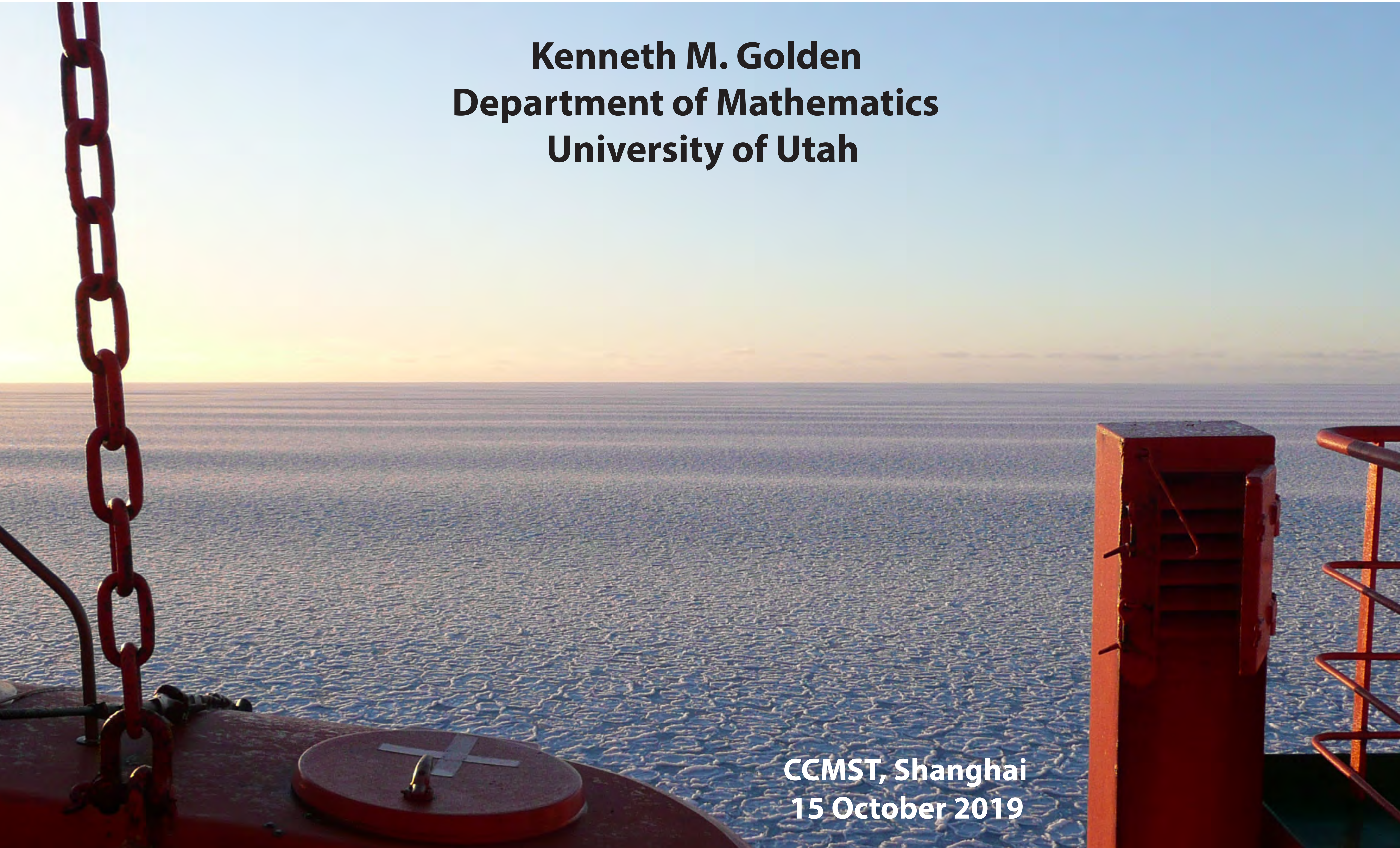
What can polar sea ice tell us about modeling the properties of composite materials?

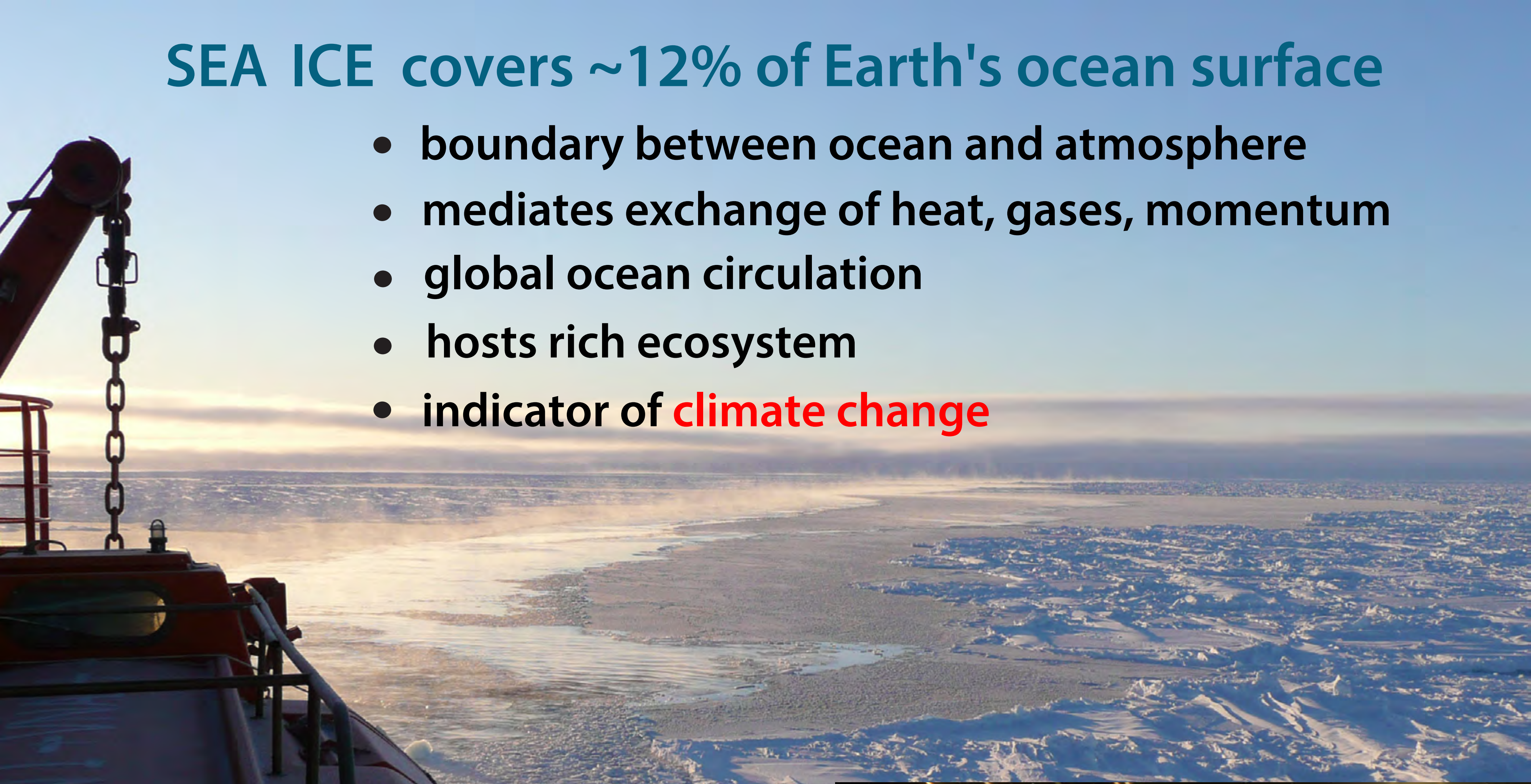


Kenneth M. Golden
Department of Mathematics
University of Utah

CCMST, Shanghai
15 October 2019

SEA ICE covers ~12% of Earth's ocean surface

- boundary between ocean and atmosphere
- mediates exchange of heat, gases, momentum
- global ocean circulation
- hosts rich ecosystem
- indicator of climate change



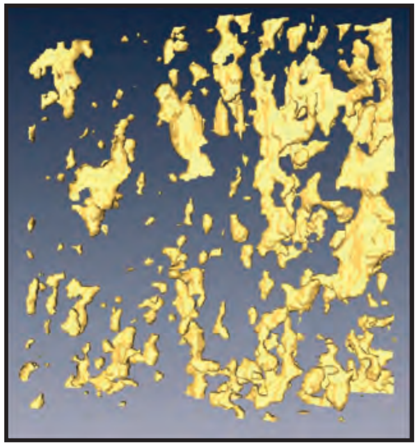
polar ice caps critical
to climate in reflecting
sunlight during summer

Sea Ice is a Multiscale Composite Material

sea ice microstructure

brine inclusions

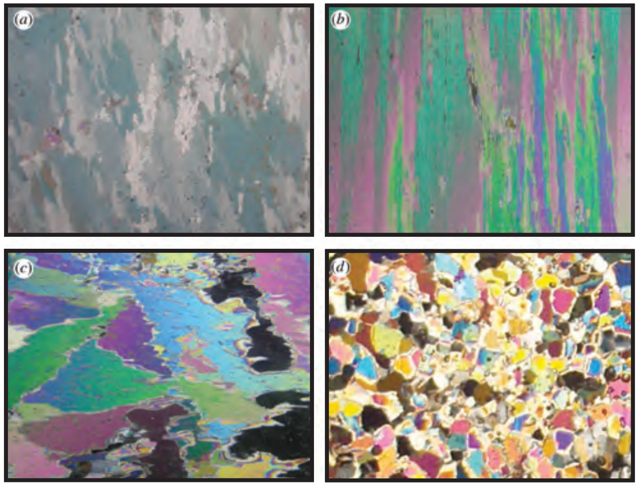
Weeks & Assur 1969



H. Eicken

Golden *et al. GRL* 2007

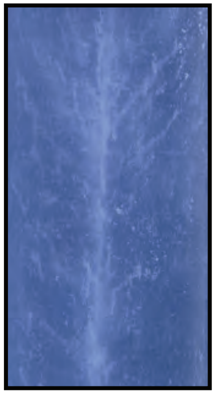
polycrystals



Gully *et al. Proc. Roy. Soc. A* 2015

brine channels

D. Cole



K. Golden

millimeters

centimeters

---

sea ice mesostructure

Arctic melt ponds

K. Frey

Antarctic pressure ridges

K. Golden

sea ice macrostructure

sea ice floes

J. Weller

sea ice pack

NASA

meters

kilometers

What is this talk about? HOMOGENIZATION

What is the role of microstructure in determining effective properties?

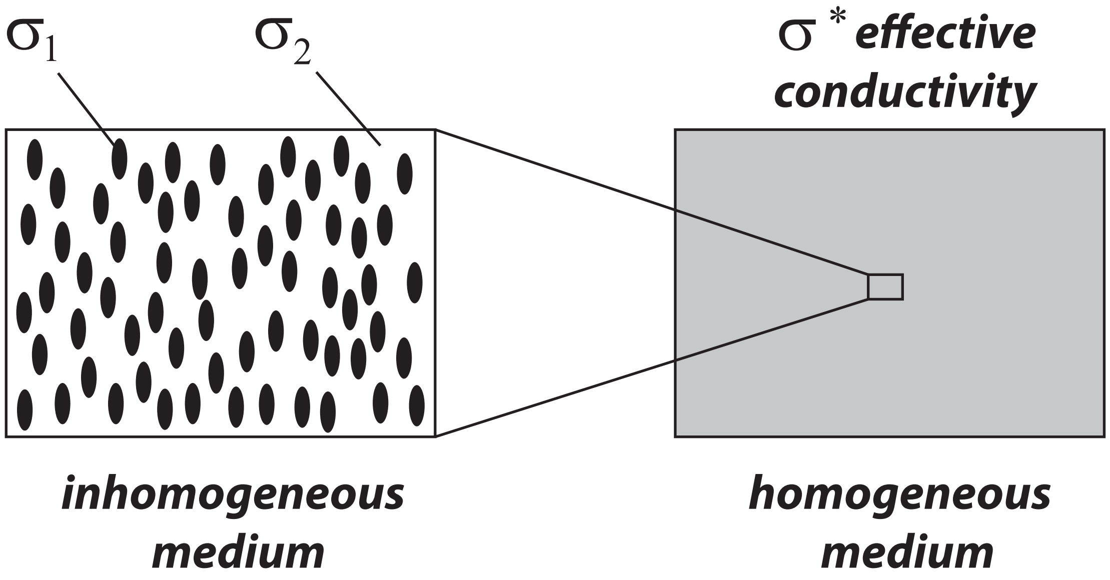
Using methods of statistical physics and homogenization to **LINK SCALES** ***in the sea ice system ... rigorously compute effective behavior and improve climate models.***

1. ***Sea ice microphysics and fluid transport***
2. ***Analytic Continuation Method, integral representations***
3. ***Extension of ACM to advection diffusion, waves in sea ice***
4. ***Fractal geometry of melt pond evolution***

Solving problems in physics of sea ice drives advances in theory of composite materials.

cross - pollination

HOMOGENIZATION - Linking Scales in Composites



find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

Wiener 1912 : arithmetic and harmonic mean bounds on effective conductivity
Hashin and Shtrikman 1962 : variational bounds on effective conductivity

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

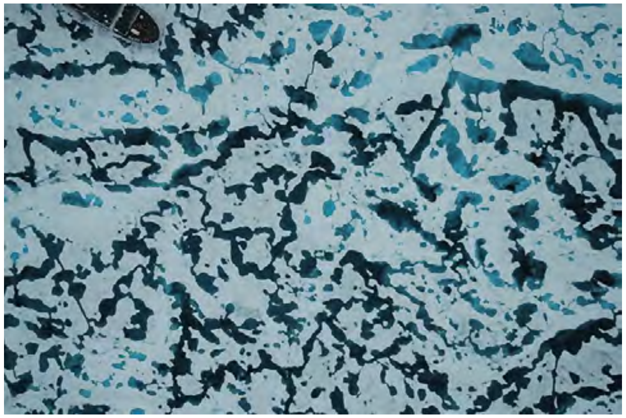
How do scales interact in the sea ice system?

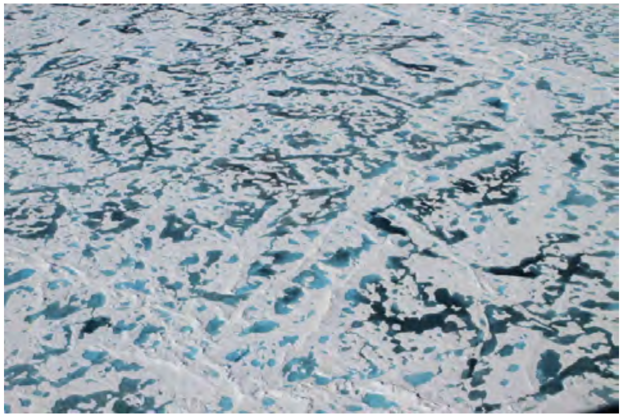

NASA

basin scale -
grid scale
albedo

Linking Scales

km
scale
melt
ponds




Perovich

km
scale
melt
ponds

Linking Scales

mm
scale
brine
inclusions

meter
scale
snow
topography

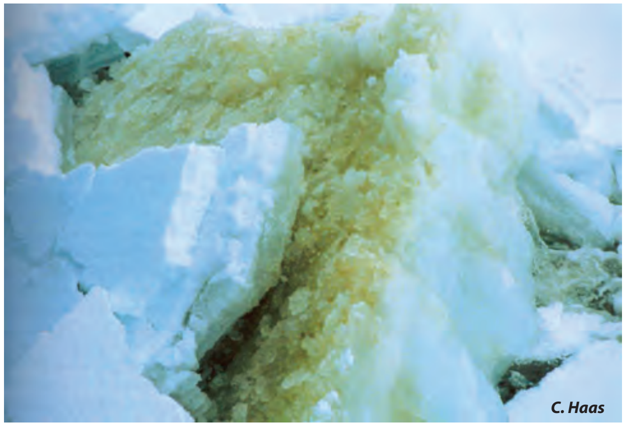
sea ice microphysics

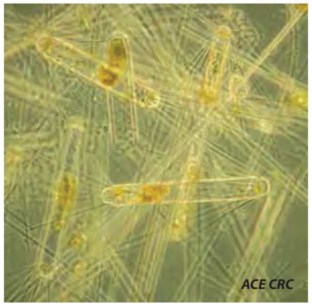
fluid transport

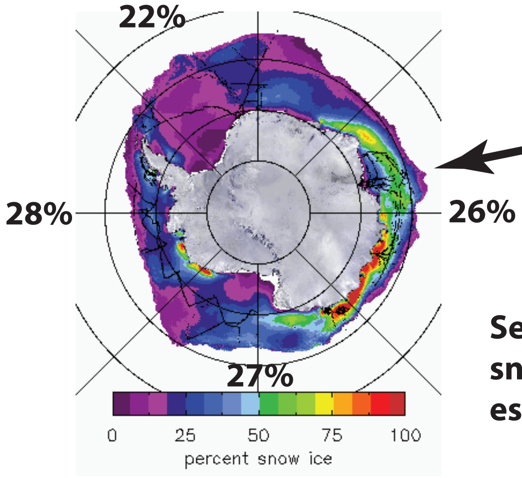
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

nutrient flux for algal communities







T. Maksym and T. Markus, 2008

September snow-ice estimates

Antarctic surface flooding and snow-ice formation

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO_2*

fluid permeability of a porous medium

how much water gets through the sample per unit time?

Darcy's Law

for slow viscous flow in a porous medium

averaged
fluid velocity

pressure
gradient

$$v = -\frac{\mathbf{k}}{\eta}\nabla p$$

viscosity

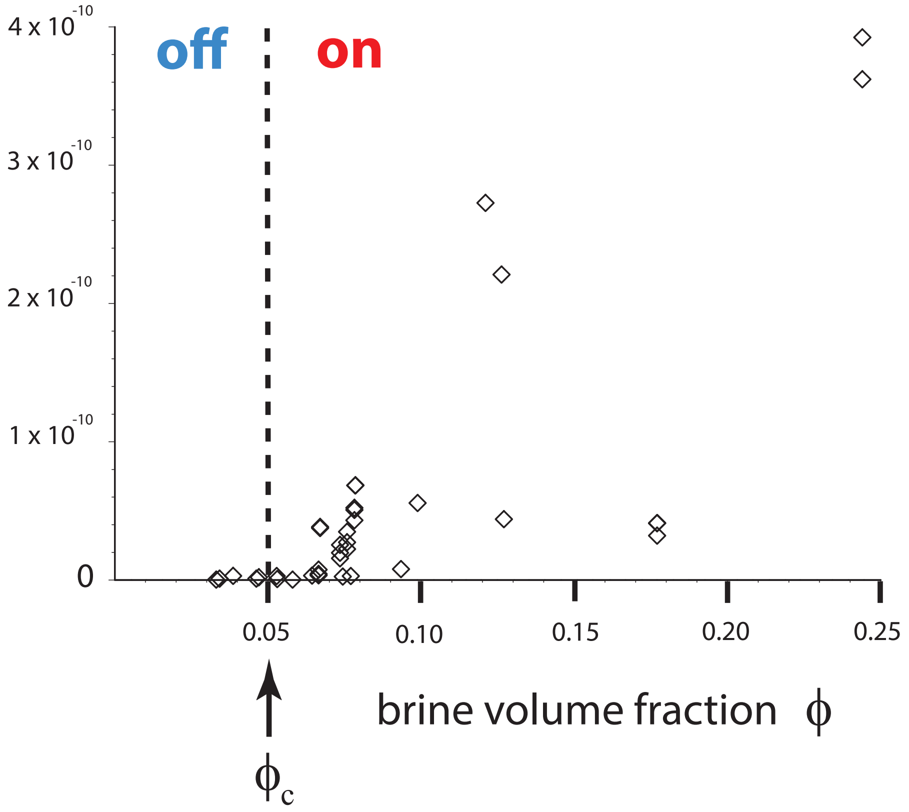
$\mathbf{k}$ = fluid permeability tensor

HOMOGENIZATION

mathematics for analyzing effective behavior of heterogeneous systems

Critical behavior of fluid transport in sea ice

impermeable permeable



Arctic field data

vertical fluid permeability k (m^2)

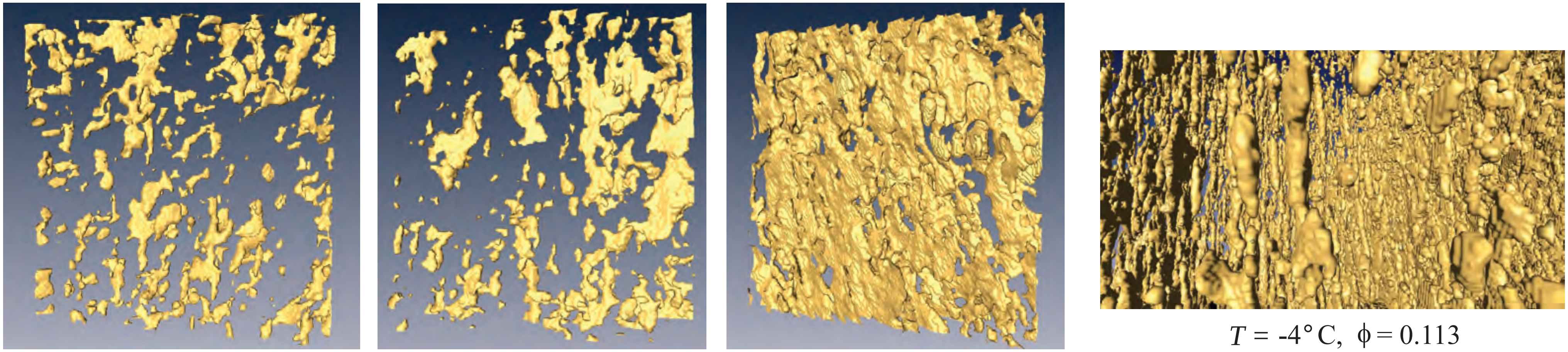
"on - off" switch for fluid flow

critical brine volume fraction $\phi_c \approx 5\%$ ⟷ $T_c \approx -5^\circ \mathrm{C}$, $S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

brine volume fraction and ***connectivity*** increase with temperature



T = -15 °C, $\phi = 0.033$

T = -6 °C, $\phi = 0.075$

T = -3 °C, $\phi = 0.143$

T = -4 °C, $\phi = 0.113$

X-ray tomography for brine phase in sea ice

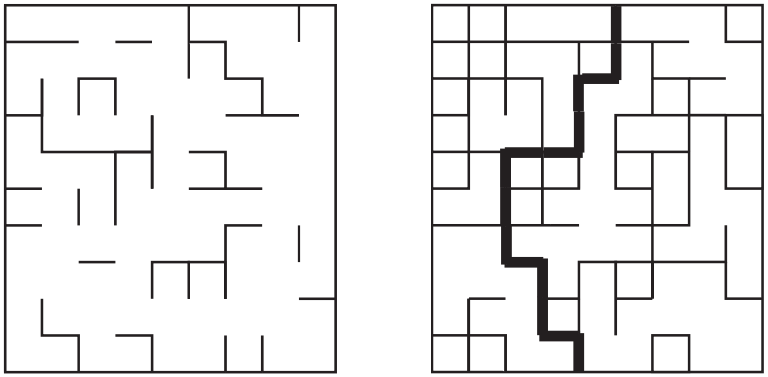
Golden, Eicken, *et al., Geophysical Research Letters* 2007

PERCOLATION THRESHOLD $\phi_c \approx 5\ \%$

Golden, Ackley, Lytle, *Science* 1998

impermeable

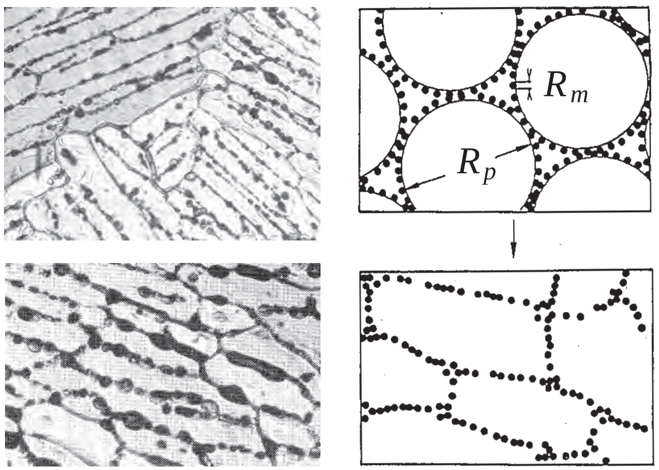
permeable



$p = 1/3$

$p = 2/3$

lattice percolation



Kusy, Turner
Nature 1971

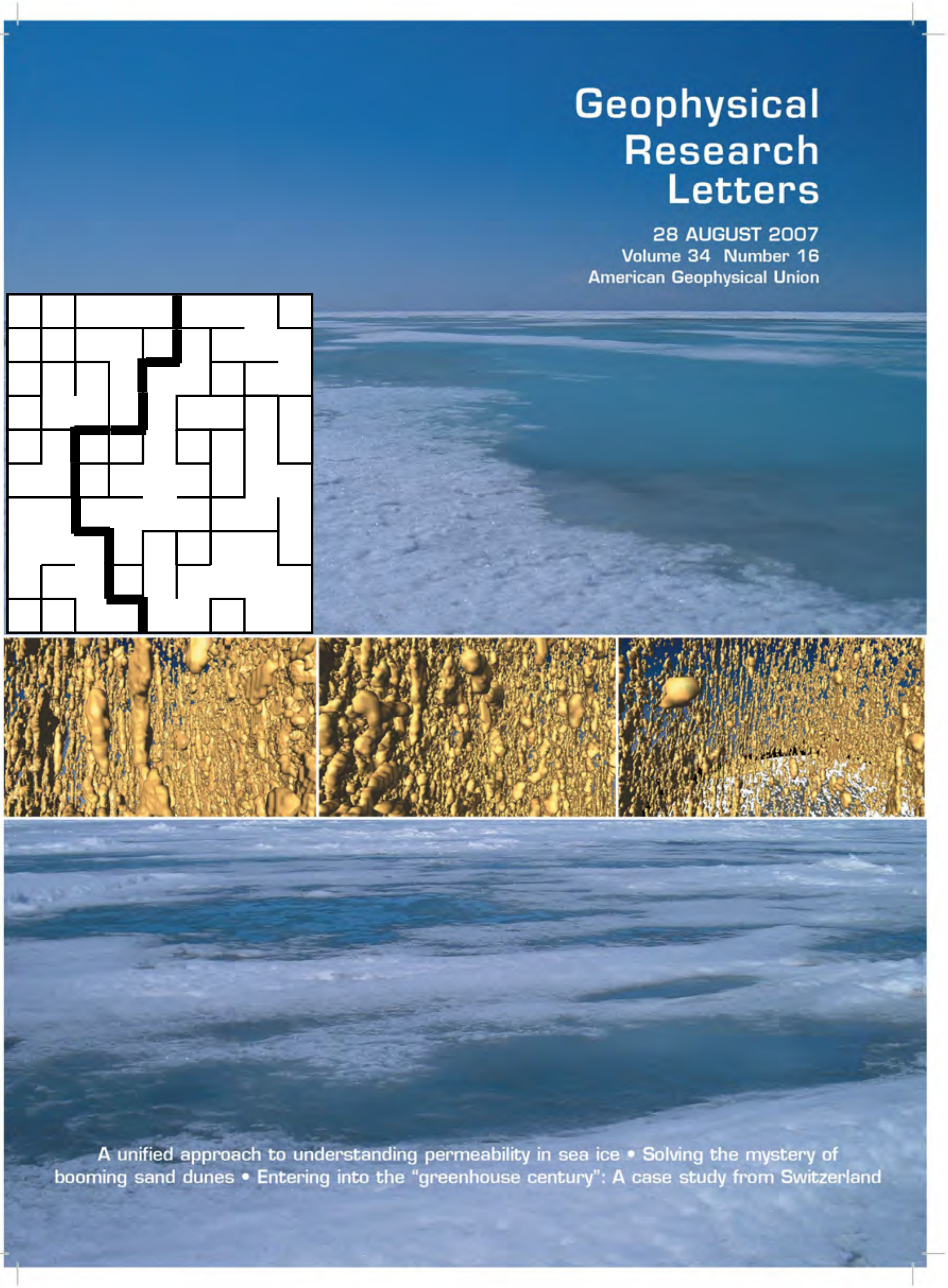
sea ice

compressed powder

continuum percolation

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007



Geophysical
Research
Letters

28 AUGUST 2007
Volume 34 Number 16
American Geophysical Union

A unified approach to understanding permeability in sea ice • Solving the mystery of booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

micro-scale

controls

macro-scale

processes

percolation theory

$$k(\phi) = k_0 (\phi - 0.05)^2 \longleftarrow$$

critical
exponent
t

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

hierarchical model
network model
rigorous bounds

agree closely
with field data

X-ray tomography for brine inclusions

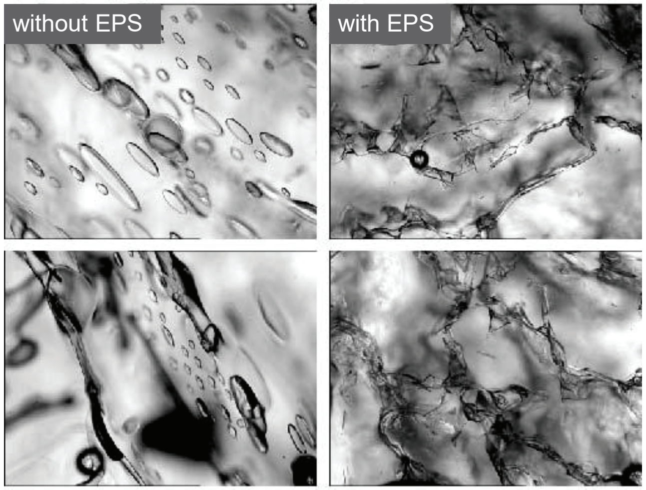
unprecedented look at thermal evolution of brine phase and its connectivity

confirms rule of fives

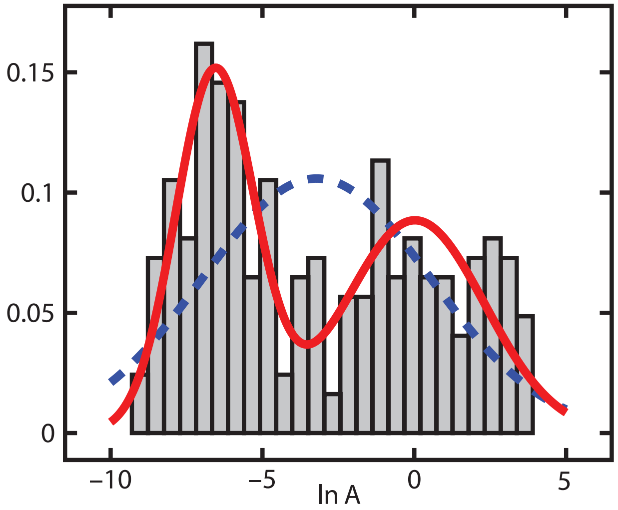
Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009

Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

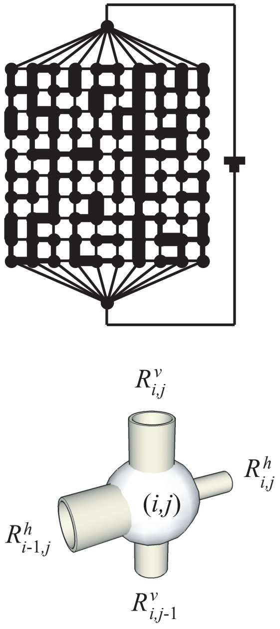
How does EPS affect fluid transport?



Krembs, Eicken, Deming, PNAS 2011



RANDOM PIPE MODEL



Zhu, Jabini, Golden,
Eicken, Morris
Ann. Glac. 2006

- **Bimodal** lognormal distribution for brine inclusions
- Develop random pipe network model with bimodal distribution;
  Use numerical methods that can handle larger variances in sizes.
- Results predict observed drop in fluid permeability *k*.
- **Rigorous bound on *k* for bimodal distribution of pore sizes**

Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
Multiscale Modeling and Simulation, 2018

How does the biology affect the physics?

Remote sensing of sea ice

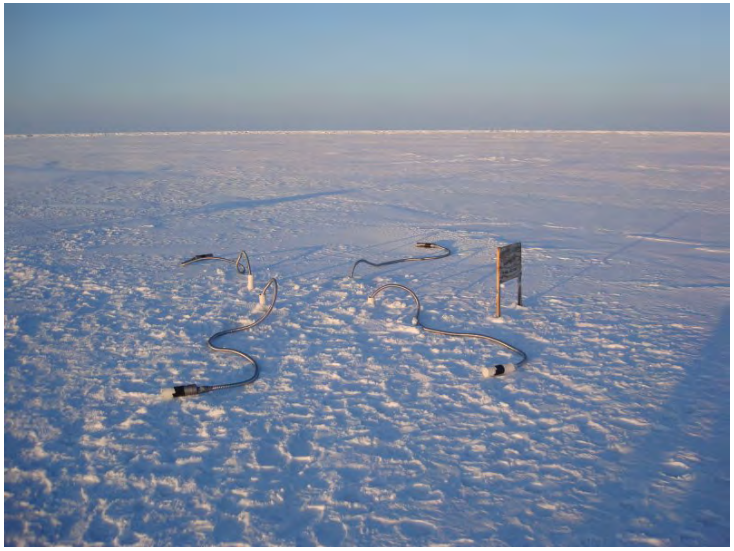
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice
properties from
electromagnetic
(EM) data

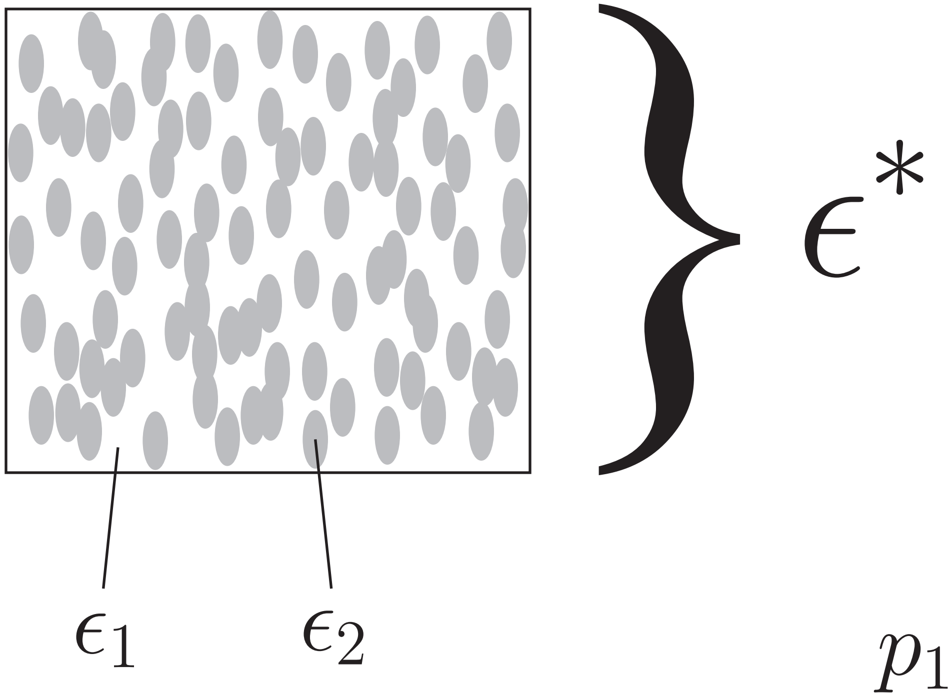
ε^*

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \text{ , composite geometry } \right)$$

What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?

Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

separates geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

material parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

$$E = s\,(s + \Gamma\chi)^{-1} e_k$$

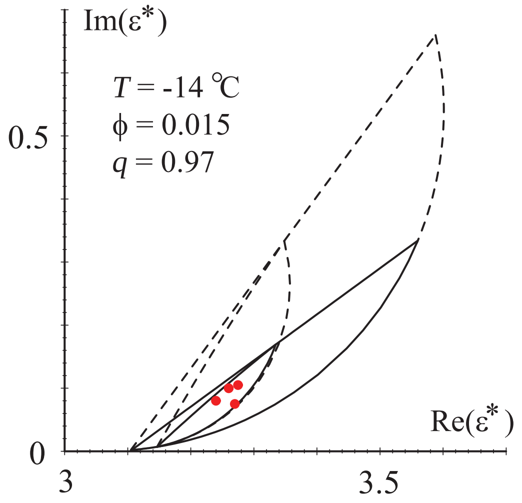
$\Gamma\chi$: **microscale** $\longrightarrow$ **macroscale**

$\Gamma\chi$ ***links scales***

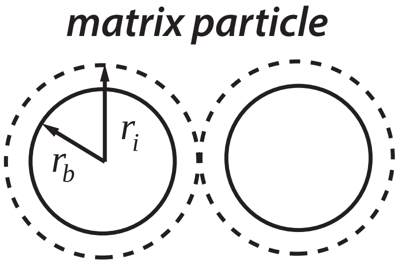
Golden and Papanicolaou, Comm. Math. Phys. 1983

forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data



$$q = r_b / r_i$$

$$0 < q < 1$$

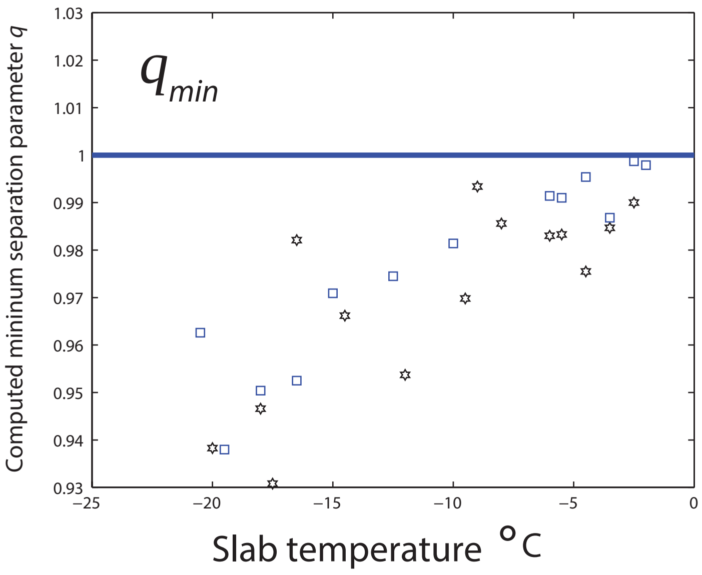
Golden 1995, 1997

Bruno 1991

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inverse bounds



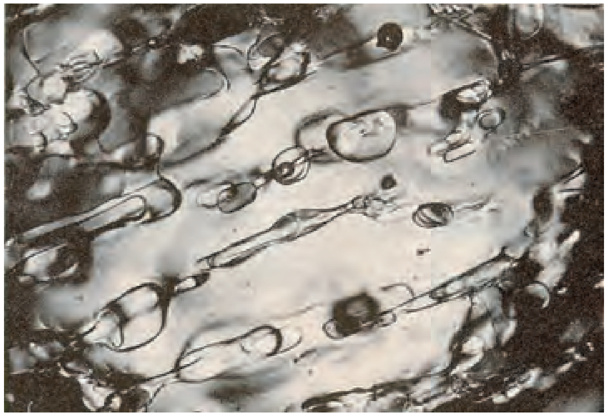
inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε*

rigorous inverse bound on spectral gap

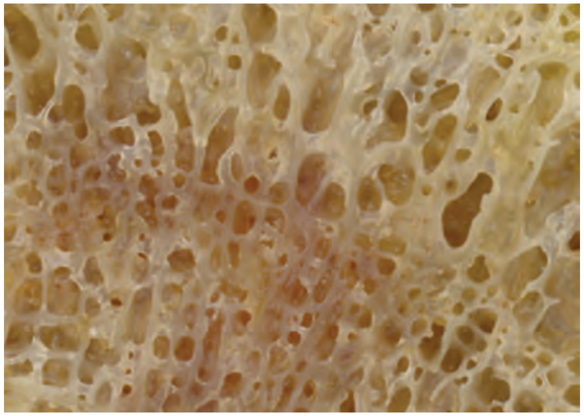
construct algebraic curves which bound admissible region in (p,q)-space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

SEA ICE

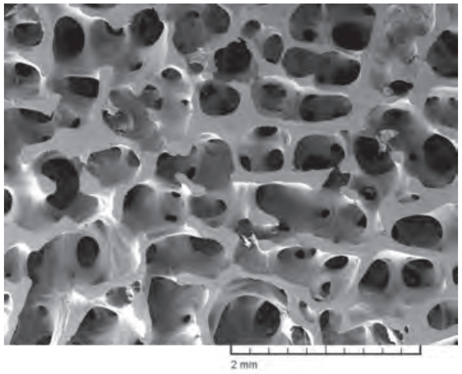


HUMAN BONE

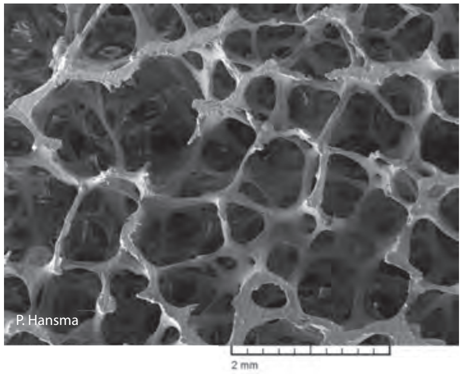


spectral characterization of porous microstructures in human bone

young healthy trabecular bone

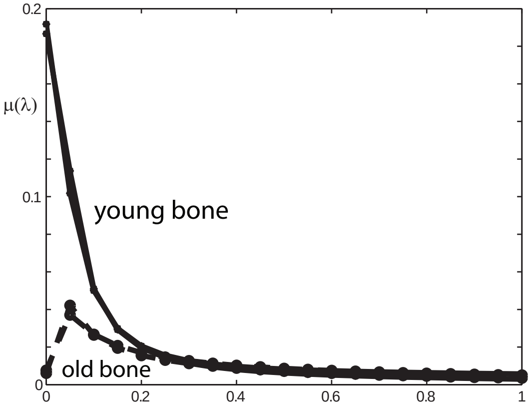


old osteoporotic trabecular bone



P. Hansma

reconstruct spectral measures from complex permittivity data



use regularized inversion scheme

apply spectral measure analysis of brine connectivity and spectral inversion to electromagnetic monitoring of osteoporosis

Golden, Murphy, Cherkaev, J. Biomechanics 2011

the math doesn't care if it's sea ice or bone!

direct calculation of spectral measures

Murphy, Hohenegger, Cherkaev, Golden, *Comm. Math. Sci.* 2015

- depends only on the composite geometry
- discretization of microstructural image gives binary network
- fundamental operator becomes a random matrix
- spectral measure computed from eigenvalues and eigenvectors

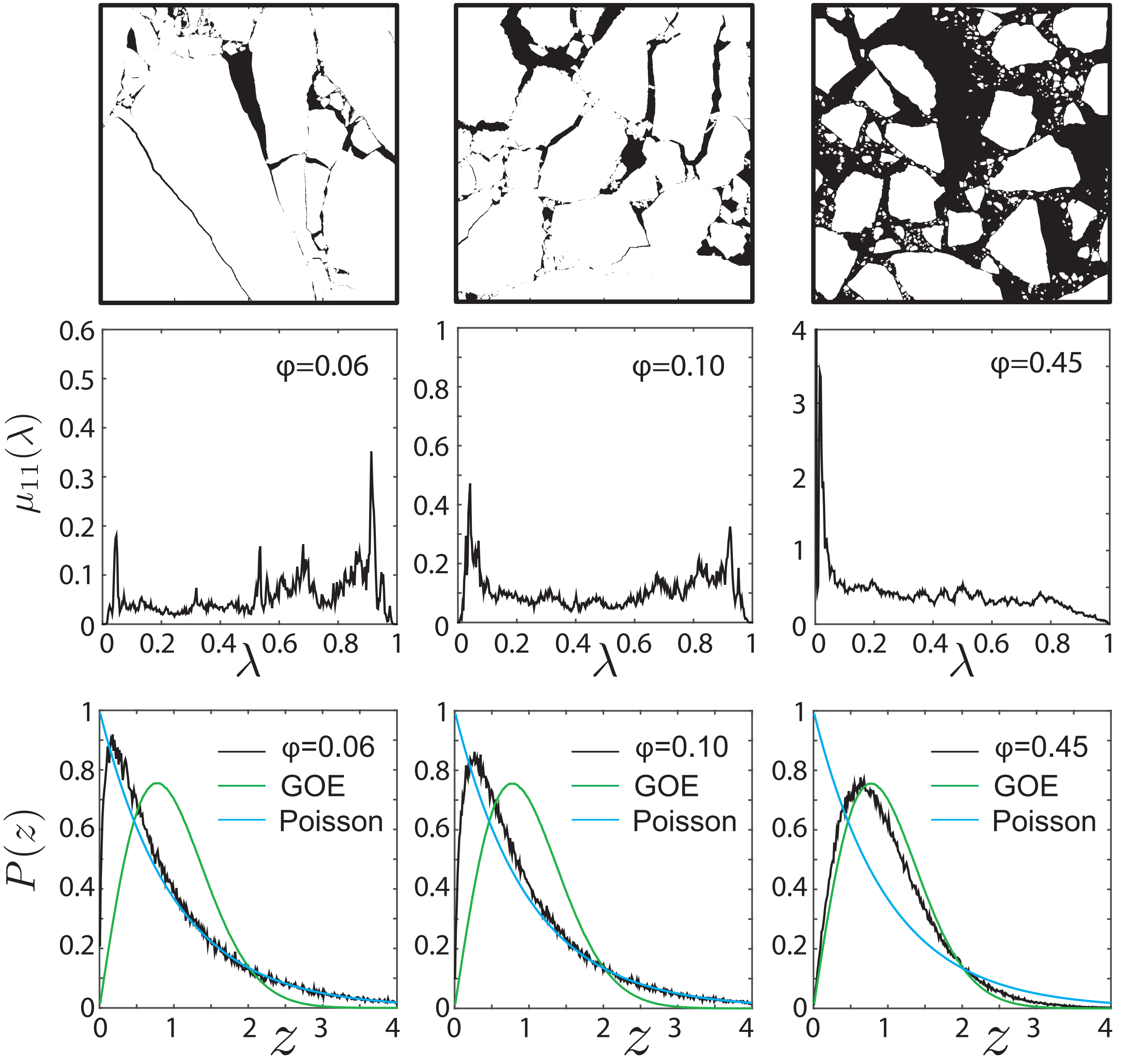
once we have the spectral measure μ it can be used in Stieltjes integrals for other transport coefficients:

electrical and thermal conductivity, complex permittivity, magnetic permeability, diffusion, fluid flow properties

earlier studies of spectral measures

Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Spectral computations for sea ice floe configurations



spectral measures

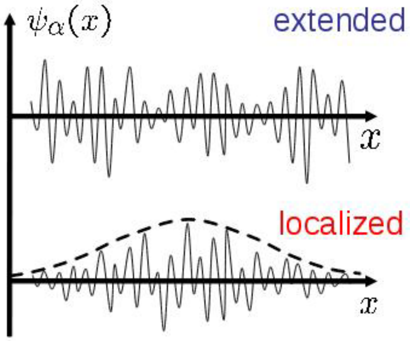
eigenvalue spacing distributions

UNIVERSAL Wigner-Dyson distribution

uncorrelated → **level repulsion**

ANDERSON TRANSITION

Murphy, Cherkaev, Golden
Phys. Rev. Lett. 2017



metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

Anderson transition in wave physics:
quantum, optics, acoustics, water waves, ...

we find a surprising analog

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

PERCOLATION
TRANSITION →

transition to universal
eigenvalue statistics (GOE)
extended states, mobility edges

-- but without wave interference or scattering effects ! --

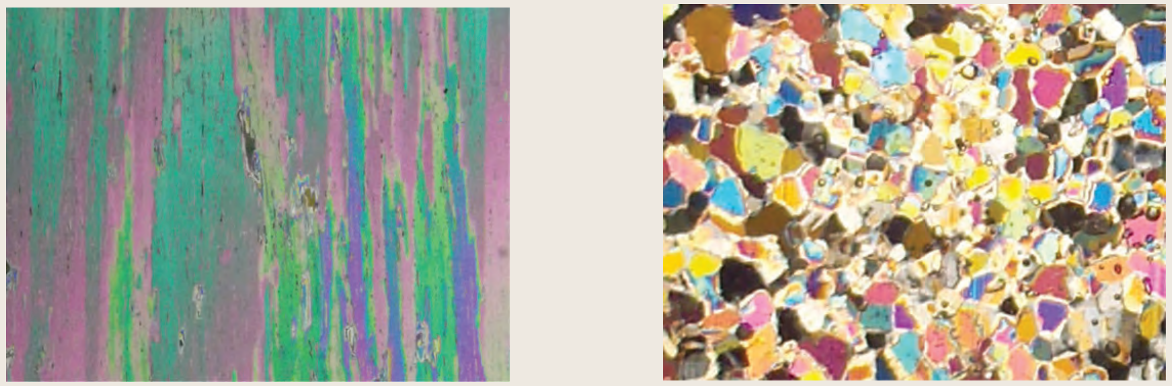
Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**

  orientation statistics
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A 8 Feb 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS
OF SCIENTIFIC
PUBLISHING

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

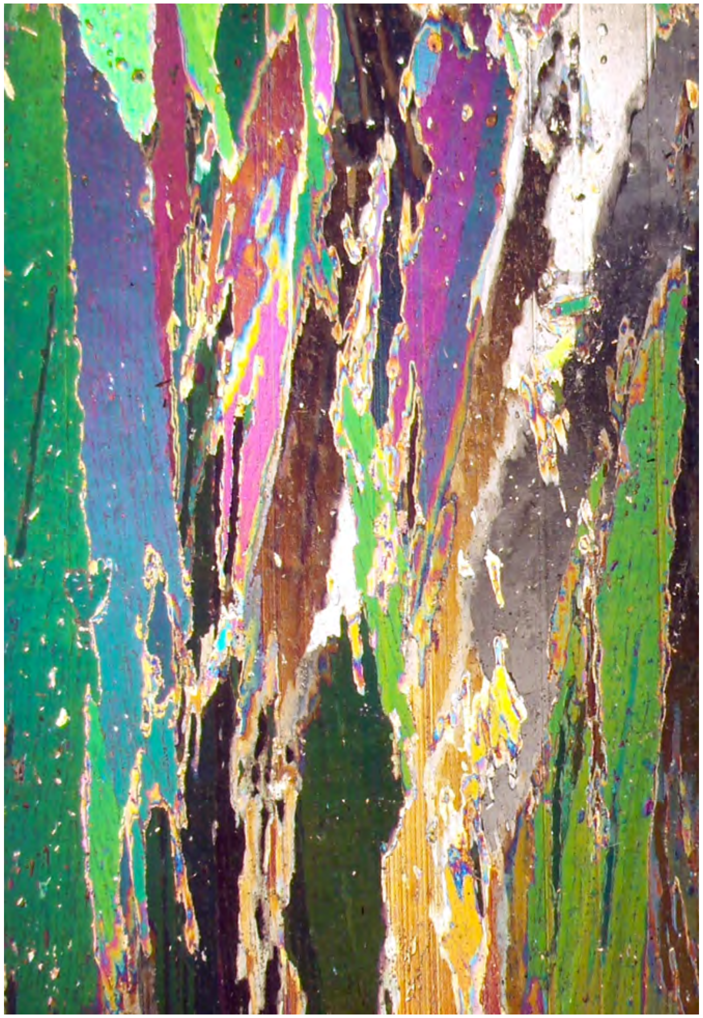
A computer model to determine how a human should walk so as to expend the least energy

THE
ROYAL
SOCIETY
PUBLISHING

21/01/15 1:47 PM

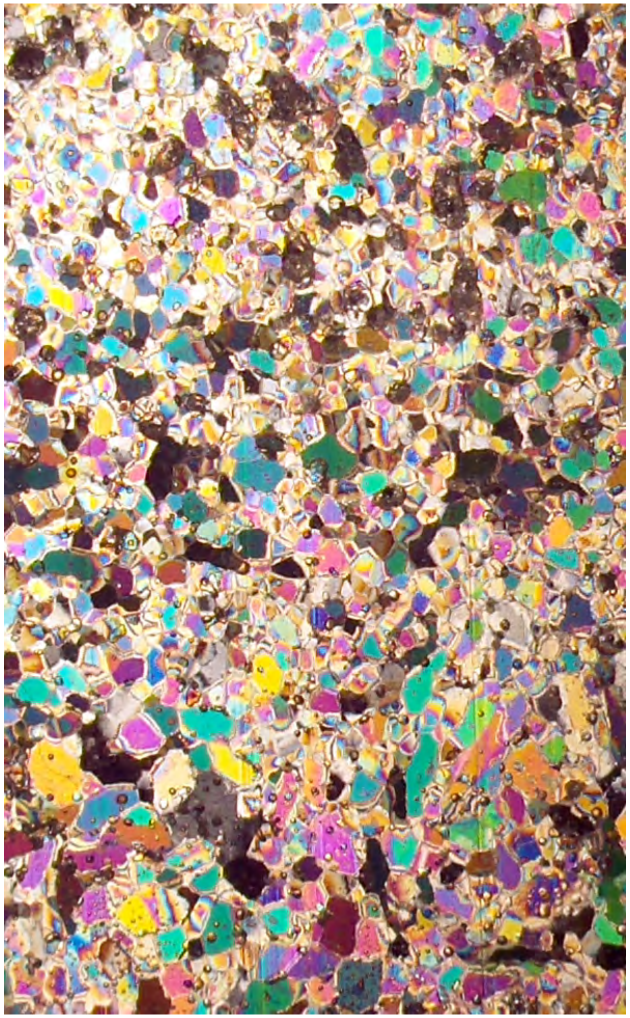
higher threshold for fluid flow in Antarctic granular sea ice

columnar



5%

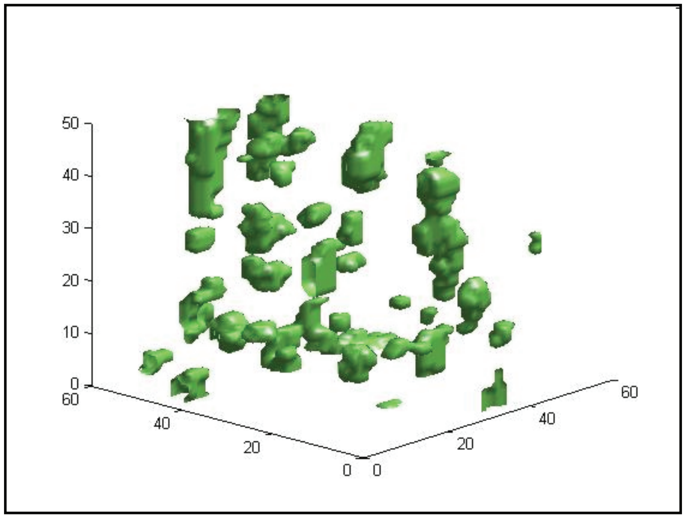
granular



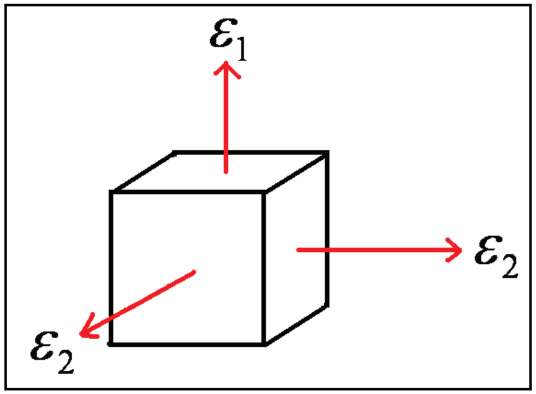
10%

Golden, Sampson, Gully, Lubbers, Tison 2019

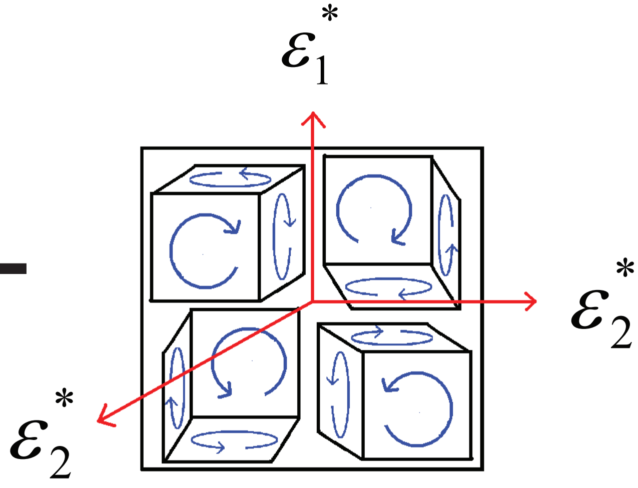
two scale homogenization for polycrystalline sea ice



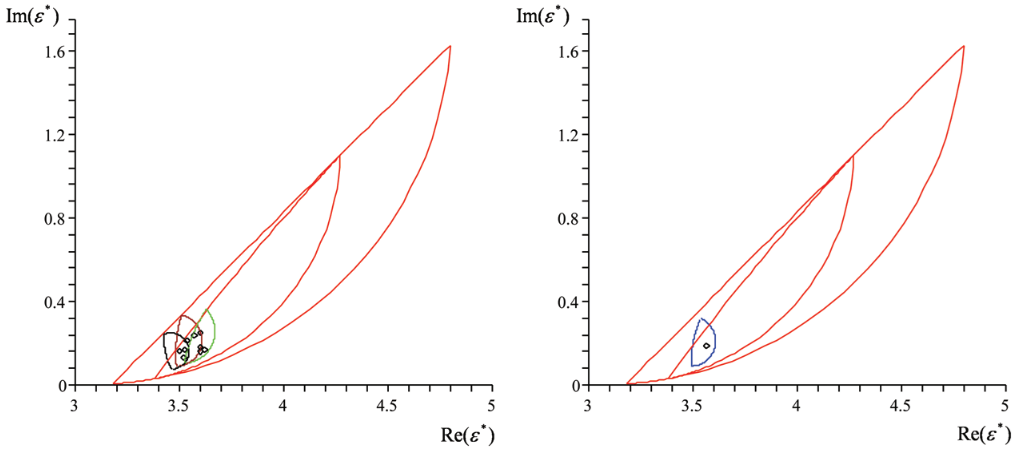
numerical homogenization for single crystal



analytic continuation for polycrystals



bounds



Gully, Lin, Cherkaev, Golden, Proc. Roy. Soc. A (and cover) 2015

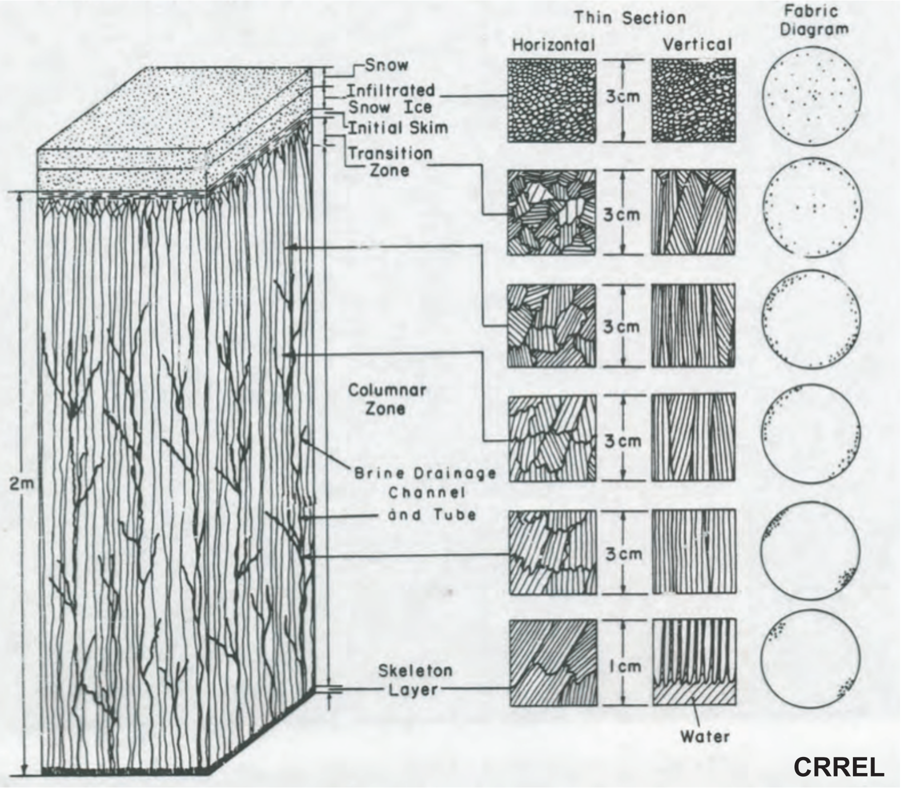
Rigorous bounds on the complex permittivity tensor of sea ice with polycrystalline anisotropy within the horizontal plane

McKenzie McLean, Elena Cherkaev, Ken Golden 2019

motivated by **Weeks and Gow, *JGR* 1979: c-axis alignment in Arctic fast ice off Barrow**

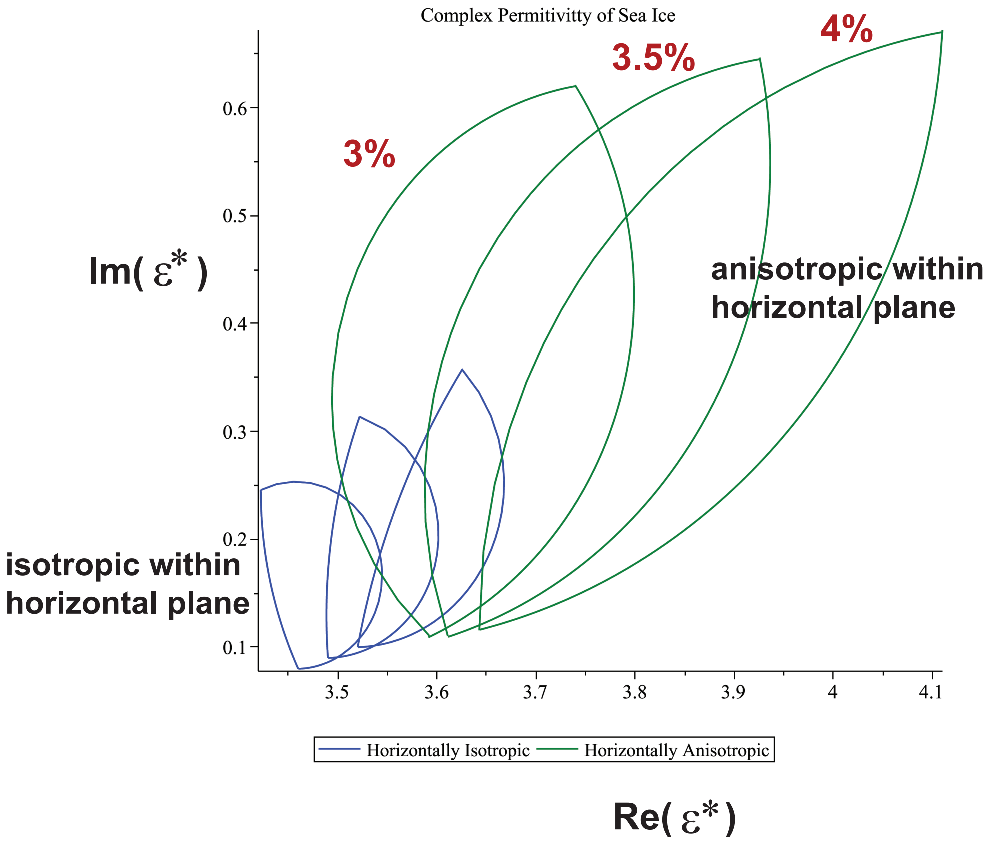
Golden and Ackley, *JGR* 1981: radar propagation model in aligned sea ice

input: orientation statistics



CRREL

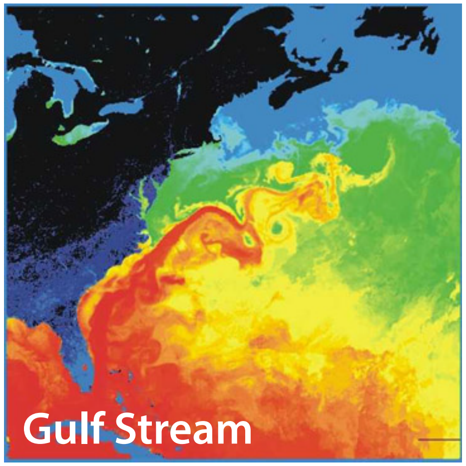
output: bounds

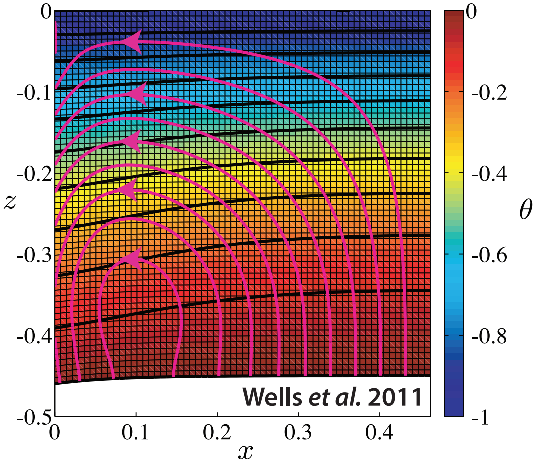


advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\Big\downarrow$$

homogenize

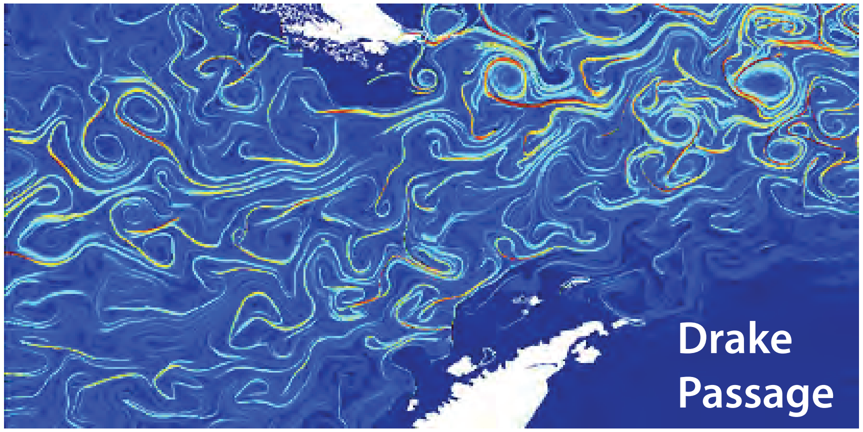
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* *effective diffusivity*

Stieltjes integral for κ^* *with spectral measure*

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019



Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2019

$$\kappa^* = \kappa \left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2} \right), \quad F(\kappa) \; = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}$$

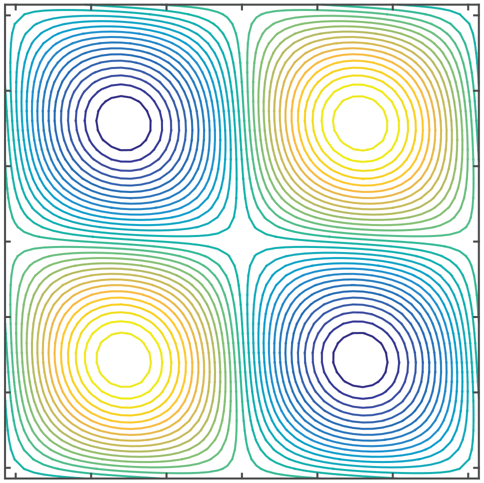
- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H\Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$, Δ is the Laplace operator
- $i\Gamma H\Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ-plane

separation of material properties and flow field

spectral measure calculations

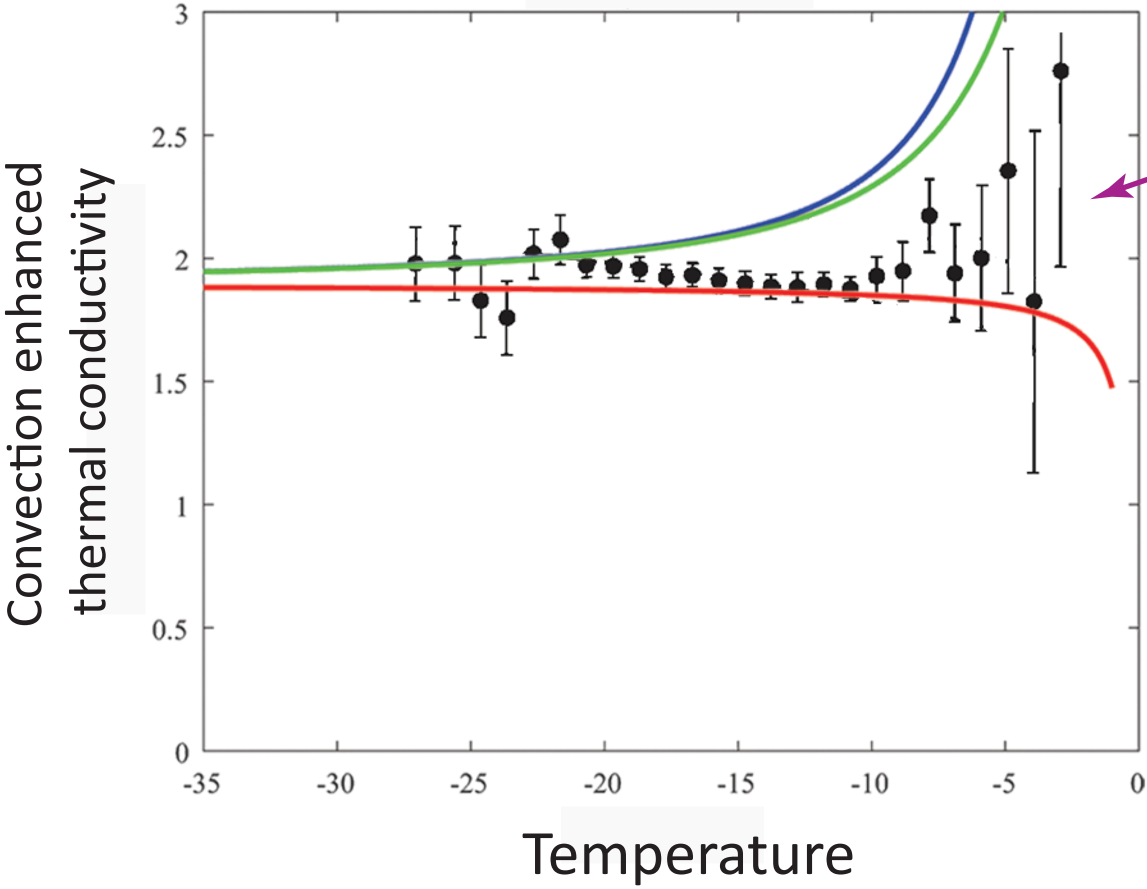
Rigorous bounds on convection enhanced thermal conductivity of sea ice

Kraitzman, Hardenbrook, Dinh, Murphy, Zhu, Cherkaev, Golden 2019



cat's eye flow model for
brine convection cells

similar bounds
for shear flows

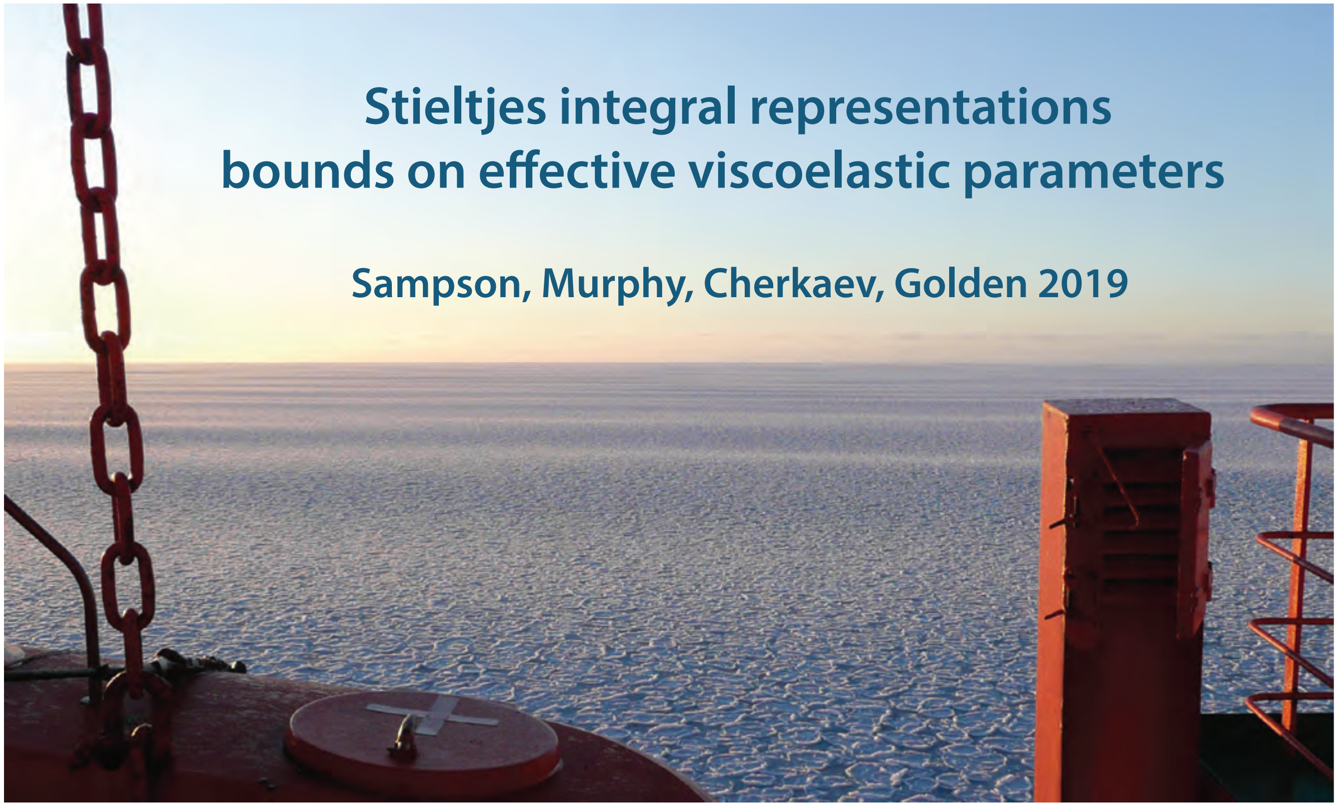


rigorous Padé bounds from Stieltjes integral +
analytical calculations of moments of measure

rigorous bounds assuming information
on flow field INSIDE inclusions

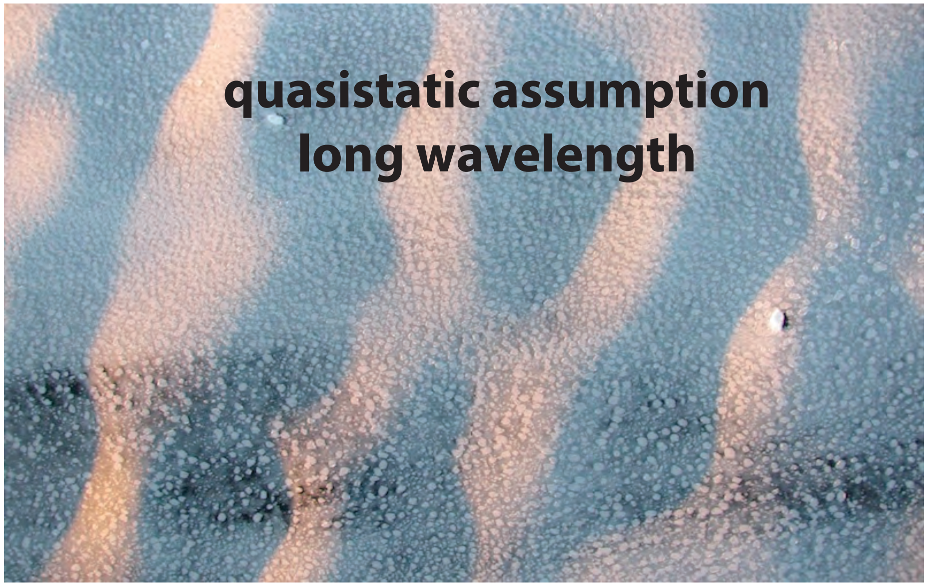
Kraitzman, Cherkaev, Golden
SIAM J. Appl. Math (in revision), 2019

wave propagation in the marginal ice zone



Stieltjes integral representations
bounds on effective viscoelastic parameters

Sampson, Murphy, Cherkaev, Golden 2019



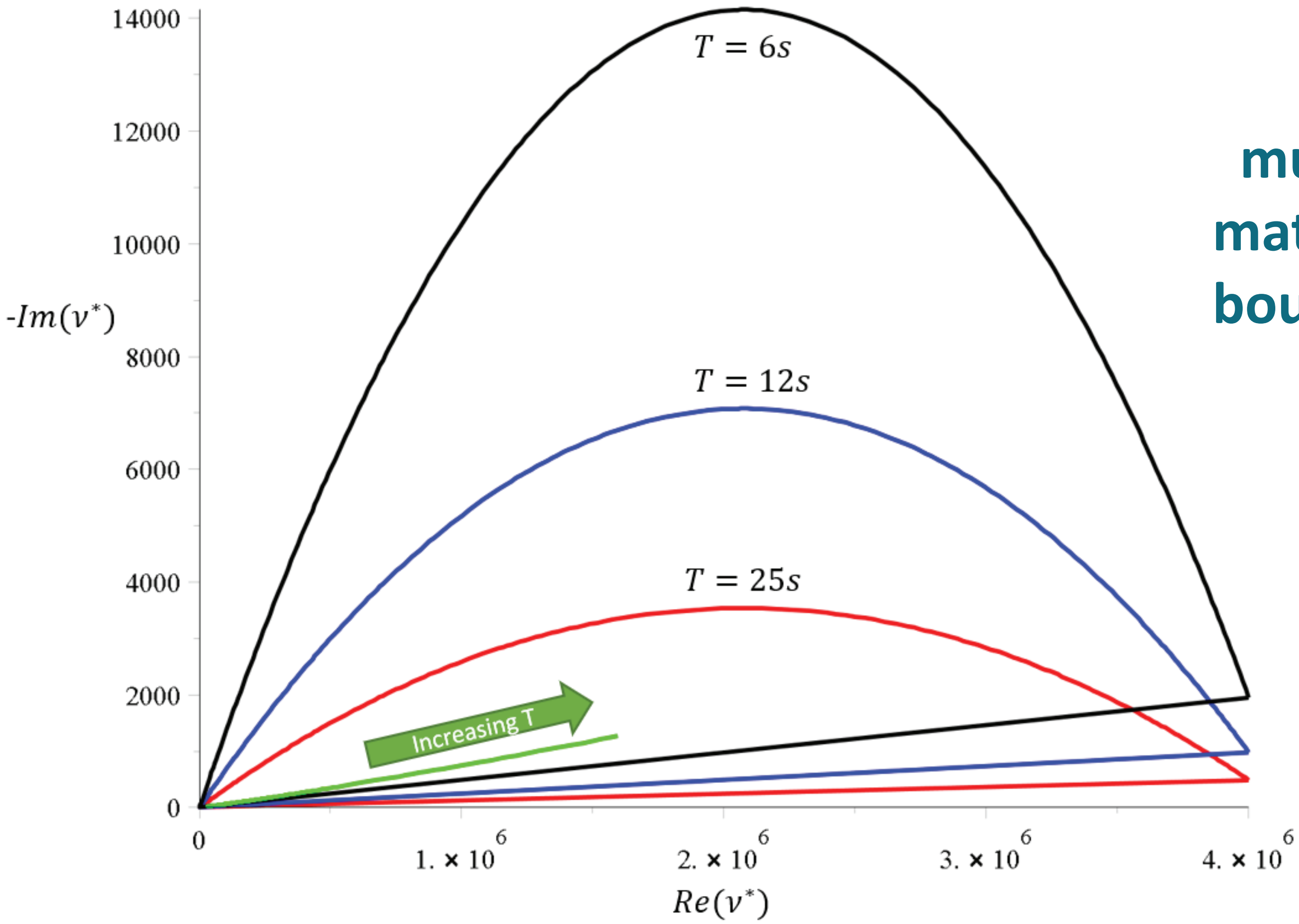
quasistatic assumption
long wavelength

bounds on the effective complex viscoelasticity

complex elementary bounds (fixed area fraction of floes)

$\nu_1 = 10^7 + i\,4875$ **pancake ice**

$\nu_2 = 5 + i\,0.0975$ **slush / frazil**



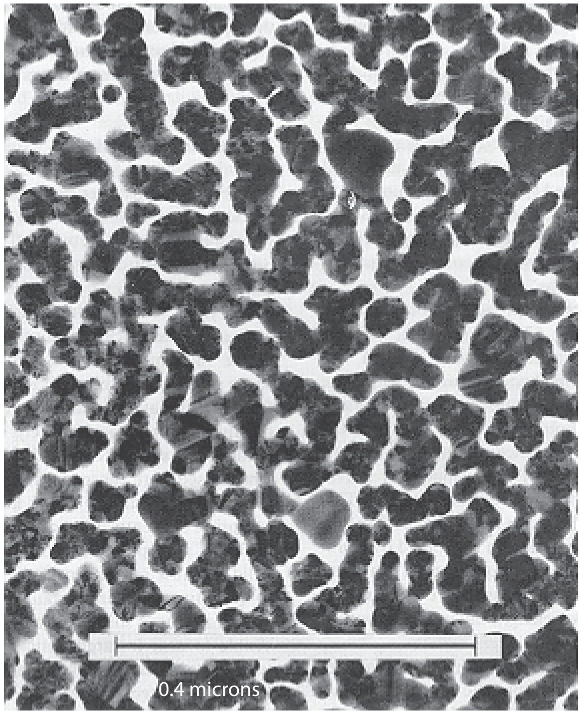
+ much tighter matrix particle bounds + data

Sampson, Murphy, Cherkaev, Golden 2019

Interaction of light with sea ice

thin silver film

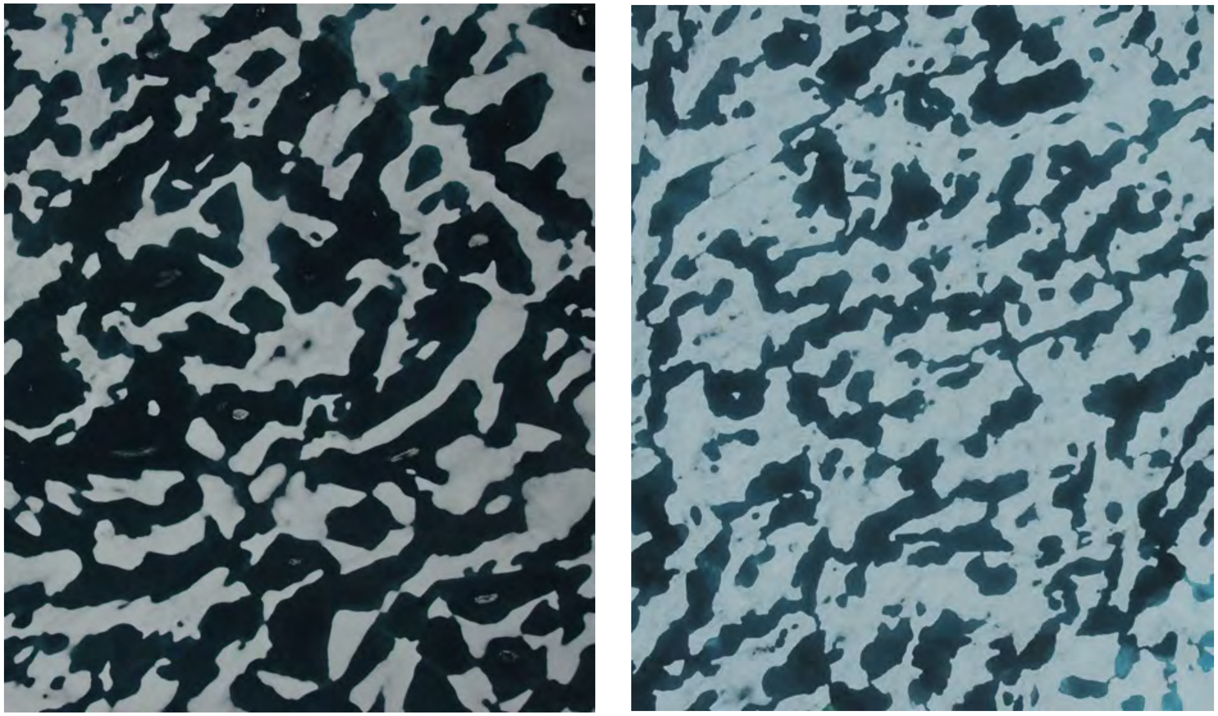
microns



(Davis, McKenzie, McPhedran, 1991)

Arctic melt ponds

kilometers



(Perovich, 2005)

optical properties

composite geometry -- area fraction of phases, connectedness, necks

melt pond formation and albedo evolution:

- *major drivers in polar climate*
- *key challenge for global climate models*

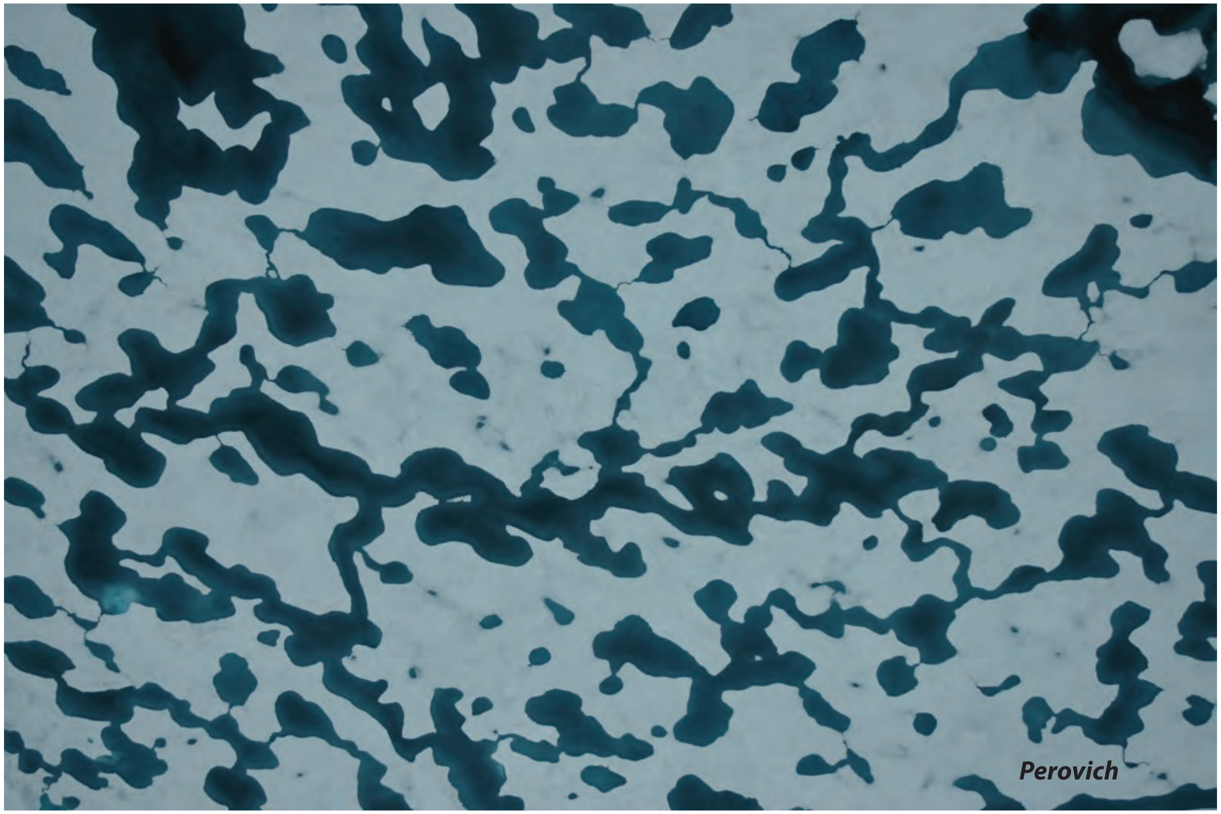
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

Flocco, Feltham, Hunke 2012


Perovich

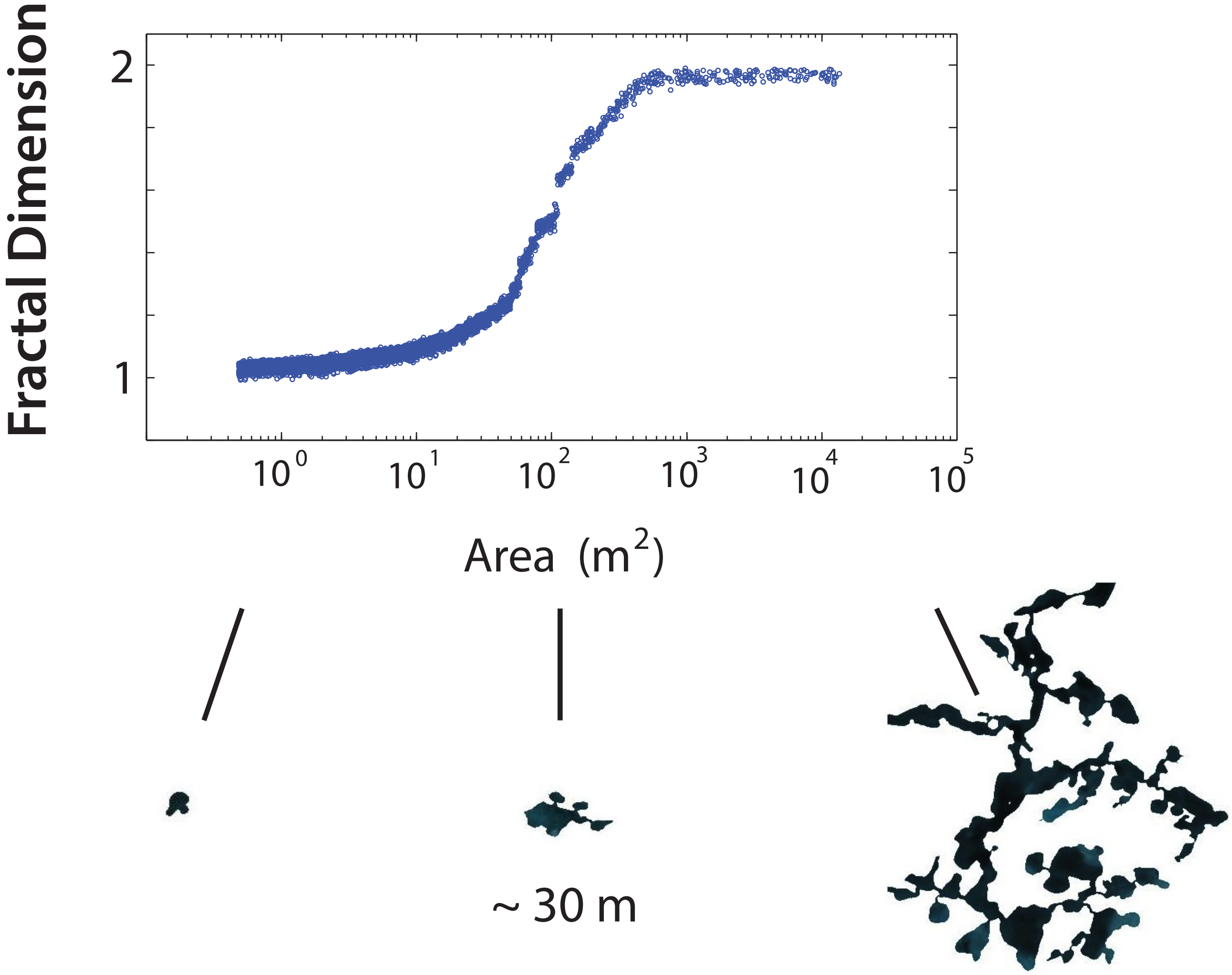
Are there universal features of the evolution similar to phase transitions in statistical physics?

Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

The Cryosphere, 2012

complexity grows with length scale

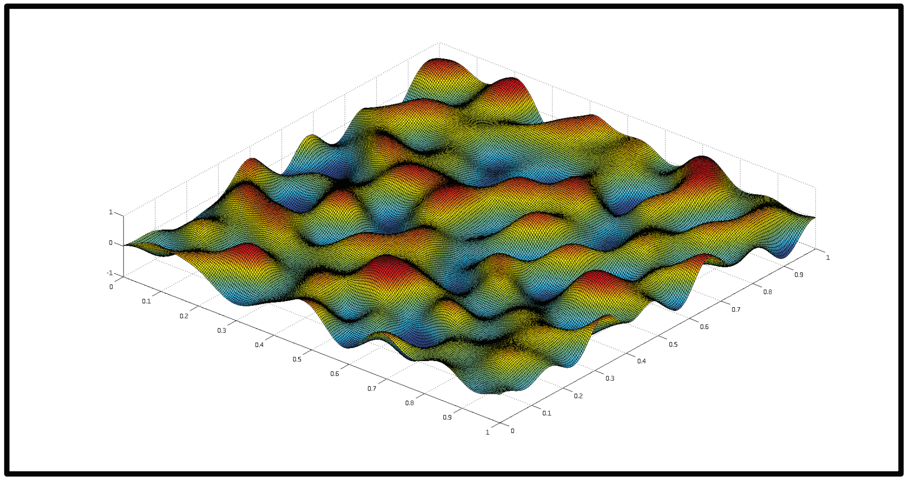


simple pond ***transitional pond*** ***complex pond***

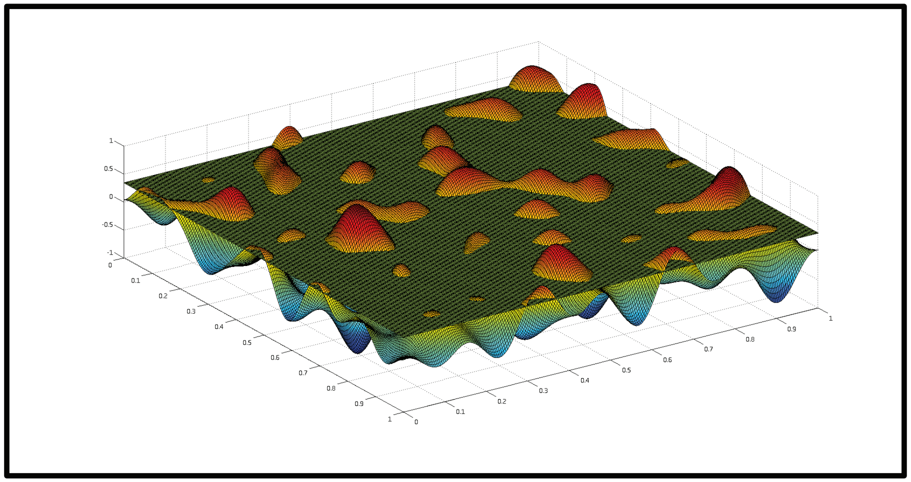
Continuum percolation model for melt pond evolution

level sets of random surfaces

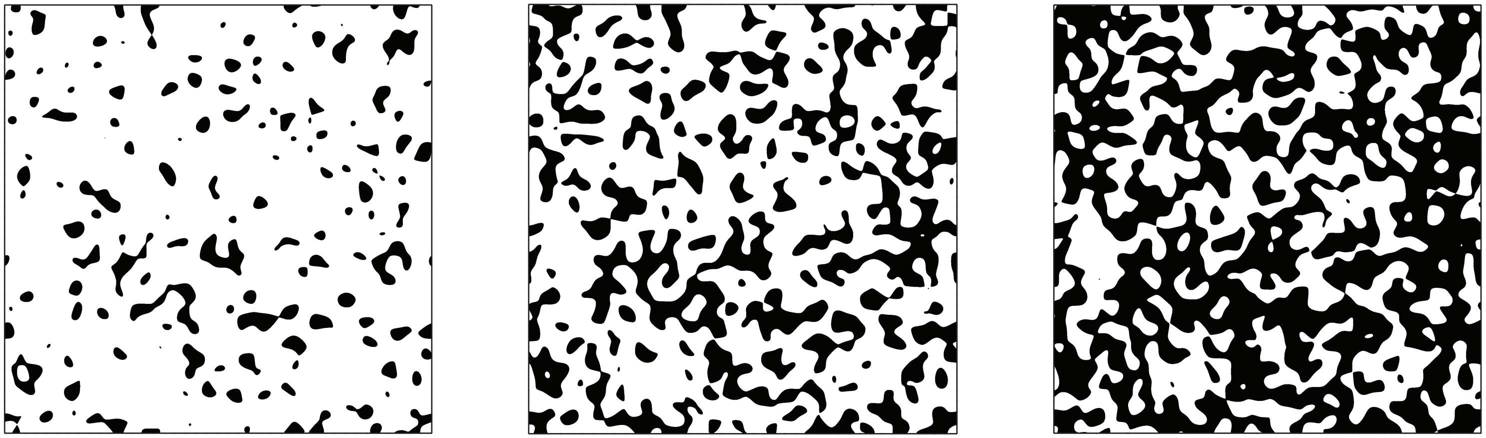
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018



random Fourier series representation of surface topography

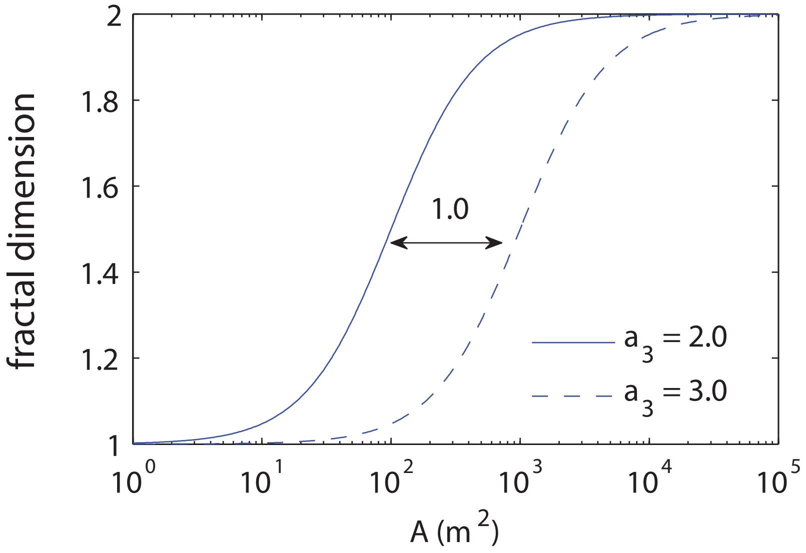


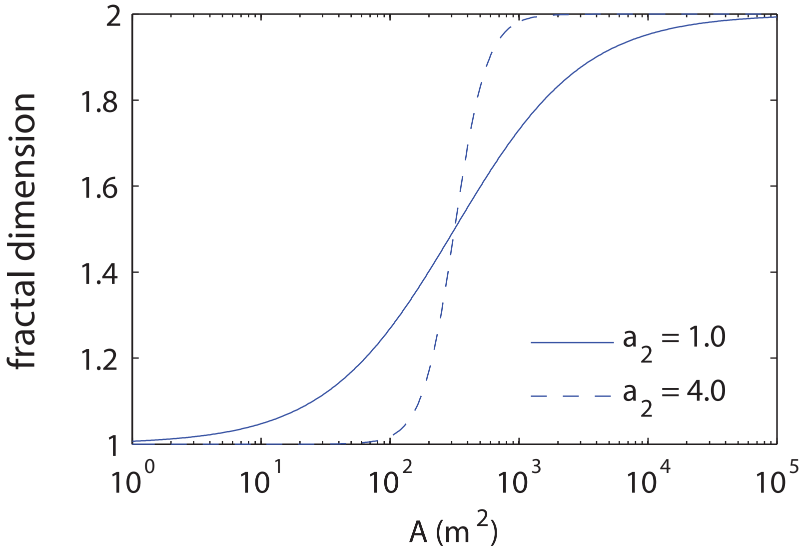
intersections of a plane with the surface define melt ponds



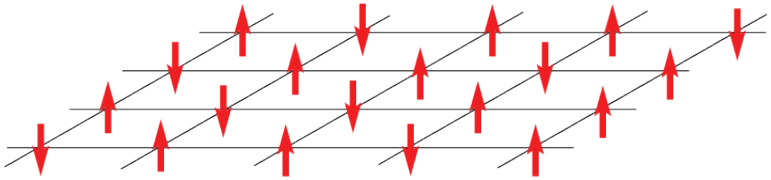
electronic transport in disordered media *diffusion in turbulent plasmas* *Isichenko, Rev. Mod. Phys., 1992*

fractal dimension curves depend on statistical parameters defining random surface





Ising Model for a Ferromagnet

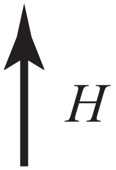


$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

blue

white

applied magnetic field

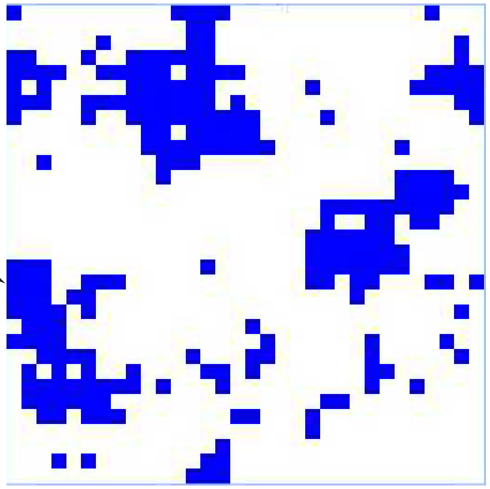


$$\mathcal{H} = -H \sum_i s_i - J \sum_{<i,j>} s_i s_j$$

nearest neighbor Ising Hamiltonian

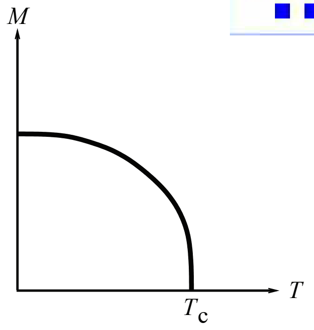
ferromagnetic interaction $J \geq 0$

islands or ponds of like spins



magnetization

$$M(T,H) = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$$



Curie point critical temperature

homogenized parameter like effective conductivity

Stieltjes integral representation for M

Baker, PRL 1968

Ising model for ferromagnets ⟶ Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.*, 2019

$$\mathcal{H} = -\sum_{i}^{N} H_i\, s_i - J \sum_{<i,j>}^{N} s_i s_j \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

random magnetic field
represents snow topography

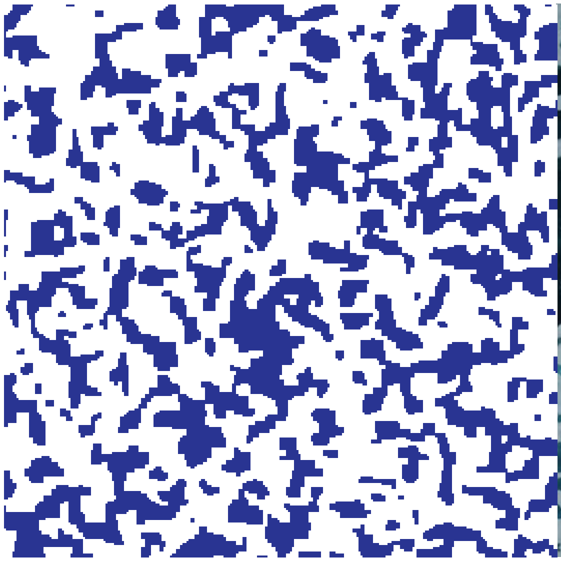
magnetization M **pond coverage** $\dfrac{(M+1)}{2}$

~ albedo

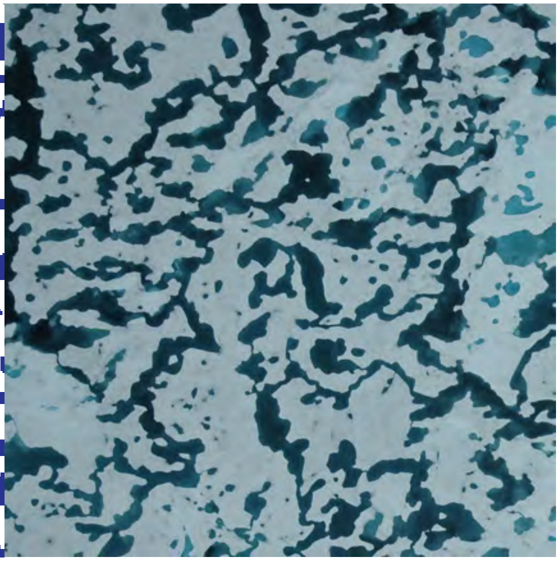
only nearest neighbor
patches interact

Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system "flows" toward metastable equilibria.

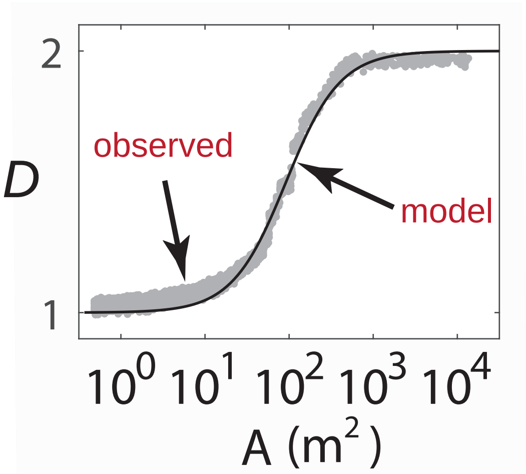
Order from Disorder



Ising model



melt pond photo (Perovich)



pond size
distribution exponent

observed	-1.5
	(Perovich, *et al.* 2002)
model	-1.58

ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data

Conclusions

1. Sea ice is a fascinating multiscale composite with structure similar to many other natural and man-made materials.

2. Mathematical methods developed for sea ice advance the theory of composites in general.

2. Homogenization and statistical physics help ***link scales in sea ice and composites***; provide rigorous methods for finding effective behavior; advance sea ice representations in climate models.

3. Fluid flow through sea ice mediates melt pond evolution and many processes important to climate change and polar ecosystems.

5. Field experiments are essential to developing relevant mathematics.

6. Our research will help to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

THANK YOU

Office of Naval Research

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Arctic and Global Prediction Program

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

UNITED STATES COAST GUARD 1790

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MCRN Mathematics and Climate Research Network

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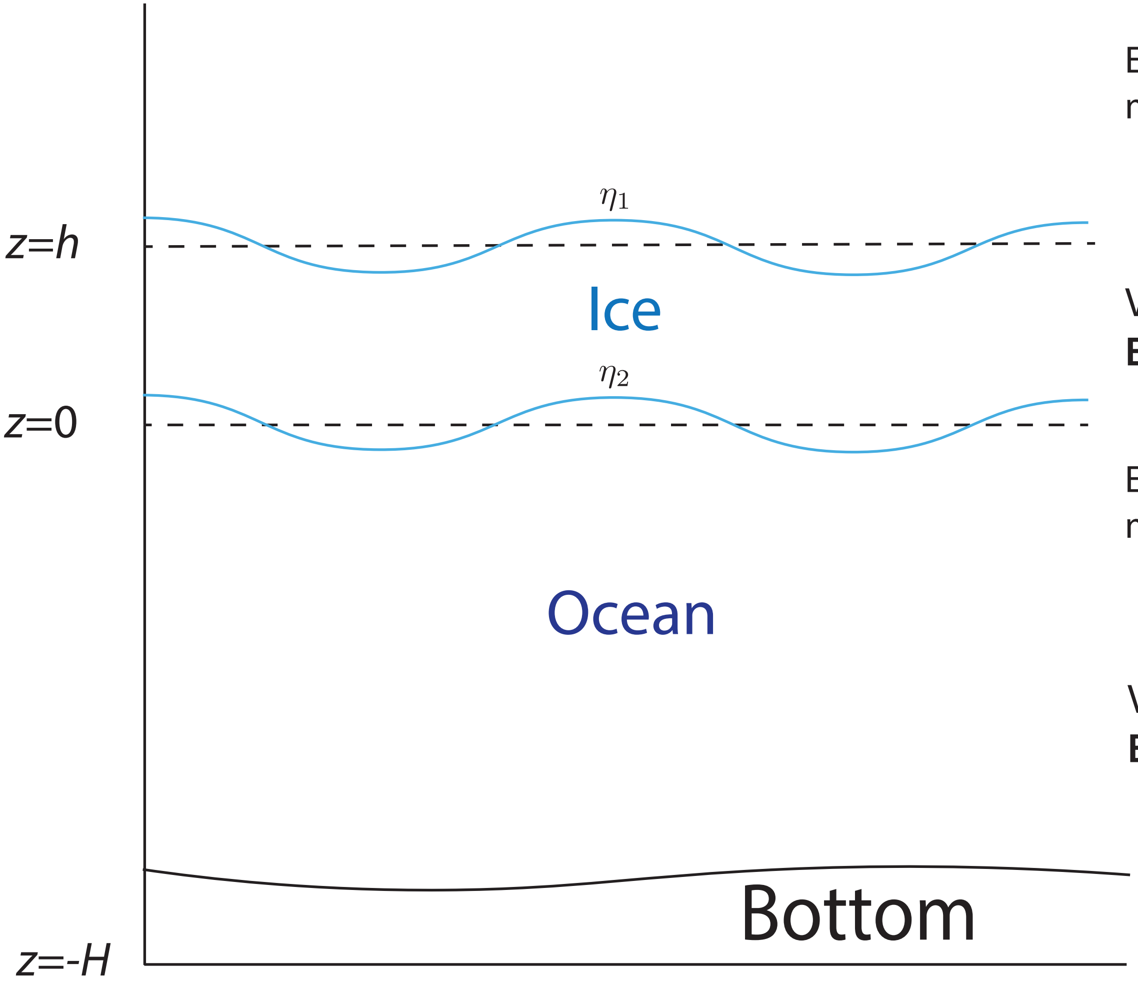
THE UNIVERSITY OF UTAH

Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999

Two Layer Models and Effective Rheological Parameters



G shear modulus

P pressure

ω angular frequency

U velocity field

ν viscosity

λ Poission ratio

ρ density

g gravity

Viscous fluid layer (Keller 1998)
Effective Viscosity $\boxed{\nu}$

Equations of motion:

$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 U + g$$

Viscoelastic fluid layer (Wang-Shen 2010)
Effective Complex Viscosity $\boxed{\nu_e = \nu + iG/\rho\omega}$

Equations of motion

$$\frac{\partial U}{\partial t} = -\frac{1}{\rho}\nabla P + \nu_e \nabla^2 U + g$$

Viscoelastic thin beam (Mosig *et al.* 2015)
Effective Complex Shear Modulus $\boxed{G_v = G - i\omega\rho\nu}$

Stieltjes integral representation for effective complex viscoelastic parameter; bounds

Sampson, Murphy, Cherkaev, Golden 2019

Homogenization for two phase viscoelastic composite

$\nu_1 = 10^7 + i\,4875$ **pancake ice**

$\nu_2 = 5 + i\,0.0975$ **slush / frazil**

microscale $\sigma = C_{ijkl}\epsilon_{kl} = C : \epsilon$

$$C = 2(\chi_1\nu_1 + \chi_2\nu_2)\Lambda_s$$

macroscale $\langle\sigma\rangle = C^* : \langle\epsilon\rangle$

$$\langle\epsilon\rangle = \epsilon^0$$

quasistatic assumption $\nabla \cdot \sigma = 0$

Strain Field

$$\epsilon = \frac{1}{2}[\nabla u + (\nabla u)^T] = \nabla^s u \quad \nabla \cdot u = 0$$

Resolvent

$$\epsilon = \left(1 - \frac{1}{s}\Gamma\chi_1\right)^{-1}\epsilon^0 \Rightarrow \boxed{\frac{\nu^*}{\nu_2} = \left(1 - \|\epsilon^0\|^{-2}F(s)\right)}$$

$$F(s) = \int_0^1 \frac{d\mu(\lambda)}{s - \lambda} \qquad s = \frac{1}{1 - \frac{\nu_1}{\nu_2}}$$

$$\Gamma = \nabla^s(\nabla \cdot \nabla^s)^{-1}\nabla \,\cdot$$