cluster

 $p_c = 1/2$  for d = 2

### order parameters in percolation theory

#### geometry

#### transport



#### **UNIVERSAL critical exponents for lattices -- depend only on dimension**

 $1 \le t \le 2$  (for idealized model), Golden, *Phys. Rev. Lett.* 1990; *Comm. Math. Phys.* 1992

#### non-universal behavior in continuum



# stealth



NW Florida Daily News, Wikimedia

*Continuum* percolation model for *stealthy* materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

 $\phi_c \approx 5\%$  Golden, Ackley, Lytle, *Science*, 1998



sea ice is radar absorbing

### order parameters in brine percolation

### geometry

#### correlation length

(characteristic scale of connectedness)

### transport

sea ice permeability

#### Arctic field data





exponent is UNIVERSAL lattice value  $t \approx 2.0$ 

Golden, Ackley, Lytle Science 1998

Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophys. Res. Lett. 2007

Pringle, Miner, Eicken, Golden, JGR (Oceans) 2009

### Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007



percolation theory for fluid permeability

$k(\phi) =$	$k_0 (\phi - 0.05)^2$	critical exponent
	$k_0 = 3 \times 10^{-8} \text{ m}^2$	t

from critical path analysis in hopping conduction

hierarchical model rock physics network model rigorous bounds

X-ray tomography for brine inclusions

confirms rule of fives

brine percolation threshold of  $\varphi=$  5% for bulk fluid flow

Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

> theories agree closely with field data

microscale governs mesoscale processes Non-universal behavior in the continuum:

critical exponents for transport in Swiss cheese model take values different than for lattices, e.g. t > 2

Halperin, Feng, Sen, Phys. Rev. Lett. 1985





 $e \neq t$ 

Swiss cheese model d = 2

conducting neck in d = 3Swiss cheese model

in general, non-universal exponents arise from a singular distribution of local conductances

# In sea ice, this distribution is lognormal. (excluding inclusions below cutoff)

Thus, the permeability exponent for sea ice is 2, the universal lattice value.

### ESTIMATE fluid conductivity scaling factor $k_0 = r^2/8$

for media with broad range of conductances

CRITICAL PATH ANALYSIS

bottlenecks control flow



critical pore

Ambegaokar, Halperin, Langer 1971: CPA in electronic hopping conduction Friedman, Seaton 1998: CPA in fluid and electrical networks Golden, Kozlov 1999: rigorous CPA on long-range checkerboard model

 $k_0 \approx r_c^2 / 8$  critical fluid conductivity

Microstructural analyses yield  $r_c \approx 0.5 \text{ mm}$ 

## Sea ice algae secrete exopolymeric substances (EPS) affecting evolution of brine microstructure.

#### How does EPS affect fluid transport? How does the biology affect the physics?



- 2D random pipe model with bimodal distribution of pipe radii
- Rigorous bound on permeability k; results predict observed drop in k

Steffen, Epshteyn, Zhu, Bowler, Deming, Golden Multiscale Modeling and Simulation, 2018

**SIAM News** 

EPS - Algae Model Jajeh, Golden, Deming, Reimer June 2024



Zhu, Jabini, Golden, Eicken, Morris *Ann. Glac*. 2006

#### electrical transport



fluid transport



electrical conductance

$$g_e = \pi r^2 \sigma$$

electrical conductivity

$$\sigma_e = \sigma$$

fluid conductance

$$g_f = \pi r^4 / 8\eta$$

fluid conductivity

$$\sigma_f = r^2/8\eta$$



#### bond lattice





### random checkerboard









### continuum

random checkerboard

 $p_c = 0.59...$ 

### Optimal Movement of a Polar Bear in a Heterogenous Icescape

Nicole Forrester, Jody Reimer, Ken Golden 2024

Polar bears expend 5X more energy swimming than walking on sea ice.

### As sea ice is lost, how do polar bears optimize their movement to save energy and survive?



### **Polar Bear Percolation**

To study the importance of ice connectedness, we exaggerate the data by setting the cost of walking on ice to 0 with the cost of swimming still at 5.





**C**<sub>i</sub>

 $C_{W}$ 

20% lce



C(p)





cores from ice sheets give climate records, e.g. CO2 data analysis of gas in bubbles

### percolation theory gives critical depth below which air bubbles are disconnected from atmosphere



Enting, Nature 1985

# Not the American Mathematical Society.

#### of the American Mathematical Society

May 2009

Volume 56, Number 5

Climate Change and the Mathematics of Transport in Sea Ice

page 562

Mathematics and the Internet: A Source of Enormous Confusion and Great Potential

page 586

#### photo by Jan Lieser

Real analysis in polar coordinates (see page 613)



measuring fluid permeability of Antarctic sea ice

**SIPEX 2007** 

### **Arctic and Antarctic field experiments**

develop electromagnetic methods of monitoring fluid transport and microstructural transitions

extensive measurements of fluid and electrical transport properties of sea ice:

2007	Antarctic	SIPEX
2010	Antarctic	McMurdo Sound
2011	Arctic	<b>Barrow AK</b>
2012	Arctic	<b>Barrow AK</b>
2012	Antarctic	SIPEX II
2013	Arctic	<b>Barrow AK</b>
2014	Arctic	Chukchi Sea



### **The Melt Pond Conundrum:**

### How can ponds form on top of sea ice that is highly permeable?

C. Polashenski, K. M. Golden, D. K. Perovich, E. Skyllingstad, A. Arnsten, C. Stwertka, N. Wright

Percolation Blockage: A Process that Enables Melt Pond Formation on First Year Arctic Sea Ice

J. Geophys. Res. Oceans 2017

### 2014 Study of Under Ice Blooms in the Chuckchi Ecosystem (SUBICE) aboard USCGC Healy





#### electrical measurements



Section 12

#### Wenner array



#### vertical conductivity

Zhu, Golden, Gully, Sampson *Physica B* 2010 Sampson, Golden, Gully, Worby *Deep Sea Research* 2011

### critical behavior of electrical transport in sea ice electrical signature of the on-off switch for fluid flow

same universal critical exponent as for fluid permeability



Golden, Eicken, Gully, Ingham, Jones, Lin, Reid, Sampson, Worby 2022

### PIPE BOUNDS on vertical fluid permeability k

Golden, Heaton, Eicken, Lytle, Mech. Materials 2006 Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophys. Res. Lett. 2007

> vertical pipes with appropriate radii maximize k





fluid analog of arithmetic mean upper bound for effective conductivity of composites (Wiener 1912)

optimal coated cylinder geometry



$$k \leq \frac{\phi \langle R^4 \rangle}{8 \langle R^2 \rangle} = \frac{\phi}{8} \langle R^2 \rangle e^{\sigma^2}$$

**inclusion cross sectional areas A lognormally distributed** In(*A*) normally distributed, mean  $\mu$  (increases with T) variance  $\sigma^2$  (Gow and Perovich 96)

get bounds through variational analyis of **trapping constant**  $\gamma$  for diffusion process in pore space with absorbing BC

#### Torquato and Pham, PRL 2004

 $\mathbf{k} \leq \gamma^{-1} \mathbf{I}$ 

for any ergodic porous medium (Torquato 2002, 2004)



UNIVERSAL critical exponents for lattices -- depend only on dimension

 $1 \le t \le 2$  (for idealized model), Golden, *Phys. Rev. Lett.* 1990; *Comm. Math. Phys.* 1992 *non-universal behavior in continuum* 



### level set percolation for a random potential

