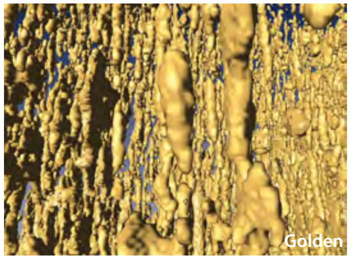
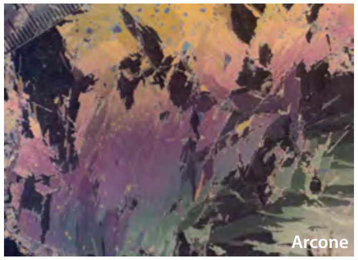
millimeters


Golden

centimeters


Arcone

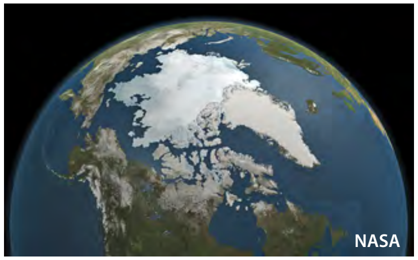
meters


Darelius

kilometers


NASA

10^3 kilometers


NASA

From Micro to Macro in Sea Ice Modeling

Ken Golden, University of Utah

Arctic Mathpedition I
May 5, 2024

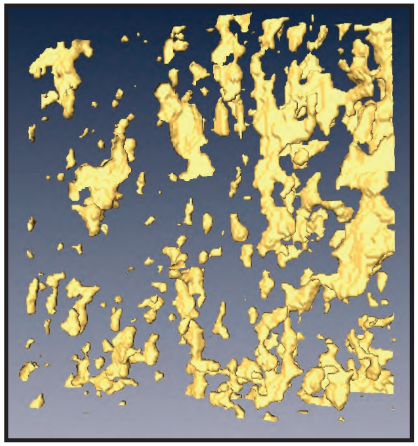
SIAM Conference on Materials Science
Emerging Trends in Multiscale Modeling
Pittsburgh, May 20, 2024

Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

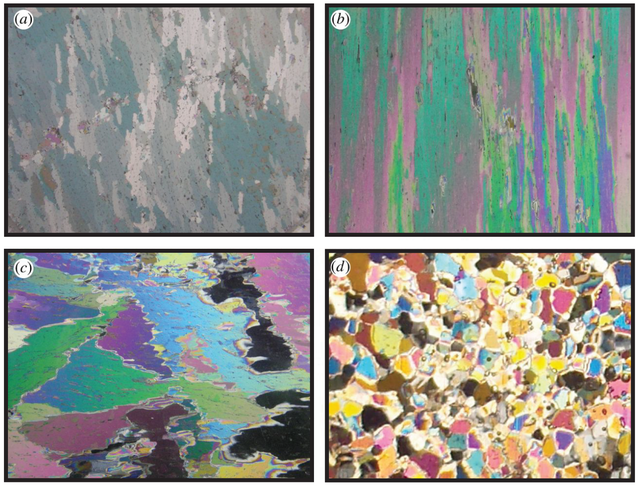
Weeks & Assur 1969



H. Eicken

Golden *et al. GRL* 2007

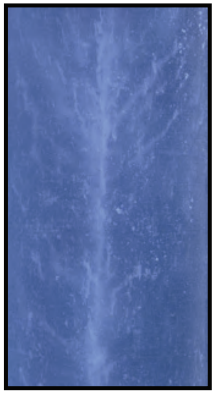
polycrystals



Gully *et al. Proc. Roy. Soc. A* 2015

brine channels

D. Cole



K. Golden

millimeters

centimeters

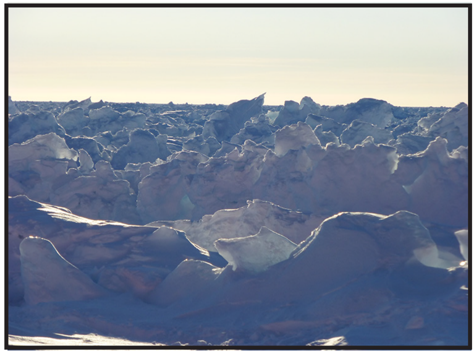
---

mesoscale

Arctic melt ponds

K. Frey

Antarctic pressure ridges



K. Golden

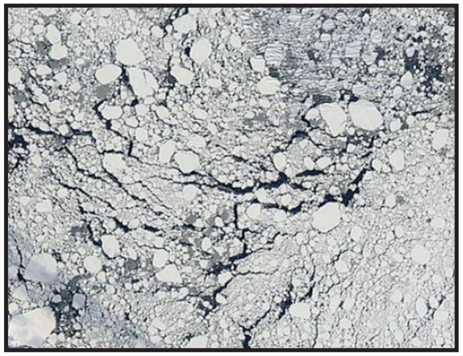
meters

macroscale

sea ice floes

J. Weller

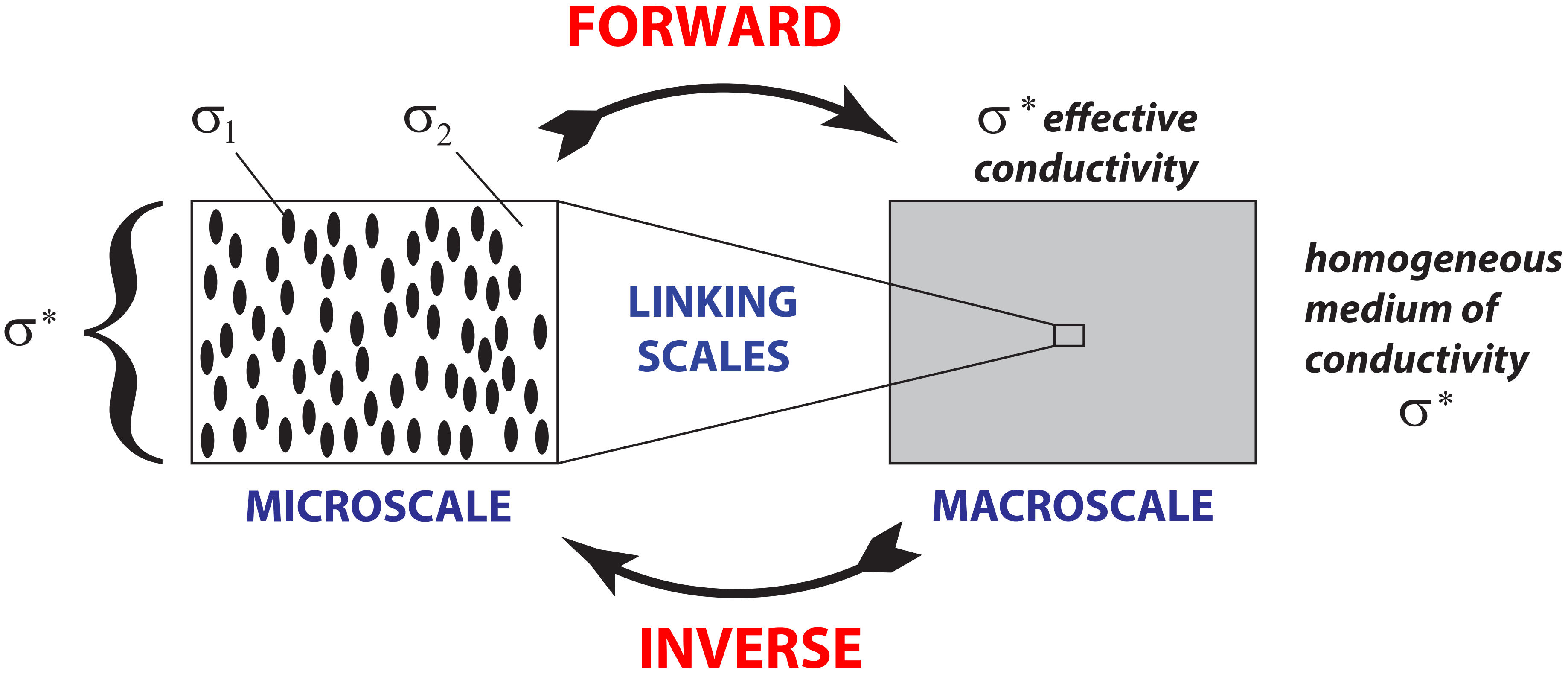
sea ice pack



NASA

kilometers

HOMOGENIZATION for Composite Materials



Maxwell 1873, Einstein 1906
Wiener 1912, Hashin and Shtrikman 1962

Polar Ecology and the Physics of Sea Ice

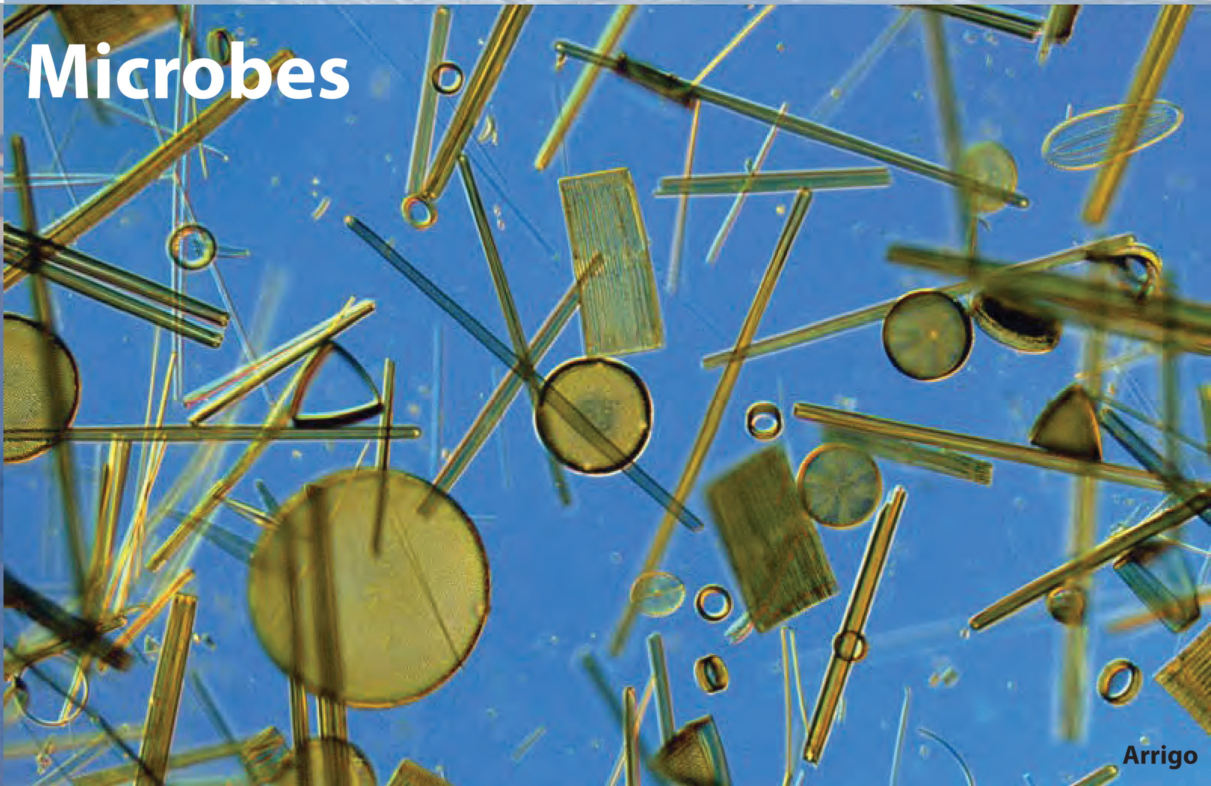
How do sea ice properties affect the life it hosts?

How does life in and on sea ice affect its physical properties?

Megafauna

Pekelo/iStockphoto

Golden



Microbes

Arrigo

What is this talk about?

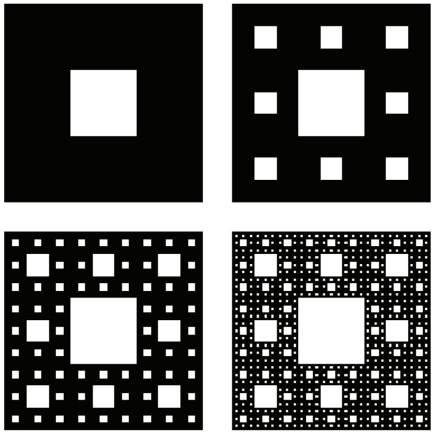
A tour of recent results on multiscale modeling of physical and biological processes in the sea ice system.

microscale

mesoscale

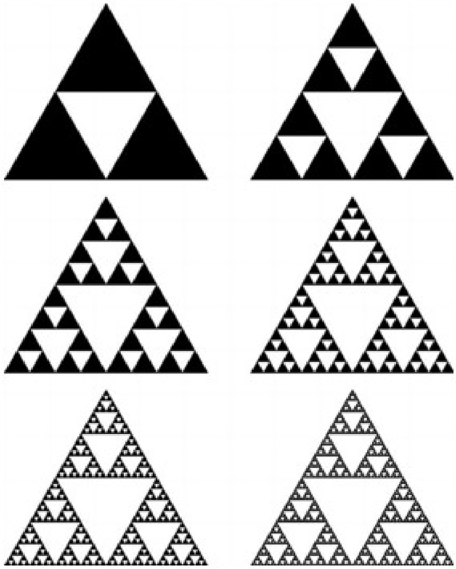
macroscale

through the lens of fractal geometry and other areas of mathematics



fractals

self-similar structure
non-integer dimension

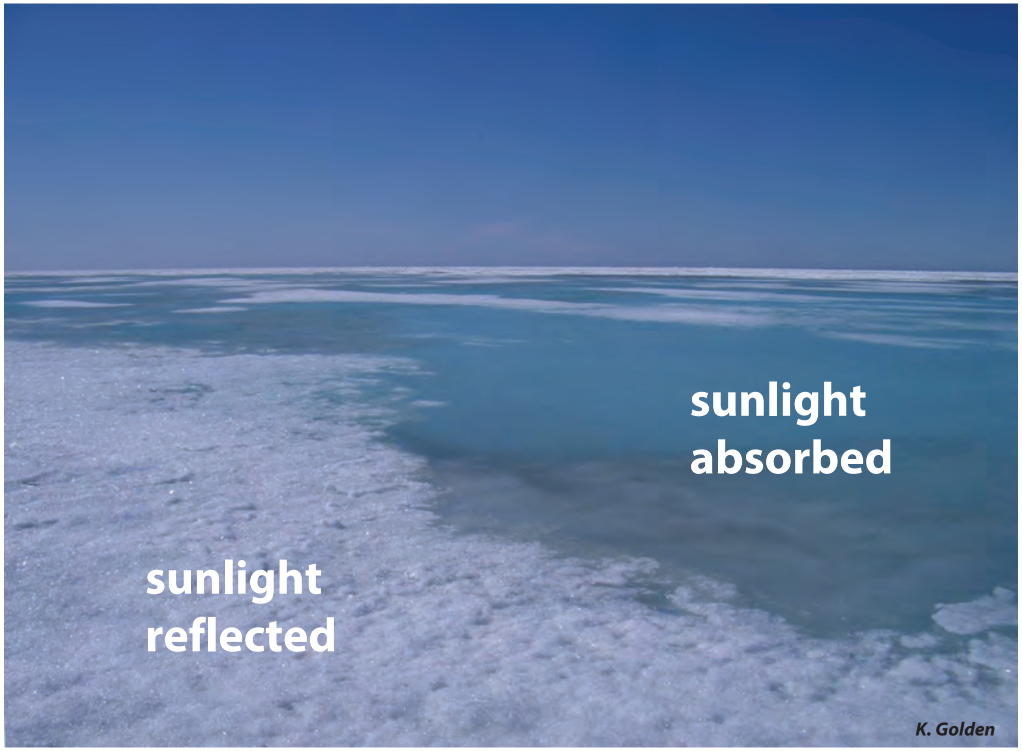


$$D = \frac{\log3}{\log2} = 1.585...$$

microscale

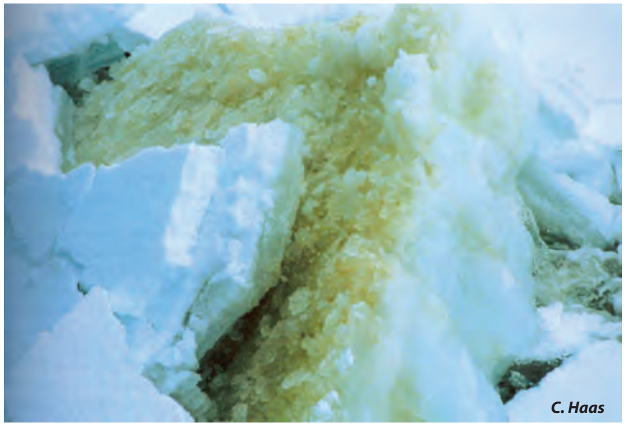
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

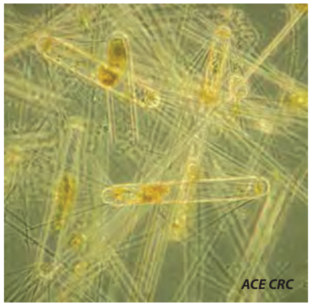


K. Golden

nutrient flux for algal communities

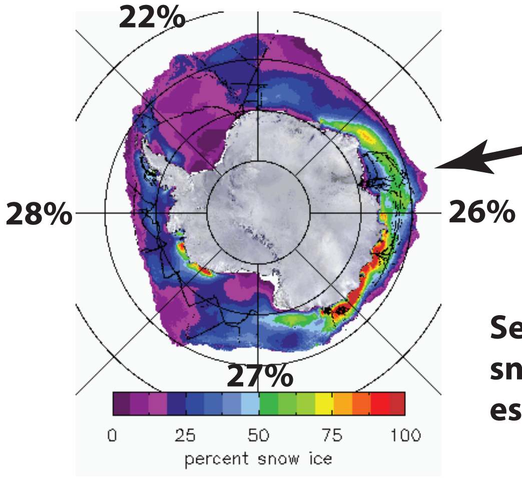


C. Haas



ACE CRC

C. Krembs



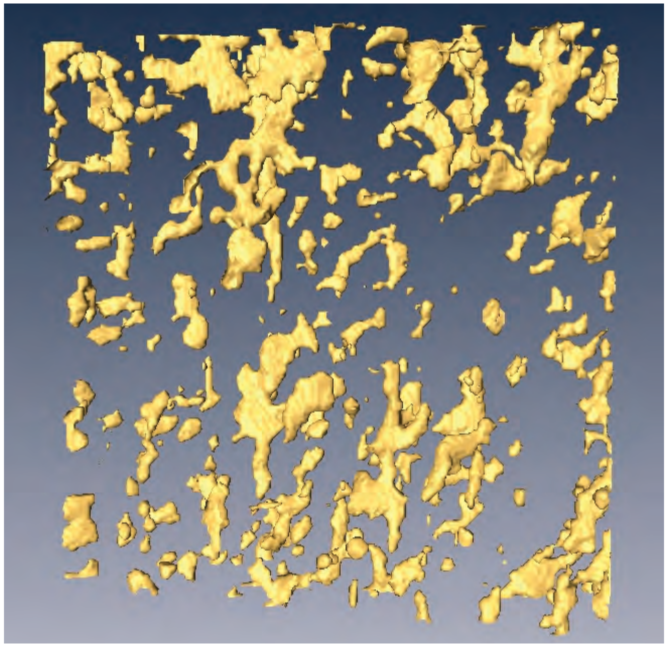
T. Maksym and T. Markus, 2008

September snow-ice estimates

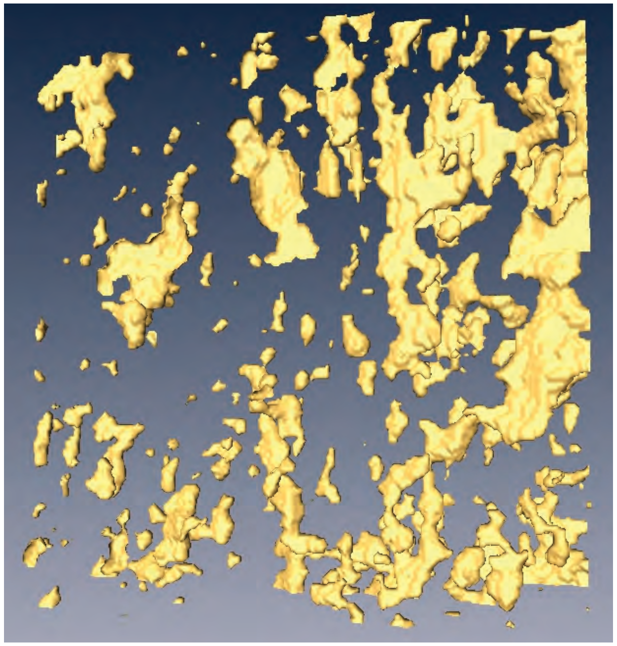
Antarctic surface flooding and snow-ice formation

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO_2*

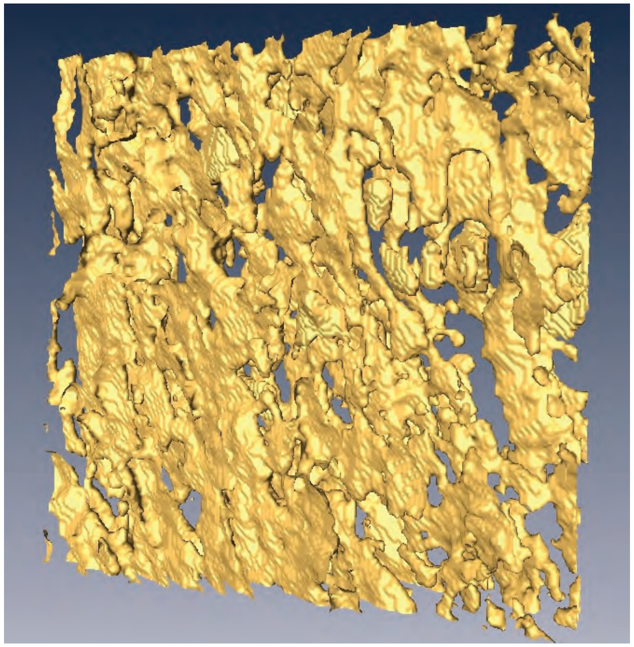
brine volume fraction and ***connectivity*** increase with temperature



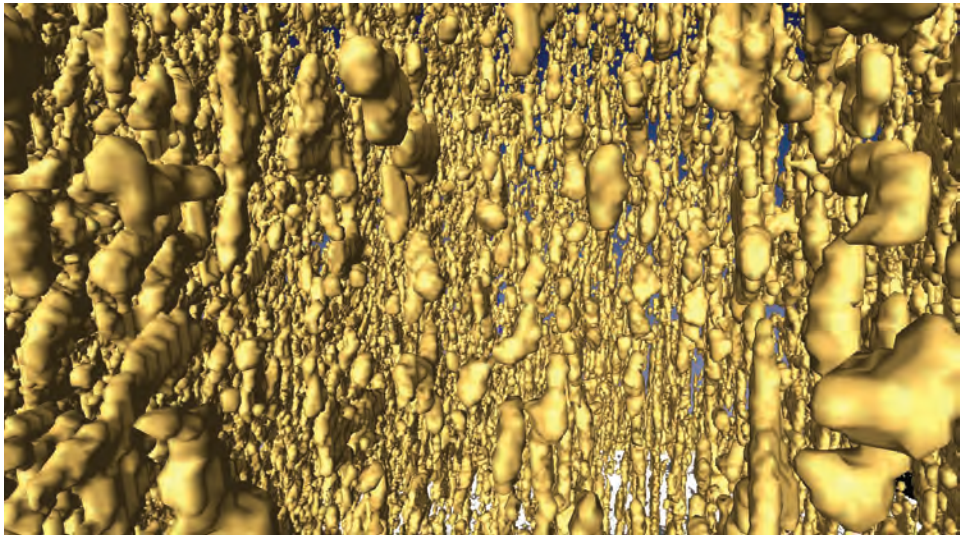
$T = -15\,^\circ\mathrm{C}, \ \phi = 0.033$



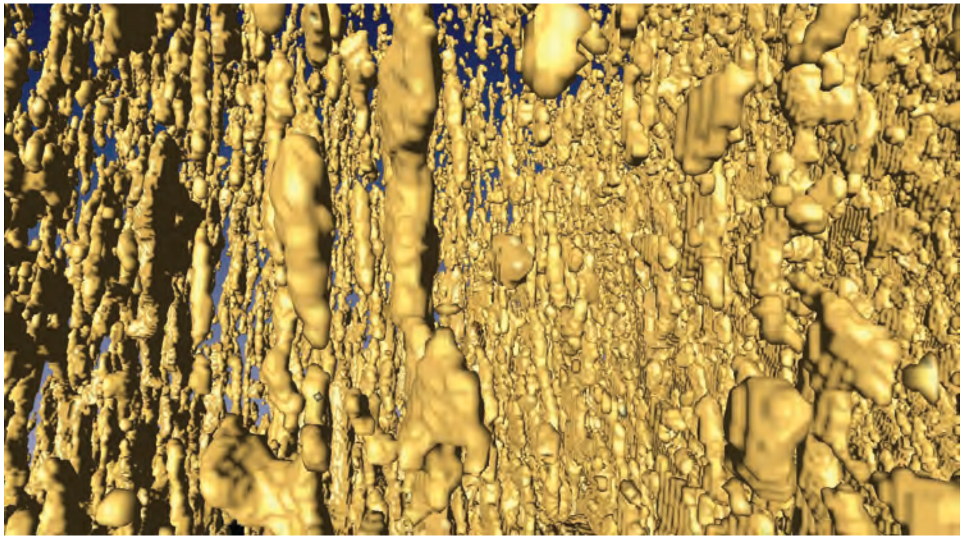
$T = -6\,^\circ\mathrm{C}, \ \phi = 0.075$



$T = -3\,^\circ\mathrm{C}, \ \phi = 0.143$



$T = -8^\circ\mathrm{C}, \ \phi = 0.057$



$T = -4^\circ\mathrm{C}, \ \phi = 0.113$

X-ray tomography for brine in sea ice

Golden *et al.*, *Geophysical Research Letters*, 2007

Critical behavior of fluid transport in sea ice

Arctic field data

vertical fluid permeability k (m^2)

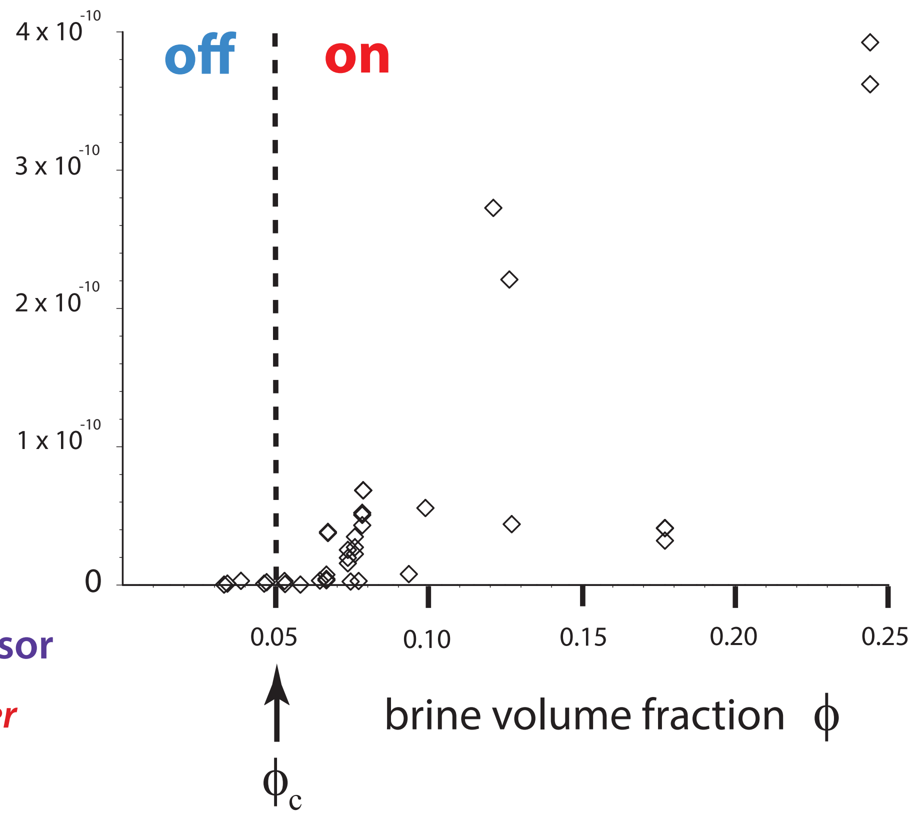
Darcy's Law

$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\nabla p$$

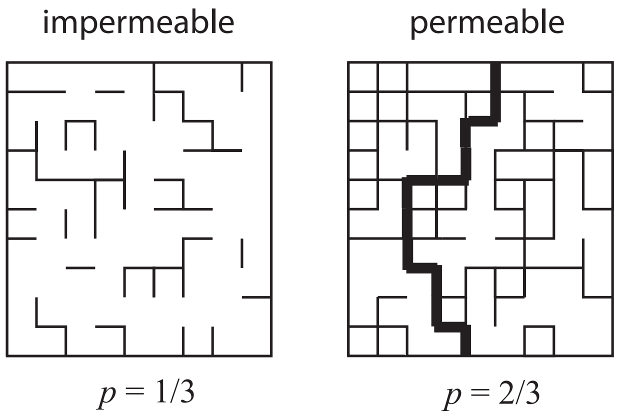
$\mathbf{k}$ = fluid permeability tensor

homogenized parameter

impermeable permeable



"on - off" switch for bulk fluid flow



lattice percolation

FRACTAL percolation clusters

PERCOLATION THRESHOLD $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ \mathrm{C},\ S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007

Geophysical
Research
Letters

28 AUGUST 2007
Volume 34 Number 16
American Geophysical Union

A unified approach to understanding permeability in sea ice • Solving the mystery of booming sand dunes • Entering into the “greenhouse century”: A case study from Switzerland

melt pond evolution

microscale

governs

mesoscale processes

percolation theory for fluid permeability

X-ray tomography for brine inclusions

confirms rule of fives

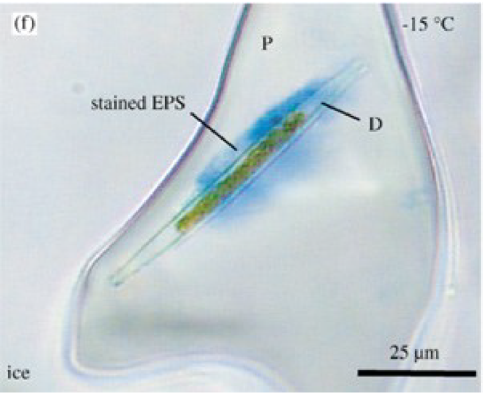
Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009

theory agrees closely with field data

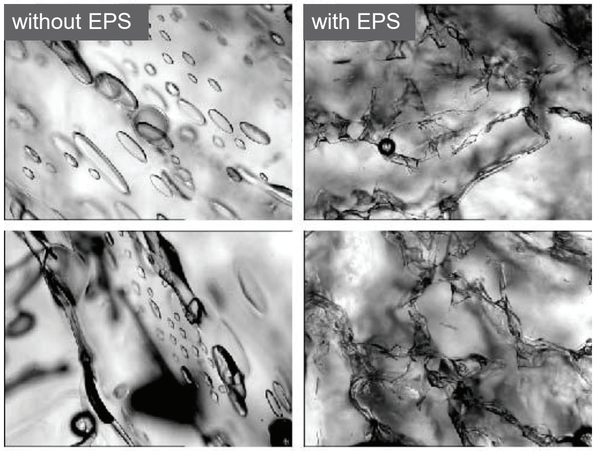
Sea ice algae secrete exopolymeric substances (EPS) affecting evolution of brine microstructure.

How does EPS affect fluid transport? **How does the biology affect the physics?**

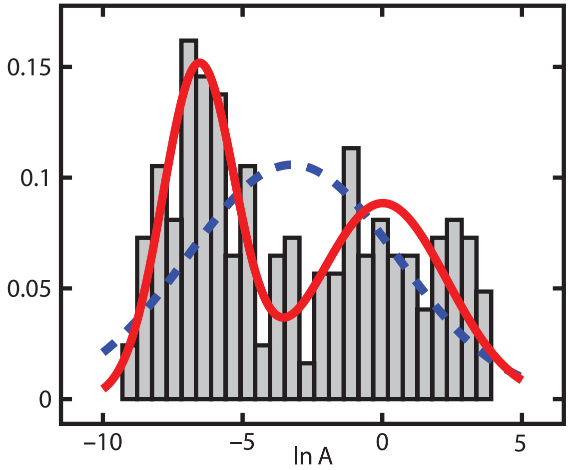
FRACTAL



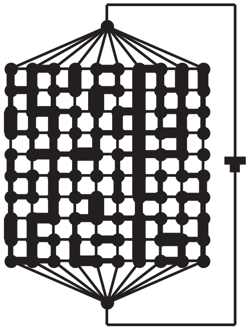
Krembs

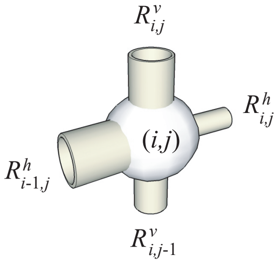


Krembs, Eicken, Deming, PNAS 2011



RANDOM PIPE MODEL





- 2D random pipe model with bimodal distribution of pipe radii
- Rigorous bound on permeability *k;* results predict observed drop in *k*

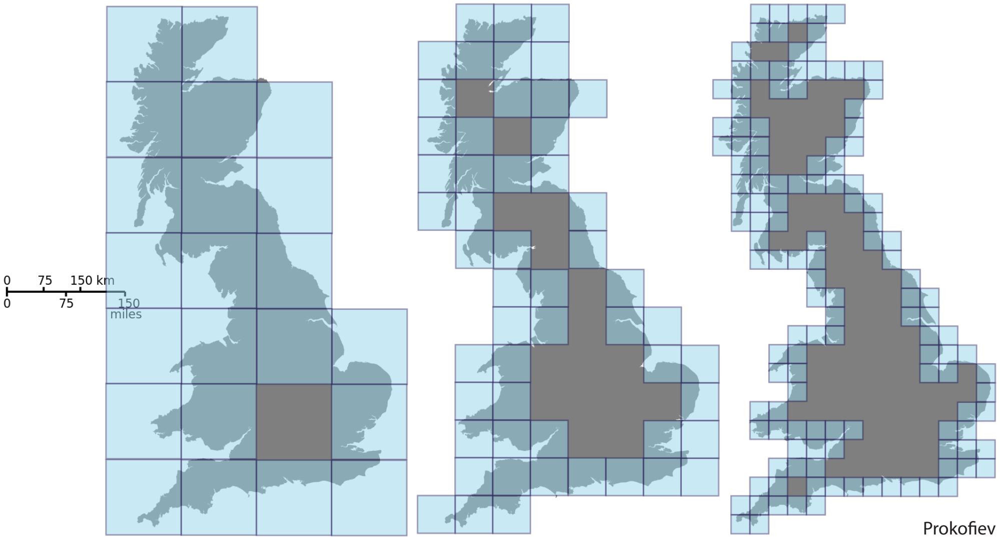
Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
***Multiscale Modeling and Simulation*, 2018**

EPS - Algae Model Jajeh, Reimer, Golden, 2024

Zhu, Jabini, Golden,
Eicken, Morris
***Ann. Glac.* 2006**

Thermal Evolution of Brine Fractal Geometry in Sea Ice

Nash Ward, Daniel Hallman, Benjamin Murphy, Jody Reimer,
Marc Oggier, Megan O'Sadnick, Elena Cherkaev and Kenneth Golden, 2024



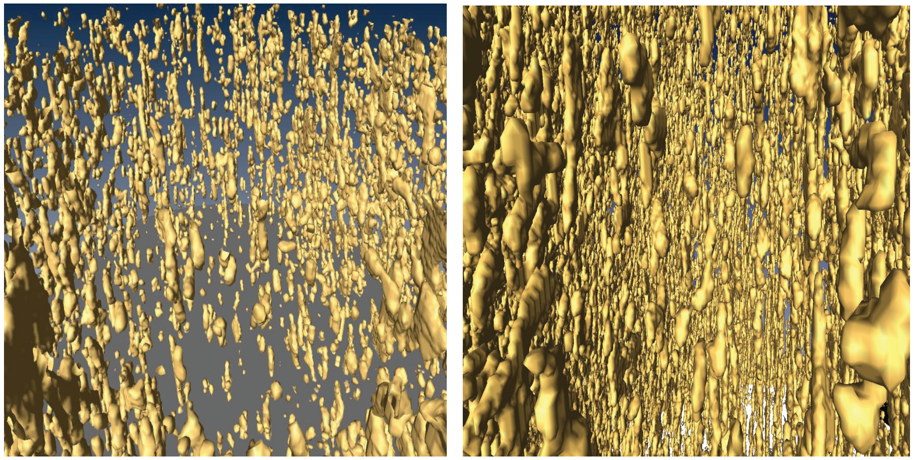
Prokofiev

fractal dimension of the coastline of Great Britain by box counting

$$N(\epsilon) \sim \epsilon^{-D}$$

brine channels and inclusions "look" like fractals (from 30 yrs ago)

$T = \text{-}12^{\circ}$ C, $\phi = 0.033$ $T = \text{-}8^{\circ}$ C, $\phi = 0.057$

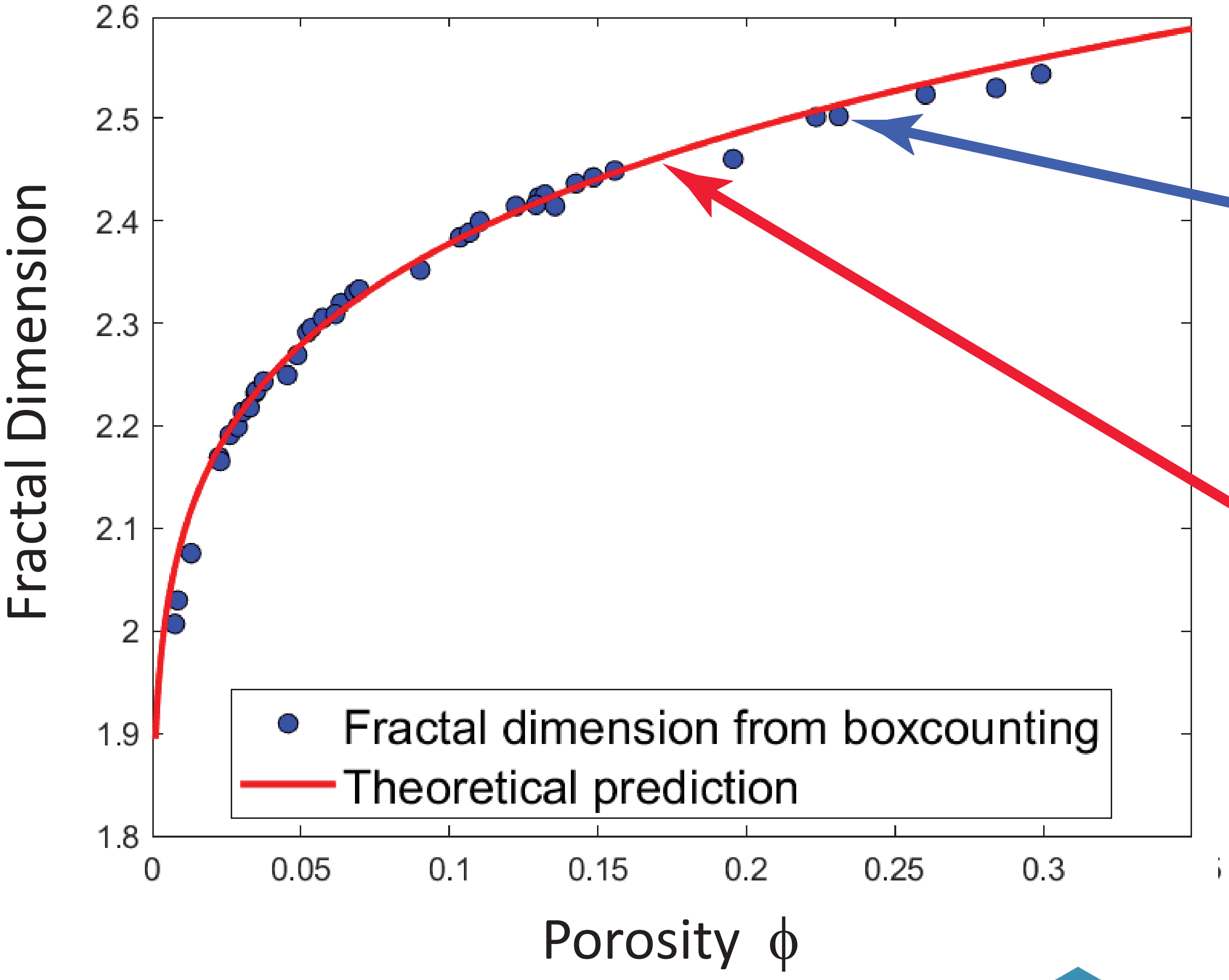


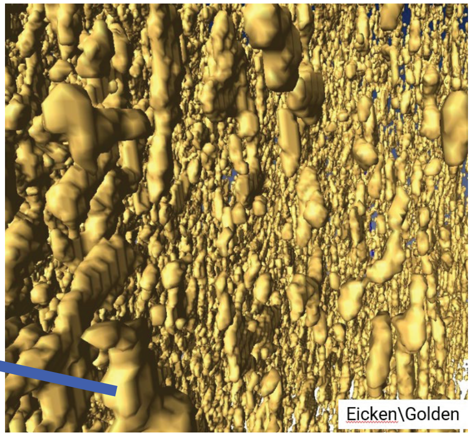
Golden, Eicken, et al. *GRL*, 2007

X-ray computed tomography of brine in sea ice

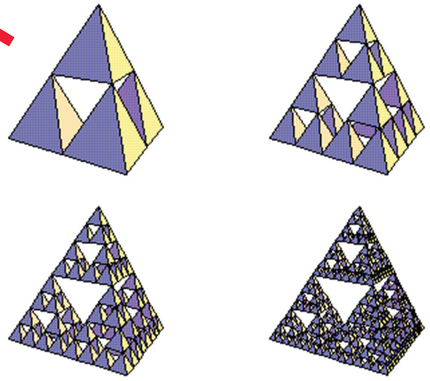
columnar and granular

The first quantitative study of the fractal dimension of brine in sea ice and its strong dependence on temperature and porosity.





Follows same curve as exactly self-similar Sierpinski tetrahedron



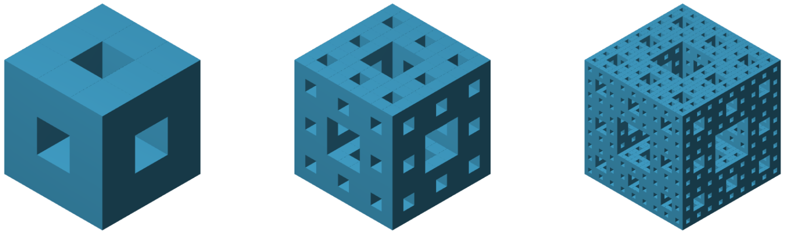
D. Eppstein

red curve

$$F_d = d_E - \frac{\ln \phi}{\ln(\lambda_{min}/\lambda_{max})}$$

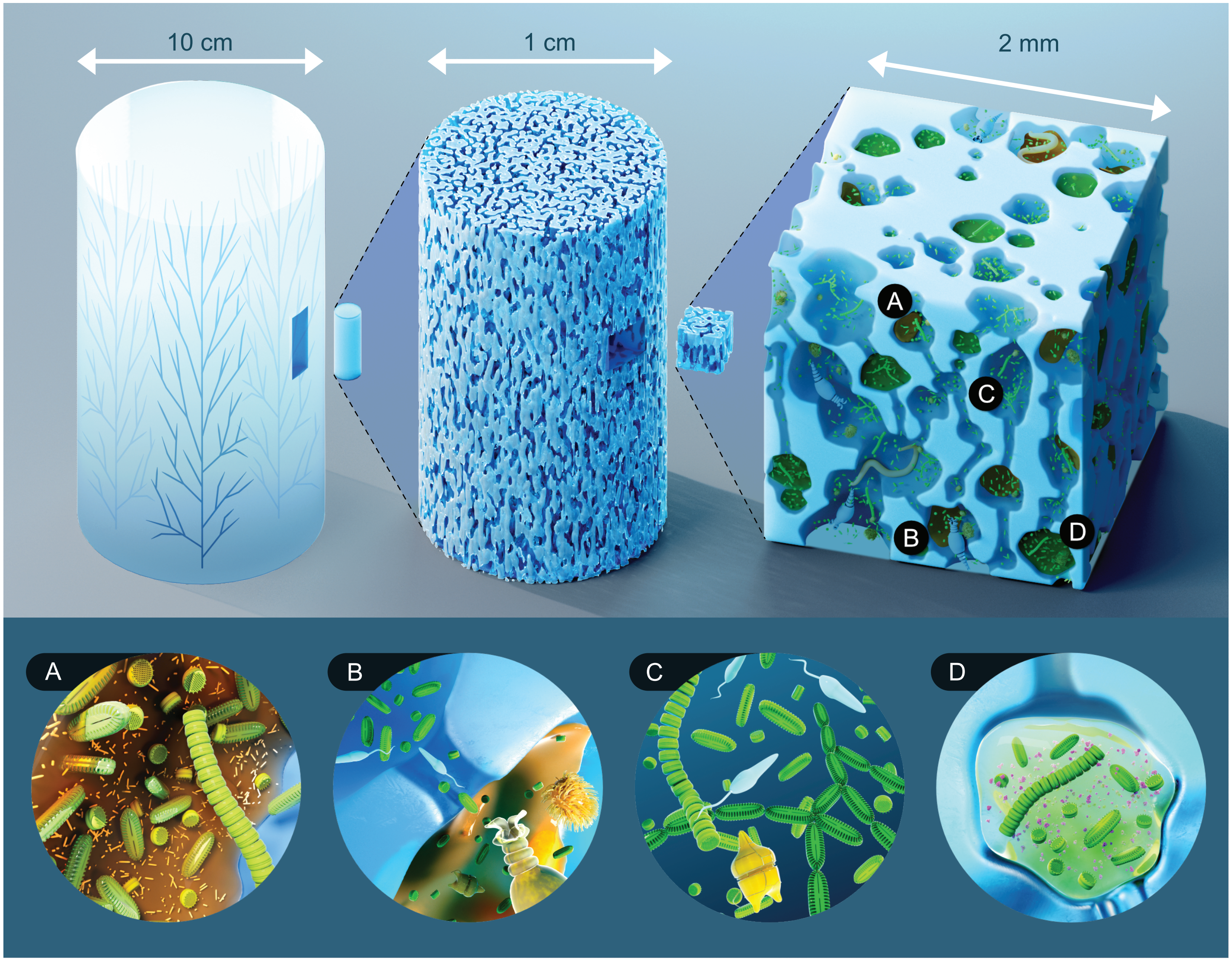
Katz and Thompson,1985; Yu and Li, 2001

discovered for sandstones
statistically self-similar porous media



Fractal geometry of brine in sea ice, Ward, *et al.* 2024

Implications of brine fractal geometry on sea ice ecology and biogeochemistry



Brine inclusions are home to ice endemic organisms, e.g., bacteria, diatoms, flagellates, rotifers, nematodes.

The habitability of sea ice for these organisms is inextricably linked to its complex brine geometry.

(A) Many sea ice organisms attach themselves to inclusion walls; inclusions with a higher fractal dimension have greater surface area for colonization.

(B) Narrow channels prevent the passage of larger organisms, leading to refuges where smaller organisms can multiply without being grazed, as in (C).

(D) Ice algae secrete extracellular polymeric substances (EPS) which alter incusion geometry and may further increase the fractal dimension.

Remote sensing of sea ice

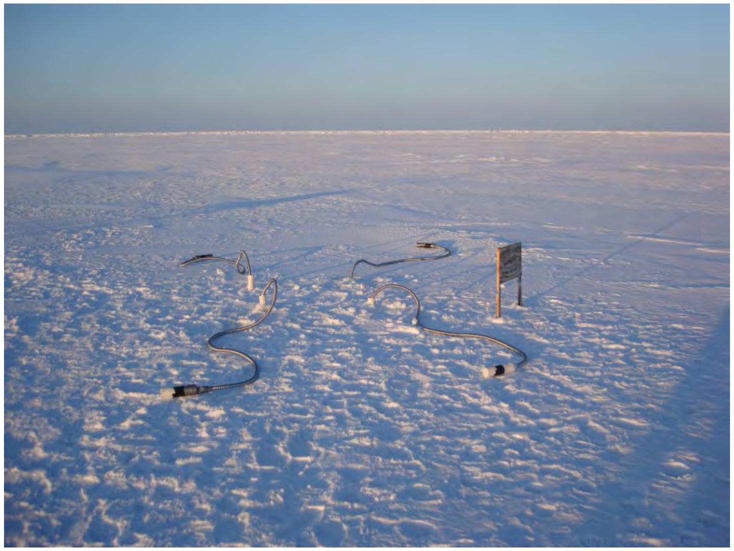
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice
properties from
electromagnetic
(EM) data

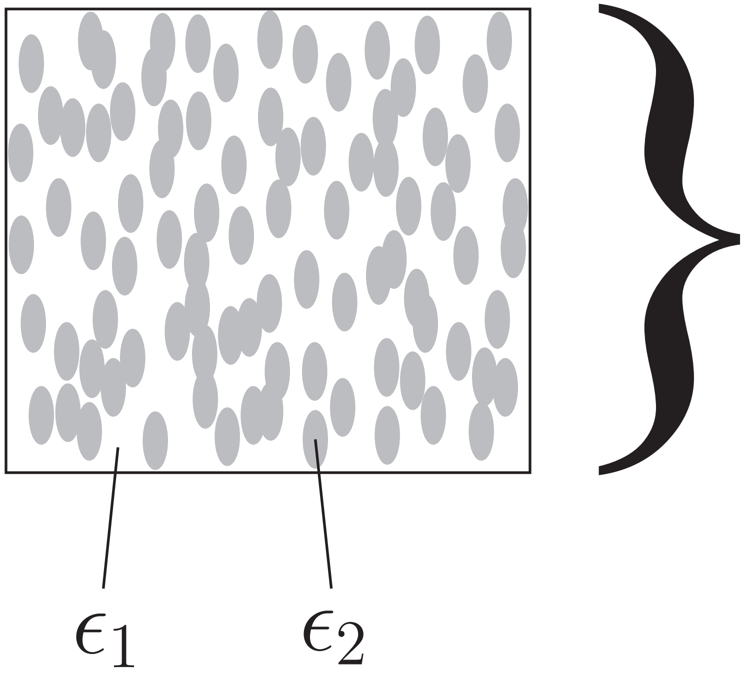
ε^*

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$\epsilon^*$$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \text{ , composite geometry } \right)$$

What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?

Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

separates geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

material parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla \cdot$$

$\chi =$ characteristic function of the brine phase

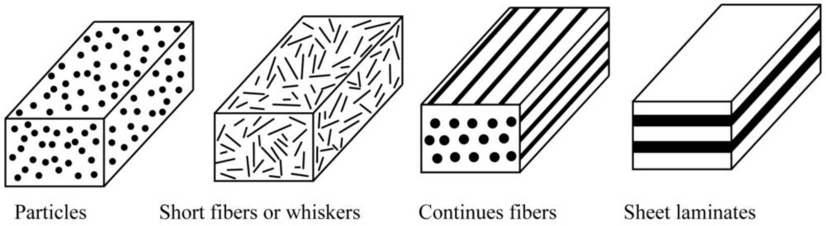
$$E = s\,(s + \Gamma\chi)^{-1} e_k$$

$\Gamma\chi$: **microscale $\longrightarrow$ macroscale**

$\Gamma\chi$ ***links scales***

Golden and Papanicolaou, Comm. Math. Phys. 1983

complexities of mixture geometry



Bartleby.com

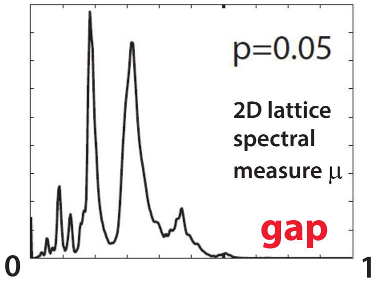
distilled

distilled

Analytic Continuation Method

Stieltjes Integral Representations
for Homogenized Parameters

Bergman 1978, Milton 1979
Golden & Papanicolaou 1983



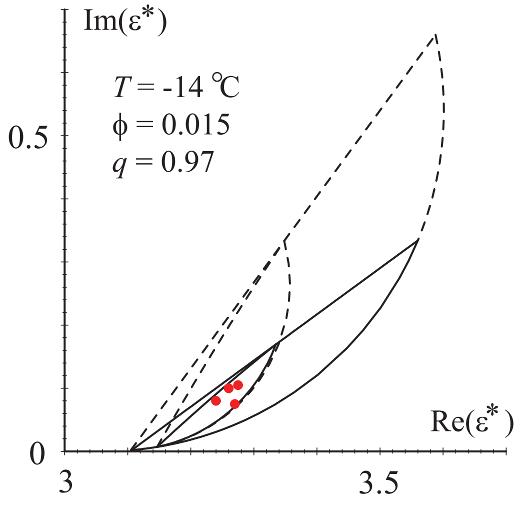
spectral properties of operator (matrix)
~ quantum states, energy levels for atoms

eigenvectors eigenvalues

EXTEND to: polycrystals, advection diffusion, waves through ice pack

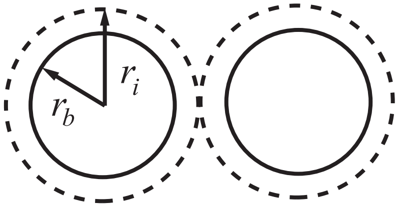
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle

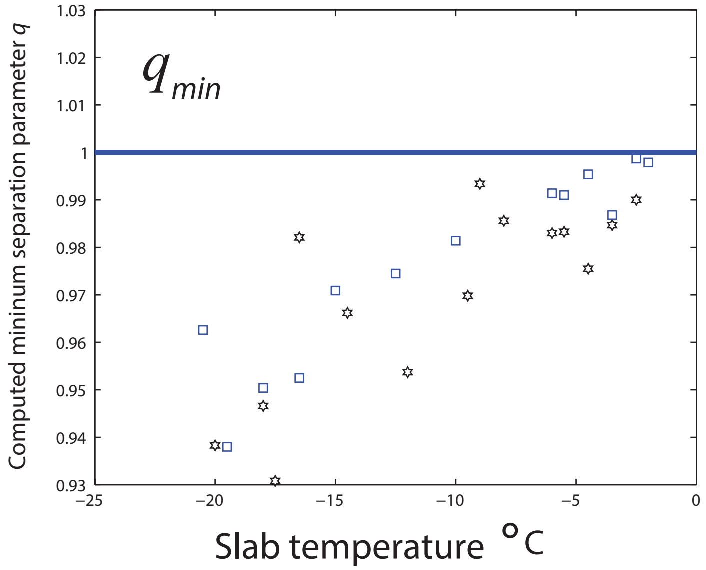


$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

inverse bounds



Inverse Homogenization

Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001), McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

ε^* → ***composite geometry***
(spectral measure μ)

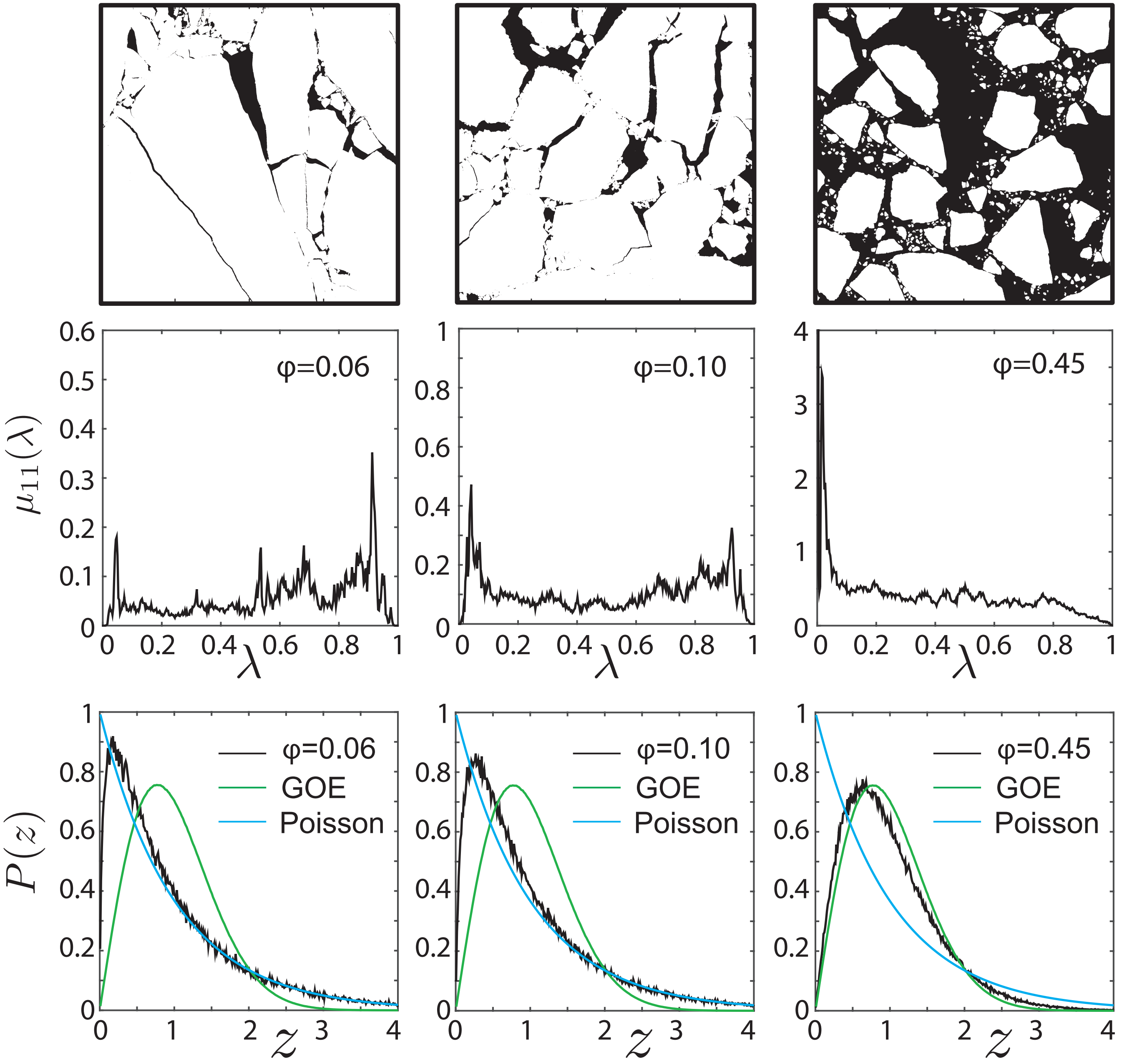
inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden Physica B, 2007

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

Spectral computations for sea ice floe configurations



spectral measures

eigenvalue spacing distributions

UNIVERSAL Wigner-Dyson distribution

uncorrelated → **level repulsion**

Murphy, Cherkaev, Golden, *Phys. Rev. Lett.* 2017; Murphy, Cherkaev, Hohenegger, Golden, *Comm. Math. Sci.* 2015

Eigenvalue Statistics of Random Matrix Theory

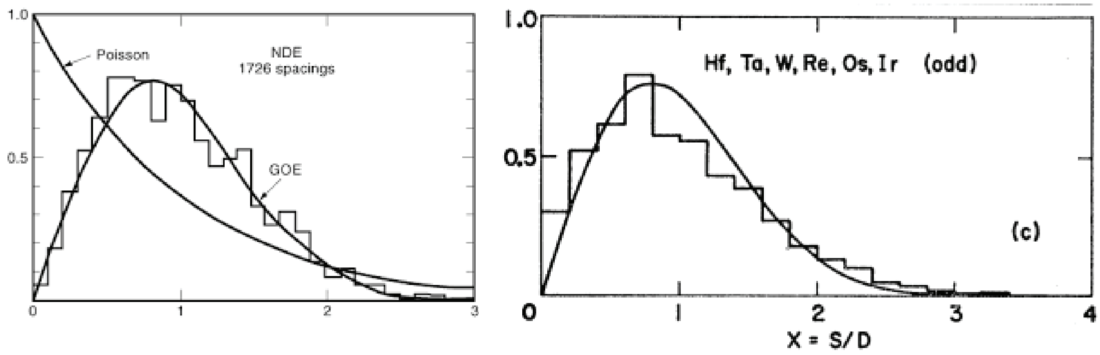
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[\mathbf{N}]_{ij} \sim N(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^{T})/2$ ***Gaussian orthogonal ensemble (GOE)***

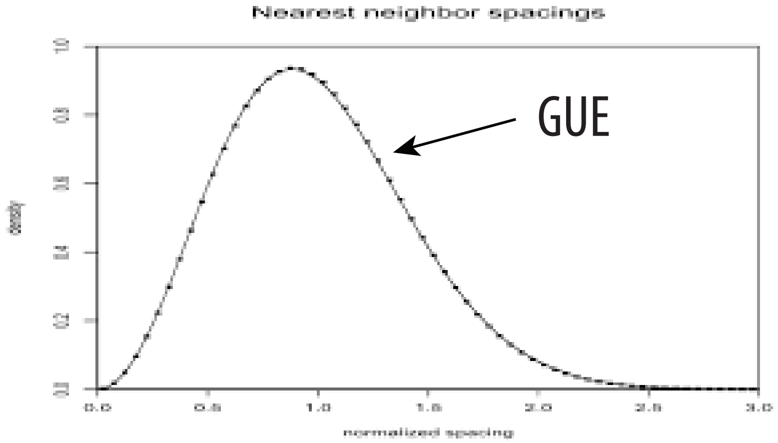
$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^{\dagger})/2$ ***Gaussian unitary ensemble (GUE)***

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics.

Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



Universal eigenvalue statistics arise in a broad range of "unrelated" problems!

Electric fields in brine microstructure of sea ice

Resolvent representation

$$\chi\vec{E} = s(sI - \chi\Gamma\chi)^{-1}\chi\vec{E}_0$$

expressed in terms of the eigenvalues λ_i *and eigenvectors* $\vec{w}_i$ *of* $\chi\Gamma\chi$*:*

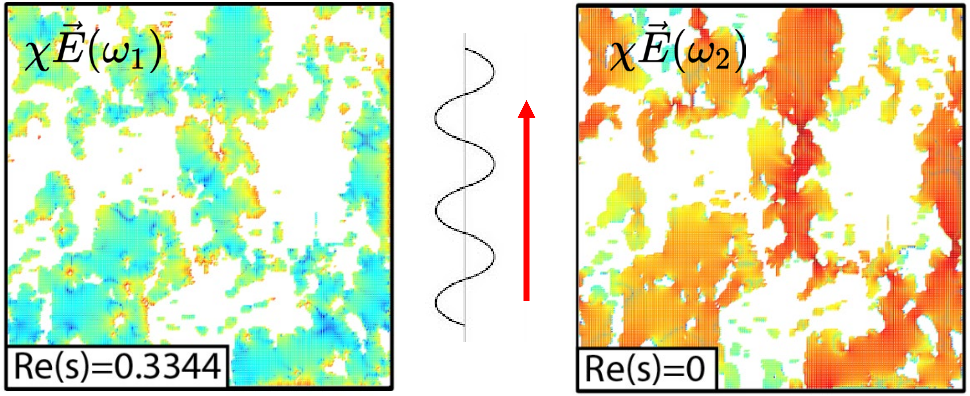
$$\chi\vec{E} = s\sum_{i=1}^{N}\frac{\vec{w}_i^{\,T}\chi\hat{e}_k}{s-\lambda_i}\vec{w}_i$$

Electric field

Constituent material properties

Geometry controls which frequencies propagate!

K. M. Golden, N. B. Murphy, D. Hallman, E. Cherkaev, Stieltjes functions and spectral analysis in the physics of sea ice, *Nonlin. Proc. Geophys., 2023*

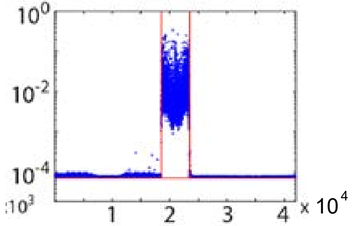


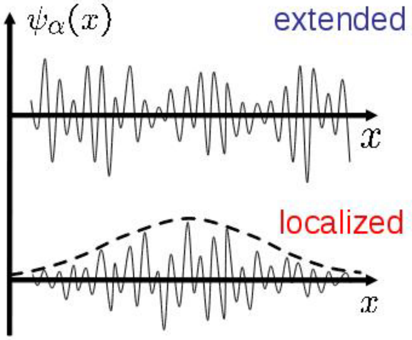
frequency = ω_1

Localized field

frequency =ω_2

Extended field





Anderson localization

disorder-driven

metal / insulator transition

Anderson 1958
Mott 1949
Evangelou 1992
Shklovshii et al 1993

propagation vs. localization in wave physics:
quantum, optics, acoustics, water waves

Wave equations

Laplace + Diffusion
equations

we find percolation-driven

Anderson transition for classical transport in composites

mobility edges, localization, universal spectral statistics

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

but no wave interference or scattering effects at play!

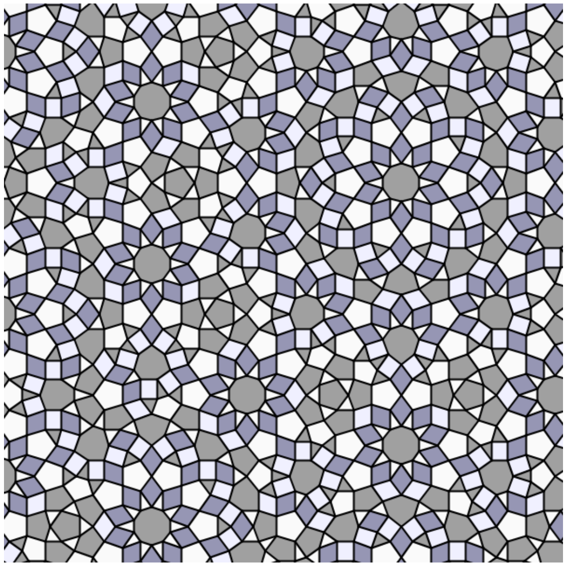
Given these findings in random systems, what class of media might we look at to design new materials with exciting properties?

Order to Disorder in Quasiperiodic Composites

D. Morison (Physics), N. B. Murphy, E. Cherkaev, K. M. Golden, *Communications Physics* 2022

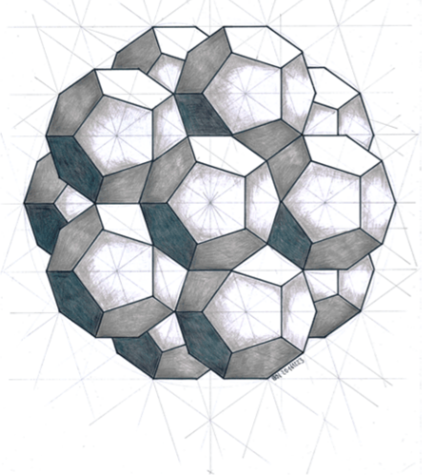
quasiperiodic crystal

quasicrystal



quasiperiodic checkerboard

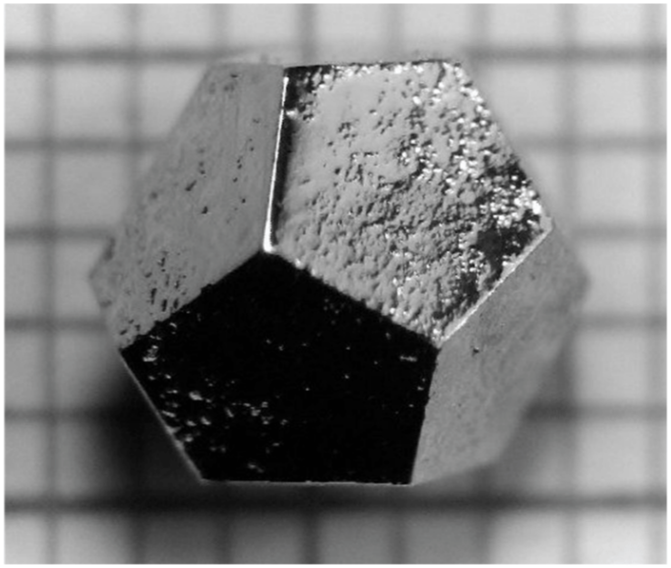
Stampfli, 2013



dense packing of dodecahedra

3D Penrose tiling

Tripkovic, 2019



Holmium–magnesium–zinc quasicrystal

ordered but aperiodic

lacks translational symmetry

Shechtman et al., *Phys. Rev. Lett.*, 1984

Levine & Steinhardt, *Phys. Rev. Lett.*, 1984

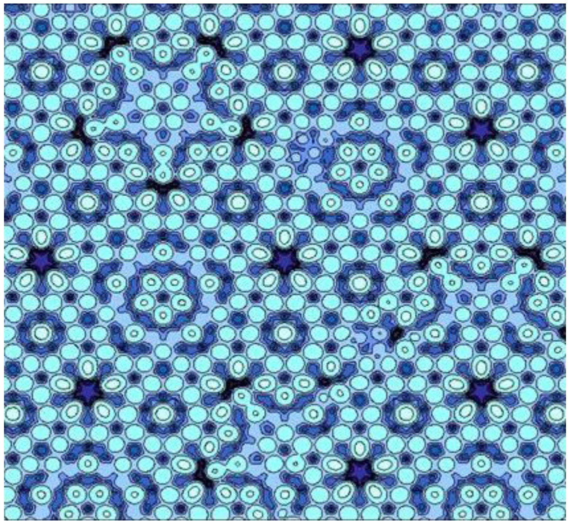
---

classical transport in quasiperiodic media

Golden, Goldstein & Lebowitz, *Phys. Rev. Lett.*, 1985

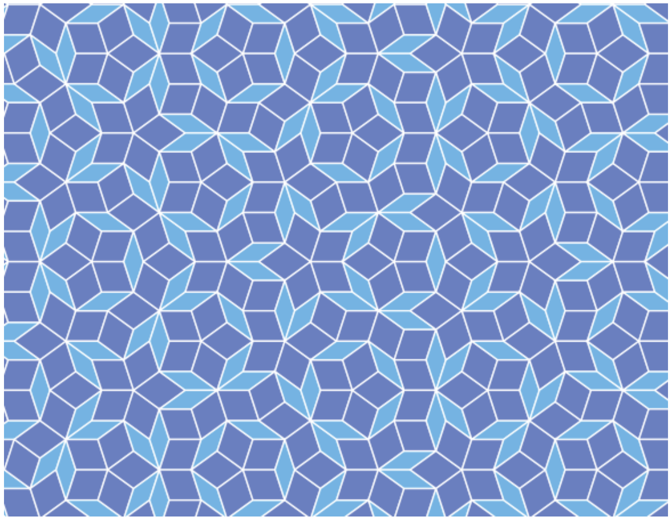
Golden, Goldstein & Lebowitz, *J. Stat. Phys.*, 1990

⋮



energy surface Al-Pd-Mn quasicrystal

Unal et al., 2007



aperiodic tiling of the plane - R. Penrose 1970s

1D, 2D inhomogeneous materials - quasiperiodic

$$\sigma(x) = 3 + \cos x + \cos kx$$

effective conductivity

$$\sigma^*(k) = \begin{cases} \text{constant} & k \ \text{irrational} \quad \textbf{quasiperiodic} \\ f(k) & k \ \text{rational} \quad \textbf{periodic} \end{cases}$$

Golden, Goldstein, Lebowitz
Classical transport in modulated structures, *Phys. Rev. Lett.* 1985

...

G. Bouchitté, S. Guenneau, F. Zolla, *SIAM Multiscale Modeling & Simulation*, 2010

E. Cherkaev, S. Guenneau, N. Wellander, *IEEE Metamaterials, 2017*

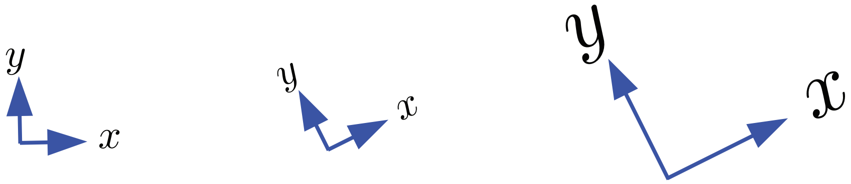
N. Wellander, S. Guenneau, E. Cherkaev, *Math. Methods in the Applied Sci.*, 2017

Session on Analysis, Homogenization, and Spectral Problems in Materials Science

Organizers: Wellander, Cherkaev, Guenneau

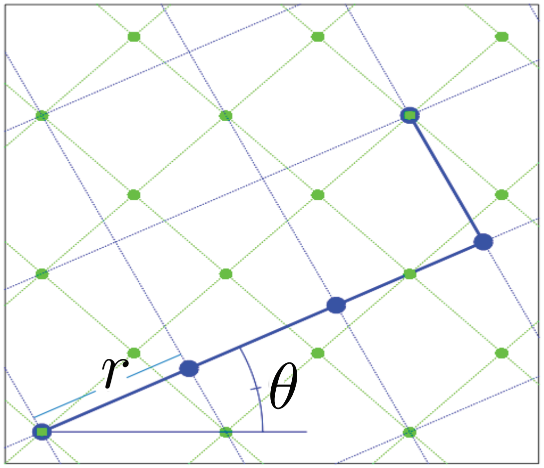
Flodén, Johnsen, Persson, ...

Moiré patterns generate two component composites **on any scale**



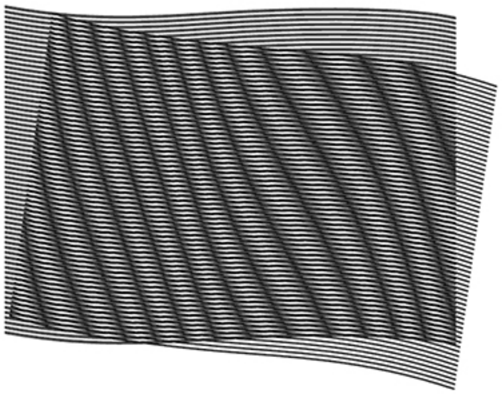
rotation
dilation

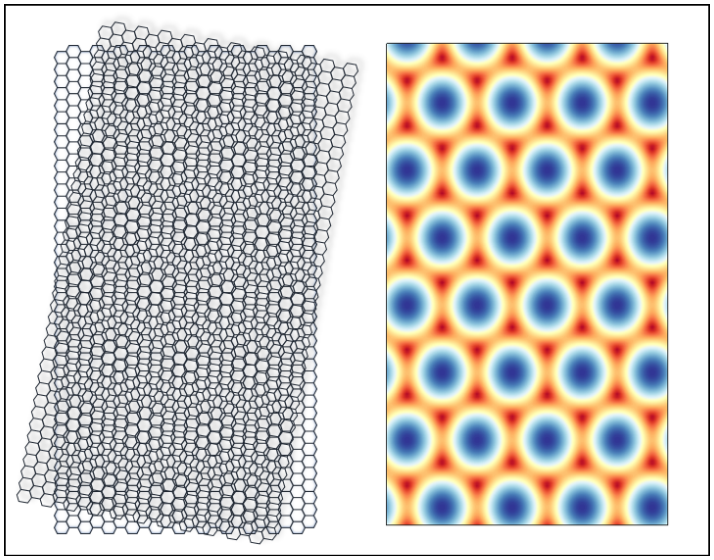
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



$$\psi\left(x', y'\right) = \cos 2\pi x' \cos 2\pi y'$$

$$\chi = \begin{cases} 1, & \psi \geq 0 \\ 0, & \psi < 0 \end{cases}$$





quantum dots
artificial atoms

Tran et al.
Nature 2019

Order to disorder in quasiperiodic composites

Morison, Murphy, Cherkaev, Golden, Comm. Phys. 2022

sea ice inspired - twisted bilayer composites

tunable quasiperiodic composites with exotic properties

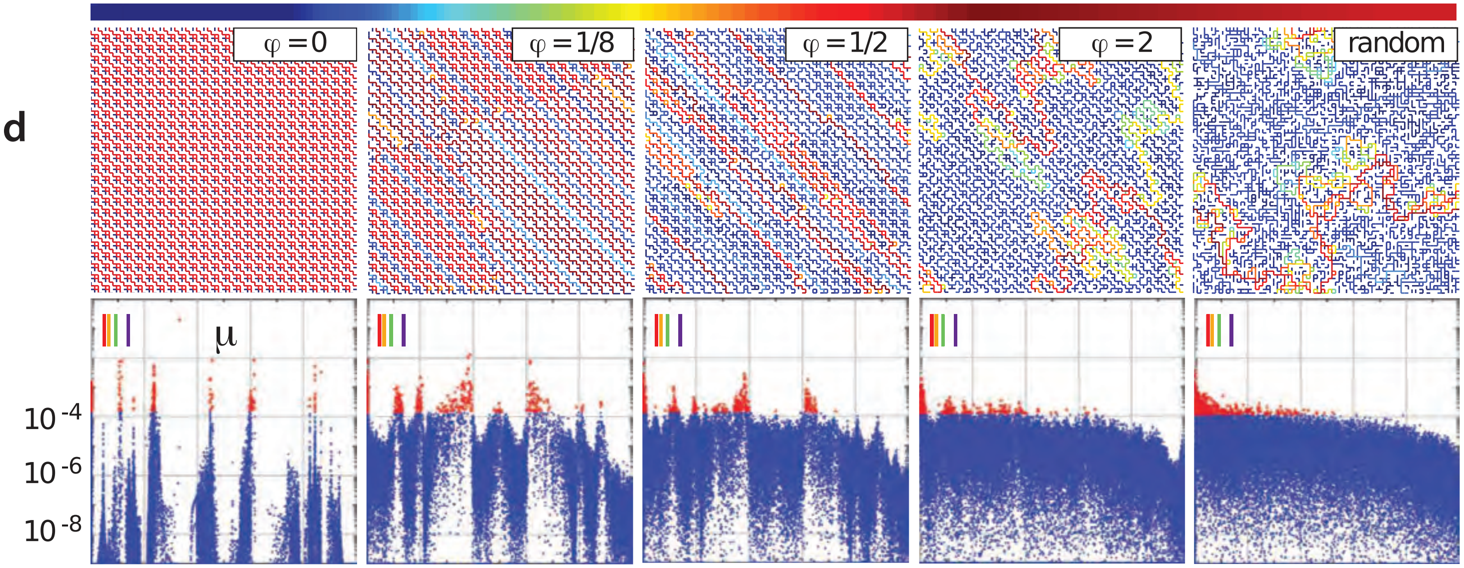
(optical, electrical, thermal) Anderson localization; our Moiré patterned geometries are similar to twisted bilayer graphene

increasing twist angle between two lattices

periodic → quasiperiodic

electric field strength

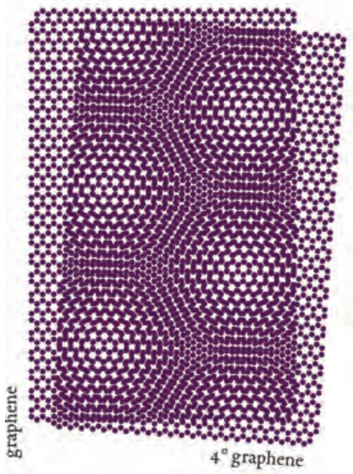
spectral measure

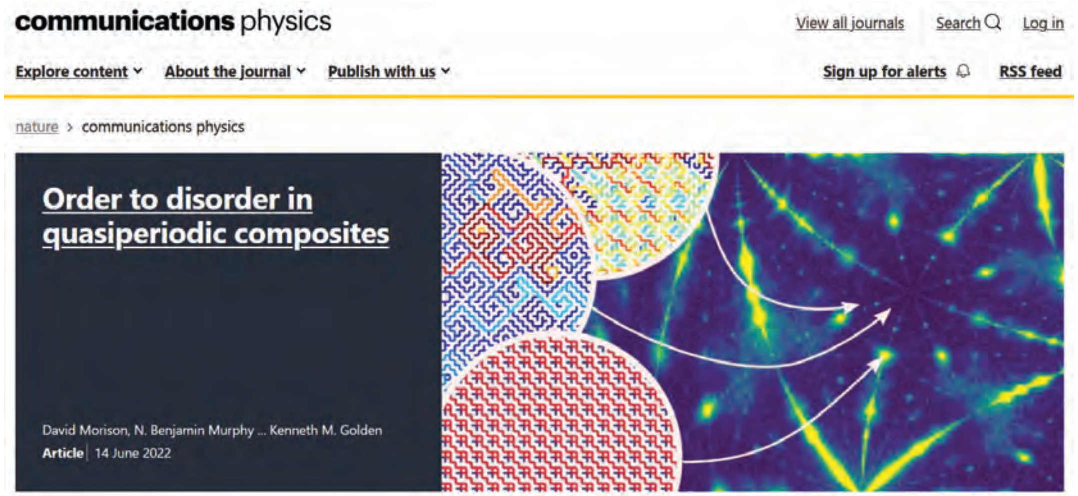


RRN at percolation threshold

twisted bilayer graphene

superconducting magic twist angle





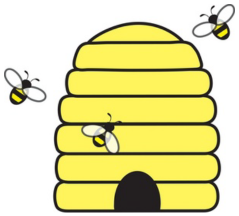
Order to disorder in quasiperiodic composites

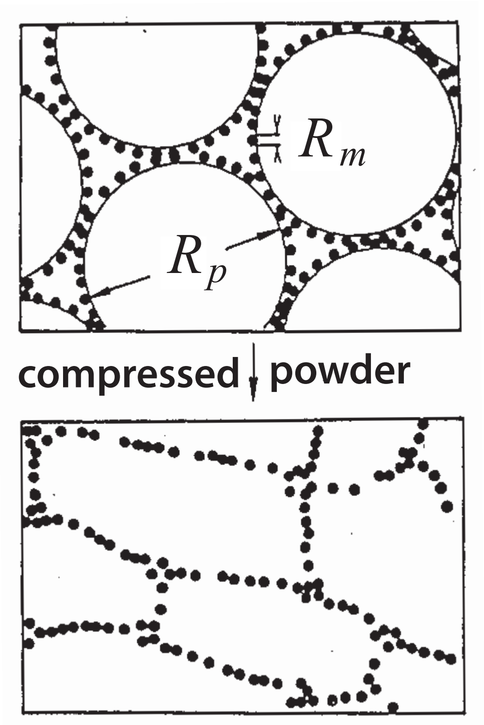
David Morison, N. Benjamin Murphy ... Kenneth M. Golden

Article | 14 June 2022

constellation of periodic systems in a sea of randomness

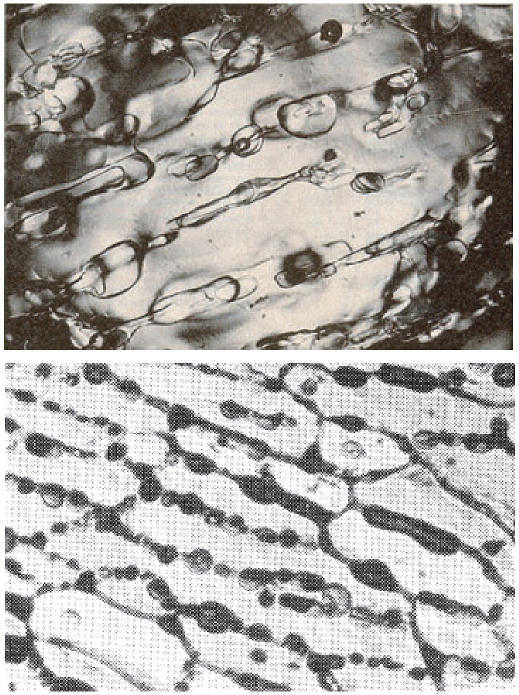
cross pollination





radar absorbing coating

Kusy & Turner
Nature 1971

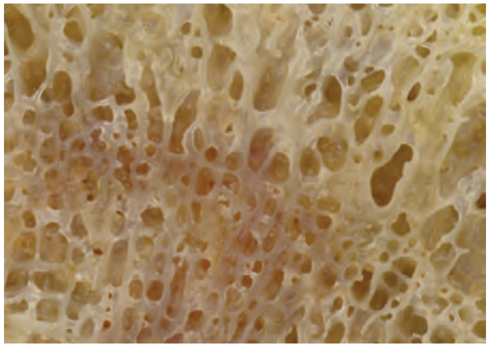


sea ice

Golden, Ackley, Lytle
Science 1998

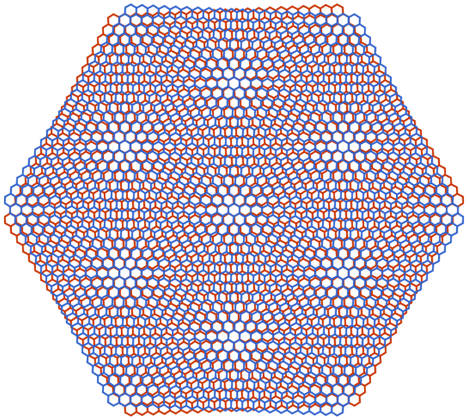
Rule of Fives
fluid flow

spectral analysis & RMT



human bone

Golden, Murphy, Cherkaev
J. Biomechanics 2011



twisted bilayer materials

Morison, Murphy, Cherkaev, Golden
Communications Physics 2022

stealth technology, climate science, medical imaging, twistronics

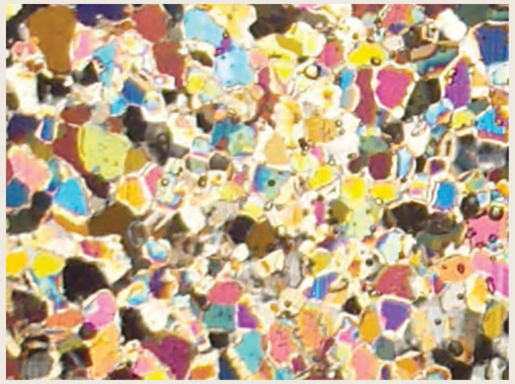
Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**

  orientation statistics
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A   8 Feb 2015

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS
OF SCIENTIFIC
PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

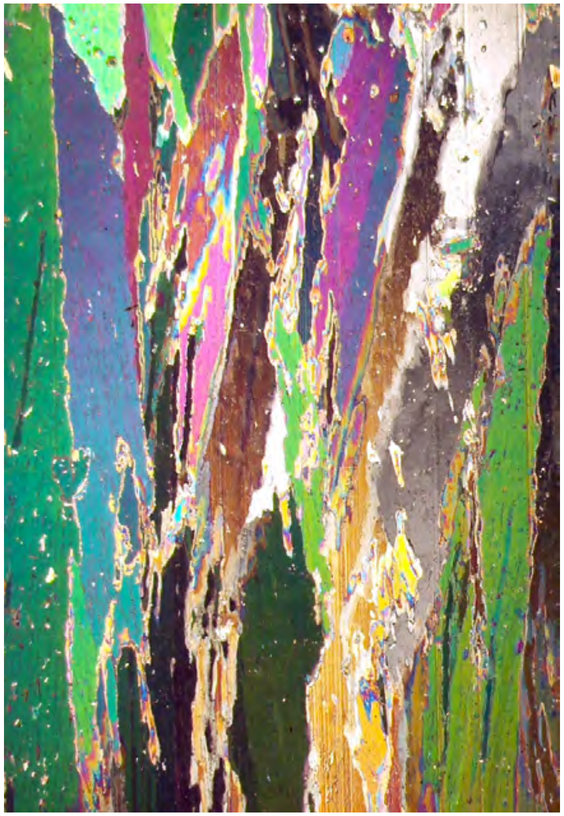
THE
ROYAL
SOCIETY
PUBLISHING

21/01/15 1:47 PM

higher threshold for fluid flow in granular sea ice

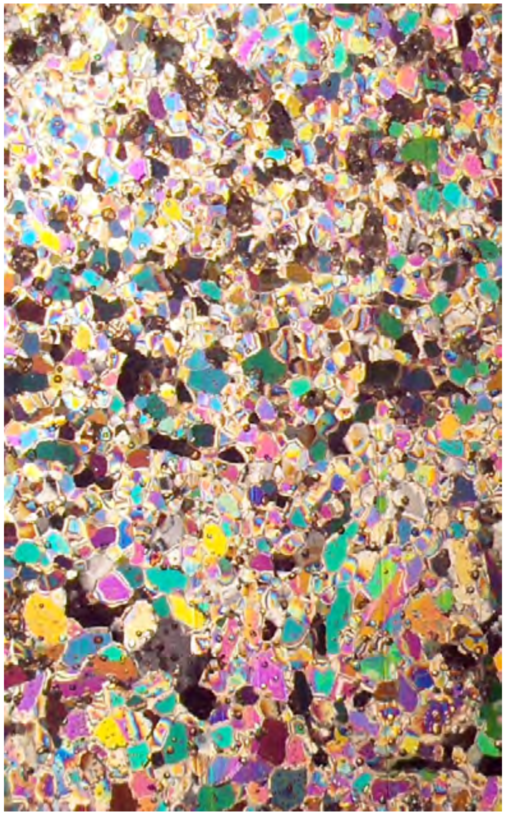
microscale details impact "mesoscale" processes

nutrient fluxes for microbes
melt pond drainage
snow-ice formation



columnar

5%



granular

10%

Golden, Sampson, Gully, Lubbers, Mosier, Tison 2024

electromagnetically distinguish ice types
inverse homogenization for polycrystals

SEA ICE ALGAE high level of local heterogeneity

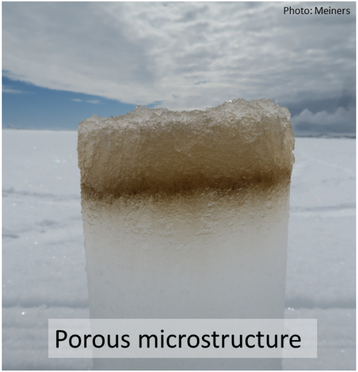


Photo: Meiners

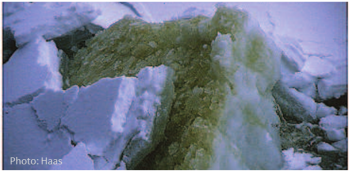
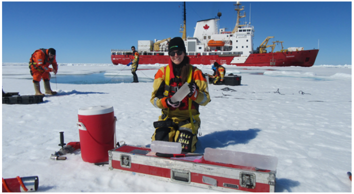


Photo: Haas



Can we improve agreement between algae models and data?

80% of polar bear diet can be traced to ice algae*.

---

* Brown TA, et al. (2018). *PloS one*, 13(1), e0191631

Received: 7 June 2022 | Revised: 1 August 2022 | Accepted: 1 August 2022

DOI: 10.1111/ele.14095

ECOLOGY LETTERS WILEY

METHOD

Uncertainty quantification for ecological models with random parameters

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[1]Department of Mathematics, University of Utah, Salt Lake City, Utah, USA

[2]School of Biological Sciences, University of Utah, Salt Lake City, Utah, USA

[3]Scientific Computing and Imaging Institute, University of Utah, Salt Lake City, Utah, USA

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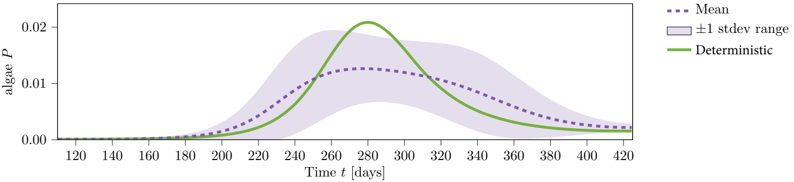
Jody R. Reimer, Department of Mathematics and School of Biological Sciences, University of Utah, Salt Lake City, Utah, USA.
Email: reimer@math.utah.edu

Abstract

There is often considerable uncertainty in parameters in ecological models. This uncertainty can be incorporated into models by treating parameters as random variables with distributions, rather than fixed quantities. Recent advances in uncertainty quantification methods, such as polynomial chaos approaches, allow for the analysis of models with random parameters. We introduce these methods with a motivating case study of sea ice algal blooms in heterogeneous environments. We compare Monte Carlo methods with polynomial chaos techniques to help understand the dynamics of an algal bloom model with random parameters.

Introduce polynomial chaos approach to widely used ecological ODE models, but with random parameters.

ECOLOGICAL INSIGHTS



- lower peak bloom intensity
- longer bloom duration
- able to compare variance to data

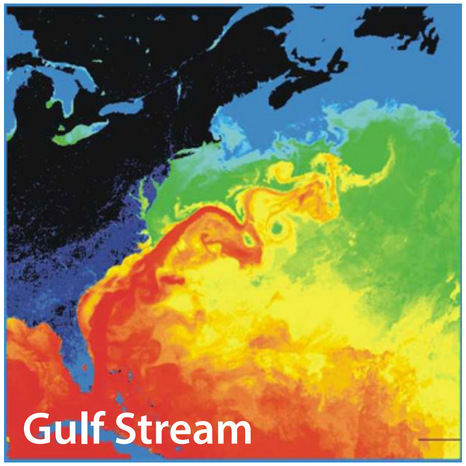
Inverse Problem: given algal and nutrient data, recover growth rate distribution
Anthony Lee, Jody Reimer, Akil Narayan, Ken Golden 2024

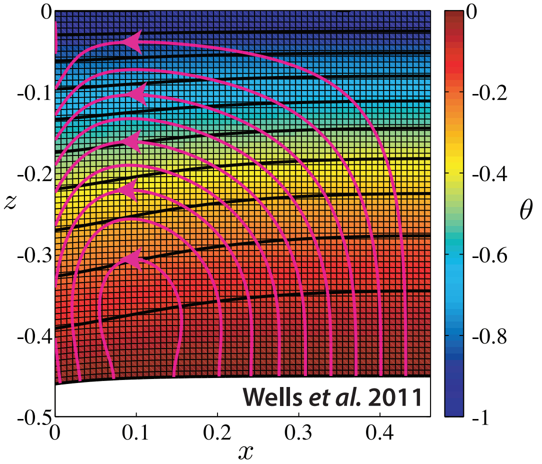
mesoscale

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

homogenize

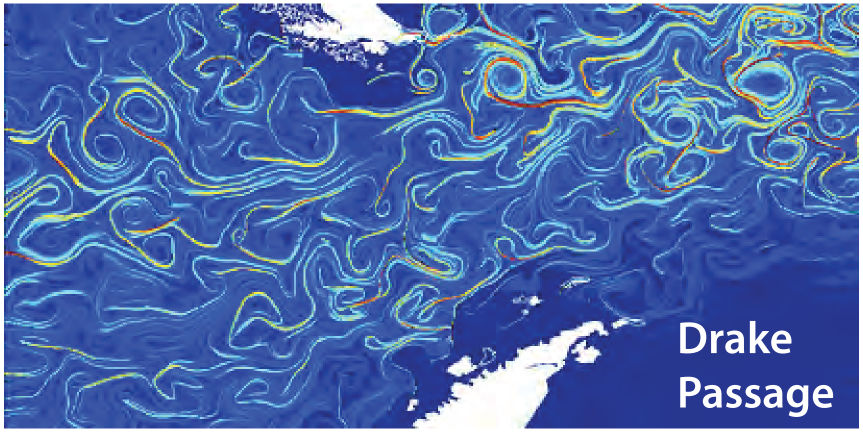
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* ***effective diffusivity***

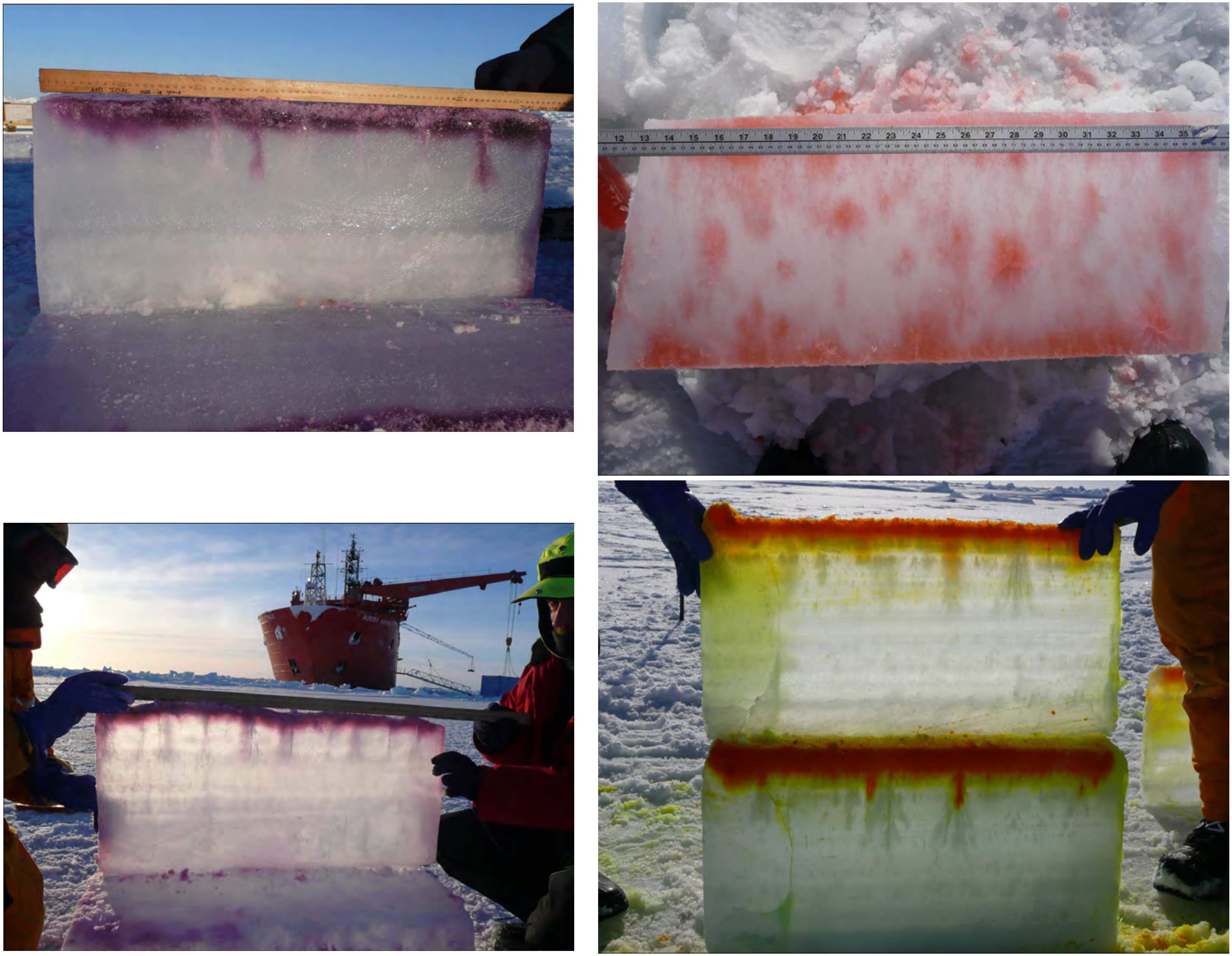
Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020



tracers flowing through inverted sea ice blocks



Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

$$\kappa^* = \kappa \left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2} \right), \quad F(\kappa) = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}$$

- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H \Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$, Δ is the Laplace operator
- $i\Gamma H \Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ-plane

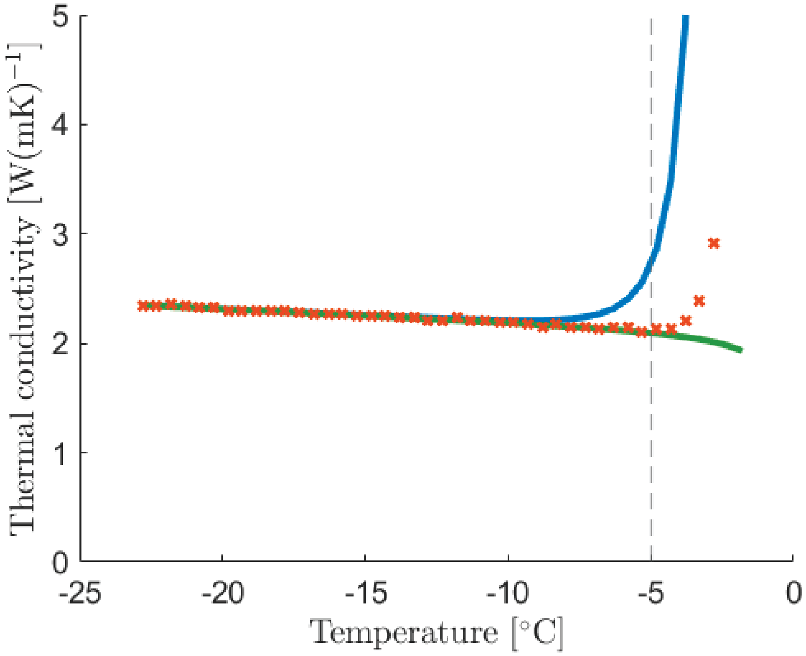
rigorous framework for numerical computations of spectral measures and effective diffusivity for model flows

new integral representations, theory of moment calculations

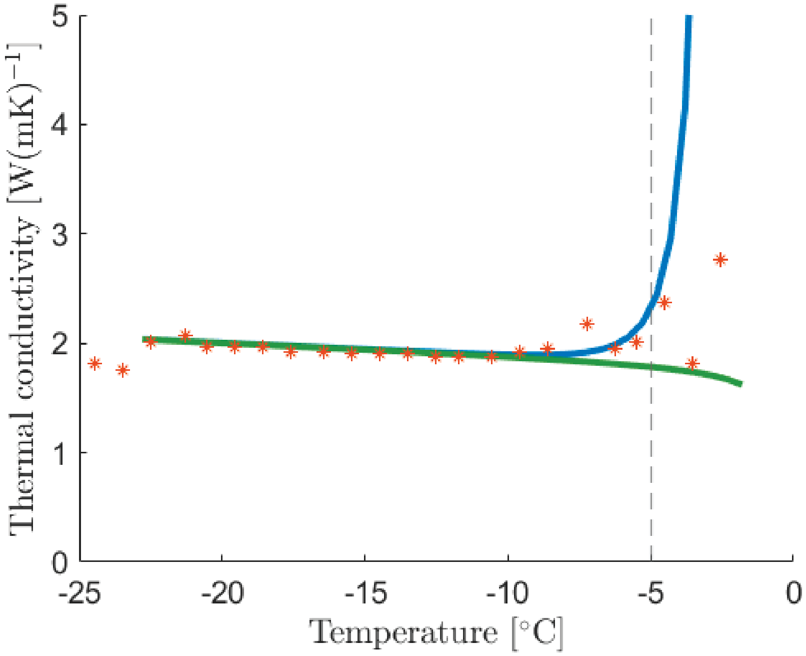
separation of material properties and flow field

Bounds on Convection Enhanced Thermal Transport

simulations



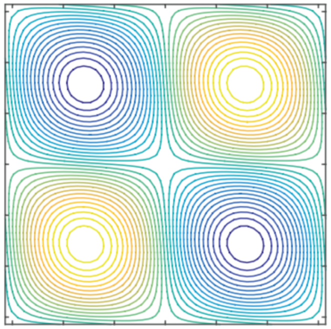
data [Trodahl et al., 2001]



Monte-Carlo simulations of SDE with temperature dependent Péclet number P

strength of advection B = κP/2π

Euler-Maruyama and subsampling methods for SDE



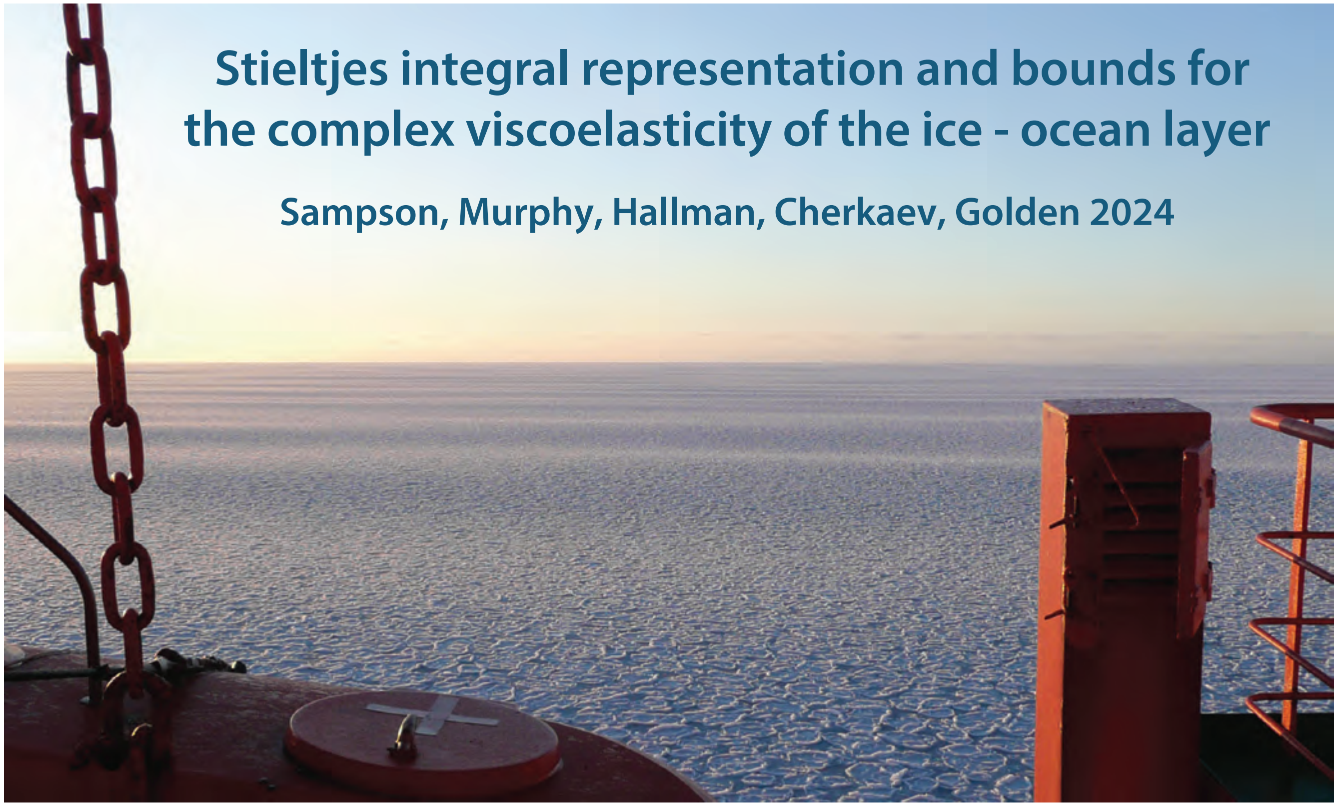
cat's eye flow model for brine convective flow

Rigorous Padé approximant bounds in terms of P using Stieltjes integral + analytic continuation method for the measure

Darcy velocity v = 0.5 [m/s]

Kraitzman, Hardenbrook, Dinh, Murphy, Cherkaev, Zhu, & Golden, 2024

wave propagation in the marginal ice zone (MIZ)



Stieltjes integral representation and bounds for the complex viscoelasticity of the ice - ocean layer

Sampson, Murphy, Hallman, Cherkaev, Golden 2024

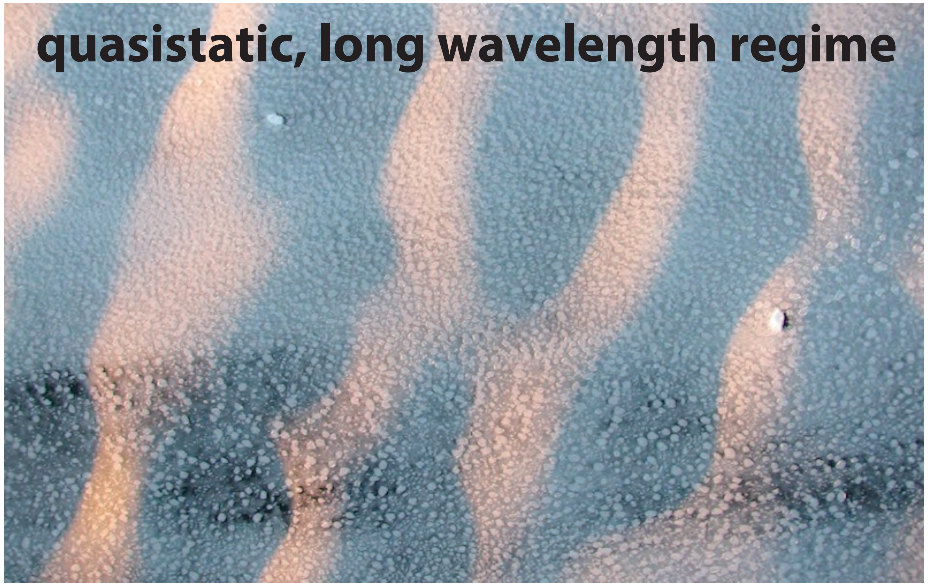
first theory of key parameter in wave-ice interactions only fitted to wave data before

Keller, 1998
Mosig, Montiel, Squire, 2015
Wang, Shen, 2012

Analytic Continuation Method

Bergman (78) - Milton (79)
integral representation for ε^*
Golden and Papanicolaou (83)

Milton, *Theory of Composites* (02)



quasistatic, long wavelength regime

homogenized parameter depends on sea ice concentration and ice floe geometry

like EM waves

Strain fields in two-component viscoelastic materials

Electromagnetic waves (old)

Mechanical waves (NEW!)

$$\nabla \times \vec{E} = 0,\ \nabla \cdot \vec{J} = 0,\ \vec{J} = \boldsymbol{\sigma}\vec{E} \longrightarrow \nabla \cdot \sigma = 0,\ \ \epsilon = \epsilon_0 + \overbrace{\nabla^S \vec{u}}^{\epsilon_f},\ \ \sigma = 2\nu\epsilon$$

Resolvent representation of the ***electric*** *field*

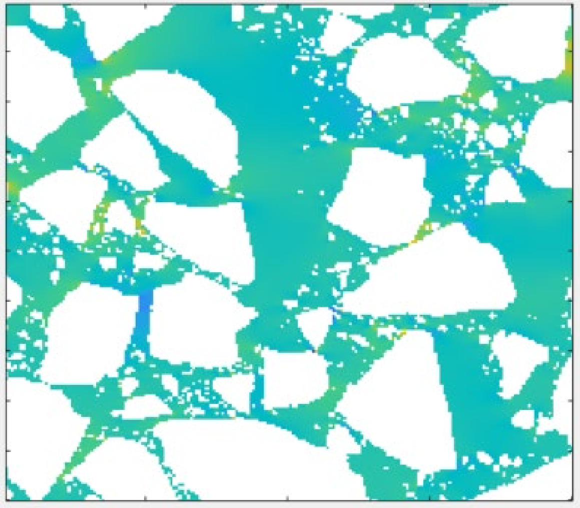
$$\chi\vec{E} = s(sI - \chi\Gamma\chi)^{-1}\chi\vec{E}_0$$

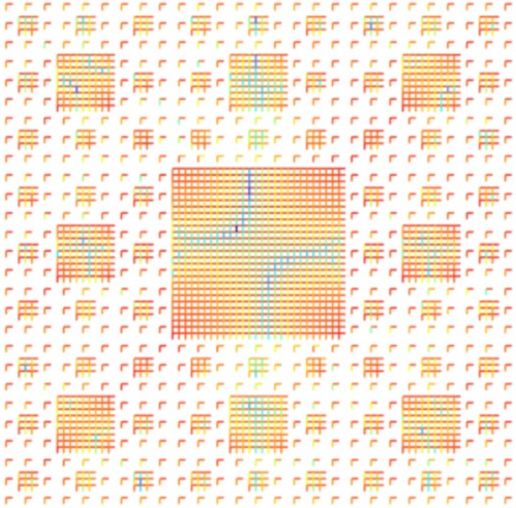
Resolvent representation of the ***strain*** *field*

$$\chi_1\epsilon = s\left(sI - \chi_1\Gamma^S\chi_1\right)^{-1}\chi_1\epsilon_0$$

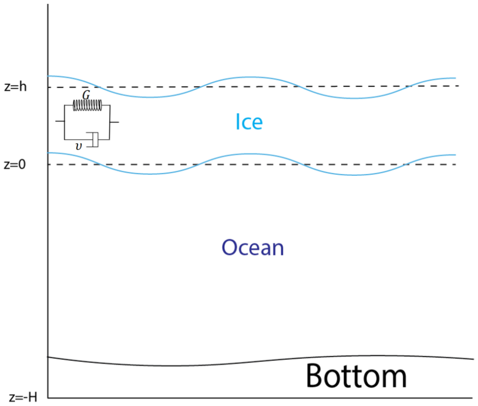
Preliminary results:
strain fields in pack ice and fractals

"Waves in sea ice"





Open problem in climate modeling: *How do ice conditions and sea ice floe geometry influence the* ***attenuation*** *of oceanic waves?*

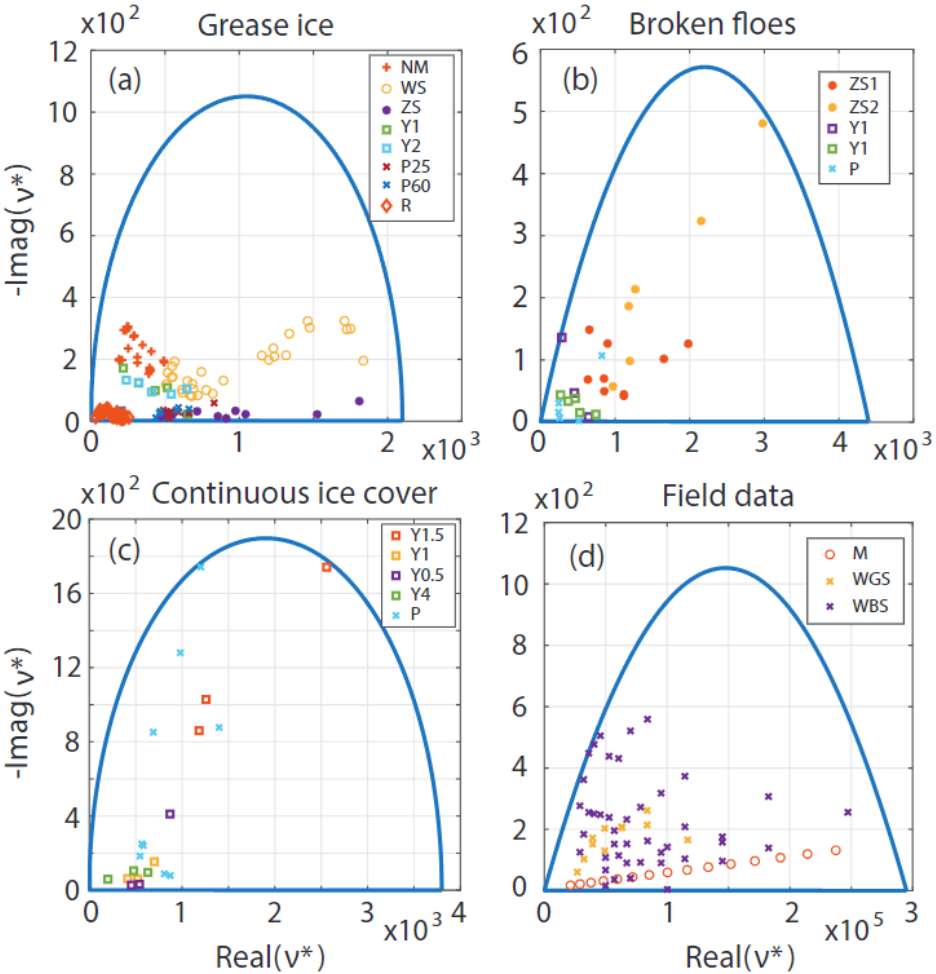


Christian Sampson (2017)

We model the top ocean-ice layer as a homogeneous, isotropic Kelvin-Voigt material with complex viscoelasticity ν^*

In this model, wave propagation & attenuation are influenced by ν^*

Forward bounds for ν^* have been obtained which are fitted to multiple well known wave-ice datasets, including *Wadhams et al., 1988, Newyear & Martin, 1997, Wang & Shen, 2010, Meylan et al., 2014* and several others!

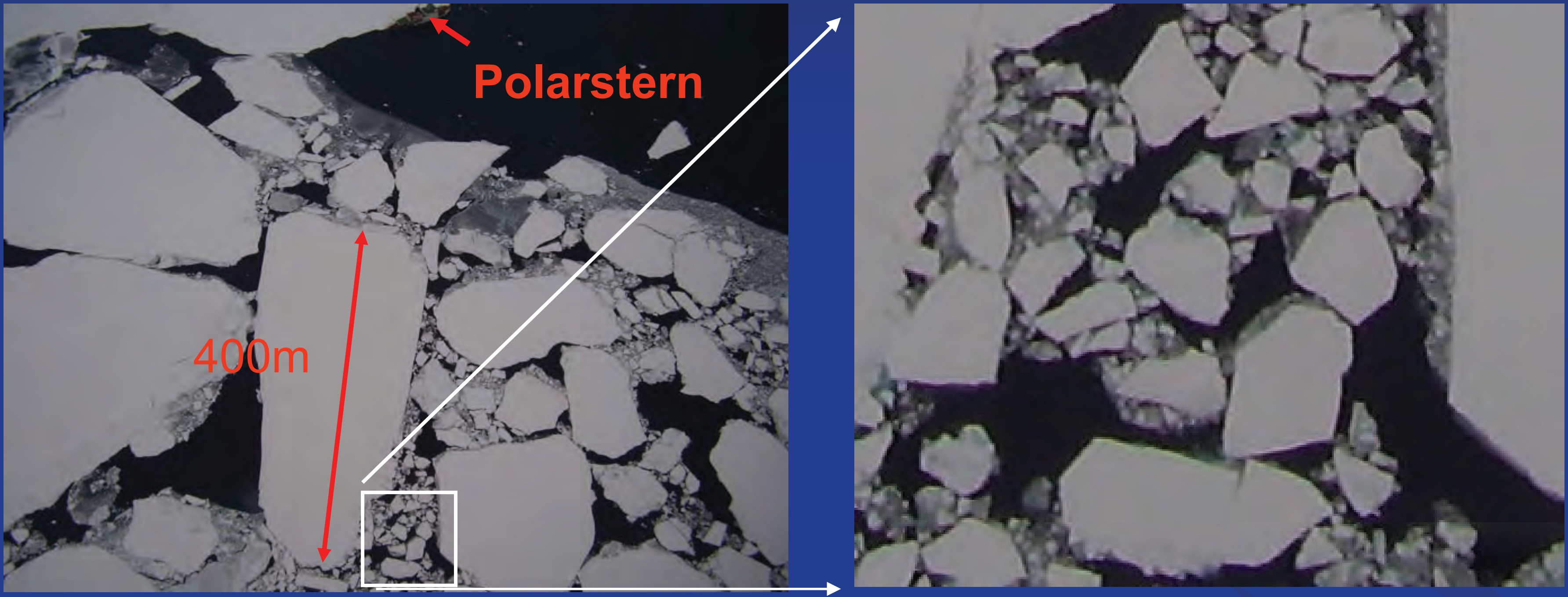


C. Sampson, D. Hallman, N. B. Murphy, K. Golden, E. Cherkaev (2024)
Bounds on the complex viscoelasticity for surface waves on ice-covered seas

The sea ice pack has fractal structure.

Self-similarity of sea ice floes

Weddell Sea, Antarctica



fractal dimensions of Okhotsk Sea ice pack
smaller scales D~1.2, larger scales D~1.9

fractal dim. *vs.* floe size exponent
Adam Dorsky, Nash Ward, Ken Golden 2024

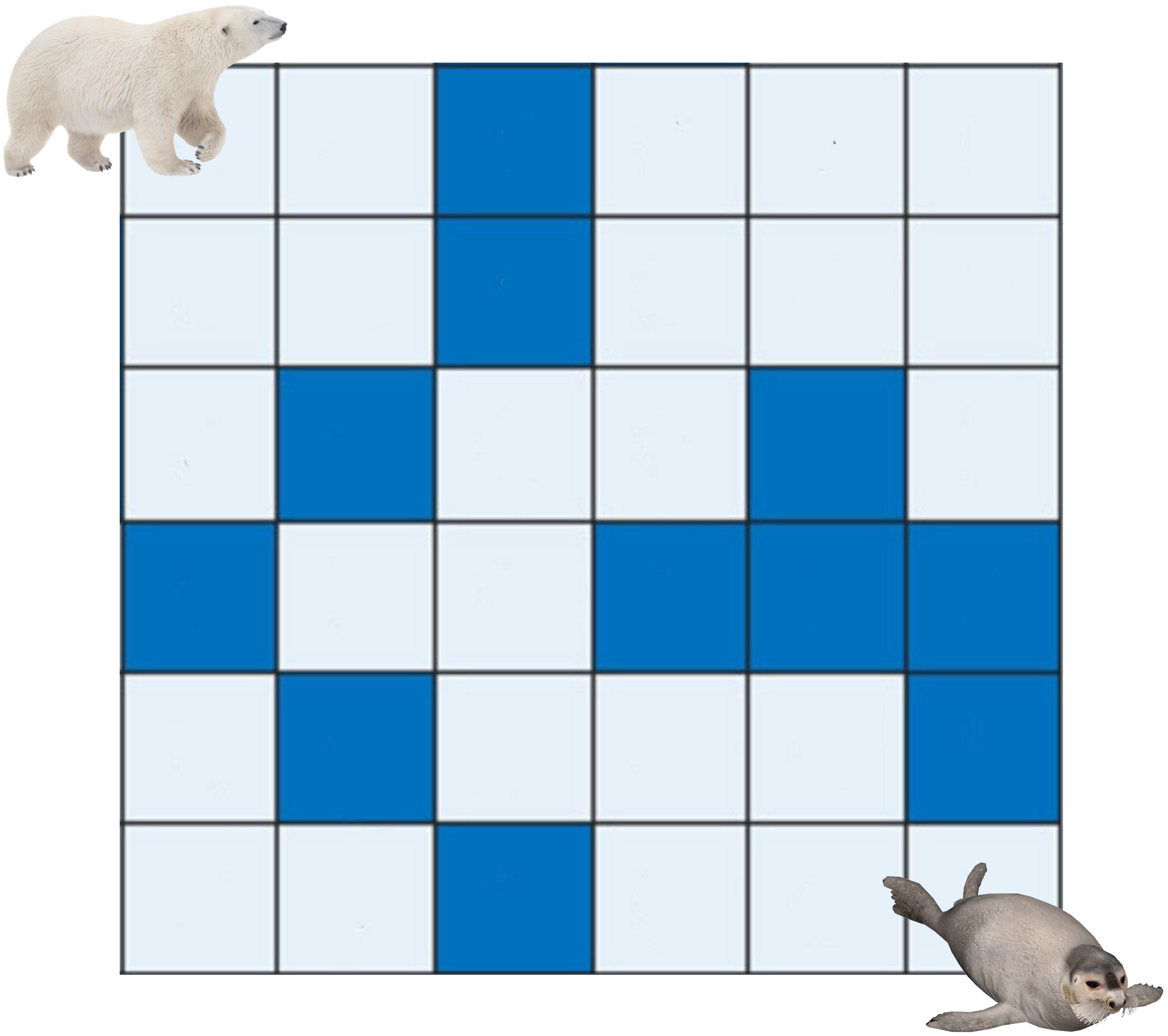
Toyota, *et al. Geophys. Res. Lett.* 2006
Rothrock and Thorndike, *J. Geophys. Res. 1984*

Optimal Movement of a Polar Bear in a Heterogenous Icescape

Nicole Forrester, Jody Reimer, Ken Golden 2024

Polar bears expend 5X more energy swimming than walking on sea ice.

As sea ice is lost, how do polar bears optimize their movement to save energy and survive?

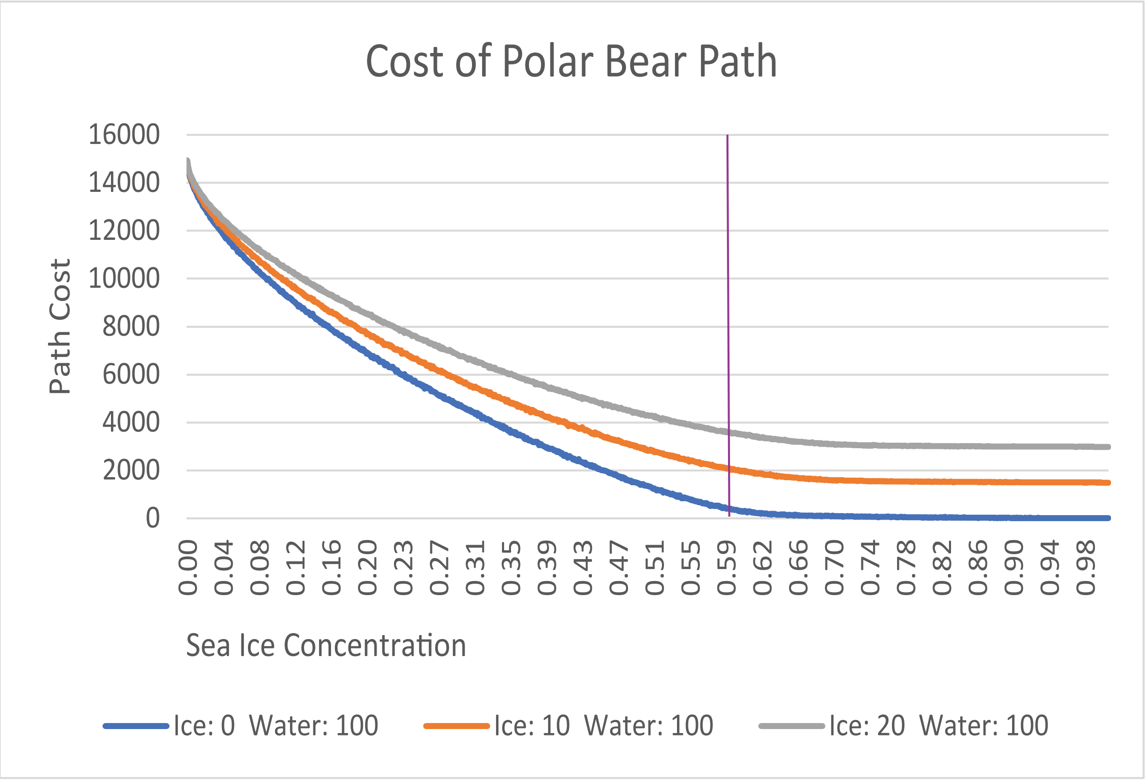


https://pluspng.com/img-png/polar-bear-png-polar-white-bear-png-1000.png
https://zt2downloadlibrary.fandom.com/wiki/Bearded_Seal_(Tamara_Henson)

Polar Bear Percolation

To study the importance of ice connectedness, we exaggerate the data by setting the cost of walking on ice to 0 with the cost of swimming still at 5.

$C(p)$



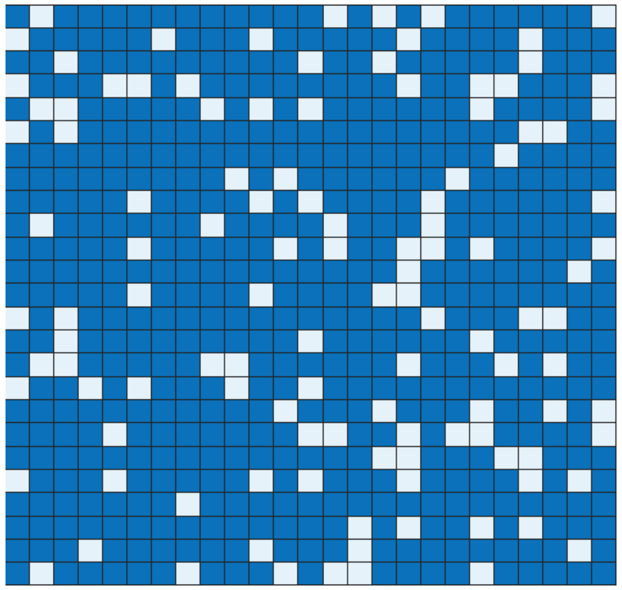
← $h = 0.2$

← $h = 0.1$

← $h = 0$

$$h = \frac{c_i}{c_w}$$

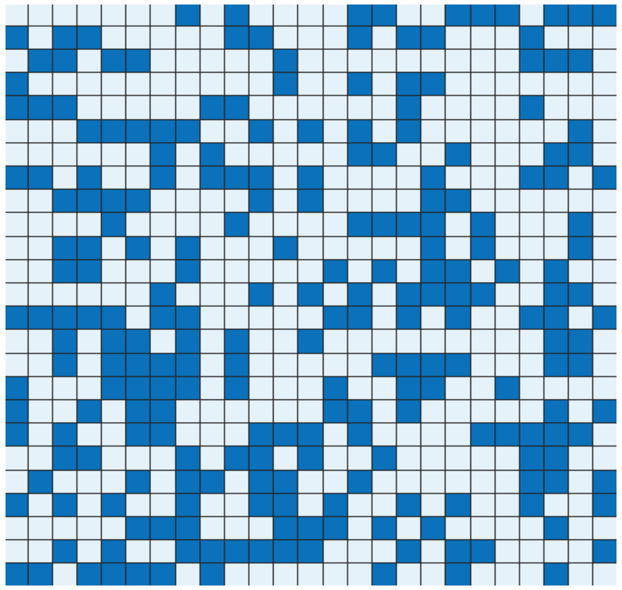
ratio of local “conductivities”



20% Ice

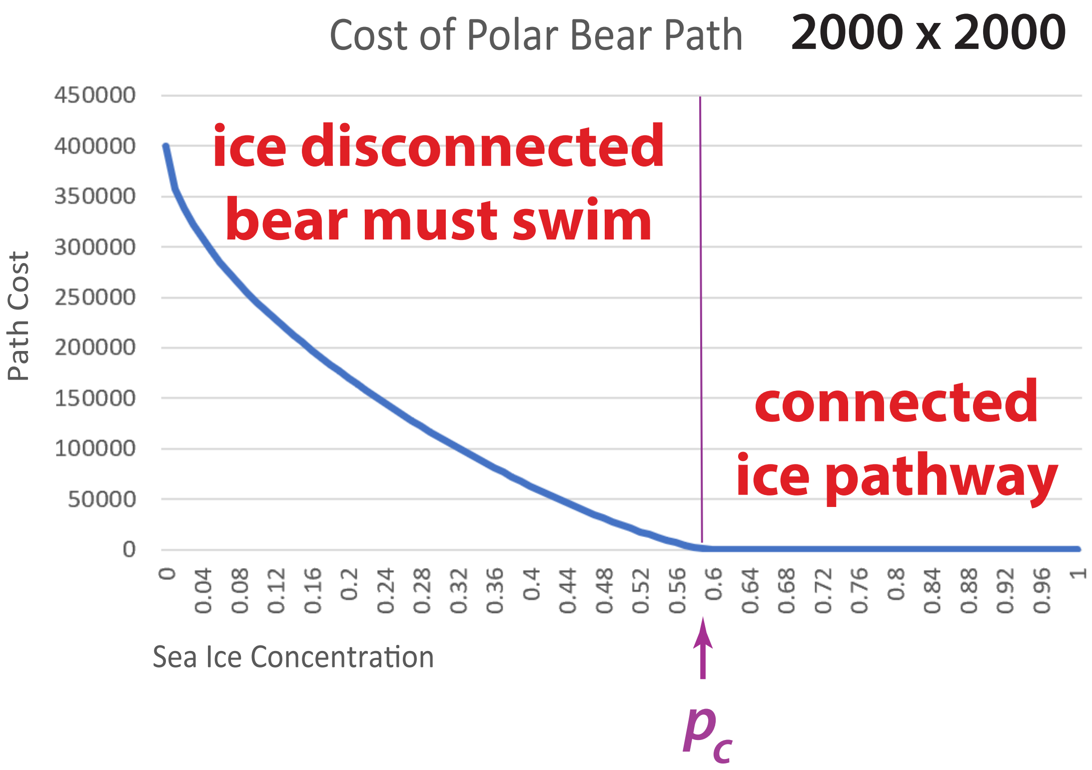
site percolation threshold

$p_c = 0.59 \;\; for \;\; d = 2$



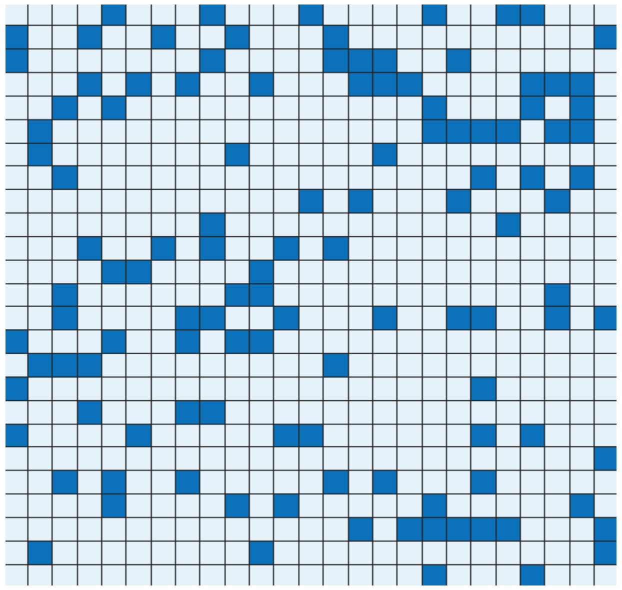
60% Ice

$C(p)$



← $h = 0$

Polar Bear Critical Exponent



80% Ice

melt pond formation and albedo evolution:

- *major drivers in polar climate*
- *key challenge for global climate models*

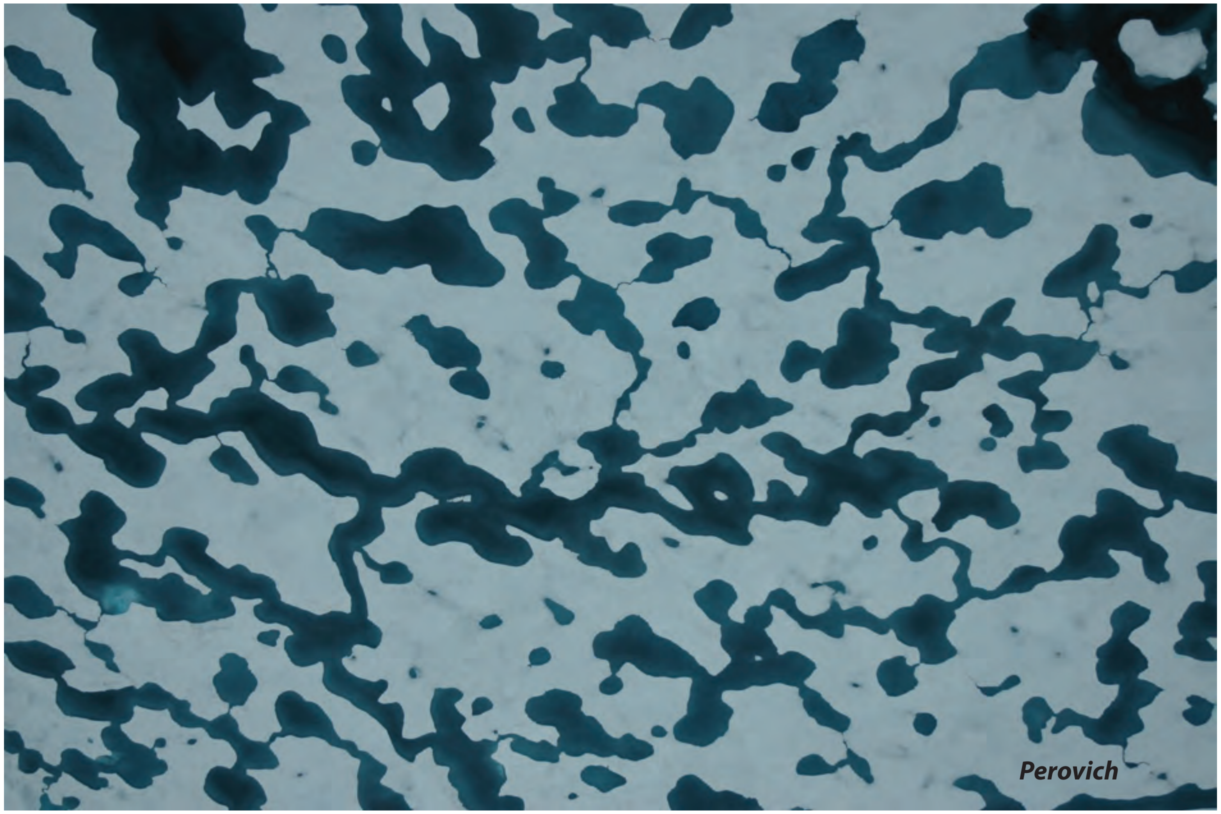
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

Flocco, Feltham, Hunke 2012


Perovich

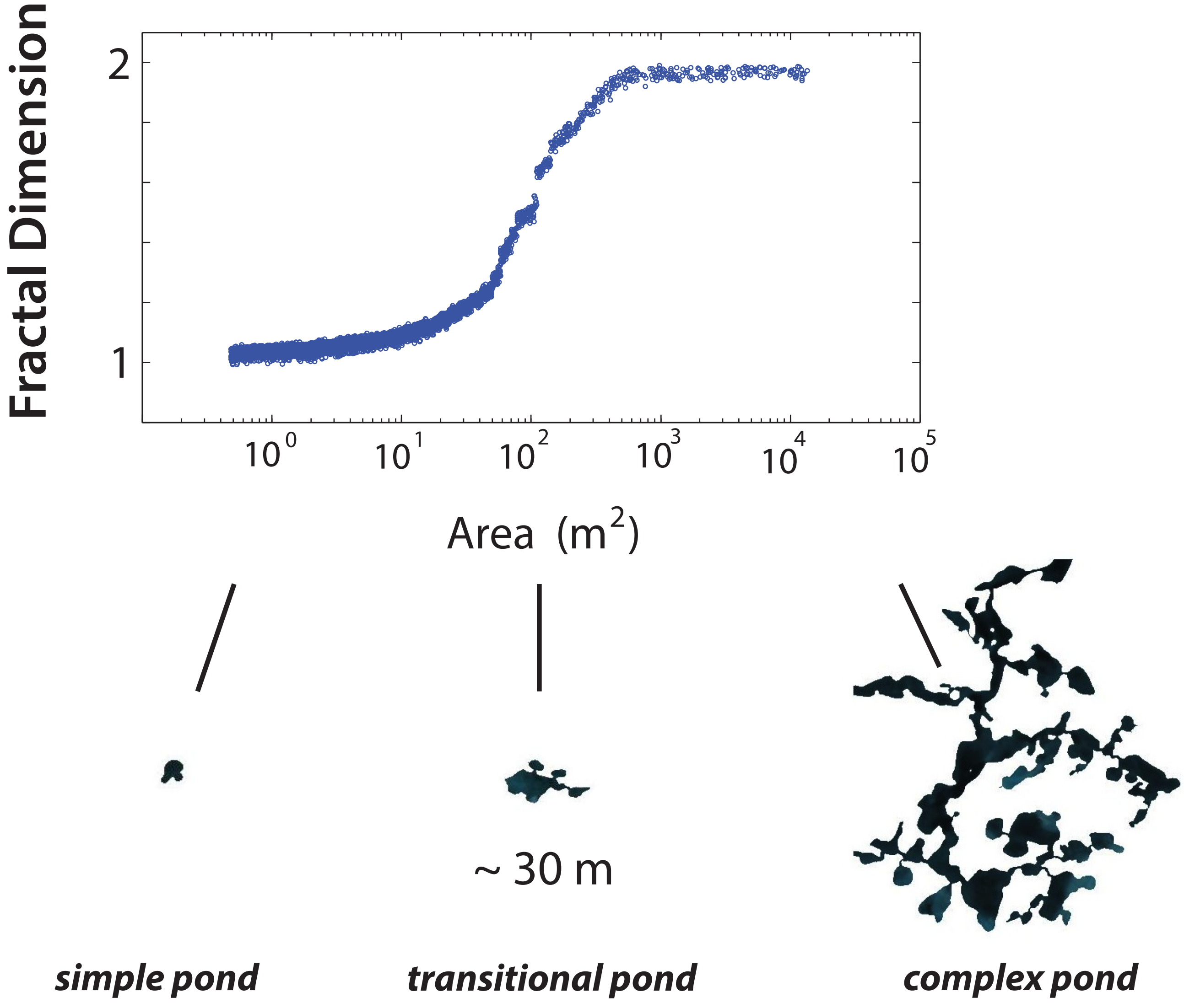
Are there universal features of the evolution similar to phase transitions in statistical physics?

Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

The Cryosphere, 2012

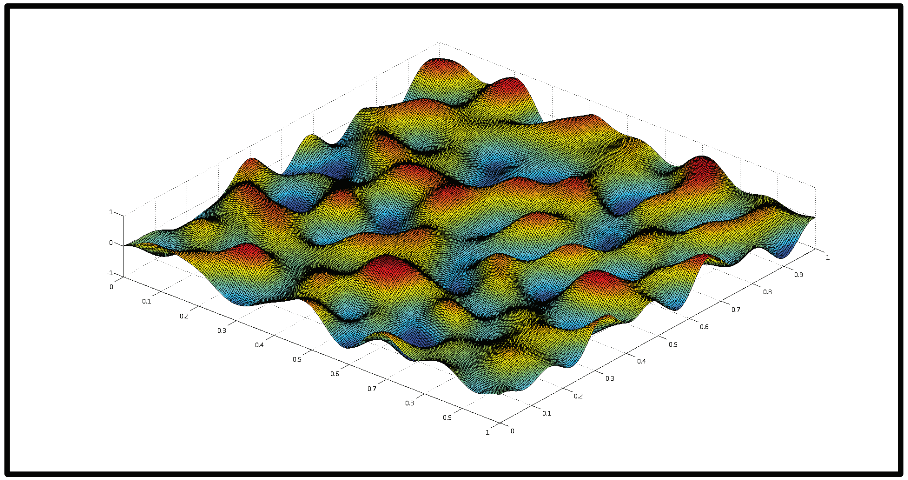
complexity grows with length scale



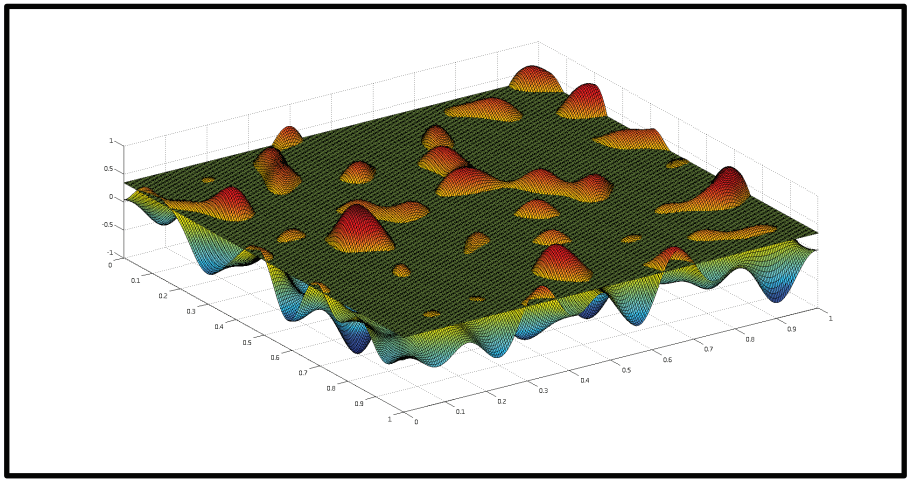
Continuum percolation model for melt pond evolution

level sets of random surfaces

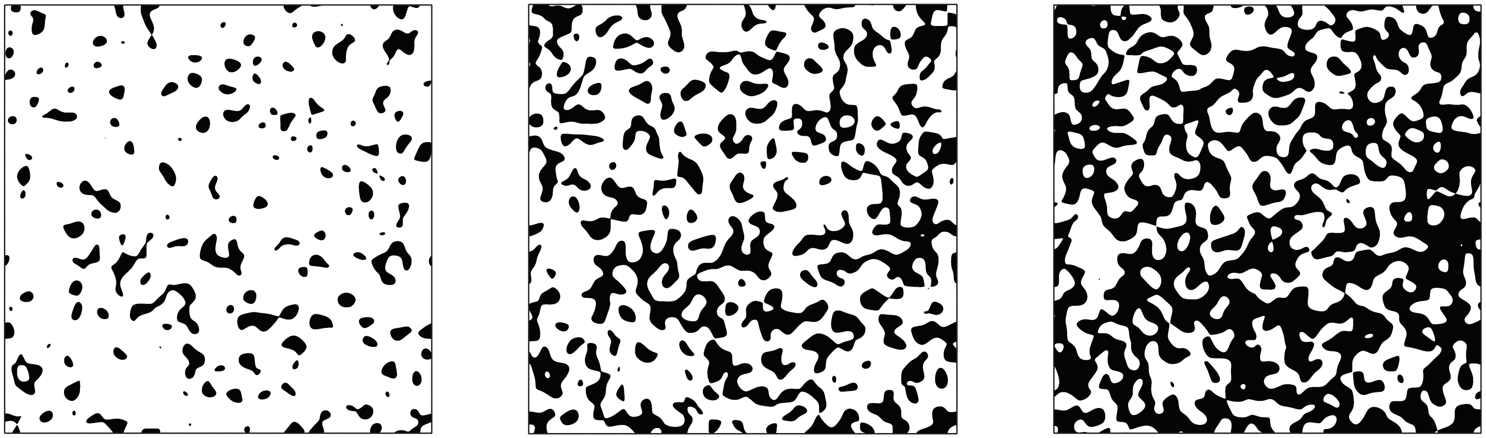
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018



random Fourier series representation of surface topography



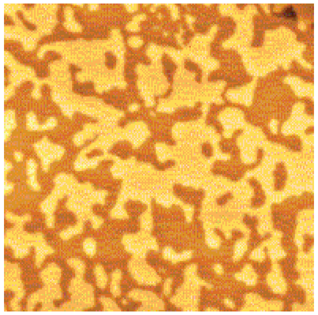
intersections of a plane with the surface define melt ponds



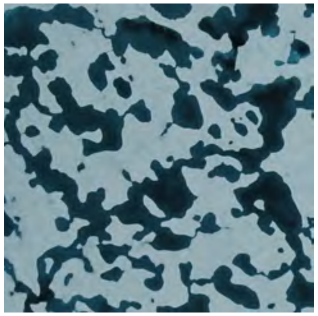
electronic transport in disordered media *diffusion in turbulent plasmas* *Isichenko, Rev. Mod. Phys., 1992*

From magnets to melt ponds

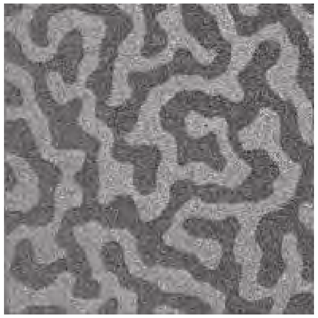
100 year old model for magnetic materials used to explain melt pond geometry



magnetic domains in cobalt



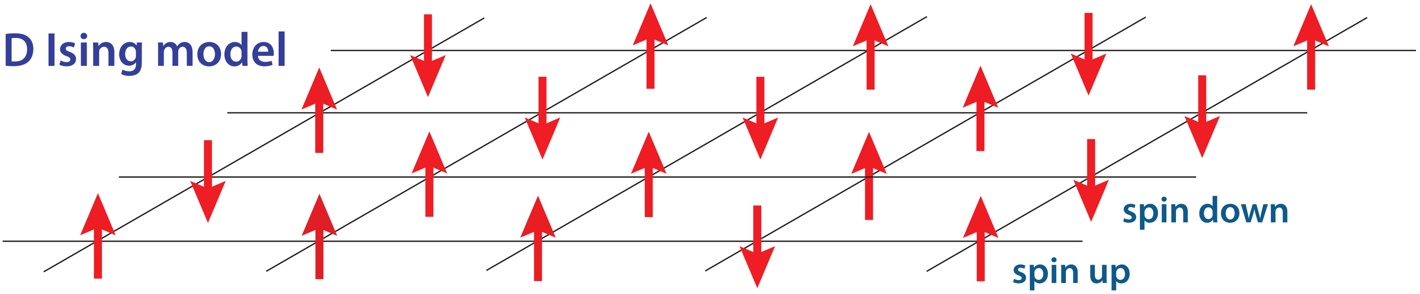
Arctic melt ponds

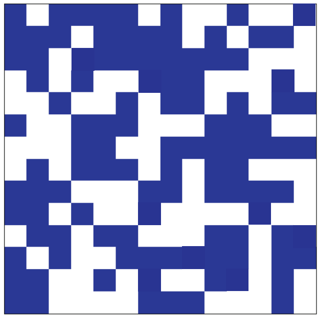


magnetic domains in cobalt-iron-boron

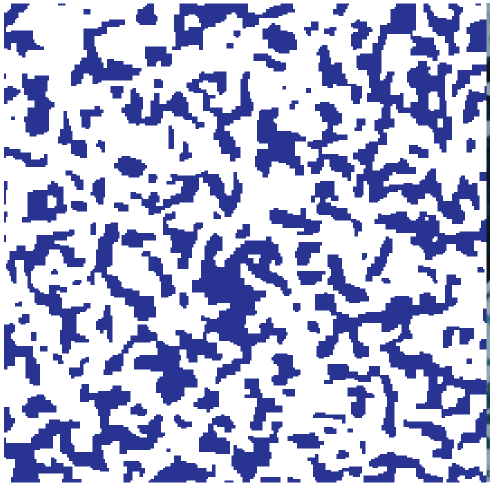
Arctic melt ponds

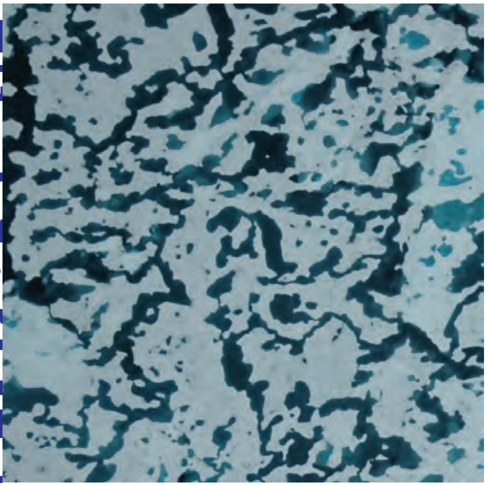
2D Ising model





model





real ponds (Perovich)

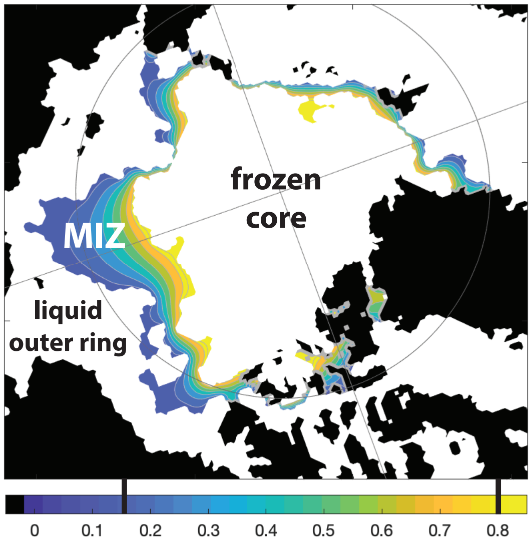
Ma, Sudakov, Strong, Golden, *New J. Phys.* 2019

Scientific American, EOS, PhysicsWorld, ...

macroscale

Model larger scale effective behavior with partial differential equations that *homogenize* complex local structure and dynamics.

Arctic MIZ



sea ice concentration ψ

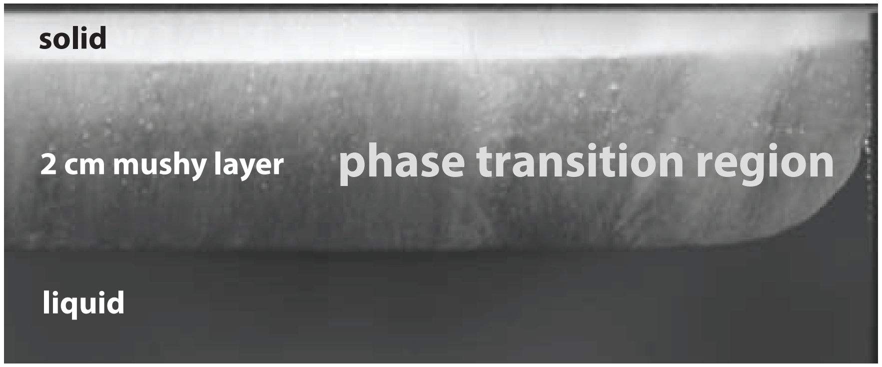
Partial differential equation models and deep learning for the sea ice concentration field, 2024

Delaney Mosier, Eric Brown, Court Strong,
Jingyi Zhu, Bao Wang, Ken Golden

advection diffusion model

Predict MIZ width and location with basin-scale phase change model.

seasonal and long term trends



NaCl-H2O in lab
(Peppin et al., 2007;, J. Fluid Mech.)

Arctic marginal ice zone annual cycle explained by ocean-scale mushy layer model, 2024

C. Strong, E. Cherkaev, and K. M. Golden

northward 1600 km & widens by factor of 4

MIZ as a moving phase transition region

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + S$$

$$S = [\rho(c_l - c_s)T + \rho L] \frac{\partial \psi}{\partial t}$$

$$\psi = 1 - \left(\frac{T - T_s}{T_l - T_s} \right)^{\alpha}$$

$$k_x = \left(\frac{\psi}{k_s} + \frac{1 - \psi}{k_l} \right)^{-1}$$

$$k_z = \psi k_s + (1 - \psi) k_l$$

homogenization

ρ effective density

T temperature

c specific heat

L latent heat of fusion

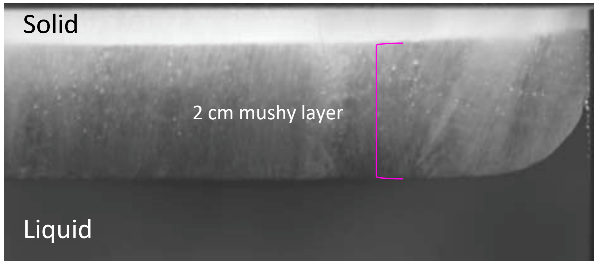
S models nonlinear phase change

ψ sea ice concentration

k effective diffusivity

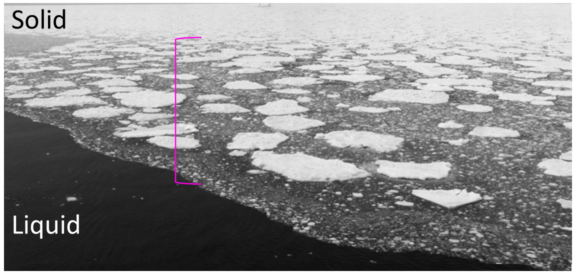
l liquid, s solid

Classical small-scale application



NaCl-H_2O in lab
(Peppin et al., 2007;, J. Fluid Mech.)

Macroscale application

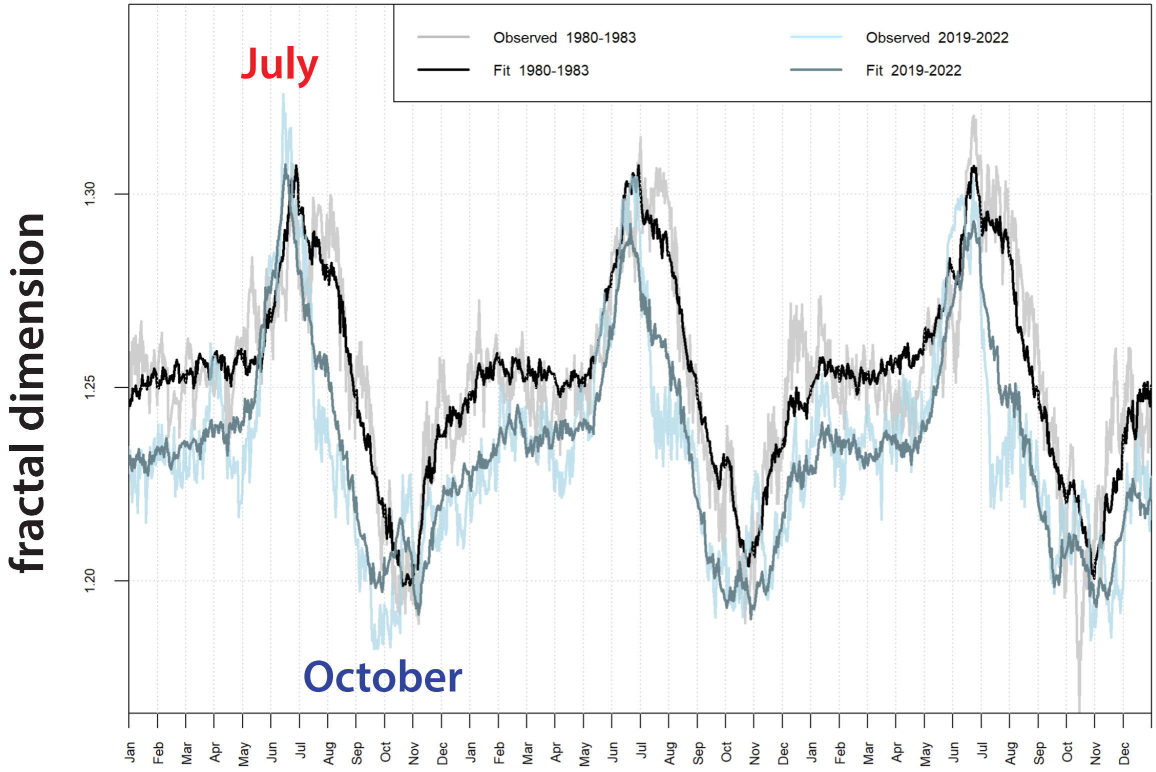


- Develop multiscale PDE model for simulating phase transition fronts to predict MIZ seasonal cycles and decadal trends
- Model simulates MIZ as a large-scale mushy layer with effective thermal conductivity derived from physics of composite materials

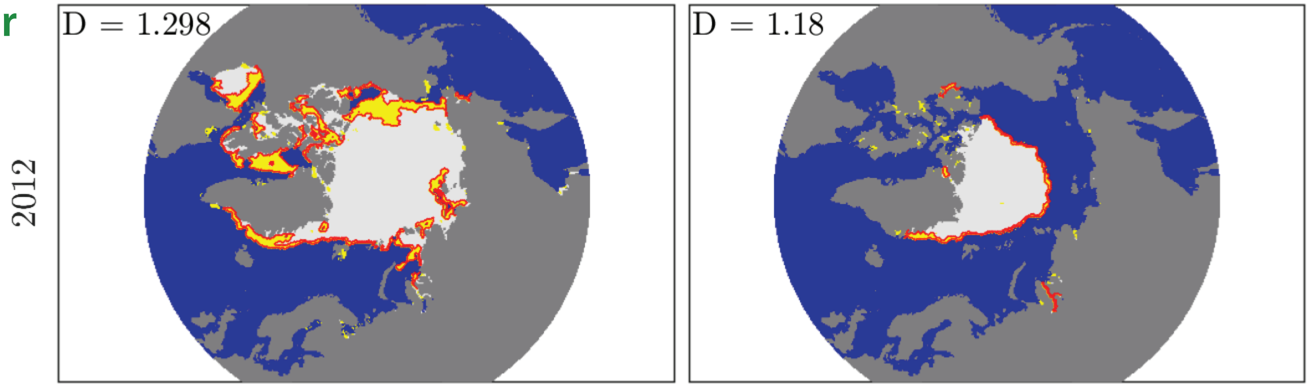
Evolution of the Fractal Geometry of the Arctic Marginal Ice Zone

Julie Sherman, Court Strong, Ken Golden 2024

Compute the fractal dimension of the boundary of the Arctic MIZ by boxcounting methods; analyze seasonal cycle and long term trends.



early summer



early autumn

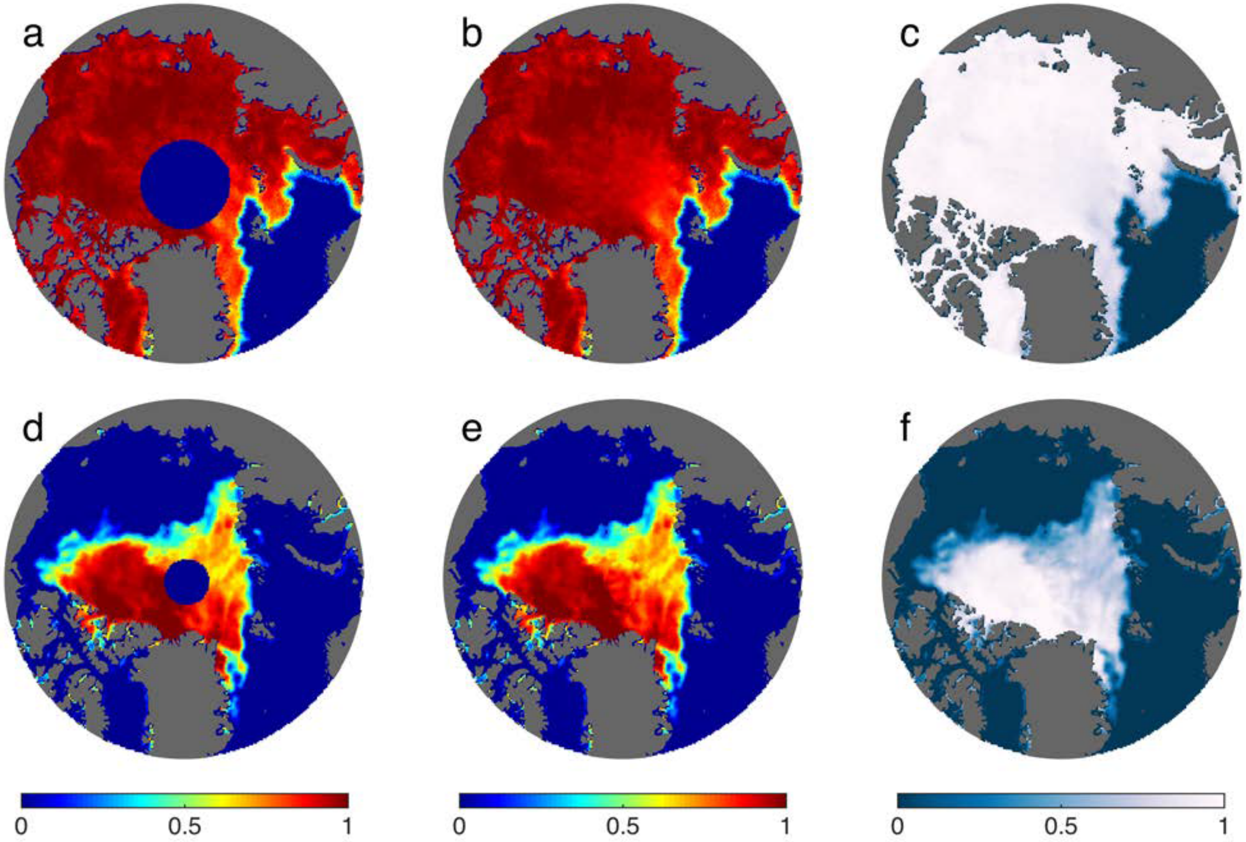
Filling the polar data gap with partial differential equations

hole in satellite coverage of sea ice concentration field

previously assumed ice covered

Gap radius: 611 km
06 January 1985

Gap radius: 311 km
30 August 2007



$$\Delta\psi=0$$

fill = harmonic function satisfying satellite BC's plus learned stochastic term

Strong and Golden, *Remote Sensing* 2016
Strong and Golden, *SIAM News* 2017

Global Sea Ice Concentration Climate Data Records, 2022
Lavergne, Sorensen, et al., Norwegian Met. Inst., ... OSI SAF

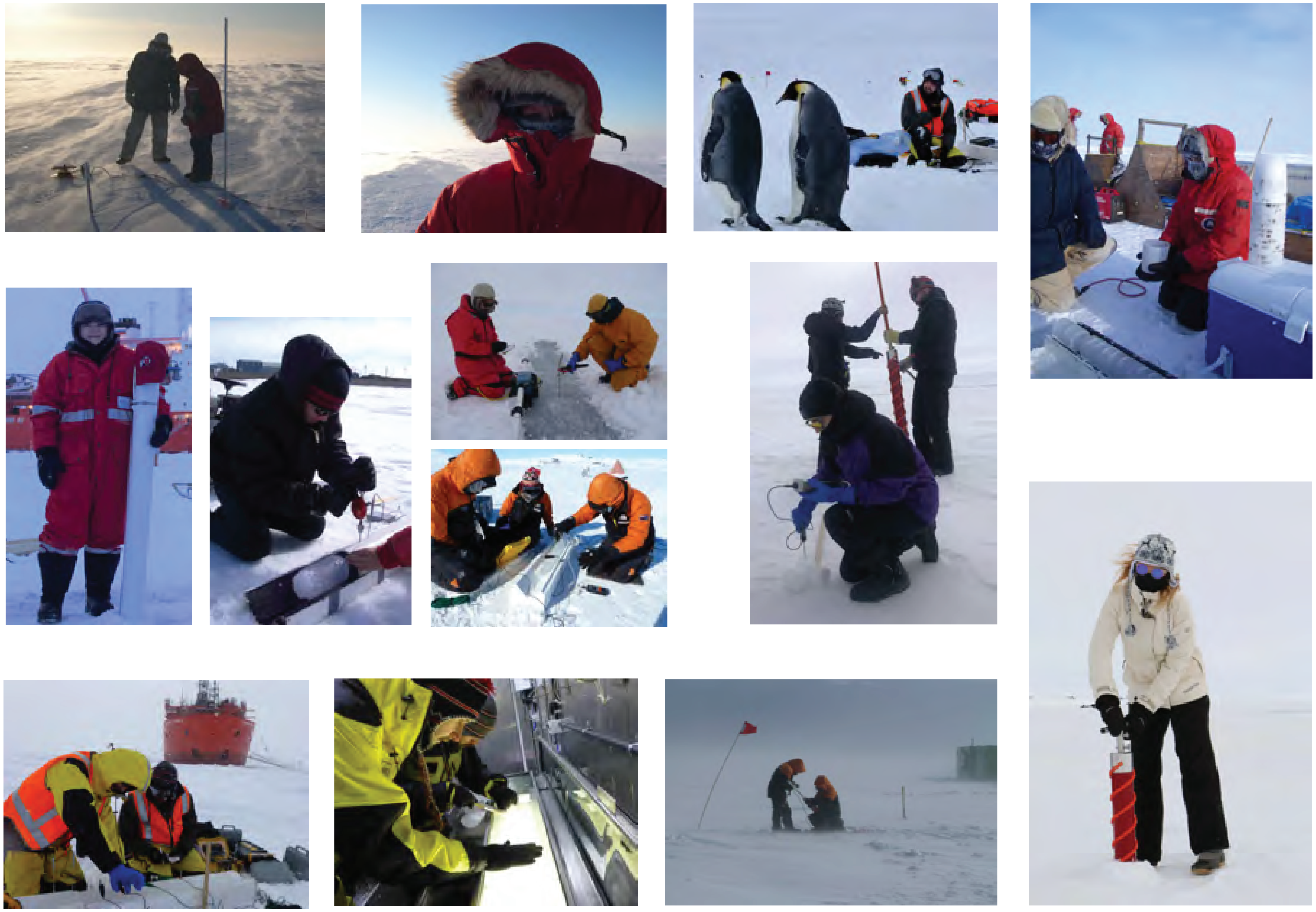
Conclusions

Our research is helping to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

Mathematics for sea ice advances the theory of composites, inverse problems, and other areas of science and engineering.

Modeling sea ice leads to unexpected areas of math and physics.

Thank you to so many postdocs, graduate students, undergraduates, high school students and colleagues who contributed to this work!



U. of Utah students in the Arctic and Antarctic (2003-2022): closing the gap between theory and observation - making math models come alive and experiencing climate change firsthand.

ISSN 0002-9920 (print)
ISSN 1088-9477 (online)

Notices

of the American Mathematical Society

November 2020

Volume 67, Number 10

AMERICAN
MATHEMATICAL
SOCIETY
Advancing research. Creating connections.

The cover is based on "Modeling Sea Ice," page 1535.

NSF Research Training Grant (RTG) with 15 Applied Math faculty:

optimization and inverse problems

July 2022 - June 2027

Overall goal: Build an advanced, competitive U.S. STEM workforce.

- Strengthen our graduate and postdoctoral programs in applied math to attract top students in the nation, and place them in top jobs.
- Provide transformative experiences that draw students into math.

Arctic Mathpeditions - May 2024 & 2026

OPEN POSITIONS:
Postdoctoral, Ph.D., Undergraduate

THANK YOU

Office of Naval Research

Applied and Computational Analysis Program

Arctic and Global Prediction Program

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

UNITED STATES COAST GUARD
1790

DEPARTMENT OF THE NAVY
ONR
Science & Technology

MCRN
Mathematics and Climate Research Network

Australian Government
Department of the Environment and Water Resources
Australian Antarctic Division

ace CRC
ANTARCTIC CLIMATE & ECOSYSTEMS
COOPERATIVE RESEARCH CENTRE

NSF

Antarctica New Zealand

UAF
UNIVERSITY OF ALASKA FAIRBANKS

THE UNIVERSITY OF UTAH

Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999

2—THE MERCURY Thursday, July 23, 1998

Fire endangers Hobart's ice ship

By DAVID CARRIGG

AN engine-room fire has left the Hobart-based Antarctic research ship Aurora Australis without power in dangerous sea ice off the Antarctic coast.

None of the 79 people on board was injured in the blaze, which broke out early yesterday morning while the ship was in deep water 185km off the coast.

The extent of the damage is not known.

Australian Antarctic Division director Rex Moncur said the fire was extinguished by flooding the engine room with an inert gas.

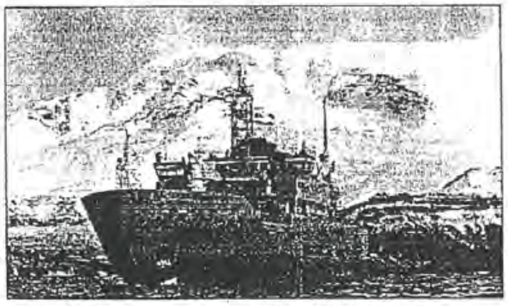
The gas had to be cleared before crew wearing breathing apparatus could enter and assess the situation.

He said it could be some time before the extent of damage was known.

The 25 crew and 54 expeditioners, mostly from Hobart, would wear thermal clothing and stay below decks to keep warm.

"There is always a risk of becoming ice-bound in these waters at this time of the year but at this stage we don't expect to launch a rescue mission from Hobart," Mr Moncur said.

The ship was in regular radio contact with the Antarctic Division's Hobart office.



A file photo of the Aurora Australis in Antarctica.

He expected the expeditioners and crew to abandon the pioneering winter voyage and return the ship to Hobart for repairs in about a week.

The Antarctic Division, which hires the ship from P&O Australia, would not be hiring another vessel for the expedition.

"It's a pretty specialist vessel so you couldn't get the sort of research capability that this ship has got readily available," Mr Moncur said.

"We hope the next voyage can still proceed on schedule, which is early September."

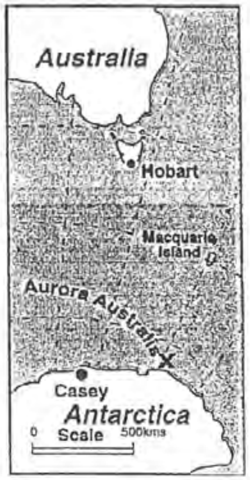
The Aurora Australis is owned by P&O Australia and charted by the Antarctic Division for about $11 million a year.

P&O Australia managing director Richard Hein said yesterday the company was assessing the situation and a number of rescue options were being considered.

It was too early to say whether P&O would be liable for the cost of the aborted mission.

The vessel left Hobart last Wednesday for a seven-week voyage mainly to study a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

The ship was nearing the polynya when the fire broke out.



Oceanographers believe a closer study of the phenomenon will lead to a better understanding of climate change.

CSIRO Marine Research oceanographer Steve Rintoul said the dense bottom water, created only in a few places in Antarctica and to a lesser extent in the North Atlantic, was critical to the chemistry and biology of the world's oceans.

THE ADVERTISER (Adelaide)
Thurs 23 July 1998 Page 14

Fire strands Antarctic ship in sea ice

AN engine room fire has disabled the icebreaker Aurora Australis in sea ice, deep in Antarctic waters.

There were no injuries and the ship was not in danger after Tuesday night's fire, Australian Antarctic Division director Mr Rex Moncur said. But Mr Moncur said he expected it would have to abandon its pioneering mid-winter voyage to the edge of the Antarctic continent and return to Hobart for repairs.

The cause of the fire was not known but the engines have been turned off, with the ship 100 nautical miles from the Antarctic coast.

THE CANBERRA TIMES
Thursday 23 July 1998 Page 4

Antarctic voyage stopped by fire

HOBART: An engine room fire has disabled the Australian icebreaker Aurora Australis in sea ice, deep in Antarctic waters.

Australian Antarctic Division director Rex Moncur said there were no injuries and the ship was not in danger after Tuesday night's fire.

But Mr Moncur said he expected Aurora Australis would have to abandon its pioneering mid-winter voyage to the edge of the Antarctic continent to return to Hobart for repairs.

The fire had been extinguished and the engines were turned off, leaving the ship in sea ice about 100 nautical miles from the Antarctic coast, he said. The weather was good.

Crew had to wear breathing apparatus to enter the engine room and it was likely to be 24 hours before the damage could be fully assessed.

The Aurora, with 54 expeditioners and 25 crew, left Hobart last Wednesday for a seven-week voyage which was to have focused on a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

Mr Moncur said, the cause of the fire was not yet known.

Sydney Morning Herald
23 July, 1998

ICEBREAKER BURNS

A pioneering $2 million Australian scientific voyage to the mid-winter Antarctic pack-ice is expected to be scrapped following an engine-room fire on the Aurora Australis yesterday.

The 54 people on board were forced on deck in the

2:45 am July 22, 1998

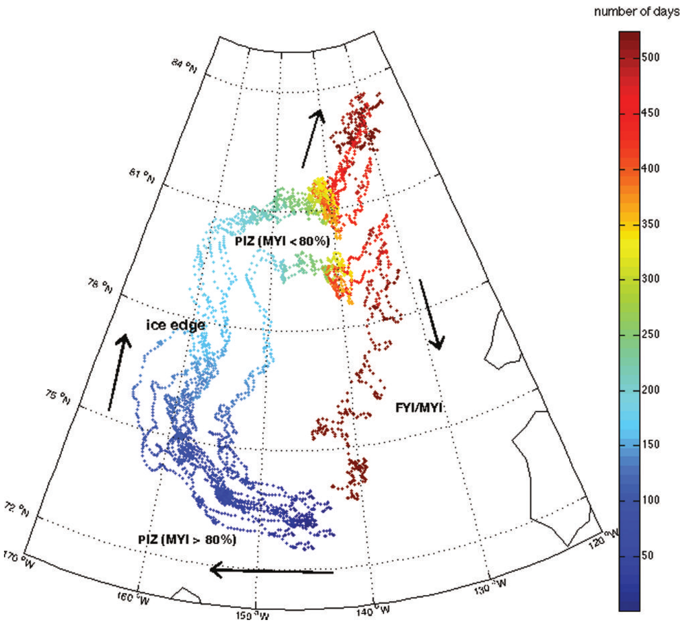
``Please don't be alarmed but we have an uncontrolled fire in the engine room ...."

about 10 minutes later ...

``Please don't be alarmed but we're lowering the lifeboats ...."

Anomalous diffusion in sea ice dynamics

Ice floe diffusion in winds and currents



observations from GPS data:

Jennifer Lukovich, Jennifer Hutchings,
David Barber, *Ann. Glac.* 2015

- On short time scales floes observed (buoy data) to exhibit Brownian-like behavior, but they are also being advected by winds and currents.
- Effective behavior is purely diffusive, sub-diffusive or super-diffusive depending on ice pack and advective conditions - Hurst exponent.

modeling:

Huy Dinh, Ben Murphy, Elena Cherkaev,
Court Strong, Ken Golden 2022

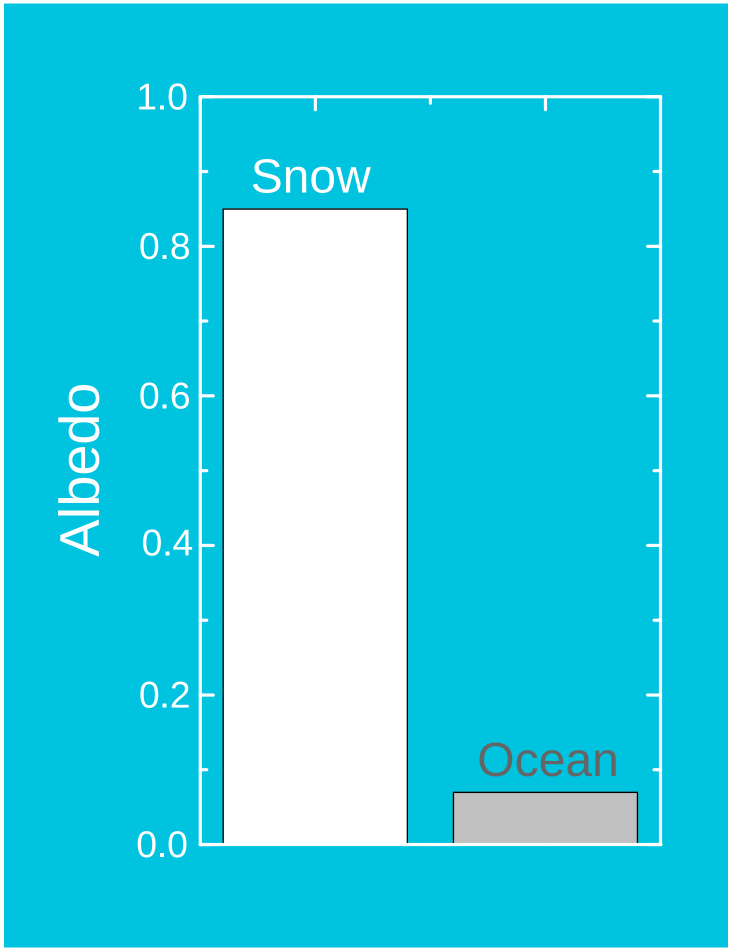
floe scale model to analyze transport regimes in terms of ice pack crowding, advective conditions

Delaney Mosier, Jennifer Hutchings, Jennifer Lukovich,
Marta D'Elia, George Karniadakis, Ken Golden 2022

learning fractional PDE governing diffusion from data

polar ice caps critical to global climate in reflecting incoming solar radiation

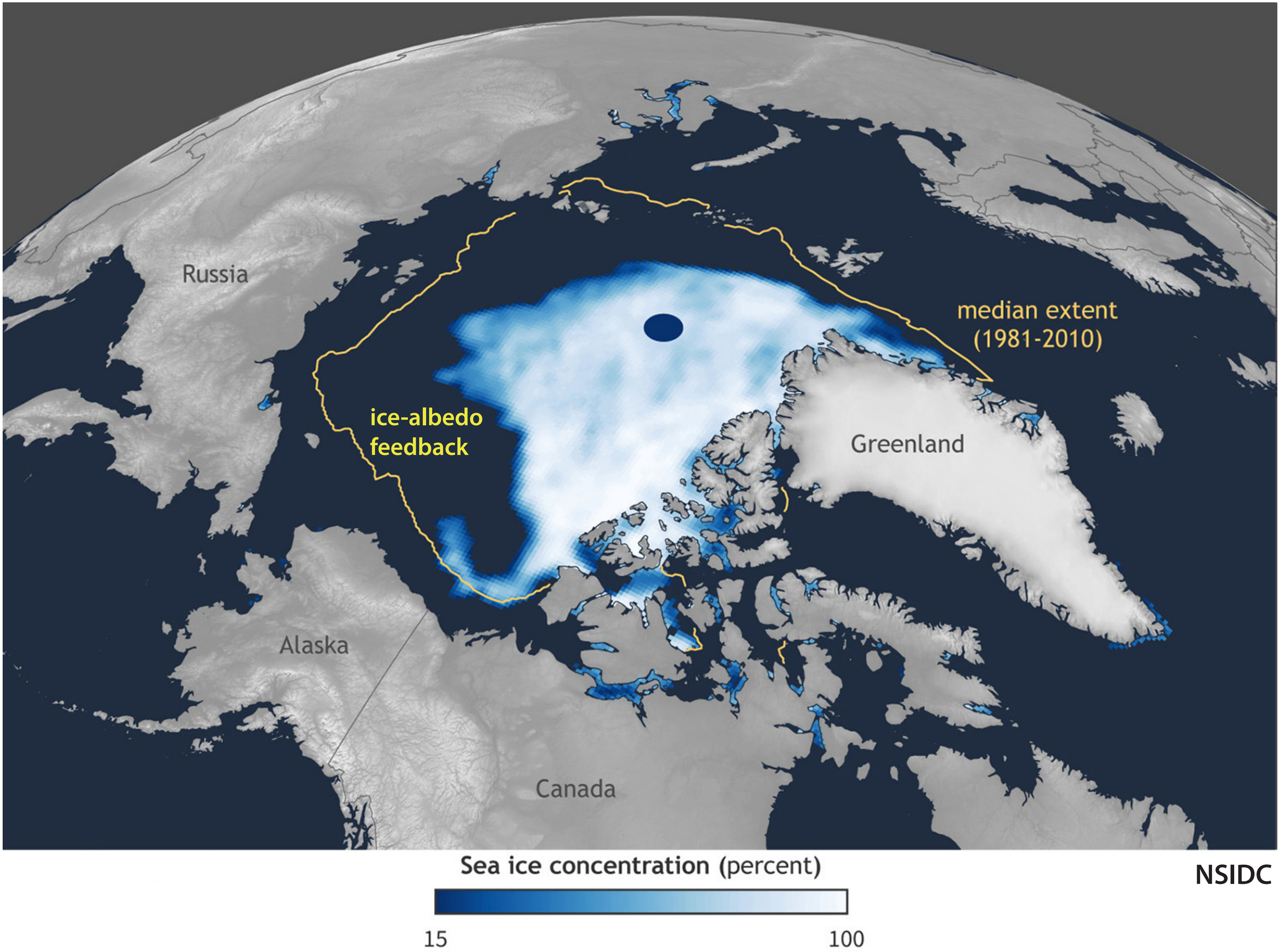
white snow and ice reflect



dark water and land absorb

$$\textit{\textbf{albedo}}\ \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

Arctic sea ice extent September 15, 2020



NSIDC

recent losses
in comparison to
the United States

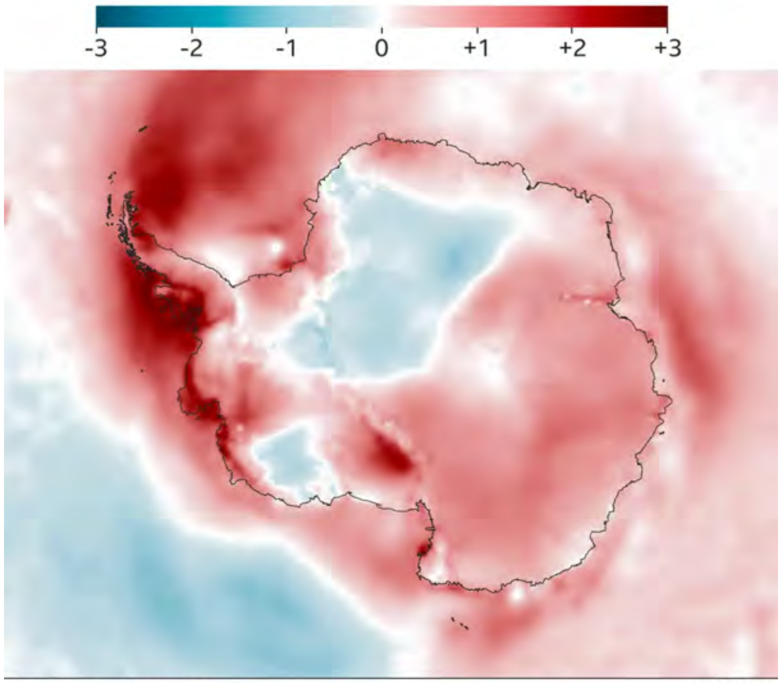
Perovich

New Record Low for Antarctic Sea Ice

February 13, 2023

Much of Antarctica warmer than average

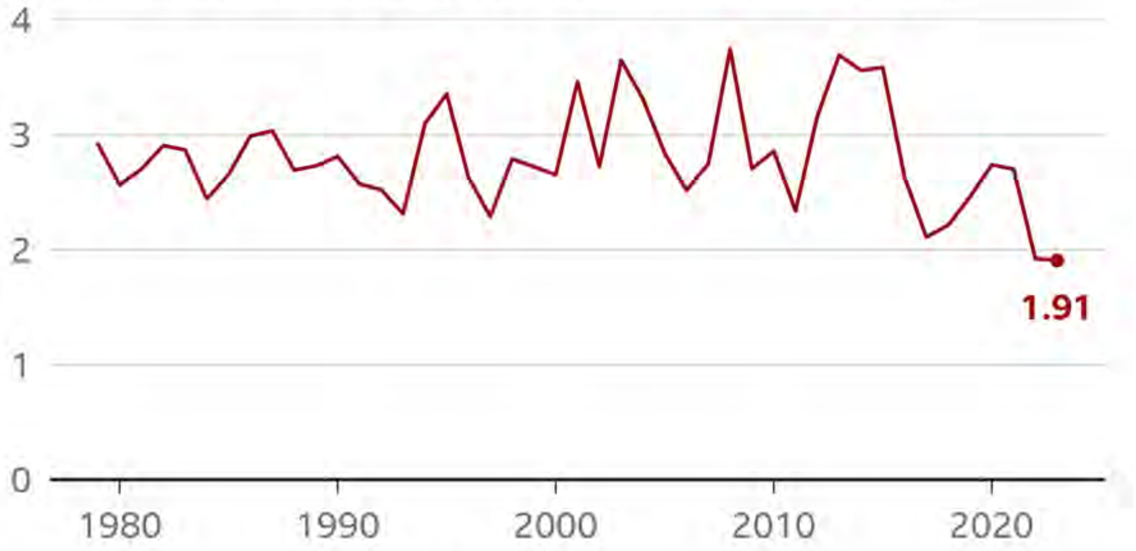
Mean 2022 surface air temp compared with 1991-2022 (°C)



Source: ECMWF ERA5

BBC

Minimum extent 1979-2023 (million sq km)

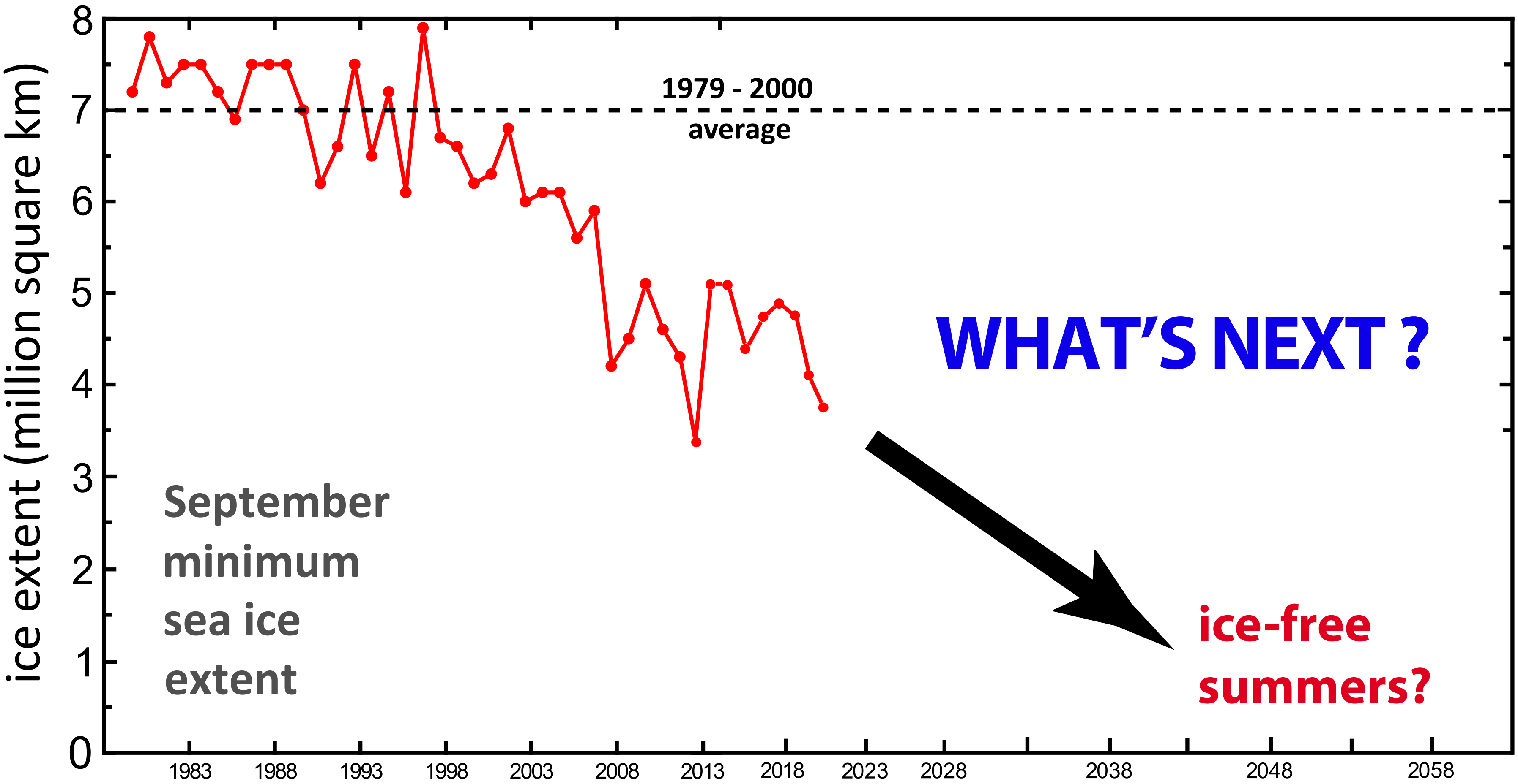


Five-day rolling average of sea-ice extent

Source: National Snow and Ice Data Center (NSIDC)

BBC

ARCTIC summer sea ice loss



September minimum sea ice extent

WHAT'S NEXT ?

ice-free summers?

National Snow and Ice Data Center (NSIDC)

predictions require lots of math modeling

sea ice algal communities

D. Thomas 2004

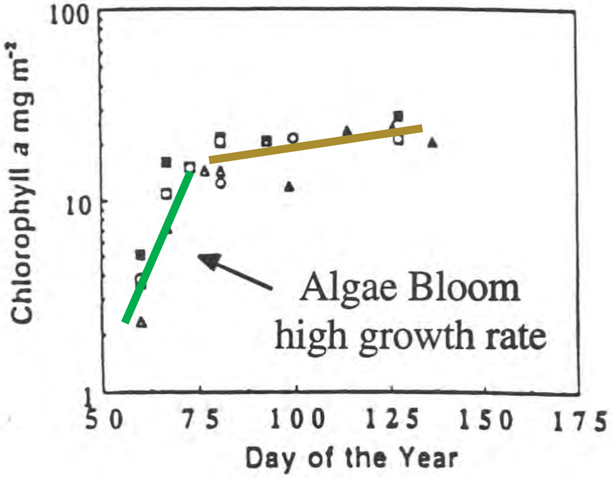
nutrient replenishment controlled by ice permeability

biological activity turns on or off according to *rule of fives*

Golden, Ackley, Lytle ***Science 1998***

Fritsen, Lytle, Ackley, Sullivan ***Science 1994***

critical behavior of microbial activity

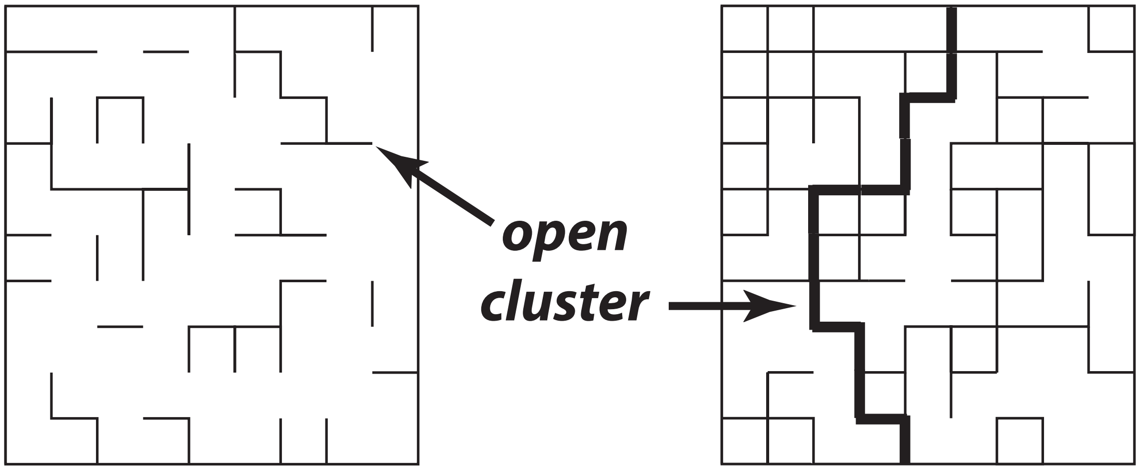


Convection-fueled algae bloom
Ice Station Weddell

percolation theory

probabilistic theory of connectedness

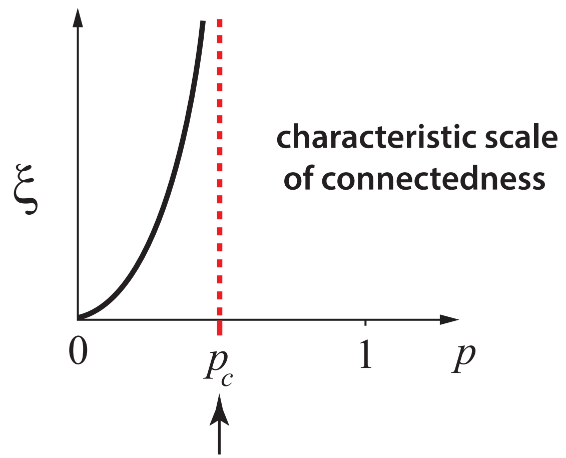
impermeable **permeable**



$p = 1/3$ $p = 2/3$

correlation length

development of long range order



percolation threshold

bond → ***open*** *with probability* p
closed *with probability* $1-p$

$$\xi(p) \sim |p - p_c|^{-\nu} \qquad p \to p_c$$

ν **universal: depends only on *d***

p_c depends on type of lattice and *d*

percolation threshold

$$p_c = 1/2 \quad \text{for} \;\; \mathrm{d} = 2$$

smallest *p* for which there is an infinite open cluster

ISSN 0002-9920

Notices

of the American Mathematical Society

May 2009

Volume 56, Number 5

Climate Change and the Mathematics of Transport in Sea Ice

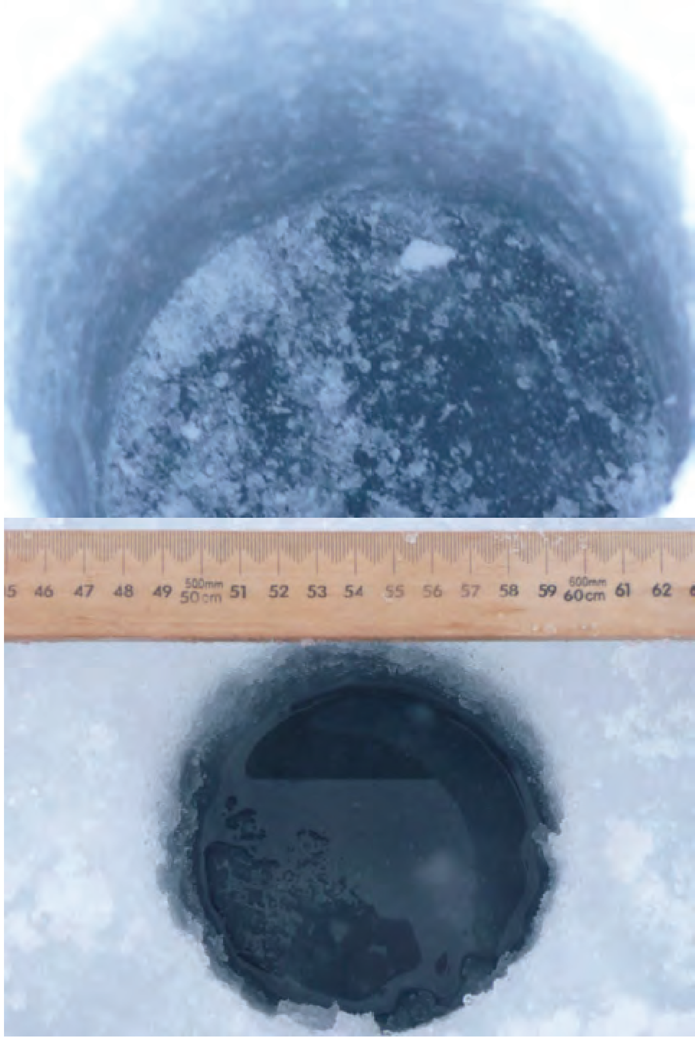
page 562

Mathematics and the Internet: A Source of Enormous Confusion and Great Potential

page 586

Real analysis in polar coordinates (see page 613)

photo by Jan Lieser



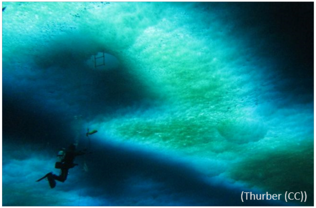
measuring fluid permeability of Antarctic sea ice

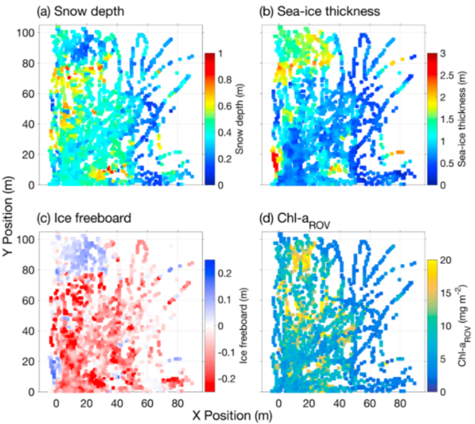
SIPEX 2007

Golden

HETEROGENEITY


(Reimer)


(Thurber (CC))



Meiners, K.M., et al. (2017). *Geophysical Research Letters*, 44(14), 7382-7390

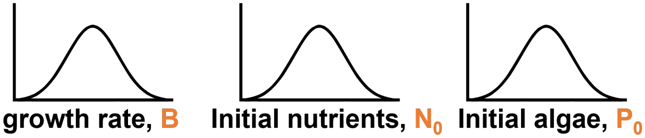
HETEROGENEITY IN INITIAL CONDITIONS

At each location within a larger region, we could consider

$$\frac{dN}{dt} = \alpha - BNP - \eta N$$
$$\frac{dP}{dt} = \gamma BNP - \delta P$$

treating parameters as random variables

$$N(0) = N_0, \qquad P(0) = P_0$$



But, Monte Carlo for Full Algae Model: 8 hours X 10,000

POLYNOMIAL CHAOS EXPANSIONS

$$N(t;B,P_0,N_0) \approx N_V(t;B,P_0,N_0) := \sum_{j=1}^{n} \widetilde{N}_j(t)\phi_j(B,P_0,N_0),$$

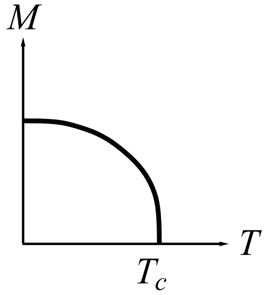
$$P(t;B,P_0,N_0) \approx P_V(t;B,P_0,N_0) := \sum_{j=1}^{n} \widetilde{P}_j(t)\phi_j(B,P_0,N_0),$$

where

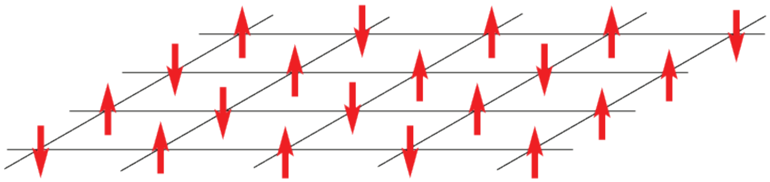
- $V := \text{span}\{\phi_j\}_{j=1}^{n}$
- ϕ_j are orthogonal polynomials that form a basis for V
- $(\widetilde{N}_j, \widetilde{P}_j)$ need to be computed

Xiu, D. (2010). Numerical methods for stochastic computations. Princeton university press.

Ising Model for a Ferromagnet



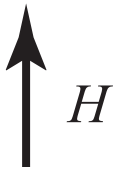
Curie point
critical temperature



$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

blue
white

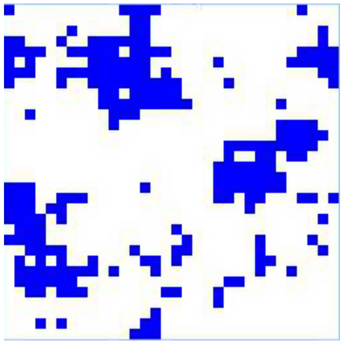
applied magnetic field



$$\mathcal{H} = -H\sum_{i} s_i - J\sum_{<i,j>} s_i s_j$$

nearest neighbor Ising Hamiltonian

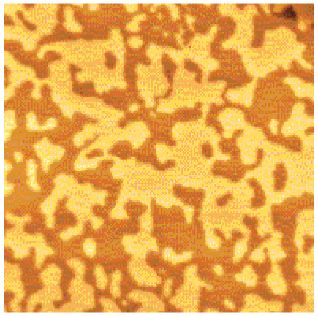
islands of like spins



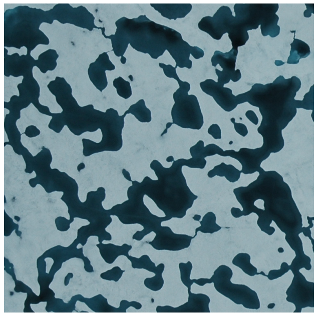
$$M(T,H) = \lim_{N\to\infty}\frac{1}{N}\left\langle \sum_j s_j \right\rangle$$

effective magnetization

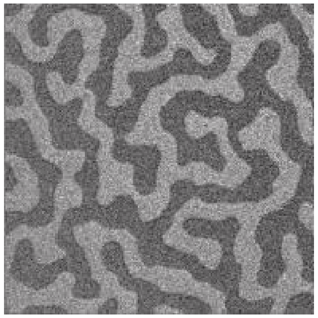
energy is lowered when nearby spins align with each other, forming **magnetic domains**



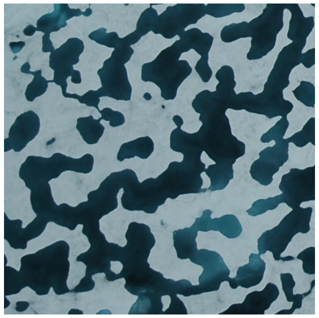
magnetic domains in cobalt



melt ponds (Perovich)



magnetic domains in cobalt-iron-boron



melt ponds (Perovich)

Ising model for ferromagnets → Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.*, 2019

$$\mathcal{H} = -\sum_{i}^{N} H_i \, s_i - J \sum_{<i,j>}^{N} s_i s_j \qquad s_i = \begin{cases} \uparrow \quad +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow \quad -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

random magnetic field represents snow topography

magnetization M

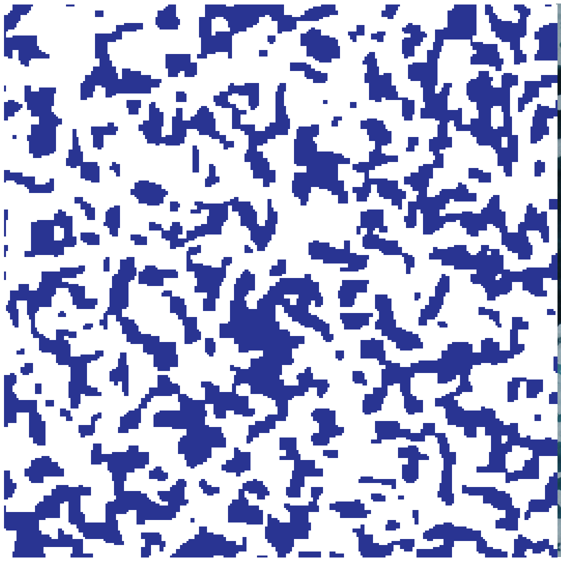
pond area fraction $F = \frac{(M+1)}{2}$

~ albedo

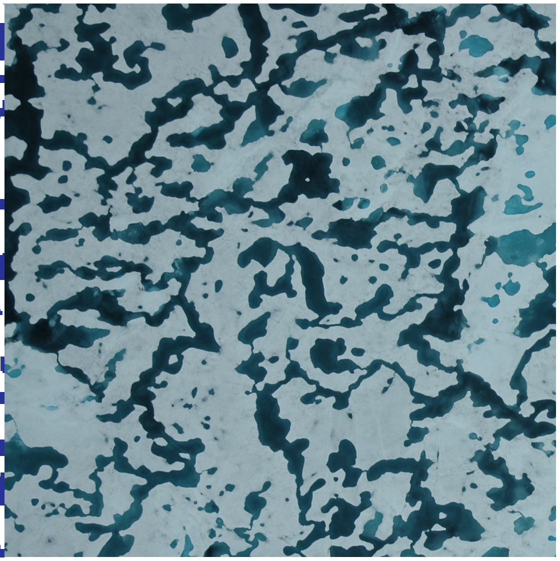
only nearest neighbor patches interact

Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system "flows" toward metastable equilibria.

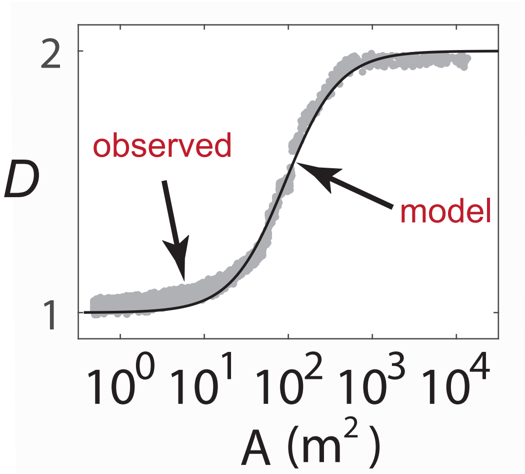
Order from Disorder



Ising model



melt pond photo **(Perovich)**



pond size distribution exponent

observed	-1.5
	(Perovich, *et al.* 2002)
model	-1.58

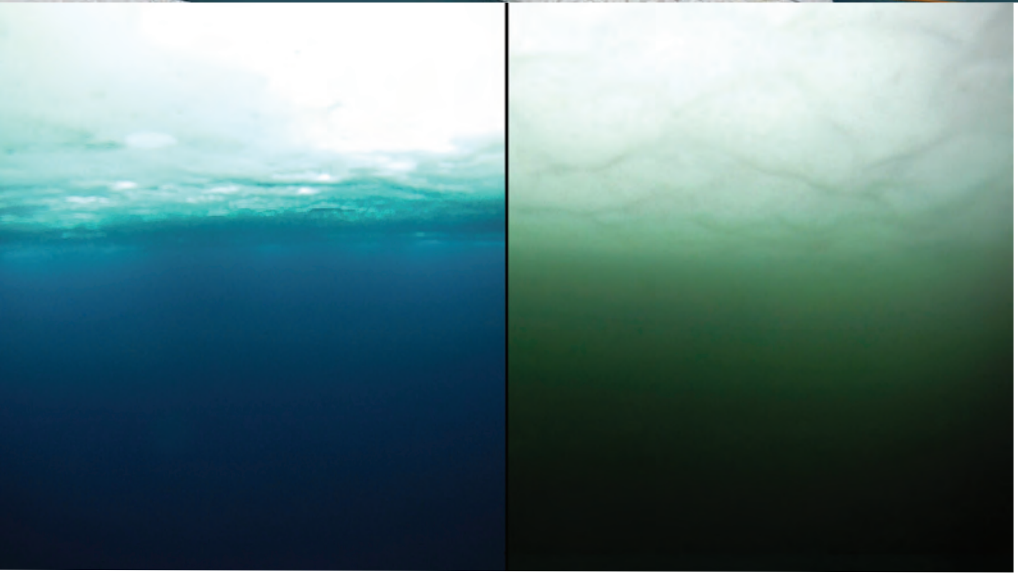
Scientific American
EOS, PhysicsWorld, ...

ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data

Melt ponds control transmittance of solar energy through sea ice, impacting upper ocean ecology.

WINDOWS

Perovich



no bloom

bloom

massive under-ice algal bloom

Arrigo et al., *Science* 2012

Have we crossed into a new ecological regime?

The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

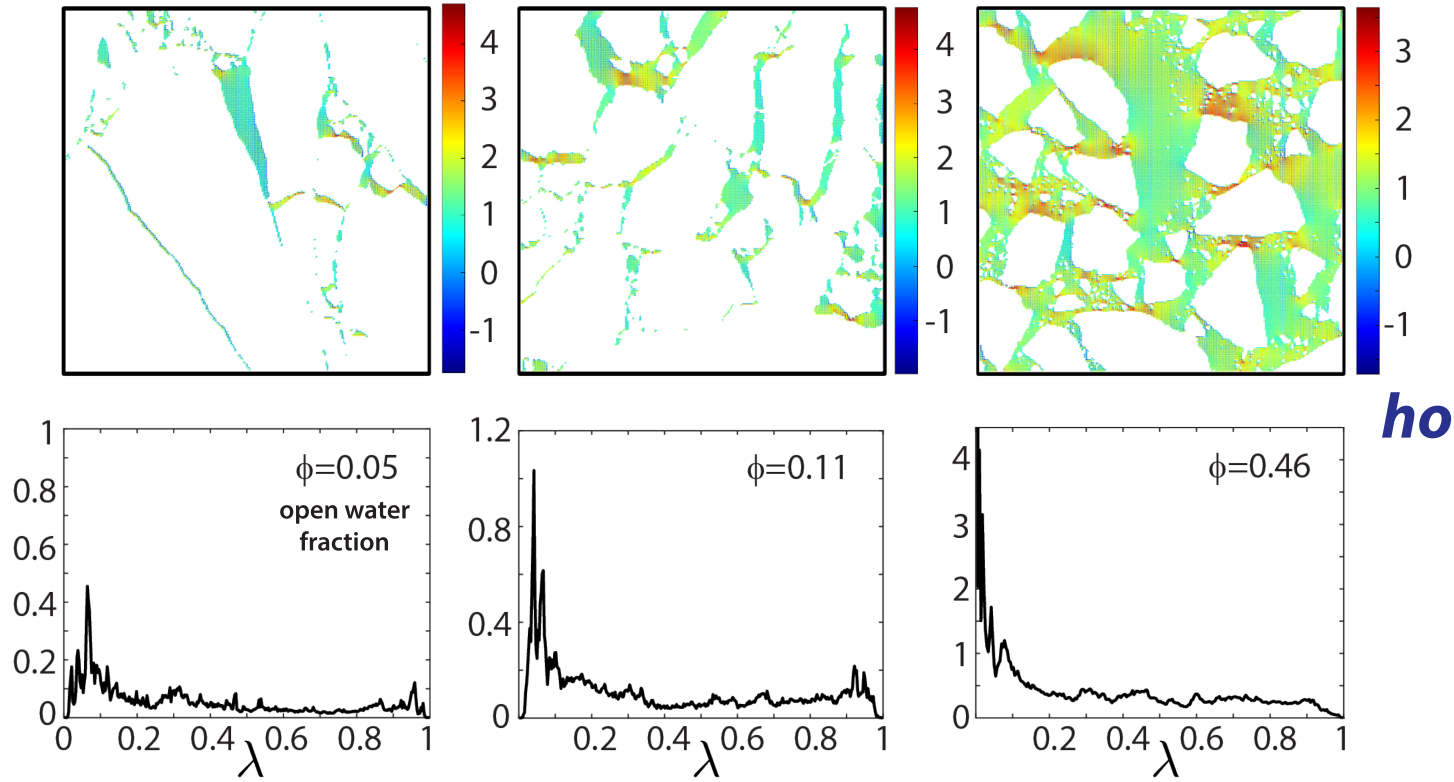
Horvat, Rees Jones, Iams, Schroeder,
Flocco, Feltham, ***Science Advances*** 2017

The effect of melt pond geometry on the distribution of solar energy under first year sea ice

Horvat, Flocco, Rees Jones, Roach, Golden
Geophys. Res. Lett. 2019

(2015 AMS MRC)

thermal flow field through the ice cover: multiscale granular composite



homogenize

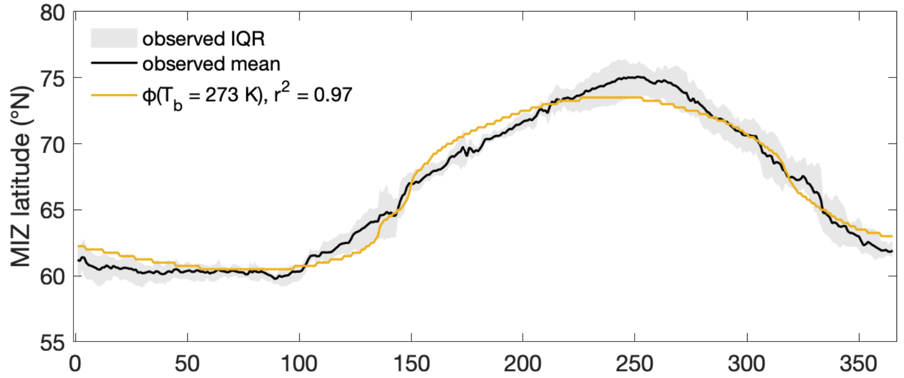
spectral measures for 2D horizontal thermal conductivity

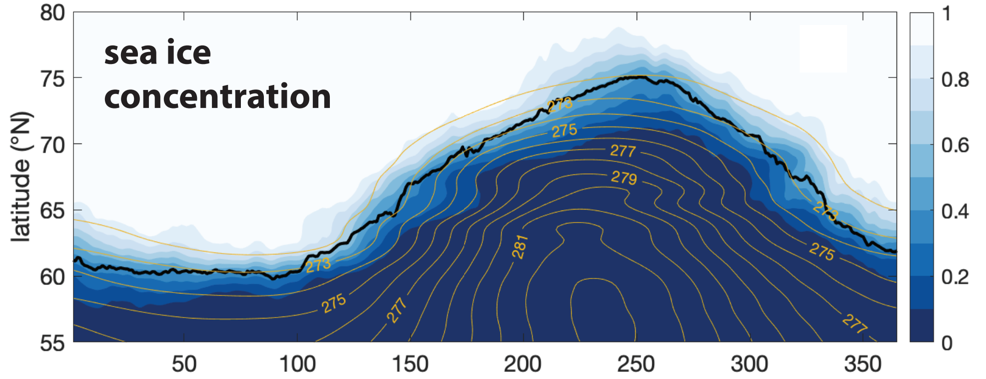
homogenized thermal conductivity is a key parameter in MIZ mushy layer model

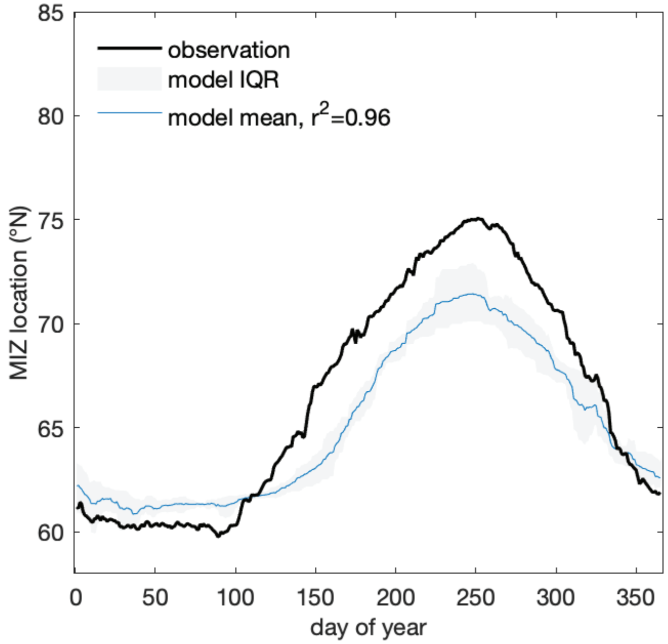
MIZ observations

MIZ model vs. observations

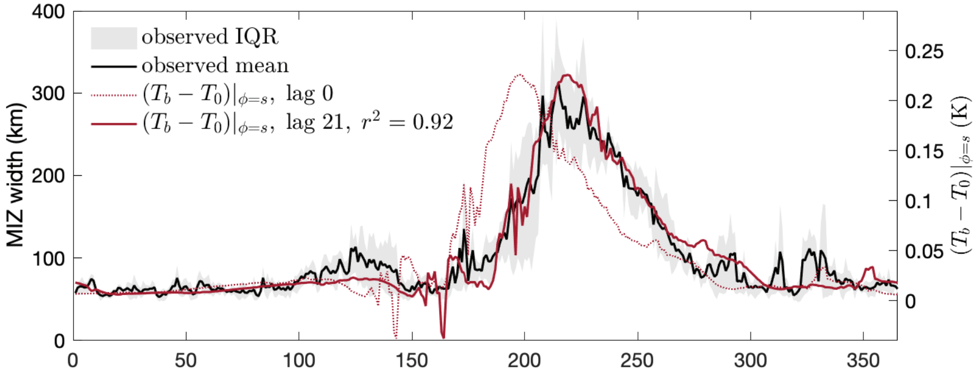
location

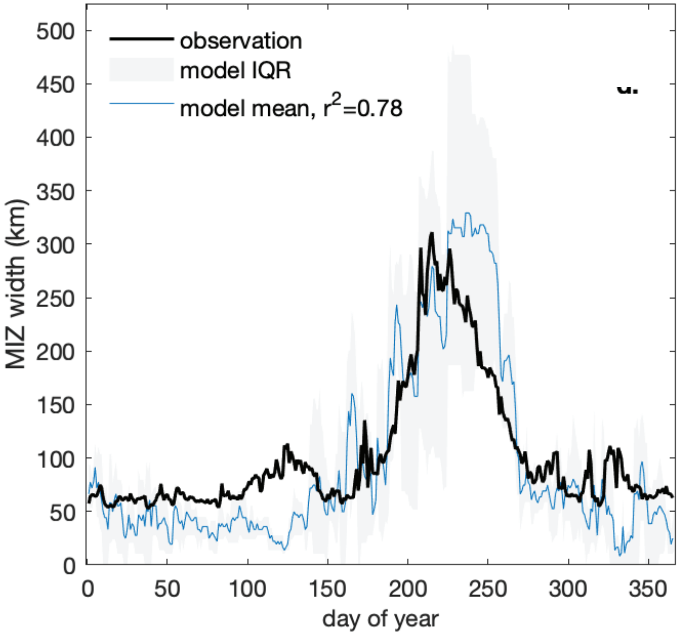






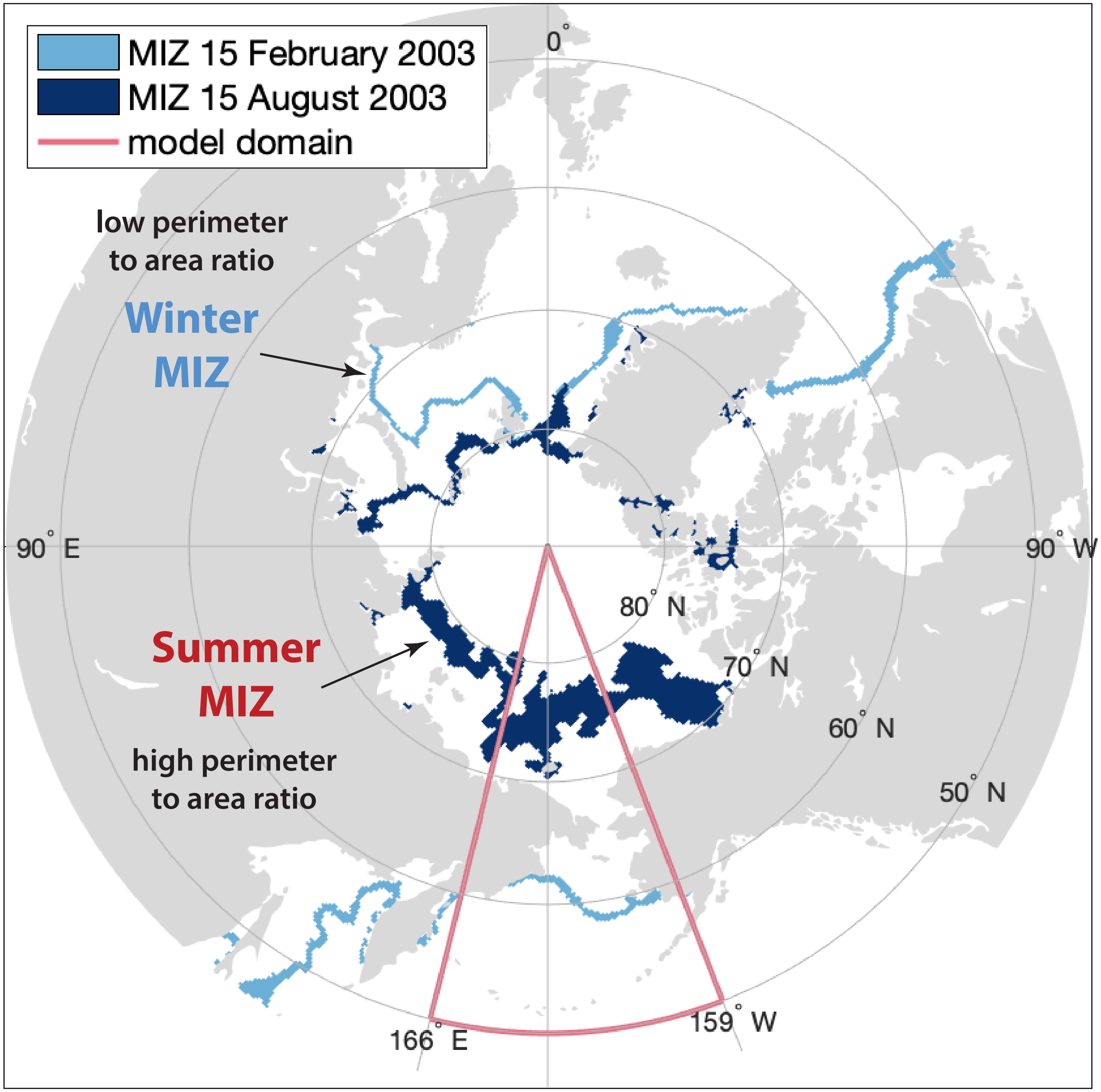
width





Model captures basic physics of MIZ dynamics.

Observed Arctic MIZ



low fractal dimension

low perimeter to area ratio

Winter MIZ

Summer MIZ

high perimeter to area ratio

high fractal dimension

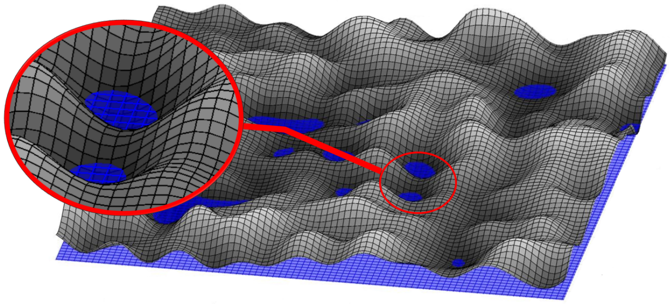
Topology of the sea ice surface and the fractal geometry of Arctic melt ponds

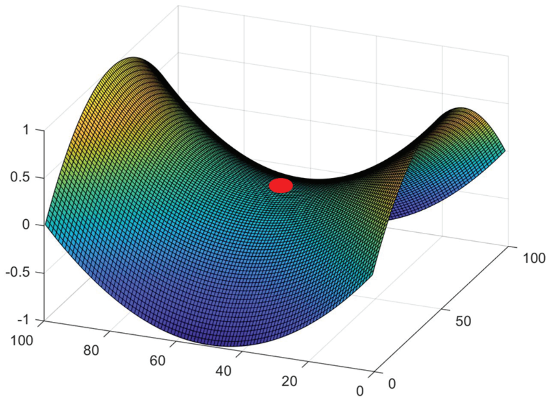
Physical Review Research (invited, under revision)

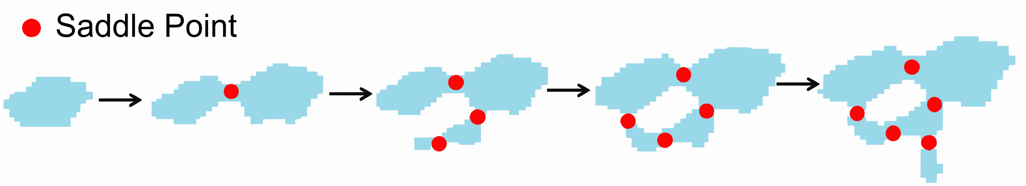
Ryleigh Moore, Jacob Jones, Dane Gollero,
Rebecca Hardenbrook, Court Strong, Ken Golden

Several models replicate the transition in fractal dimension, but none explain how it arises.

We use Morse theory applied to the random surface model to show that saddle points play the critical role in the fractal transition.







saddles drive the fractal transition

ponds coalesce
(change topology) and
complexify at saddle points

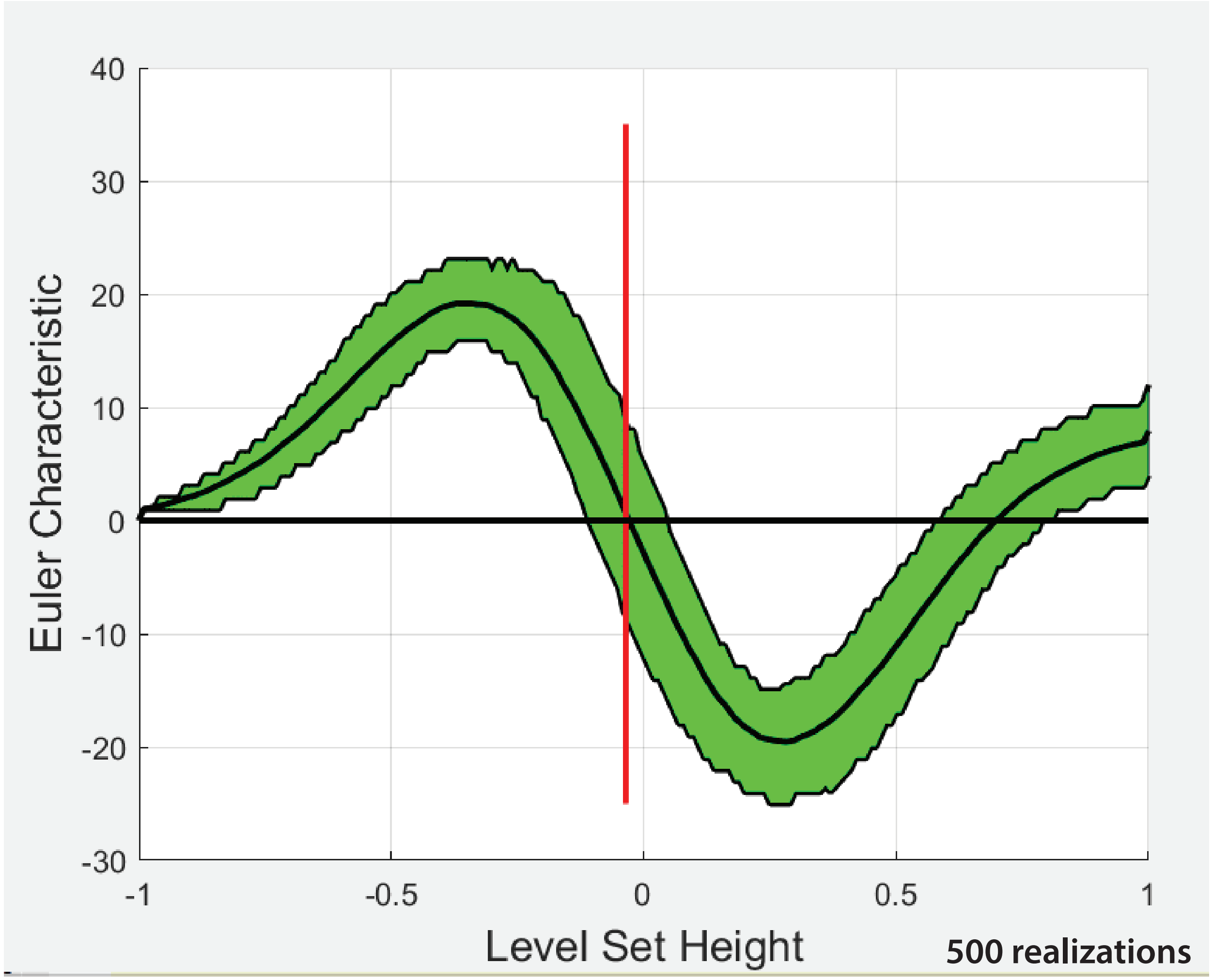
Euler characteristic = # maxima + # minima - # saddles

Topological Data Analysis

topological invariant

persistent homology

filtration - sequence of nested topological spaces, indexed by water level



Expected
Euler Characteristic Curve (ECC)

tracks the evolution of the EC of the flooded surface as water rises

zero of ECC ~ percolation

percolation on a torus
creates a giant cycle

Bobrowski & Skraba, 2020

Carlsson, 2009

Vogel, 2002 GRF

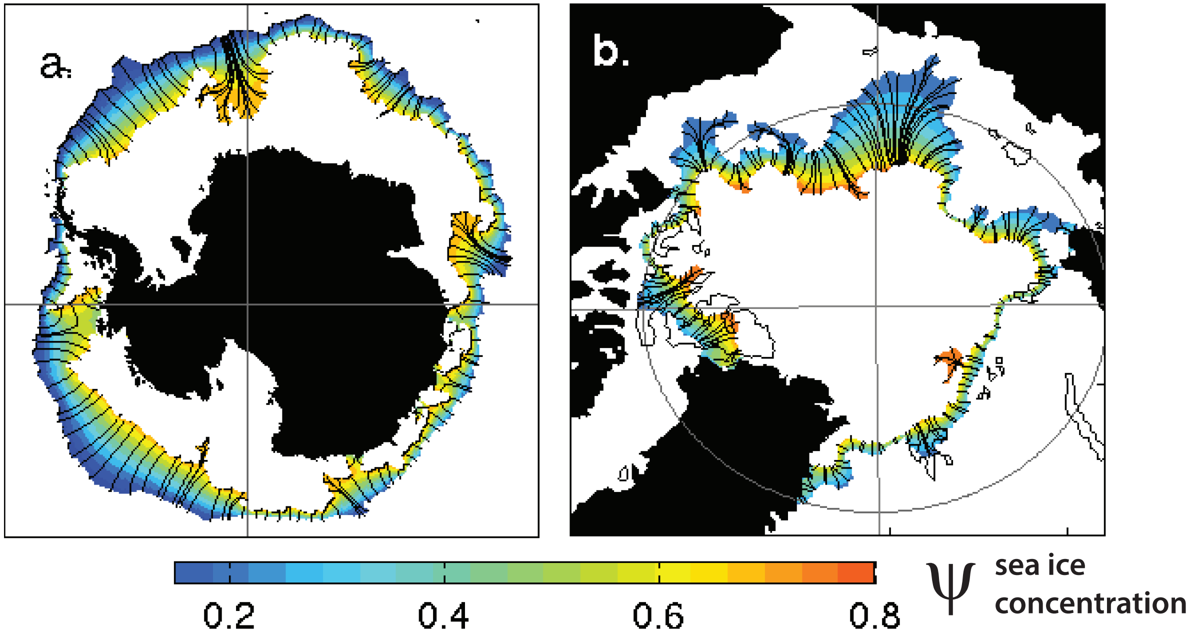
image analysis
porous media
cosmology
brain activity

Physical Review Research (invited, in revision), R. Moore, J. Jones, D. Gollero, R. Hardenbrook, C. Strong, K. M. Golden 2024

Marginal Ice Zone

MIZ

- biologically active region
- intense ocean-sea ice-atmosphere interactions
- region of significant wave-ice interactions
- dramatic seasonal cycle, 40% widening



transitional region between
dense pack ice ($c > 80\%$)
open ocean ($c < 15\%$)

How to objectively measure the width of this complex region?

$$\boxed{\nabla^2\psi = 0}$$

rat brain

↓

$$\boxed{\nabla \cdot (\sigma \, \nabla\psi) = 0}$$

MIZ WIDTH

fundamental length scale of
ecological and climate dynamics

Strong, *Climate Dynamics* 2012

Strong and Rigor, *GRL* 2013

Strong, Foster, Cherkaev, Eisenman, Golden
***J. Atmos. Oceanic Tech.* 2017**