Mathematics 1220 PRACTICE EXAM III Spring 2002

1. Use the geometric series to find the Taylor series for $f(x) = \ln(1+x)$ around $x = 0$. Determine the radius of convergence of this series. Explain your result for the radius in terms of the singularity of f. Do the same for $f(x) = 1/(1+x)^2$.

2. Find the Taylor series for $\cosh x$ around $x = 0$ by using the series for e^x. What is its radius of convergence?

3. Find the convergence set for the following power series. For (a), also analyze the type of convergence (or divergence) at the enpoints of the convergence set.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(3x+1)^n}{n\ 2^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} f_n\, x^n$, where $\{f_n\}$ is the Fibonacci sequence

4. Find the following limits. Be sure to fully justify your answers.

   (a) $\displaystyle\lim_{n\to\infty} e^{-n} \sin n$ (b) $\displaystyle\lim_{n\to\infty} (2n)^{1/2n}$

   (c) $\displaystyle\lim_{n\to\infty} \left(1 - \frac{1}{n}\right) \cos n\pi$ (d) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \left(\exp\left(\frac{k}{n}\right)^2\right) \frac{1}{n}$

5. A (zero dimensional) bull frog initially jumps a meter. On each successive jump, he can only go $\frac{3}{4}$ of the distance of the previous jump. If he takes infinitely many jumps, how far does he travel?

6. Consider solving the equation $x = e^x - 2$ for the root just to the right of $x = 1$, using the fixed point iteration scheme $x_{n+1} = g(x_n)$, where $g(x) = e^x - 2$. Even with a close initial guess of $x_1 = 1$ or $x_1 = 1.5$, the iteration scheme fails to converge, because $g'(x) > 1$. (a) Find $h(x)$ such that for the transformed problem $x = h(x)$, the iteration scheme $x_{n+1} = h(x_n)$ converges, and show that $h'(x) < 1$ for all x. (b) Sketch the simultaneous graphs of x and $h(x)$. With an initial guess of $x_1 = 2$, diagram on these graphs the convergence of the iteration scheme to the fixed point, as in Figure 4 of section 11.4. Demonstrate the divergence of the original iteration scheme $x_{n+1} = g(x_n)$ with a similar diagram.

7. Use Euler's method to discretize the logistic equation $\dfrac{dy}{dt} = ay - by^2$, and write your solution as a fixed point iteration scheme.

8. Determine whether the following infinite series converge or diverge. If a series converges, determine whether the convergence is absolute or conditional. Be sure to justify your answers completely.

   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\tan^{-1} n}{1+n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \sqrt{1 - \cos\left(\frac{1}{n}\right)}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{100}}{n!}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$ (e) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\pi}}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$