Mathematics 1220 PRACTICE EXAM II Spring 2002

ANSWERS

1. (a) Let $x = 1/n$. Then $\lim_{x\to 0} \dfrac{x^{-x}-1}{x} = -\lim_{x\to 0} x^{-x}(\ln x + 1)$ diverges, $\longrightarrow +\infty$.
   (b) Let $y = (\cos x)^{\csc x}$, and apply L'Hôpital's rule to $\ln y$, giving $y \to 1$.
   (c) Apply L'Hôpital's rule 25 times to x^{25}/e^x, giving 0.
   (d) Apply L'Hôpital's rule: $\lim_{x\to\infty} \dfrac{xe^x}{e^{x^2/2}} = \lim_{x\to\infty} e^{(\frac{-x^2}{2}+x)}\left(1+\dfrac{1}{x}\right)$. Since $\dfrac{-x^2}{2} + x = \dfrac{x}{2}(2-x) < 0$ for $x > 2$, the limit is 0.
   (e) $\lim_{x\to-\infty}(e^{-x} - x) = \lim_{x\to+\infty}(e^x + x) \longrightarrow +\infty$.
   (f) Apply L'Hôpital's rule, and use the Second Fundamental Theorem of Calculus on the numerator, which gives $\sin 1$ as the limit.
   (g) Apply L'Hôpital's rule, and use the Second Fundamental Theorem of Calculus on the numerator. Another application of L'Hôpital's rule gives 1/3.
   (h) 3 (i) $-1/2$

2. (a) 0 for $m = n$ and $m \neq n$, use $\sin mx \cos nx = \dfrac{1}{2}[\sin{(m+n)x} + \sin{(m-n)x}]$.
   (b) π, use $\sin mx \sin nx = -\dfrac{1}{2}[\cos 2mx - 1]$ for $m = n$.
   (c) $\dfrac{u^2}{2} + C,\quad u = \ln(\cosh x)$. (d) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{u+1}{2}\right) + C,\quad u = e^x$.
   (e) $\dfrac{u}{2}\sqrt{9-u^2} + \dfrac{9}{2}\sin^{-1}\dfrac{u}{3} + C,\quad u = (x+2),\quad \sqrt{5-4x-x^2} = \sqrt{9-u^2}$.
   (f) $\dfrac{x}{2}[\cos(\ln x) + \sin(\ln x)]$, use two integration by parts, the first with $u = \cos(\ln x)$, $dv = dx$, to solve for the integral.
   (g) $\dfrac{1}{2}(A\ln|x| + B\ln|x+3|) + C,\quad \dfrac{1}{x(x+3)} = \dfrac{A}{x} + \dfrac{B}{x+3}$, $A = 1/3, B = -1/3$.

3. (a) Multiply $f(x) = \sum_{n=1}^{N} a_n \sin(nx)$ by $\sin(mx)$, integrate from $-\pi$ to π, and use the orthogonality relation $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0$ for $n \neq m$ and $= \pi$ for $n = m$.
   (b) Multiply $f(x) = \sum_{n=1}^{N} a_n \sin(nx)$ by $f(x) = \sum_{m=1}^{N} a_m \sin(mx)$, integrate from $-\pi$ to π, and use the orthogonality relation.

4. (a) Let $u = \ln x$, then $p = 1/2$ for infinite domain $\Rightarrow$ divergence.
   (b) Bound the integrand above with C/x^p, for any p such that $1 < p < 3/2$, like $p = 5/4 \Rightarrow$ convergence.
   (c) $|e^{-x}\cos x| \leq e^{-x} \Rightarrow$ convergence.
   (d) Near $x = 0$, $(e^{-x^2}/x^2) \sim (1/x^2) \Rightarrow$ divergence.

(e) 16,000 integration by parts reduces the integral to a purely decaying exponential $\Rightarrow$ convergence; or use comparison $x^{16{,}000}e^{-x} < e^{-x/2}$ for sufficiently large x, but you must show this!

(f) Near $x = 0$, integrand $\sim 1/x^{2/3} \Rightarrow$ convergence.

5. (a) diverges (n^{th} term test, or geometric series with $|r| > 1$)

   (b) $\sqrt{1 - \cos\left(\frac{1}{n}\right)} \sim \frac{1}{n} \Longrightarrow$ divergence

   (c) diverges $\sqrt{n}\,\sin\left(\frac{1}{n}\right) \sim \frac{1}{\sqrt{n}}$

   (d) diverges (n^{th} term test)

   (e) converges absolutely (integral test)

   (f) converges absolutely (telescopes, or p-test)

6. 4 meters