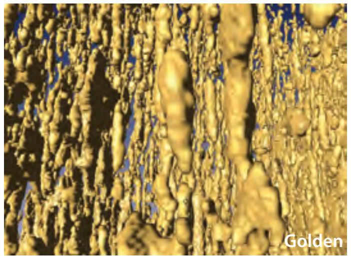
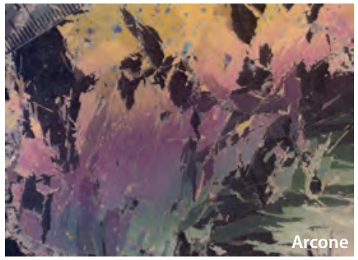
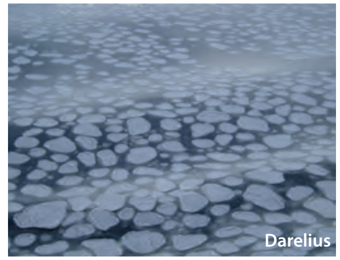
millimeters



centimeters

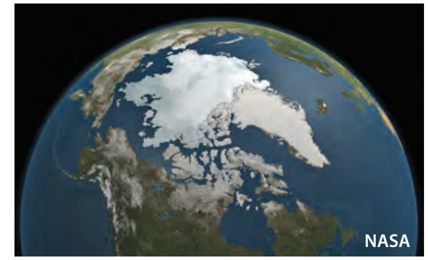


meters



kilometers

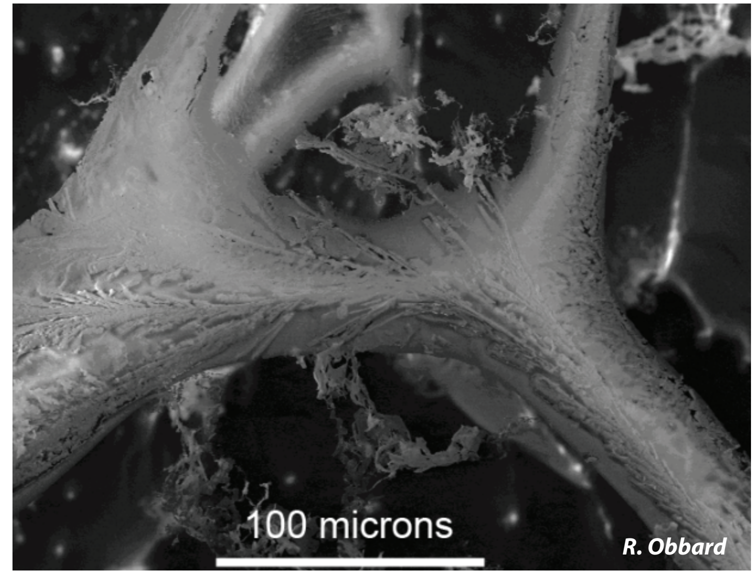
10^3 kilometers



From Micro to Macro in the Physics and Biology of Sea Ice

Kenneth M. Golden
Dept. of Mathematics, Univ. of Utah

brine inclusions in sea ice (mm)



micro - brine channel (SEM)

brine channels (cm)

sea ice is a porous composite

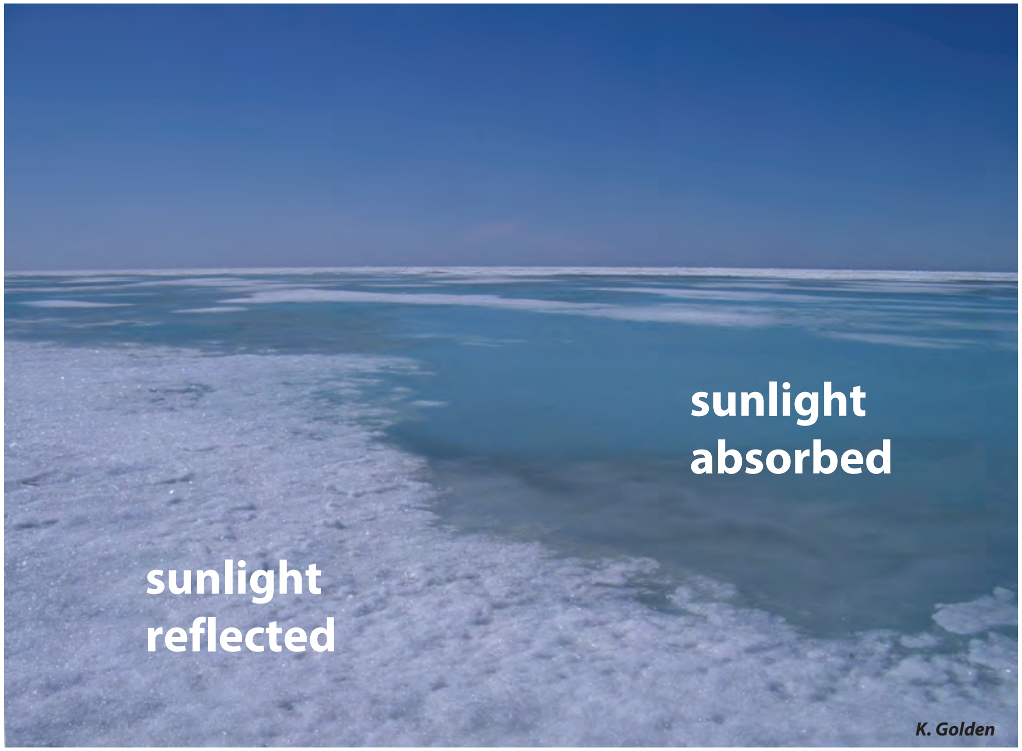
pure ice with brine, air, and salt inclusions

horizontal section

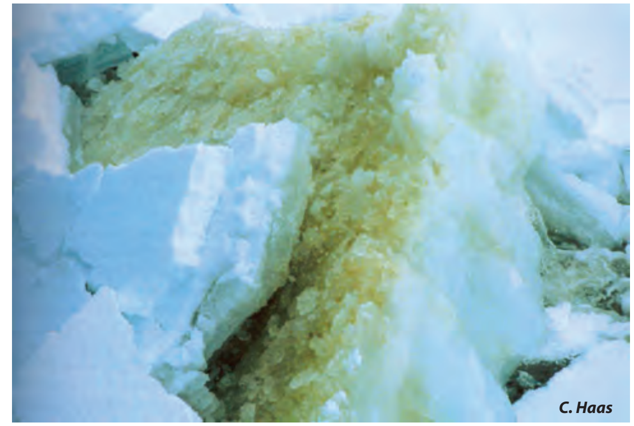
vertical section

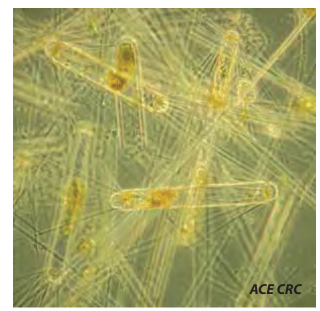
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

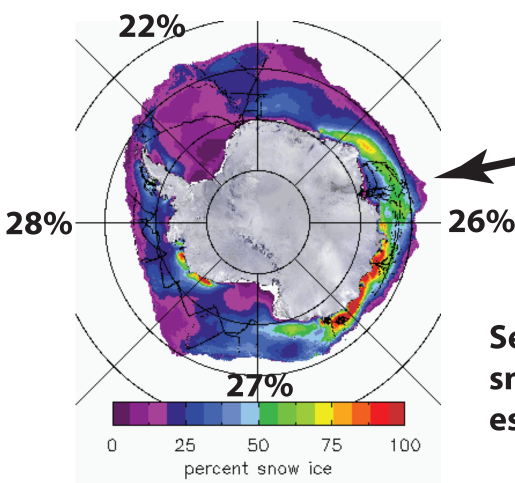
evolution of Arctic melt ponds and sea ice albedo



nutrient flux for algal communities







T. Maksym and T. Markus, 2008

September snow-ice estimates

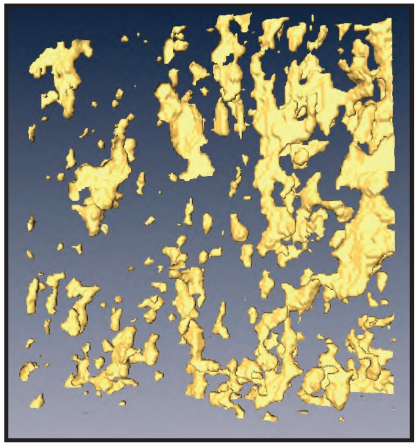
Antarctic surface flooding and snow-ice formation

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO_2*

Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

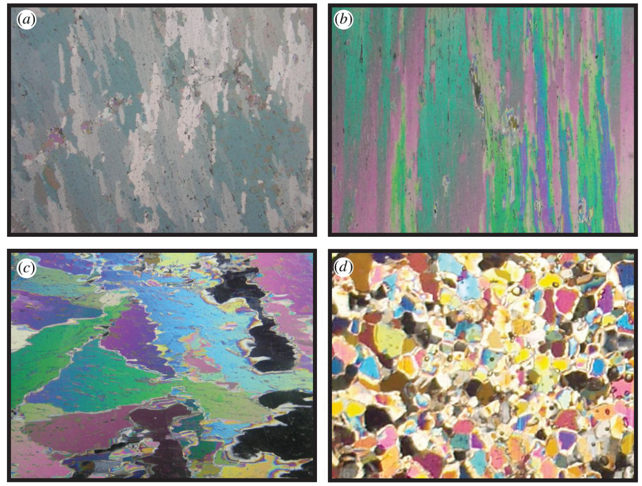


Weeks & Assur 1969

H. Eicken

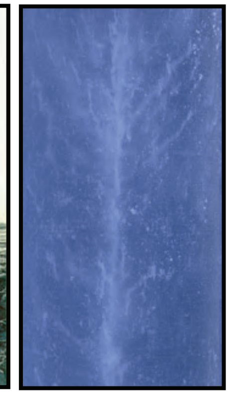
Golden *et al. GRL* 2007

polycrystals



Gully *et al. Proc. Roy. Soc. A* 2015

brine channels



D. Cole **K. Golden**

millimeters **centimeters**

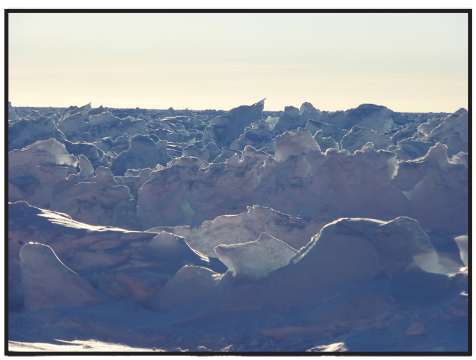
---

mesoscale

Arctic melt ponds

K. Frey

Antarctic pressure ridges



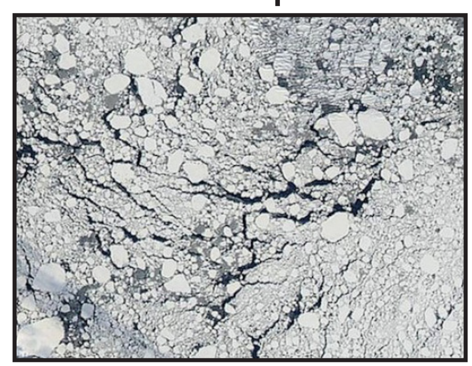
K. Golden

macroscale

sea ice floes

J. Weller

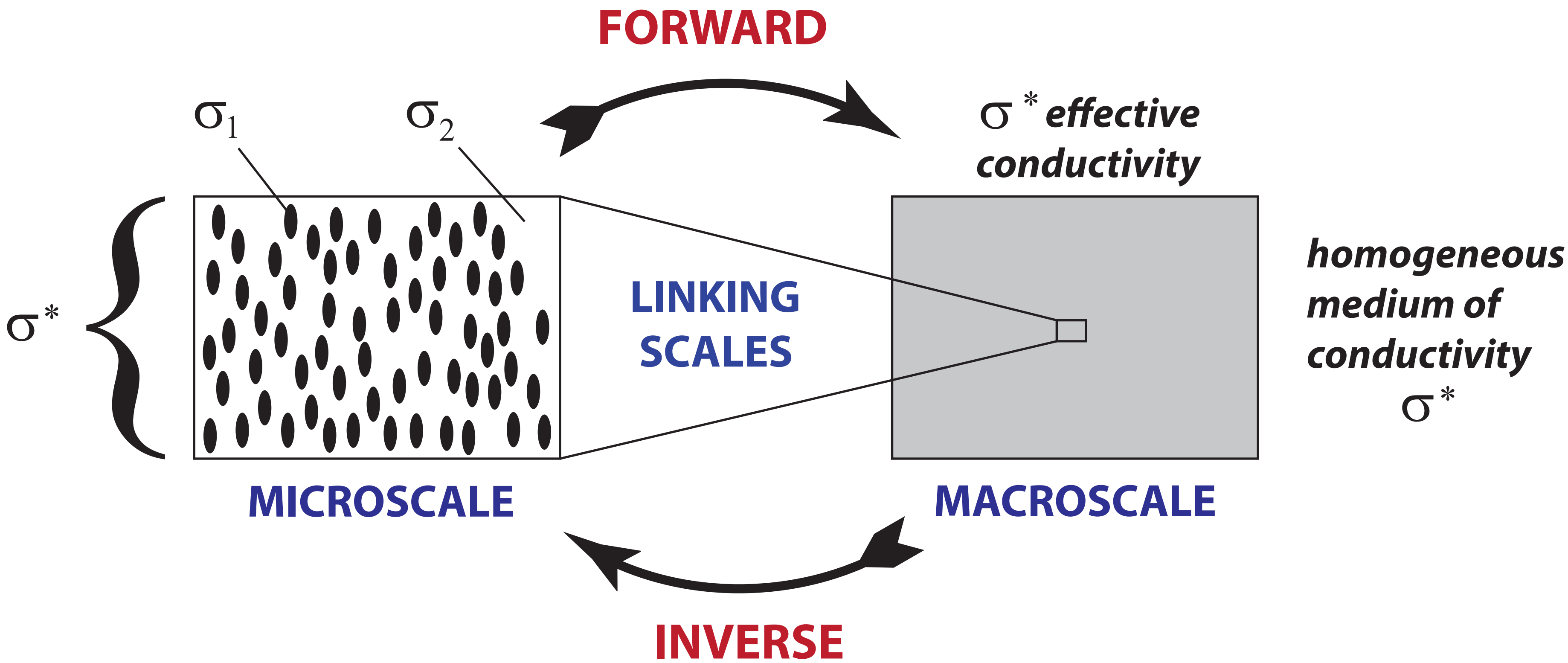
sea ice pack



NASA

meters **kilometers**

HOMOGENIZATION for Composite Materials



Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

*Wiener 1912 : arithmetic and harmonic mean **bounds** on effective conductivity*
*Hashin and Shtrikman 1962 : variational **bounds** on effective conductivity*

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

What is this talk about?

A tour of recent results on modelling macroscopic behaviour in the sea ice system, with a focus on novel mathematics.

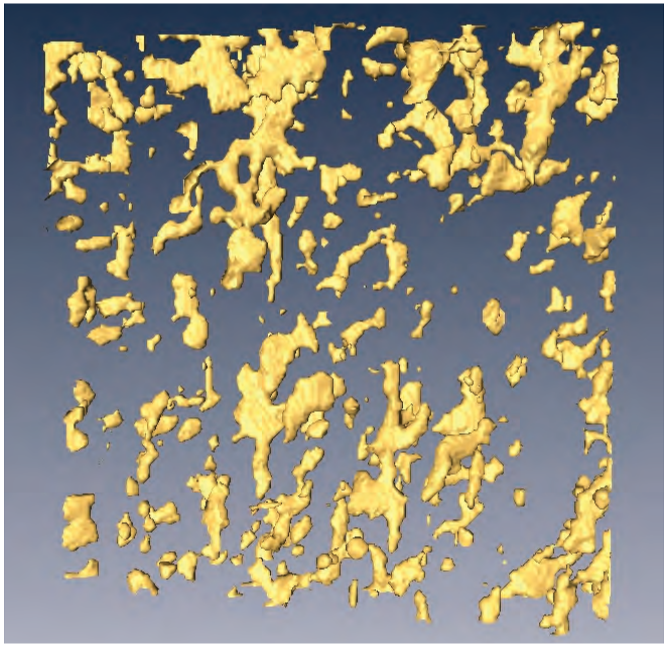
microscale

mesoscale

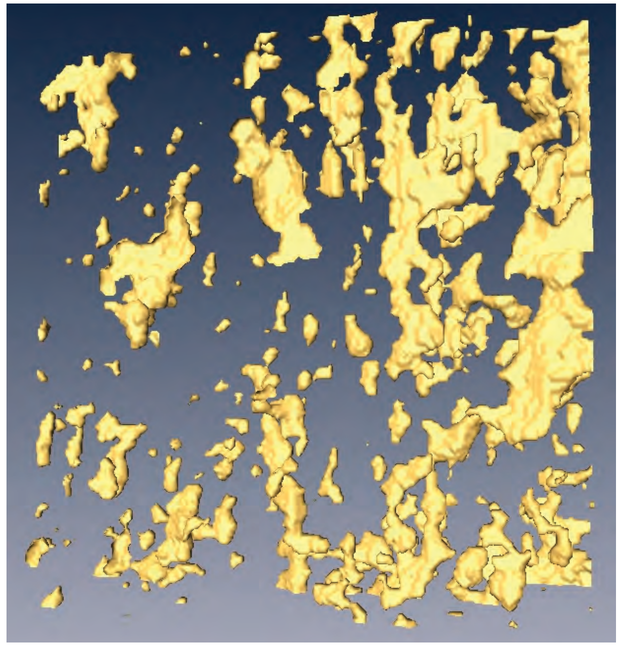
macroscale

microscale

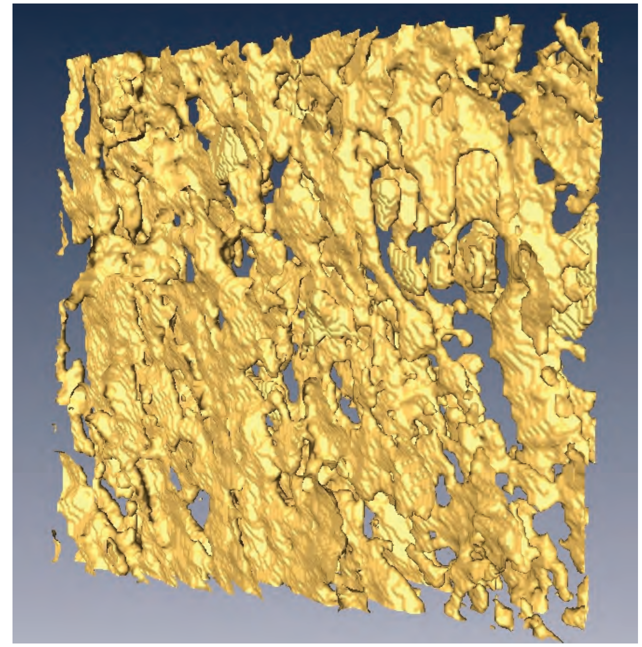
brine volume fraction and ***connectivity*** increase with temperature



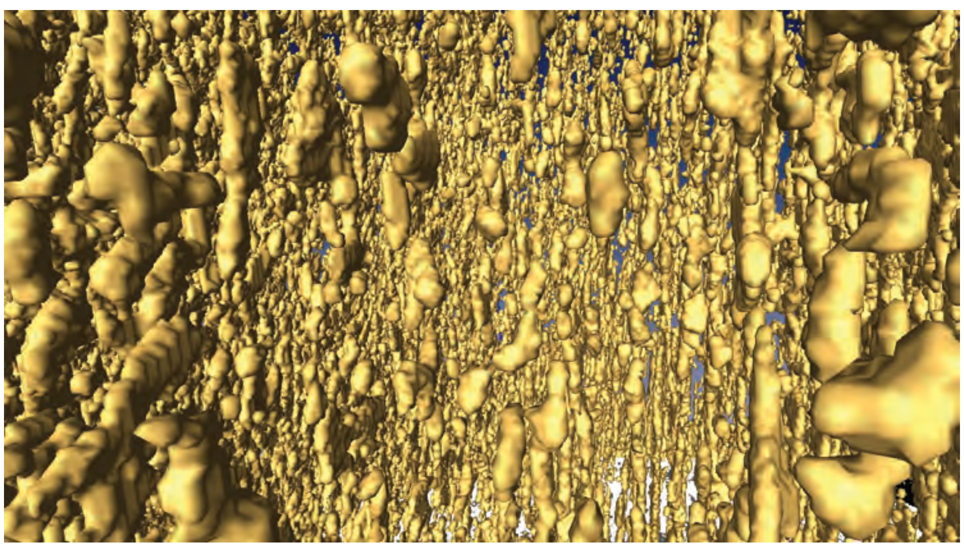
T = -15 °C, $\phi = 0.033$



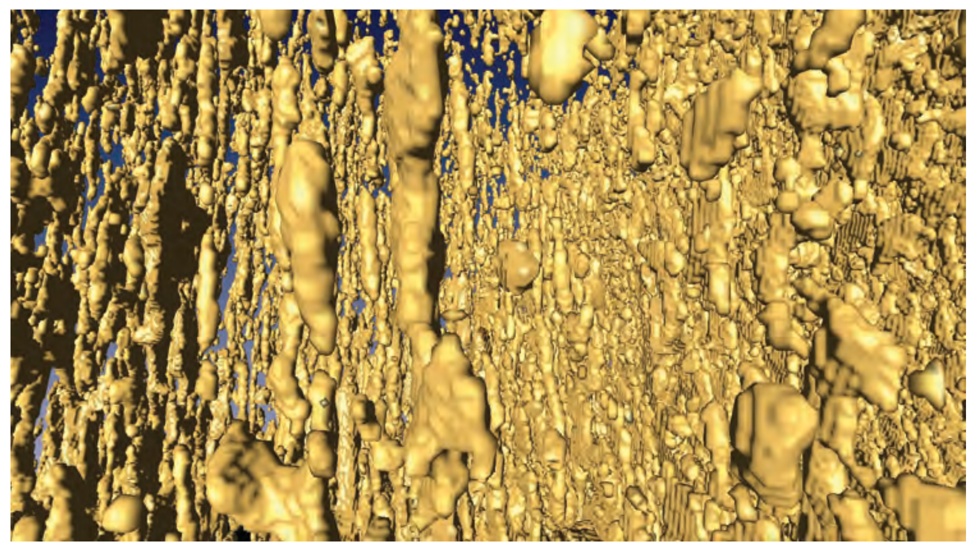
T = -6 °C, $\phi = 0.075$



T = -3 °C, $\phi = 0.143$



T = -8 °C, $\phi = 0.057$



T = -4 °C, $\phi = 0.113$

X-ray tomography for brine in sea ice

Golden *et al.*, *Geophysical Research Letters*, 2007

Critical behavior of fluid transport in sea ice

Arctic field data

vertical fluid permeability k (m^2)

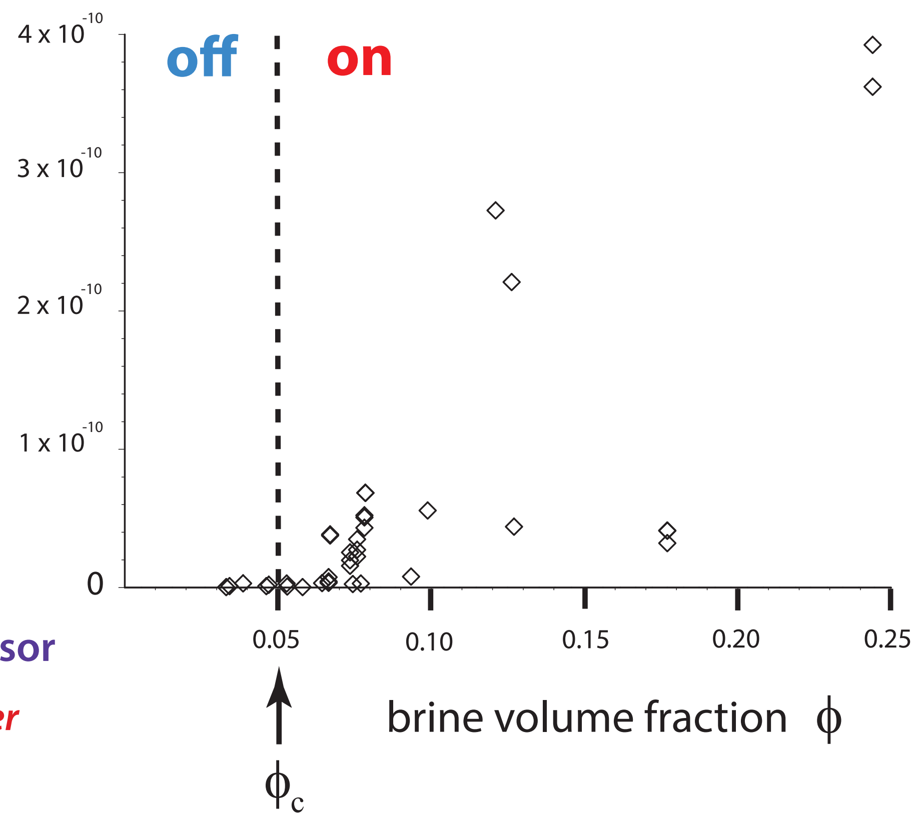
Darcy's Law

$$\mathbf{v} = -\frac{\mathbf{k}}{\eta} \nabla p$$

$\mathbf{k}$ = fluid permeability tensor

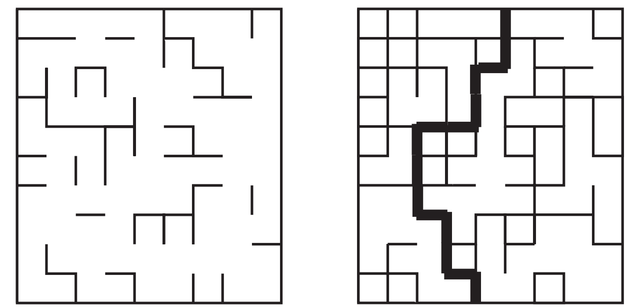
homogenized parameter

impermeable permeable



"on - off" switch for bulk fluid flow

impermeable permeable



$p = 1/3$ $p = 2/3$

lattice percolation

PERCOLATION THRESHOLD $\phi_c \approx 5\% \longleftrightarrow T_c \approx -5^\circ C,\ S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

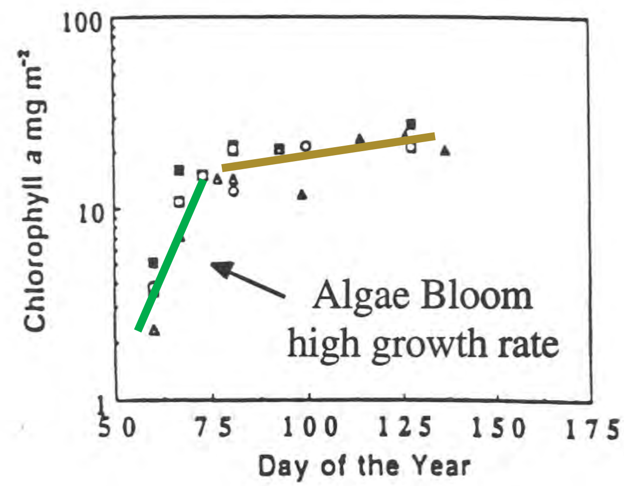
sea ice algal communities

D. Thomas 2004

nutrient replenishment controlled by ice permeability

biological activity turns on or off according to ***rule of fives***

critical behavior of microbial activity



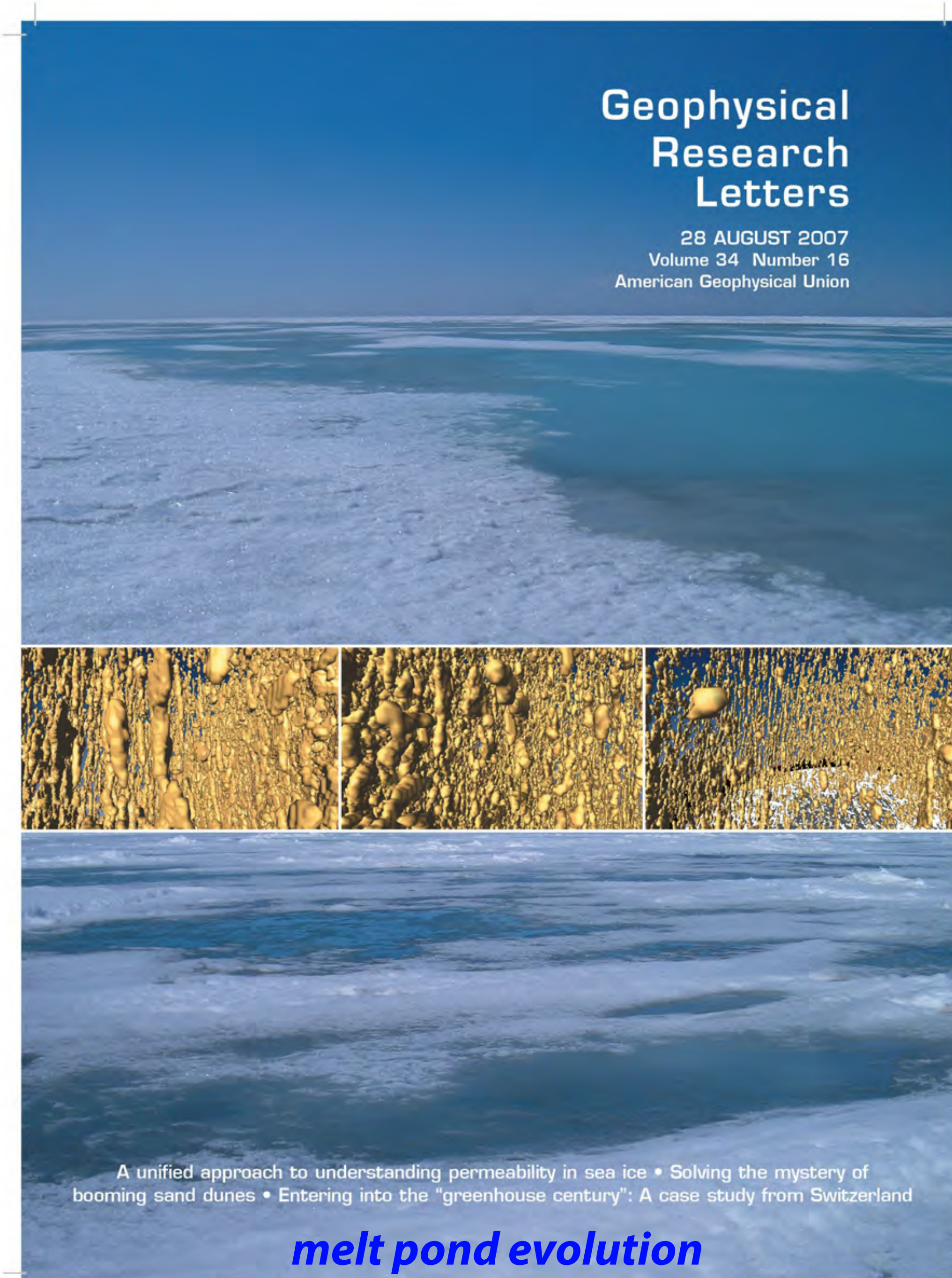
Convection-fueled algae bloom
Ice Station Weddell

Golden, Ackley, Lytle ***Science 1998***

Fritsen, Lytle, Ackley, Sullivan ***Science 1994***

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007



melt pond evolution

microscale

governs

mesoscale processes

percolation theory for fluid permeability

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical exponent t

$$k_0 = 3 \times 10^{-8} \text{ m}^2$$

from critical path analysis in hopping conduction

hierarchical model

rock physics

network model

rigorous bounds

X-ray tomography for brine inclusions

confirms rule of fives

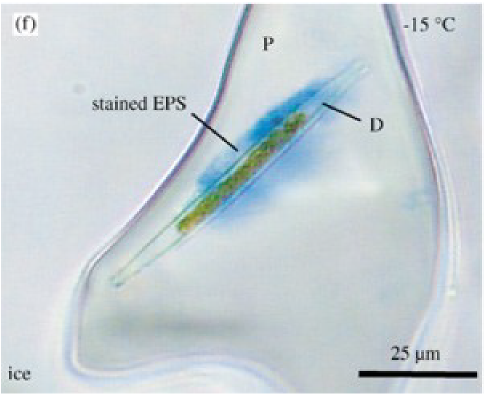
brine percolation threshold of $\phi = 5\%$ for bulk fluid flow

Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

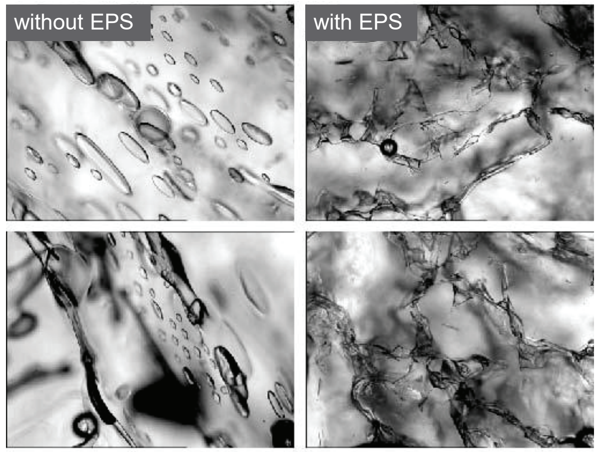
theories agree closely with field data

Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

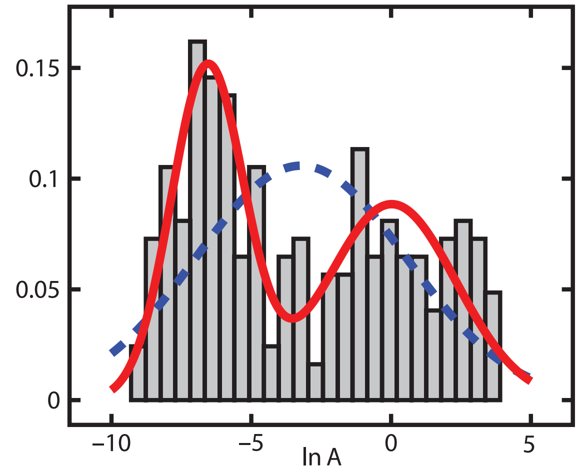
How does EPS affect fluid transport? **How does the biology affect the physics?**



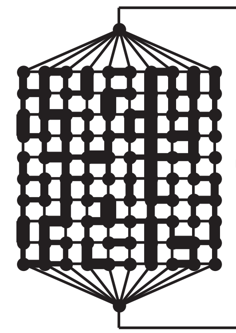
Krembs



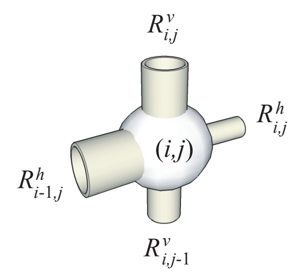
Krembs, Eicken, Deming, PNAS 2011



RANDOM PIPE MODEL



- 2D random pipe model with bimodal distribution of pipe radii
- Rigorous bound on permeability *k;* results predict observed drop in *k*

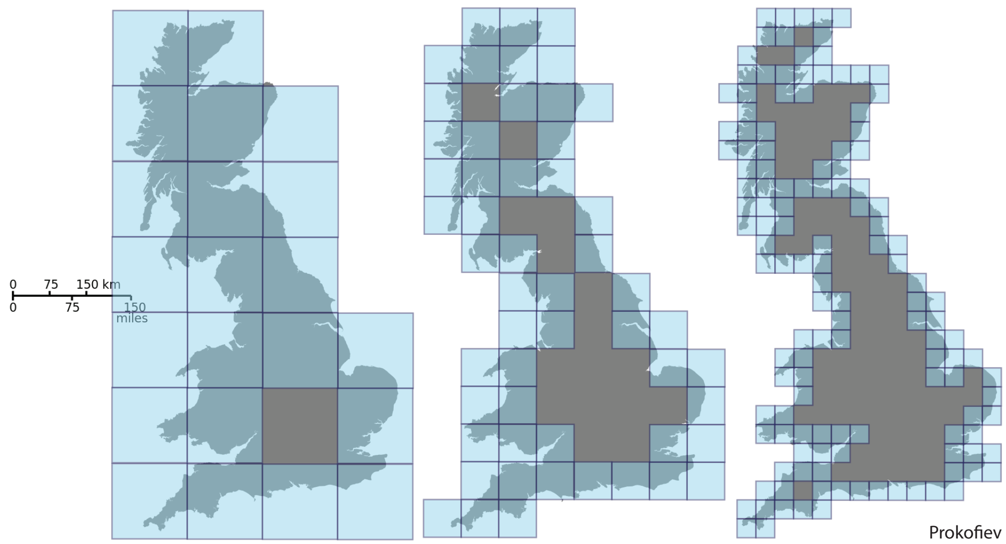


Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
***Multiscale Modeling and Simulation*, 2018**

Zhu, Jabini, Golden, Eicken, Morris
***Ann. Glac.* 2006**

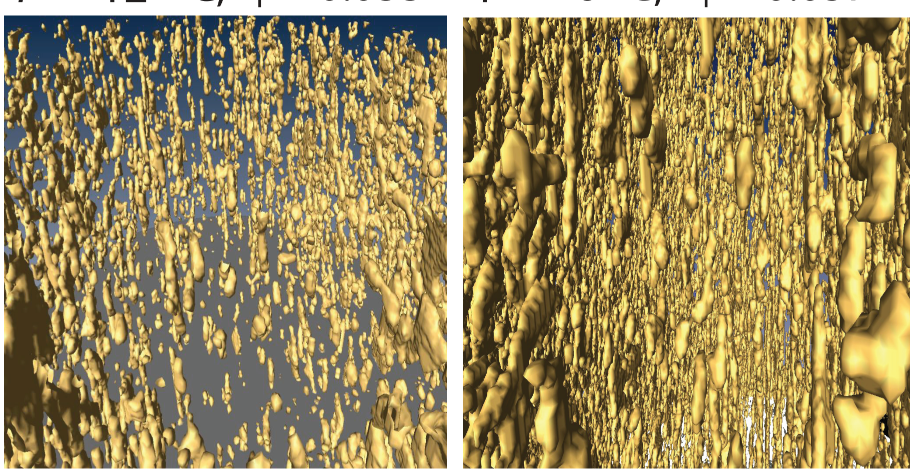
Thermal Evolution of Brine Fractal Geometry in Sea Ice

Nash Ward, Daniel Hallman, Benjamin Murphy, Jody Reimer,
Marc Oggier, Megan O'Sadnick, Elena Cherkaev and Kenneth Golden, 2022



fractal dimension of the British coastline by box counting

$T = -12^\circ$ C, $\phi = 0.033$ $T = -8^\circ$ C, $\phi = 0.057$



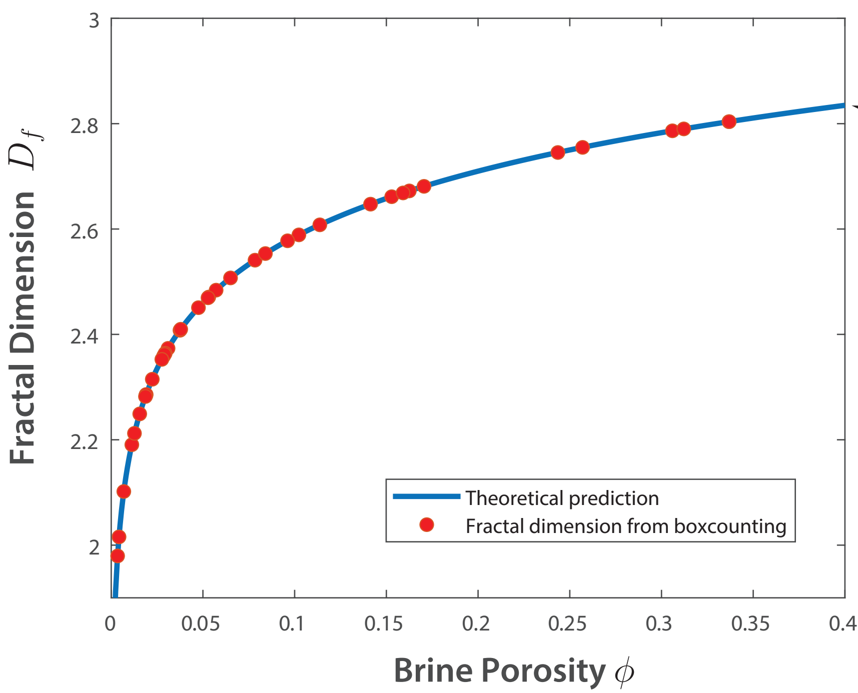
Golden, Eicken, et al. *GRL*, 2007

brine channels and inclusions "look" like fractals (from 30 yrs ago)

X-ray computed tomography of brine in sea ice

columnar and granular

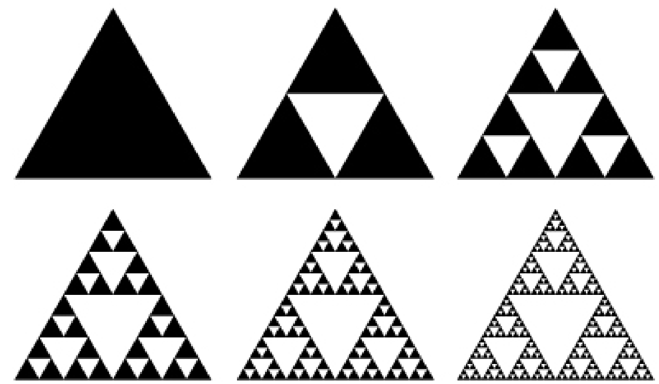
The first comprehensive, quantitative study of the fractal dimension of brine in sea ice and its strong dependence on temperature and porosity.

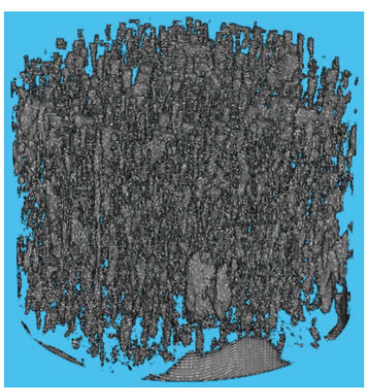


$$D_f = 3 - \frac{\ln \phi}{\ln(\lambda_{min}/\lambda_{max})}$$

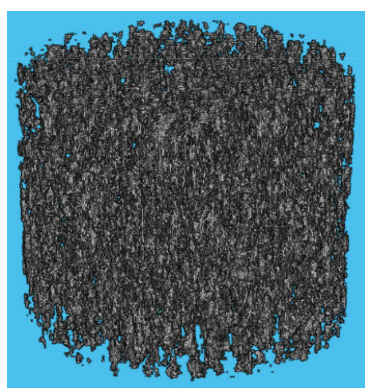
The blue curve is exact for the Sierpinski gasket (an exactly self-similar geometry); discovered for sandstones - statistically self-similar porous media.

Katz and Thompson, 1985
Yu and Li, 2001

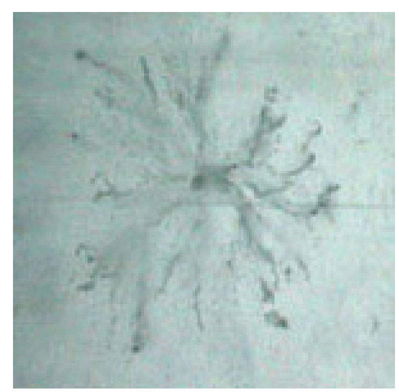




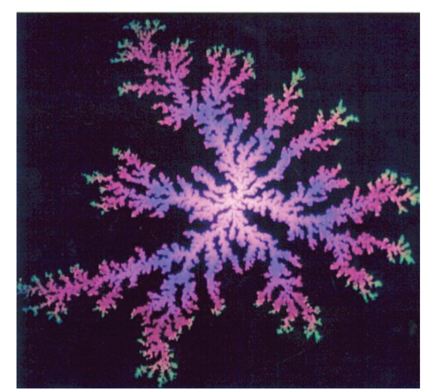
X-ray tomography



DLA model

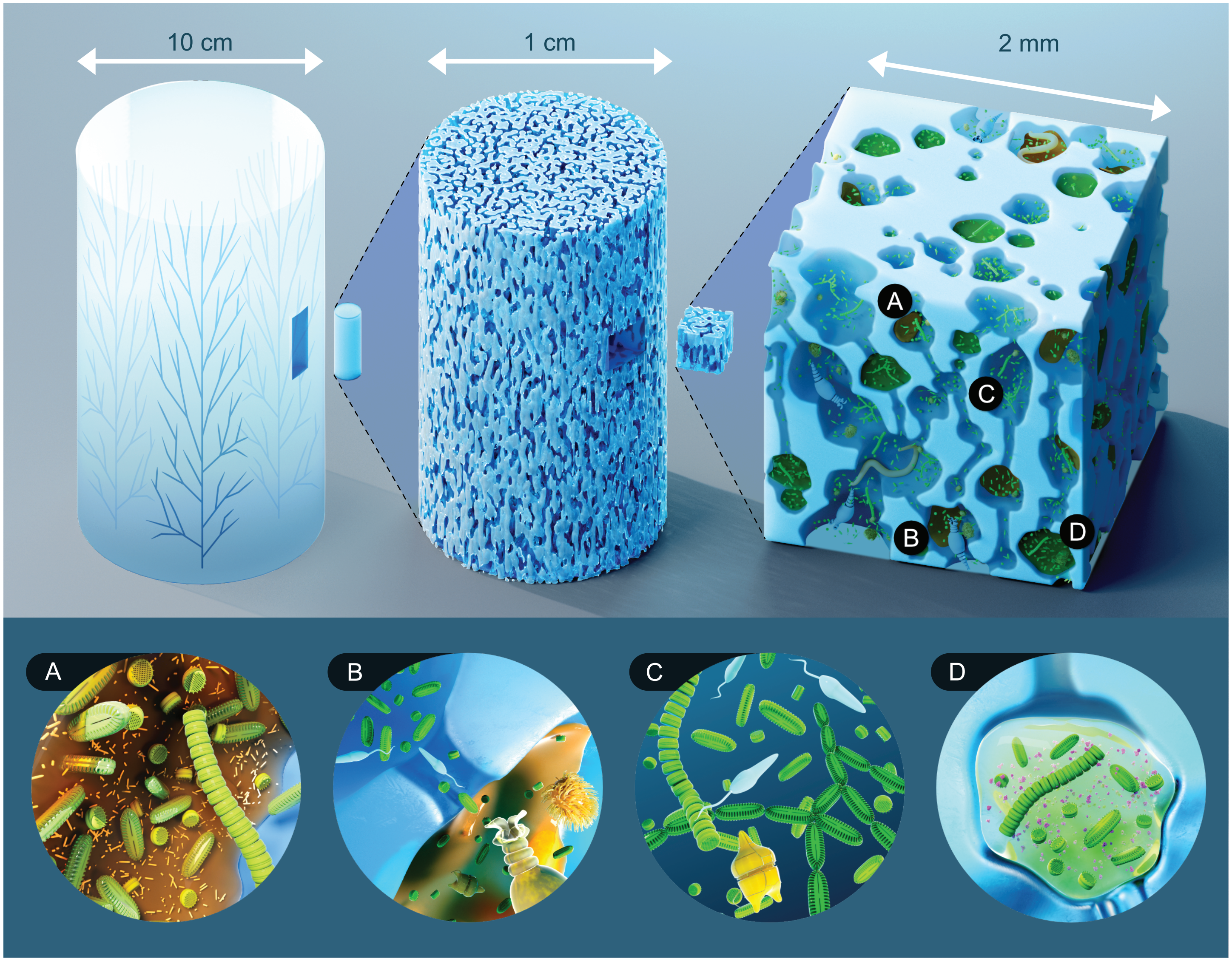


brine channel in sea ice



diffusion limited aggregation

Implications of brine fractal geometry on sea ice ecology and biogeochemistry



Brine inclusions are home to ice endemic organisms, e.g., bacteria, diatoms, flagellates, rotifers, nematodes.

The habitability of sea ice for these organisms is inextricably linked to its complex brine geometry.

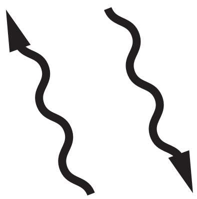
(A) Many sea ice organisms attach themselves to inclusion walls; inclusions with a higher fractal dimension have greater surface area for colonization.

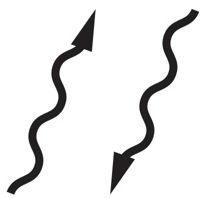
(B) Narrow channels prevent the passage of larger organisms, leading to refuges where smaller organisms can multiply without being grazed, as in (C).

(D) Ice algae secrete extracellular polymeric substances (EPS) which alter incusion geometry and may further increase the fractal dimension.


Golden

Remote sensing of sea ice





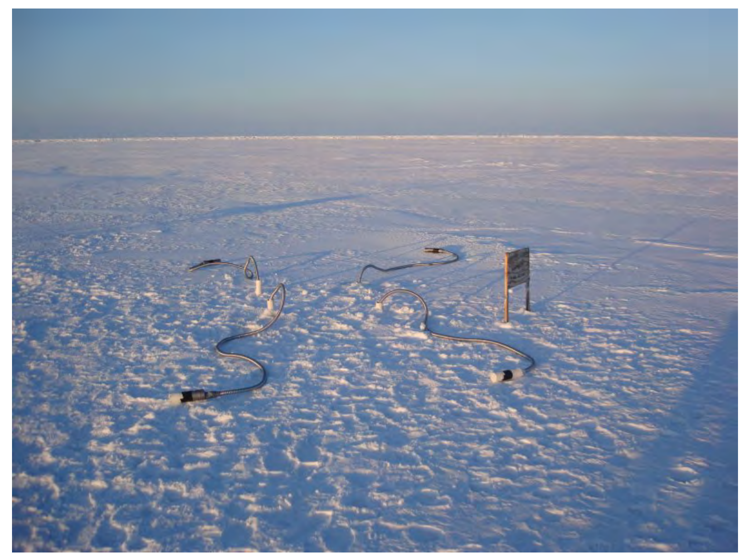
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice properties from electromagnetic (EM) data

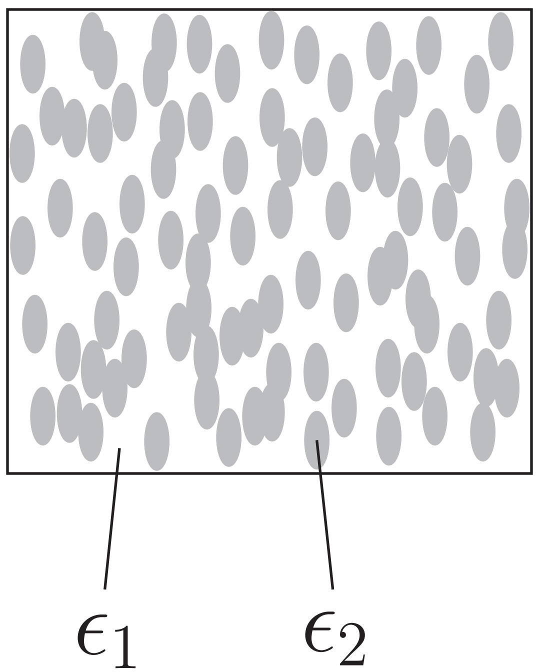
ε^*

effective complex permittivity (dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$\Big\} \ \epsilon^*$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

$p_1 \, , \, p_2$ = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \, , \text{ composite geometry} \right)$$

What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?

Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

separates geometry from parameters

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s-z}$$

geometry

material parameters

$$s = \frac{1}{1 - \epsilon_1 / \epsilon_2}$$

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla \cdot$$

$\chi =$ characteristic function of the brine phase

$$E = s\,(s + \Gamma\chi)^{-1} e_k$$

$\Gamma\chi$ **: microscale $\longrightarrow$ macroscale**

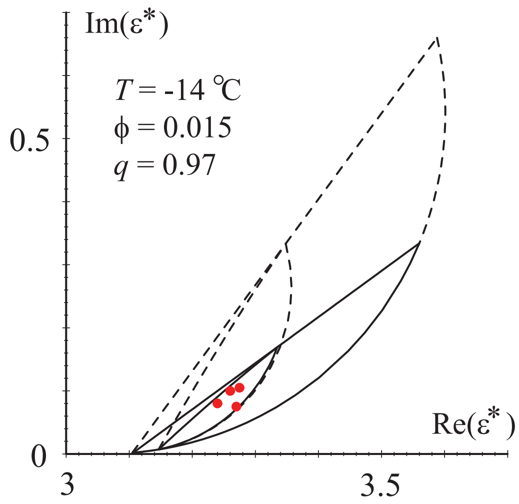
$\Gamma\chi$ ***links scales***

Golden and Papanicolaou, Comm. Math. Phys. 1983

This representation distills the complexities of mixture geometry into the spectral properties of an operator like the Hamiltonian in physics.

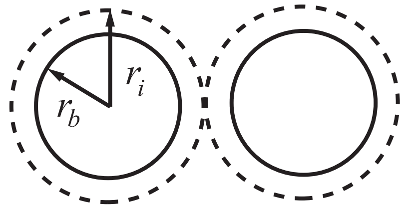
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle

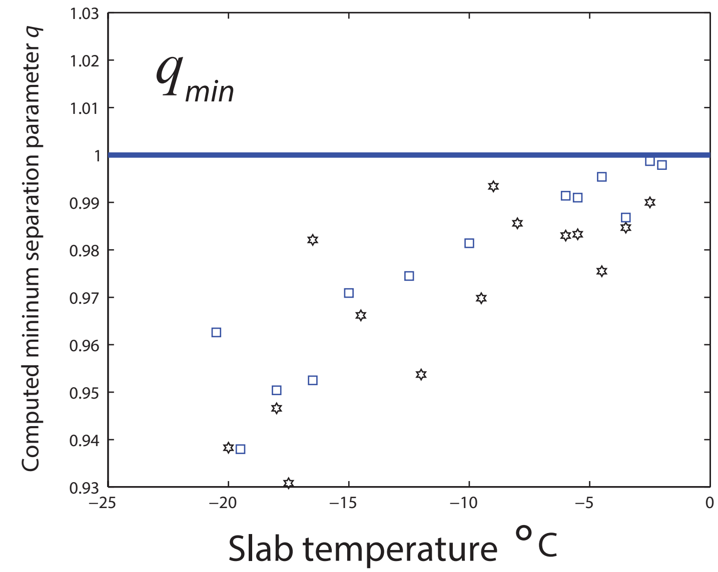


$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

inverse bounds



Inverse Homogenization

Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001), McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

ε^* → ***composite geometry*** **(spectral measure μ)**

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

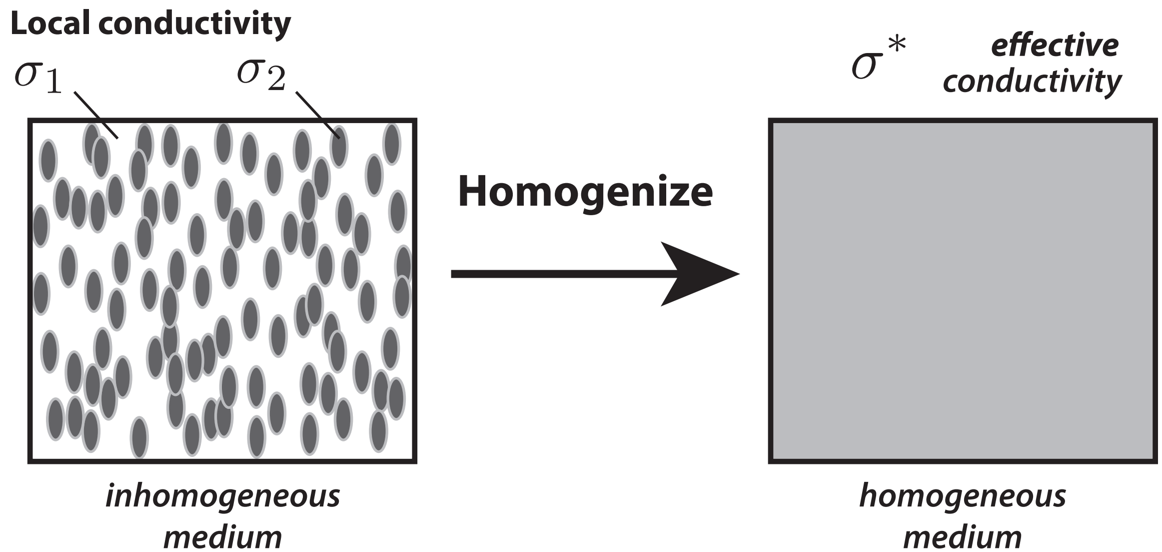
rigorous inverse bound on spectral gap

construct algebraic curves which bound admissible region in (p,q)-space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

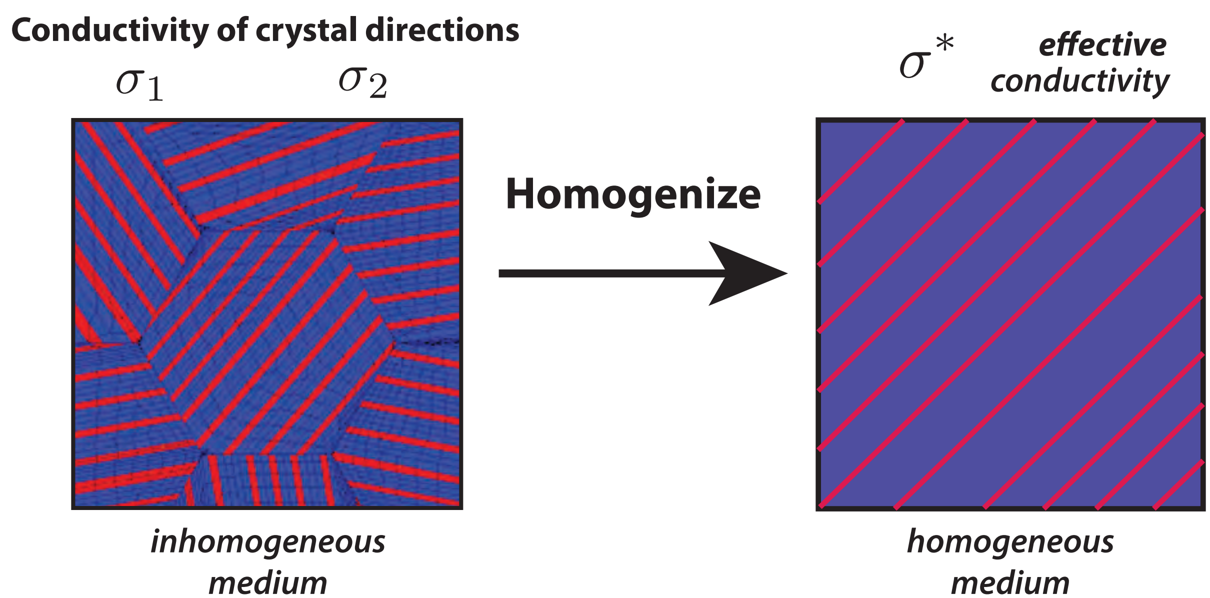
Homogenization for polycrystalline materials

Two-component composites



Find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Polycrystalline media



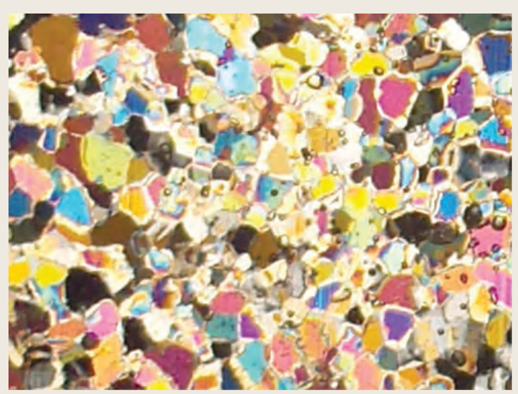
Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**

  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**

  orientation statistics
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A 8 Feb 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS OF SCIENTIFIC PUBLISHING

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

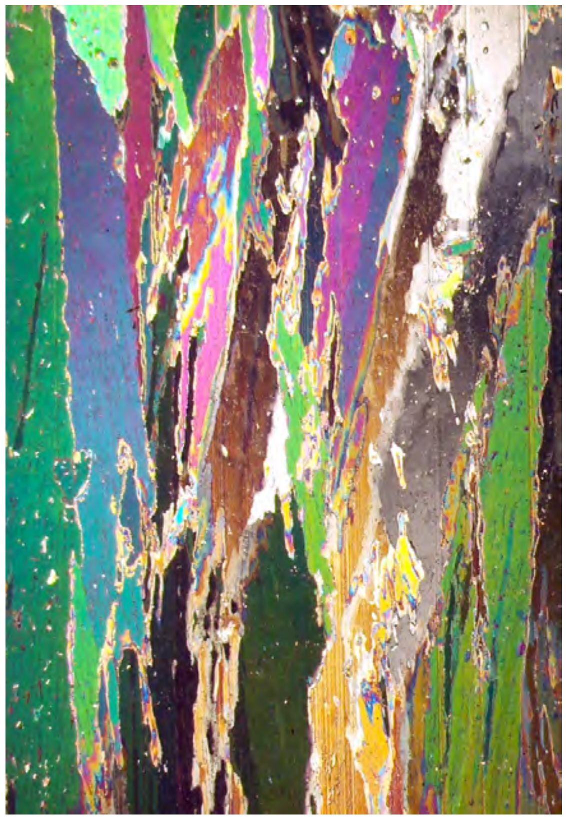
higher threshold for fluid flow in granular sea ice

microscale details impact “mesoscale” processes

nutrient fluxes for microbes
melt pond drainage
snow-ice formation

columnar | granular

5%



10%

Golden, Sampson, Gully, Lubbers, Tison 2022

electromagnetically distinguishing ice types

Kitsel Lusted, Elena Cherkaev, Ken Golden

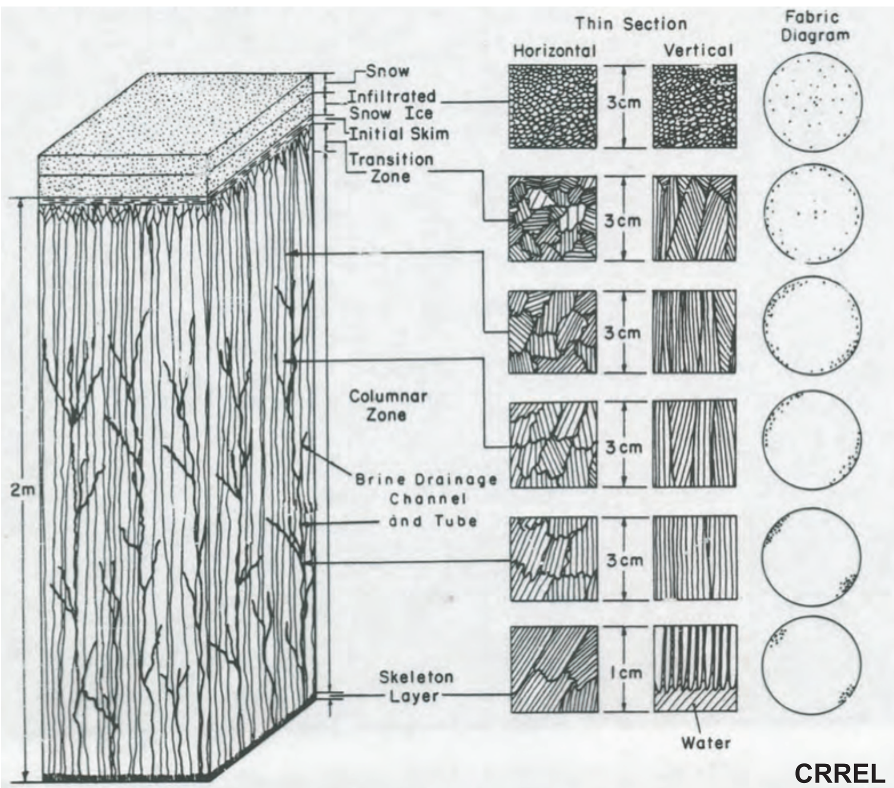
Rigorous bounds on the complex permittivity tensor of sea ice with polycrystalline anisotropy in the horizontal plane

Kenzie McLean, Elena Cherkaev, Ken Golden 2022

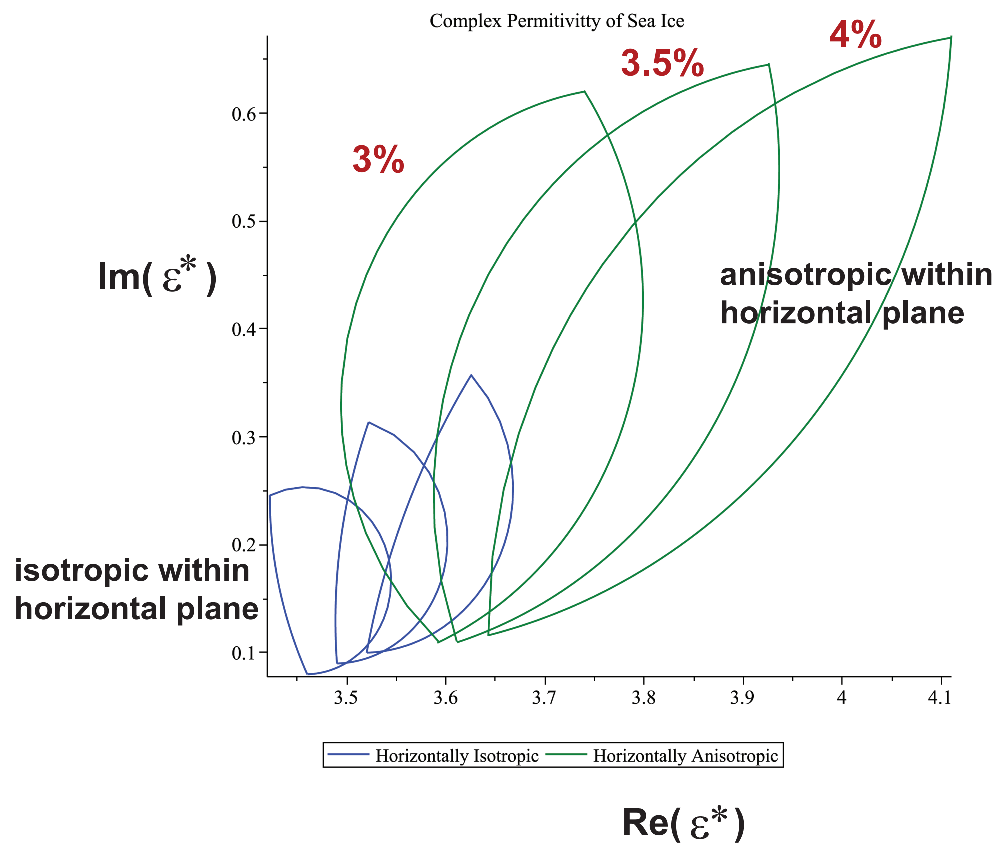
motivated by **Weeks and Gow, *JGR* 1979: c-axis alignment in Arctic fast ice off Barrow**

Golden and Ackley, *JGR* 1981: radar propagation model in aligned sea ice

input: orientation statistics



output: bounds



direct calculation of spectral measures

Murphy, Hohenegger, Cherkaev, Golden, *Comm. Math. Sci.* 2015

- depends only on the composite geometry
- discretization of microstructural image gives binary network
- fundamental operator becomes a random matrix
- spectral measure computed from eigenvalues and eigenvectors

once we have the spectral measure μ it can be used in Stieltjes integrals for other transport coefficients:

electrical and thermal conductivity, complex permittivity, magnetic permeability, diffusion, fluid flow properties

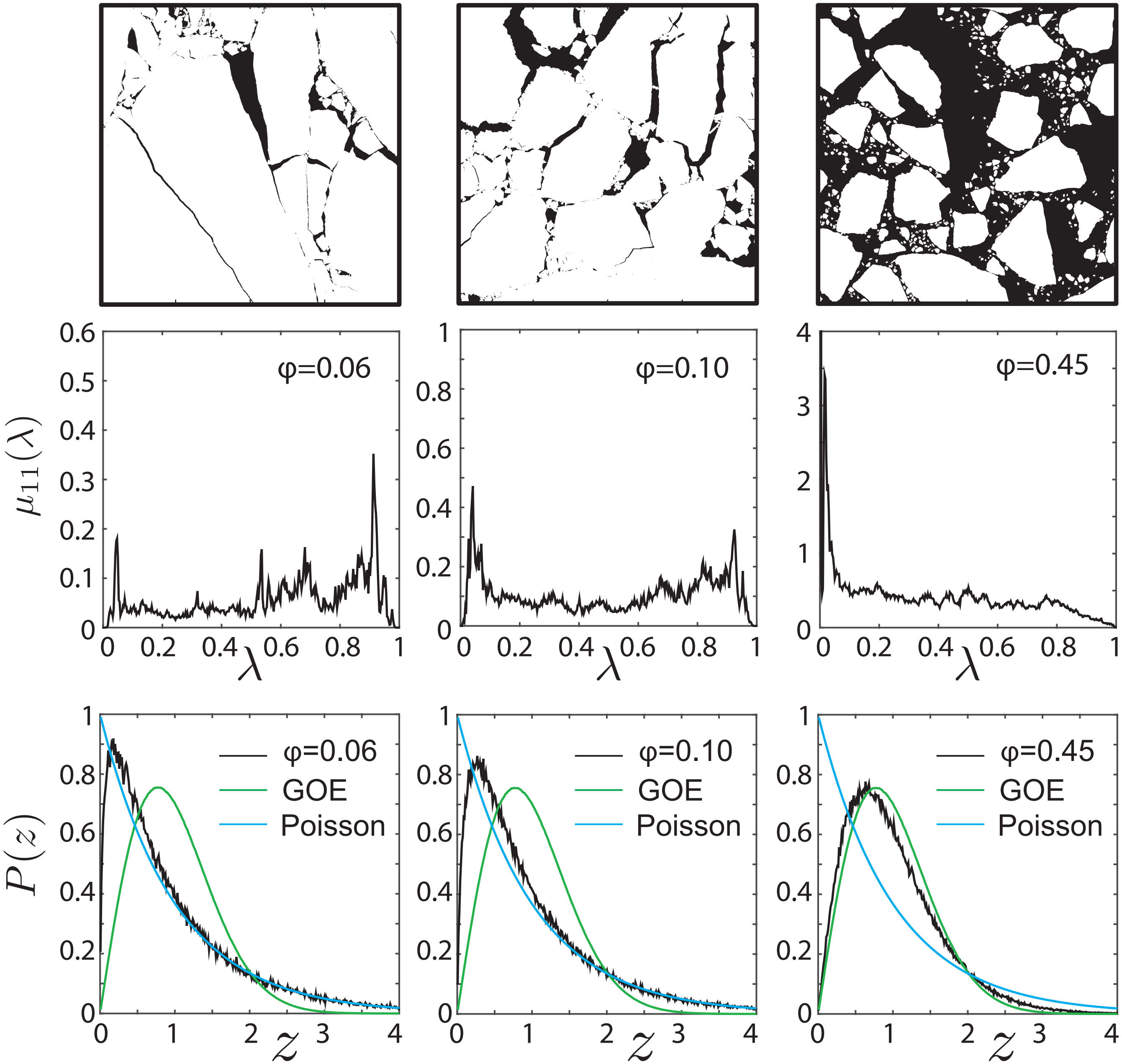
earlier studies of spectral measures

Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Spectral computations for sea ice floe configurations

spectral measures

eigenvalue spacing distributions



UNIVERSAL Wigner-Dyson distribution

uncorrelated → **level repulsion**

Eigenvalue Statistics of Random Matrix Theory

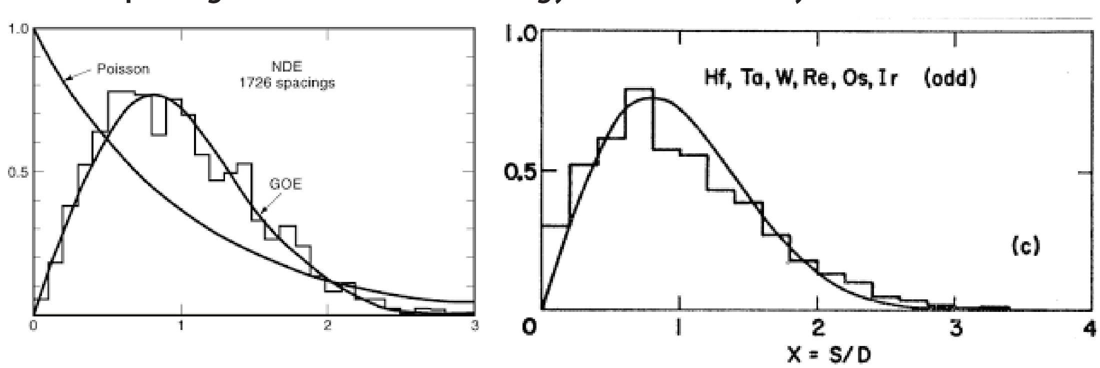
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[\mathbf{N}]_{ij} \sim N(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^T)/2$ ***Gaussian orthogonal ensemble (GOE)***

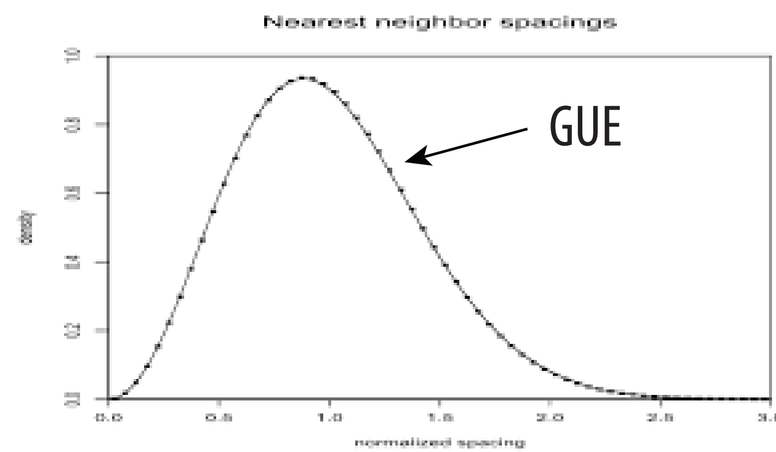
$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^\dagger)/2$ ***Gaussian unitary ensemble (GUE)***

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics.

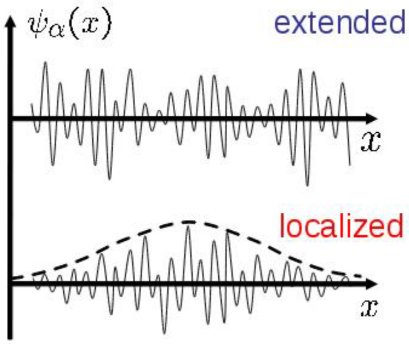
Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



Universal eigenvalue statistics arise in a broad range of "unrelated" problems!



electronic transport in semiconductors

metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

Anderson transition in wave physics:
quantum, optics, acoustics, water waves, ...

---

from analysis of spectral measures for brine, melt ponds, ice floes

we find percolation-driven

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

PERCOLATION TRANSITION → universal eigenvalue statistics (GOE)
extended states, mobility edges

-- but with NO wave interference or scattering effects ! --

Order to disorder in quasiperiodic composites

Morison, Murphy, Cherkaev, Golden, Comm. Phys. 2022

sea ice inspired - high tech spin off

tunable quasiperiodic composites with exotic properties

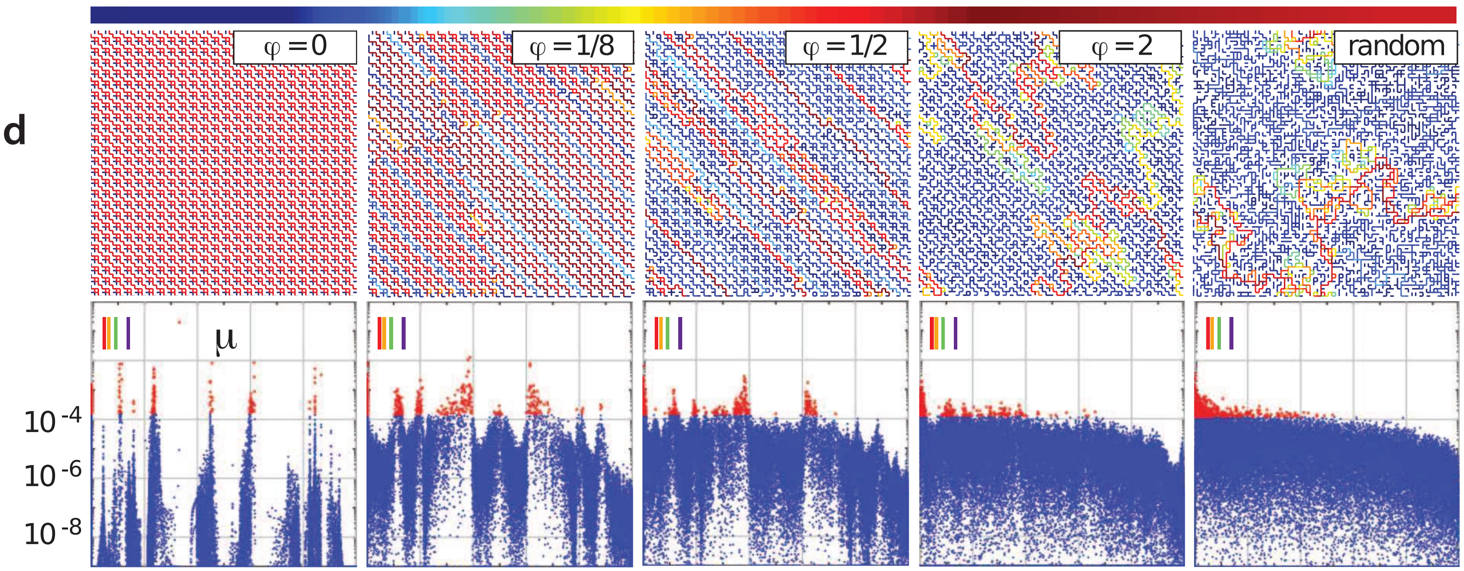
(optical, electrical, thermal, ...), Anderson localization; our Moiré patterned geometries are similar to twisted bilayer graphene

increasing twist angle between two lattices

periodic → quasiperiodic

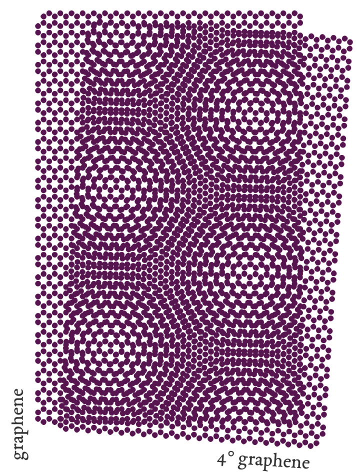
electric field strength

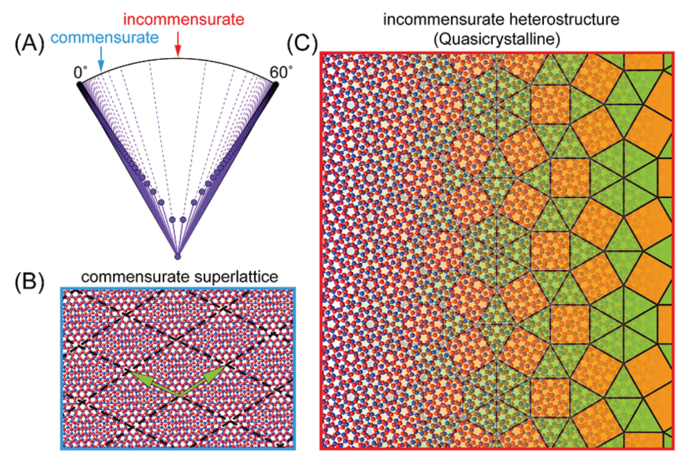
spectral measure



RRN at percolation threshold

twisted bilayer graphene





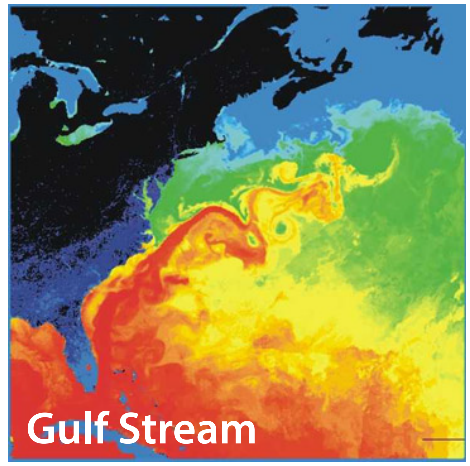
Yao et al., 2018

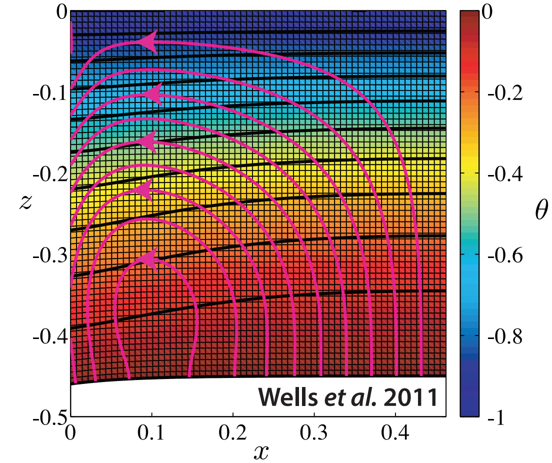
mesoscale

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

↓ ***homogenize***

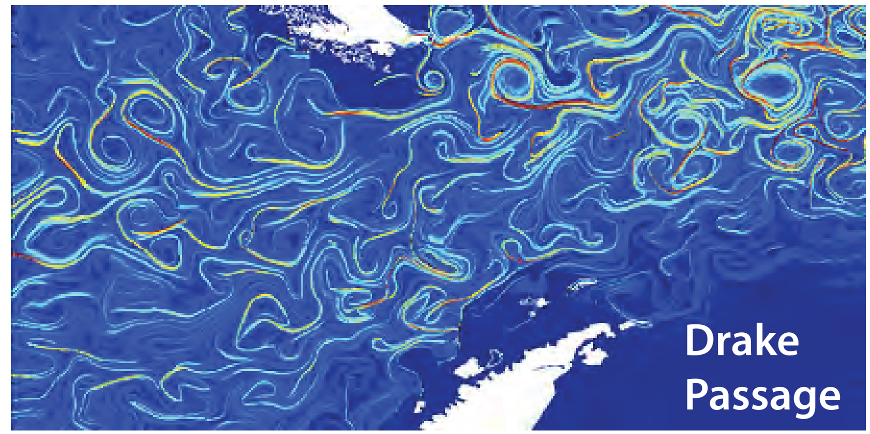
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* ***effective diffusivity***

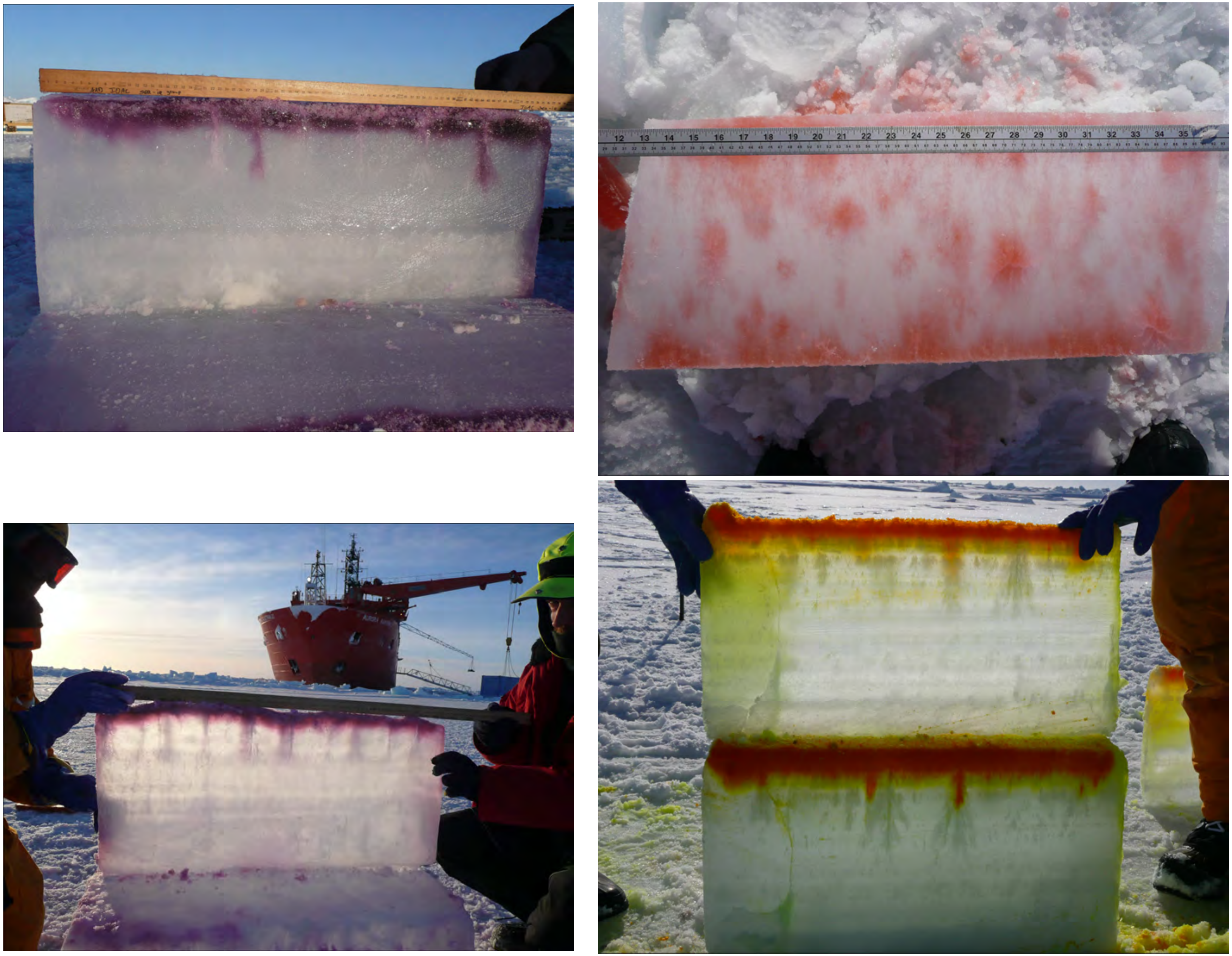
Stieltjes integral for κ^* ***with spectral measure***

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020



tracers flowing through inverted sea ice blocks



Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

$$\kappa^* = \kappa \left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2} \right), \quad F(\kappa) = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2 + \tau^2}$$

- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H \Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$, Δ is the Laplace operator
- $i\Gamma H \Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ-plane

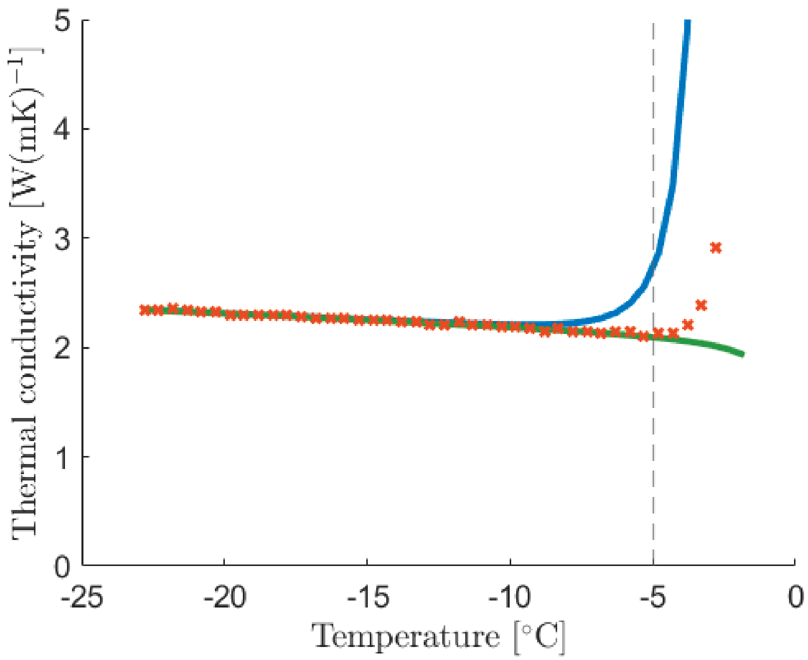
rigorous framework for numerical computations of spectral measures and effective diffusivity for model flows

new integral representations, theory of moment calculations

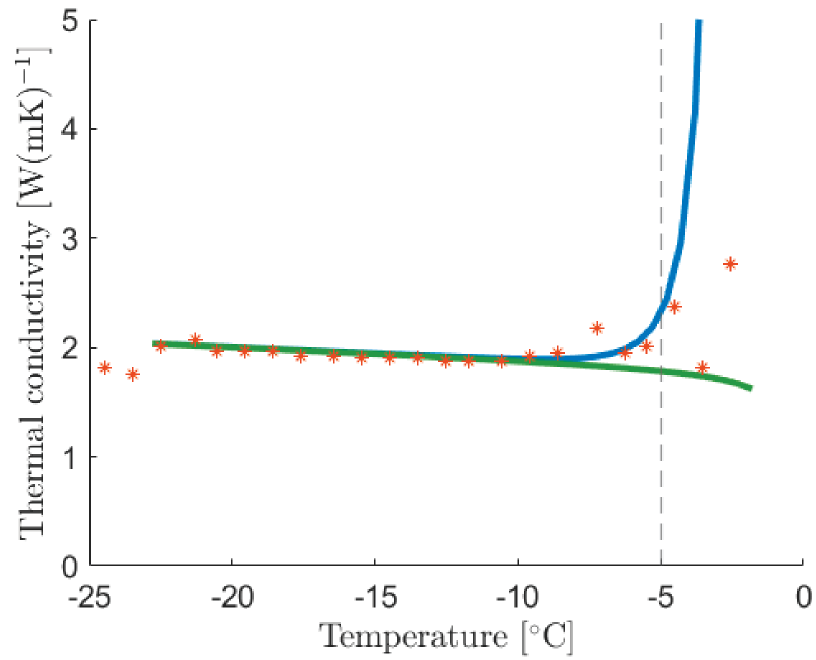
separation of material properties and flow field

Bounds on Convection Enhanced Thermal Transport

simulations



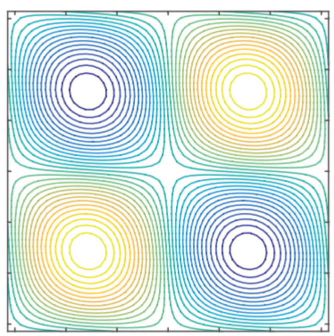
data [Trodahl et al., 2001]



Monte-Carlo simulations of SDE with temperature dependent Péclet number P

strength of advection B = κP/2π

Euler-Maruyama and subsampling methods for SDE



cat's eye flow model for brine convective flow

Rigorous Padé approximant bounds in terms of P using Stieltjes integral + analytic continuation method for the measure

Darcy velocity v = 0.5 [m/s]

Kraitzman, Hardenbrook, Dinh, Murphy, Cherkaev, Zhu, & Golden, 2022

wave propagation in the marginal ice zone (MIZ)

Stieltjes integral representation and bounds for the complex viscoelasticity of the ice - ocean layer

Sampson, Murphy, Cherkaev, Golden 2022

first theory of key parameter in wave-ice interactions only fitted to wave data before

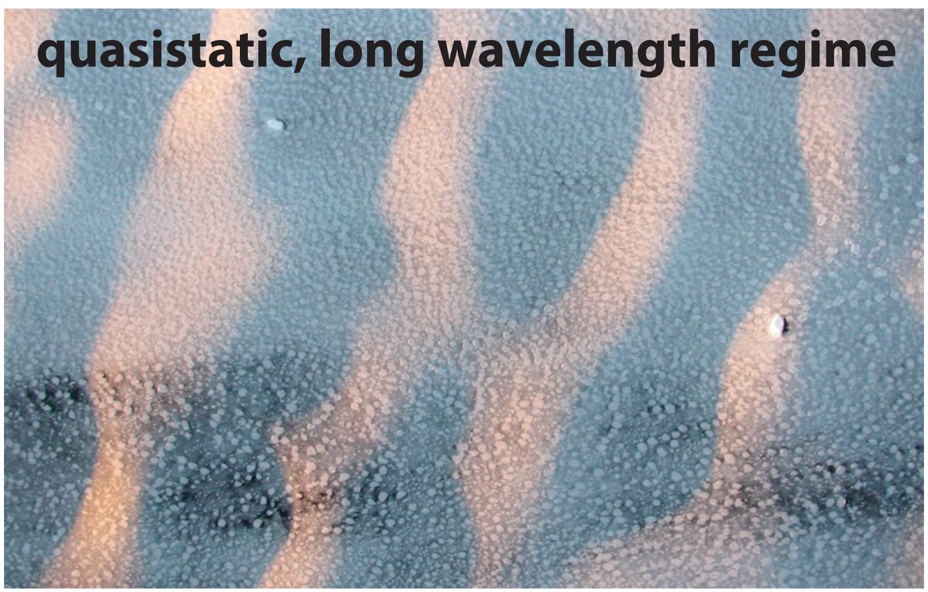
Keller, 1998
Mosig, Montiel, Squire, 2015
Wang, Shen, 2012

Analytic Continuation Method

Bergman (78) - Milton (79) integral representation for ε^*

Golden and Papanicolaou (83)

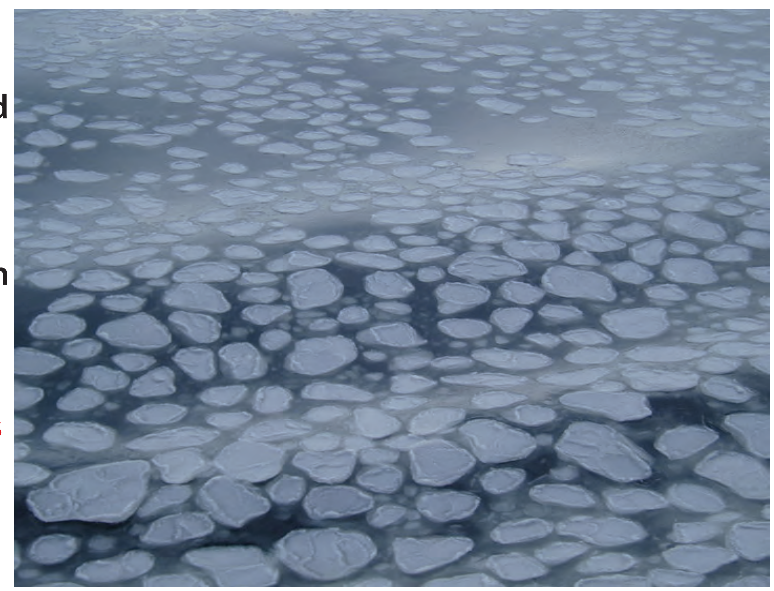
Milton, *Theory of Composites* (02)



quasistatic, long wavelength regime

homogenized parameter depends on sea ice concentration and ice floe geometry

like EM waves



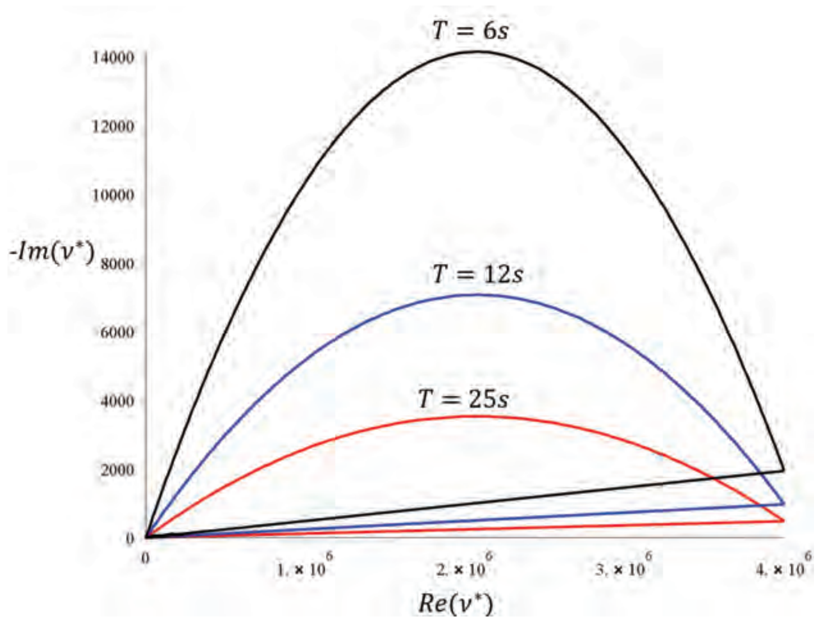
bounds on the effective complex viscoelasticity

$\nu_1 = 10^7 + i\,4875$ pancake ice

$\nu_2 = 5 + i\,0.0975$ slush / frazil

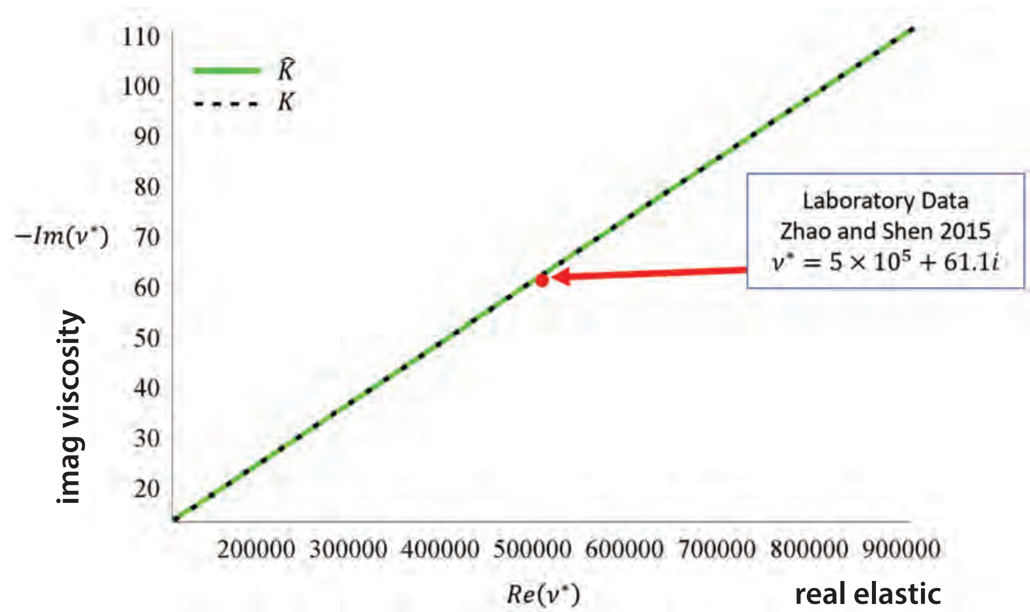
high contrast

complex elementary bounds (fixed area fraction of floes)



Elementary bounds for wave periods T.

matrix-particle bounds



Golden

melt pond formation and albedo evolution:

- *major drivers in polar climate*
- *key challenge for global climate models*

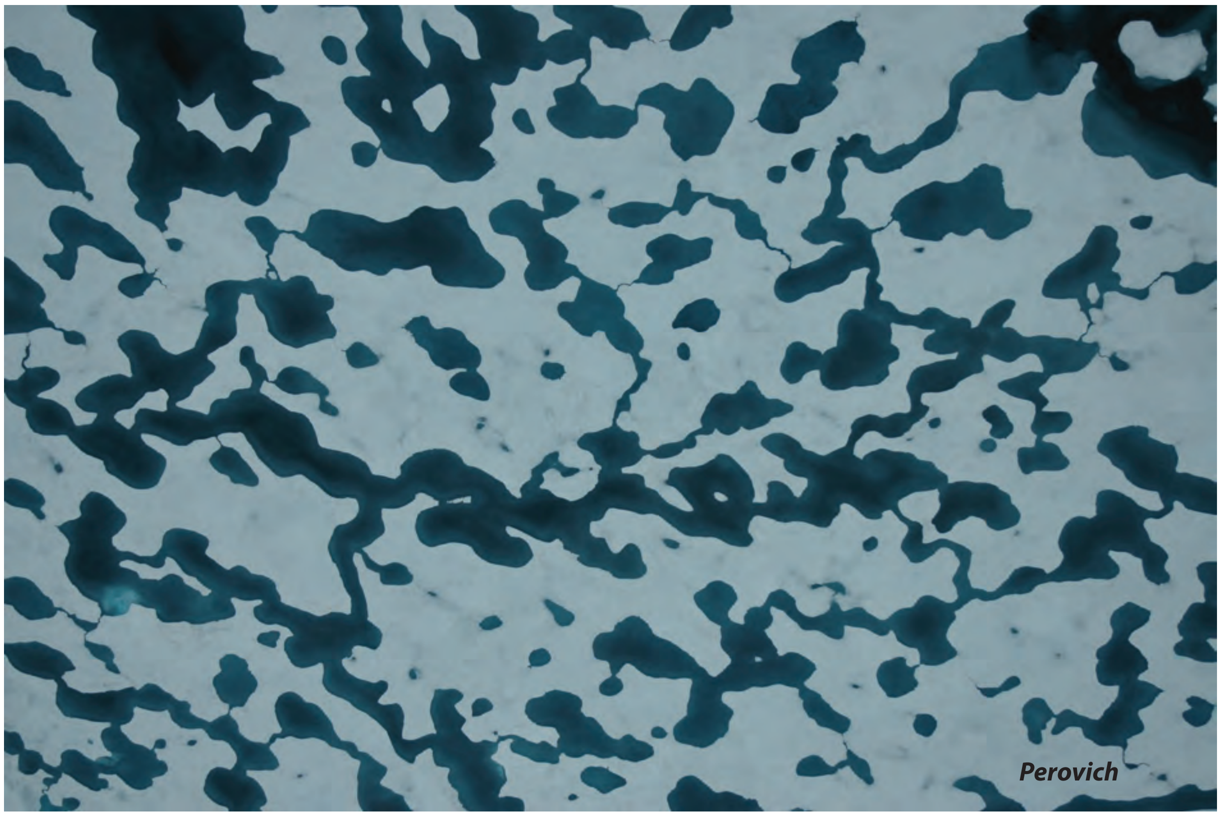
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

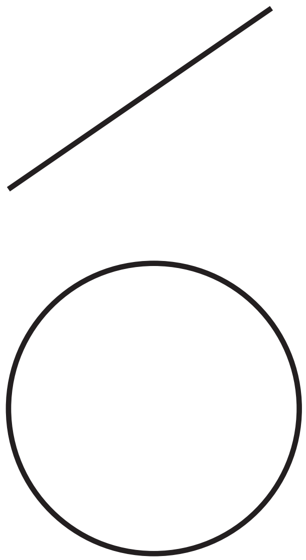
Flocco, Feltham, Hunke 2012



Are there universal features of the evolution similar to phase transitions in statistical physics?

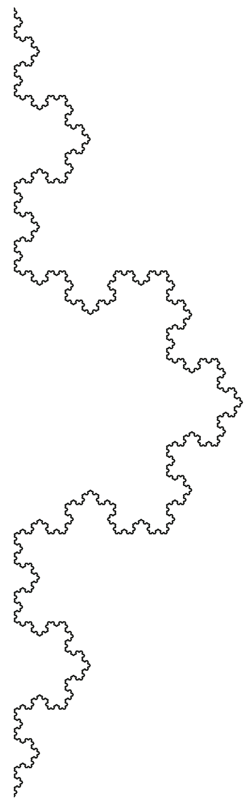
fractal curves in the plane

they wiggle so much that their dimension is >1



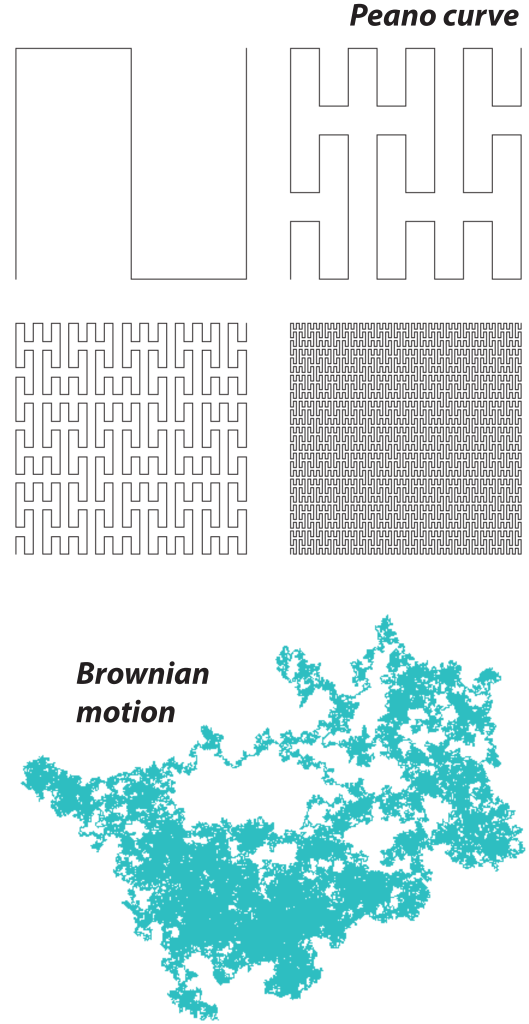
simple curves

D = 1



Koch snowflake

D = 1.26



space filling curves

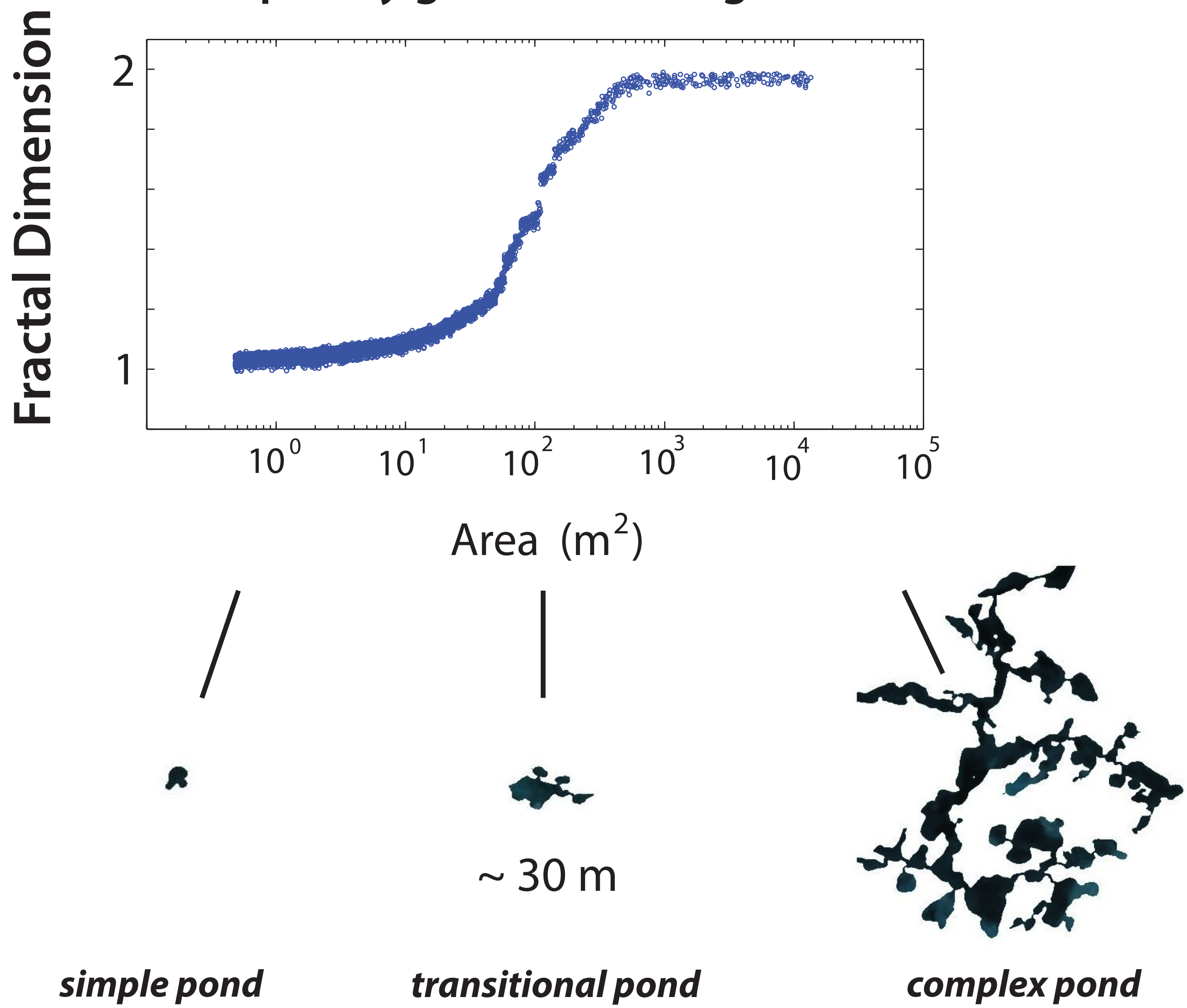
D = 2

Transition in the fractal geometry of Arctic melt ponds

Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

The Cryosphere, 2012

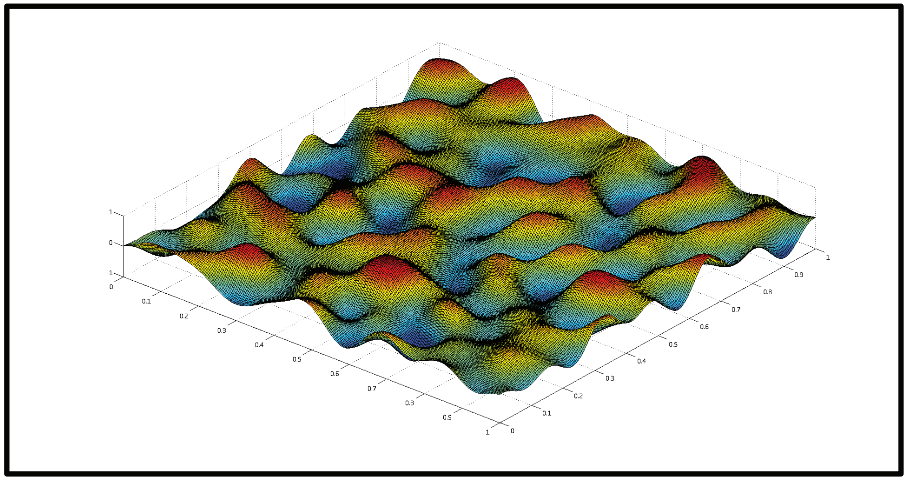
complexity grows with length scale



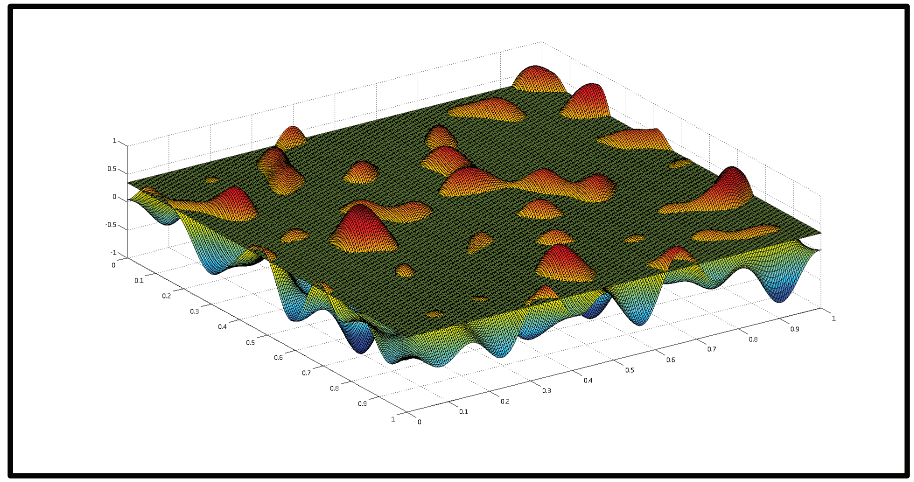
Continuum percolation model for melt pond evolution

level sets of random surfaces

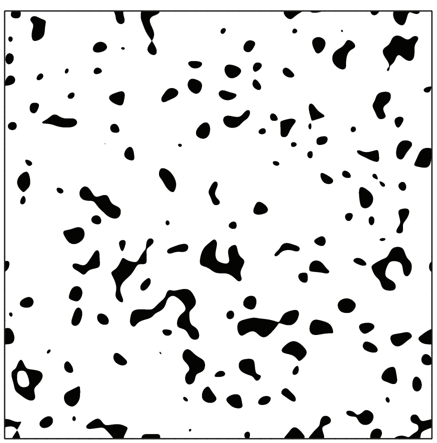
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018

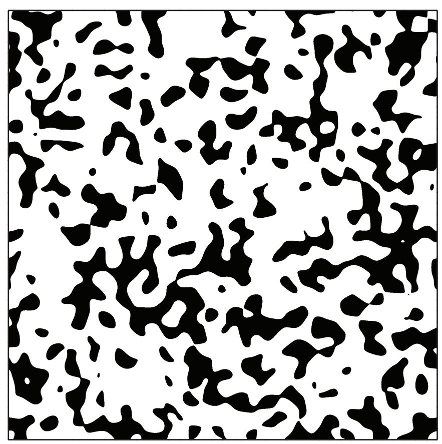


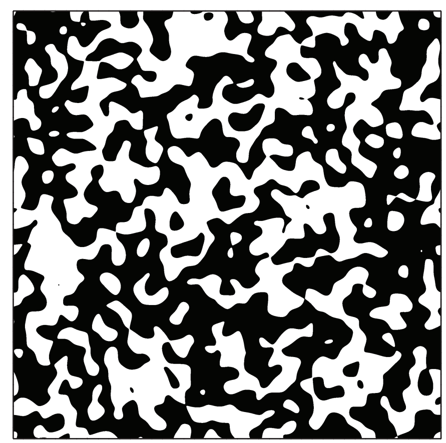
random Fourier series representation of surface topography



intersections of a plane with the surface define melt ponds

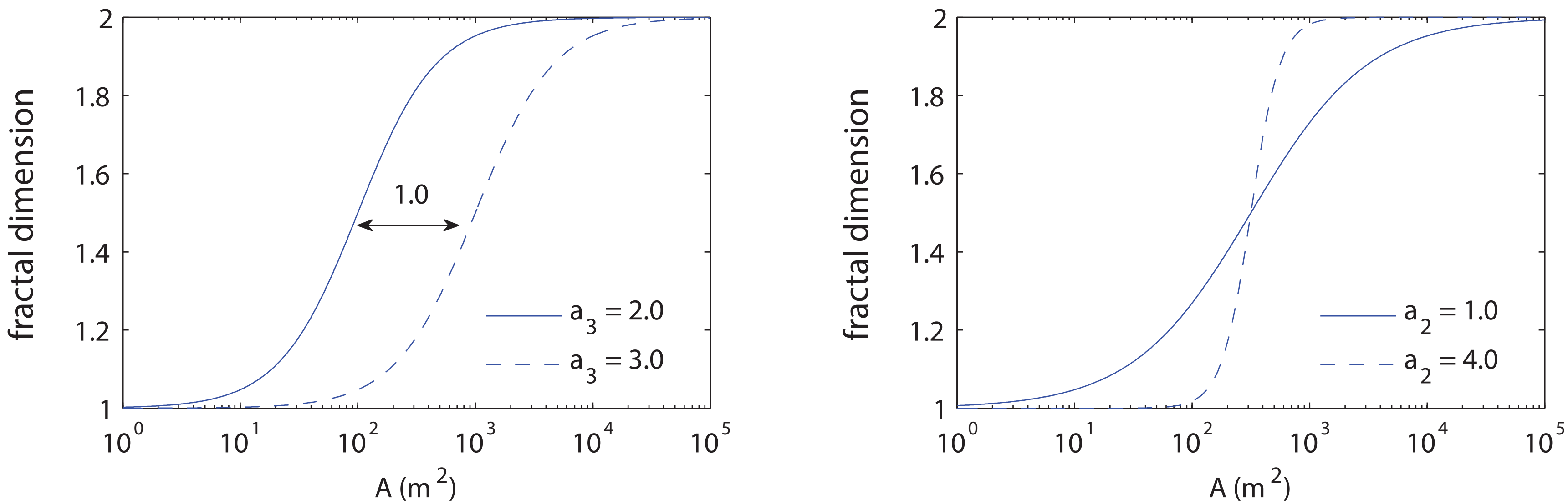






electronic transport in disordered media　　*diffusion in turbulent plasmas*　　*Isichenko, Rev. Mod. Phys., 1992*

fractal dimension curves depend on statistical parameters defining random surface



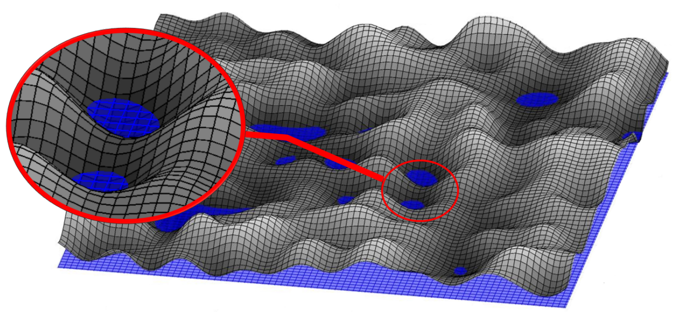
Topology of the sea ice surface and the fractal geometry of Arctic melt ponds

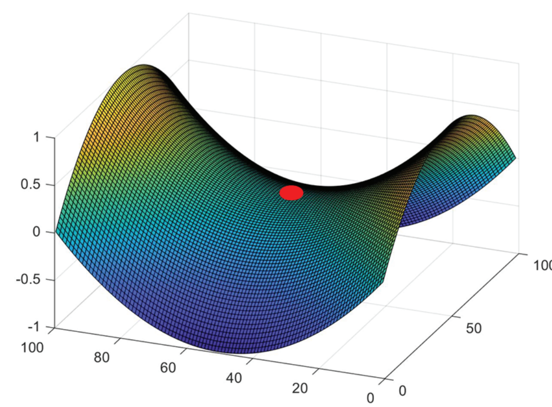
Physical Review Research (invited, under revision)

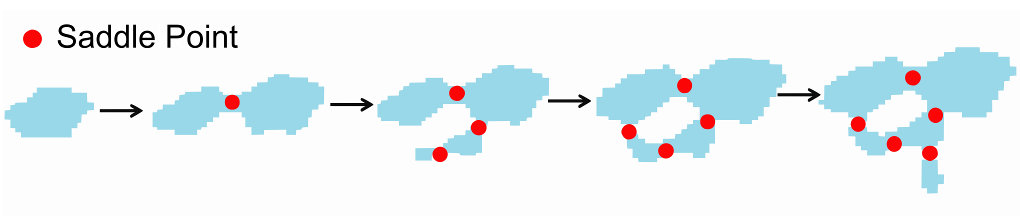
Ryleigh Moore, Jacob Jones, Dane Gollero,
Court Strong, Ken Golden

Several models replicate the transition in fractal dimension, but none explain how it arises.

We use Morse theory applied to the random surface model to show that saddle points play the critical role in the fractal transition.

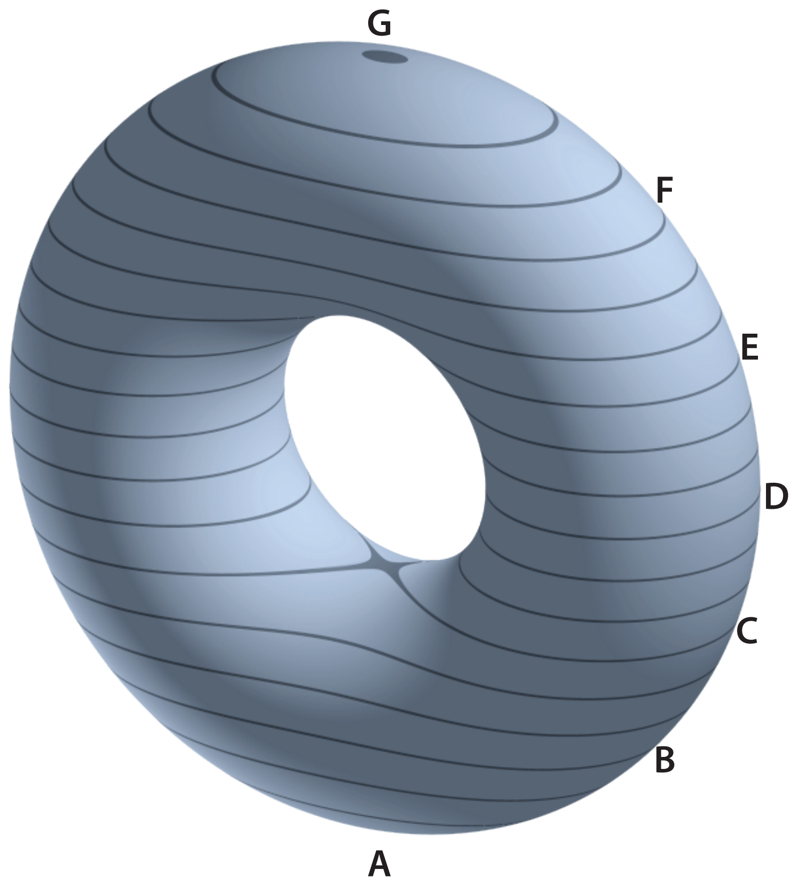






ponds coalesce (change topology) and complexify at saddle points

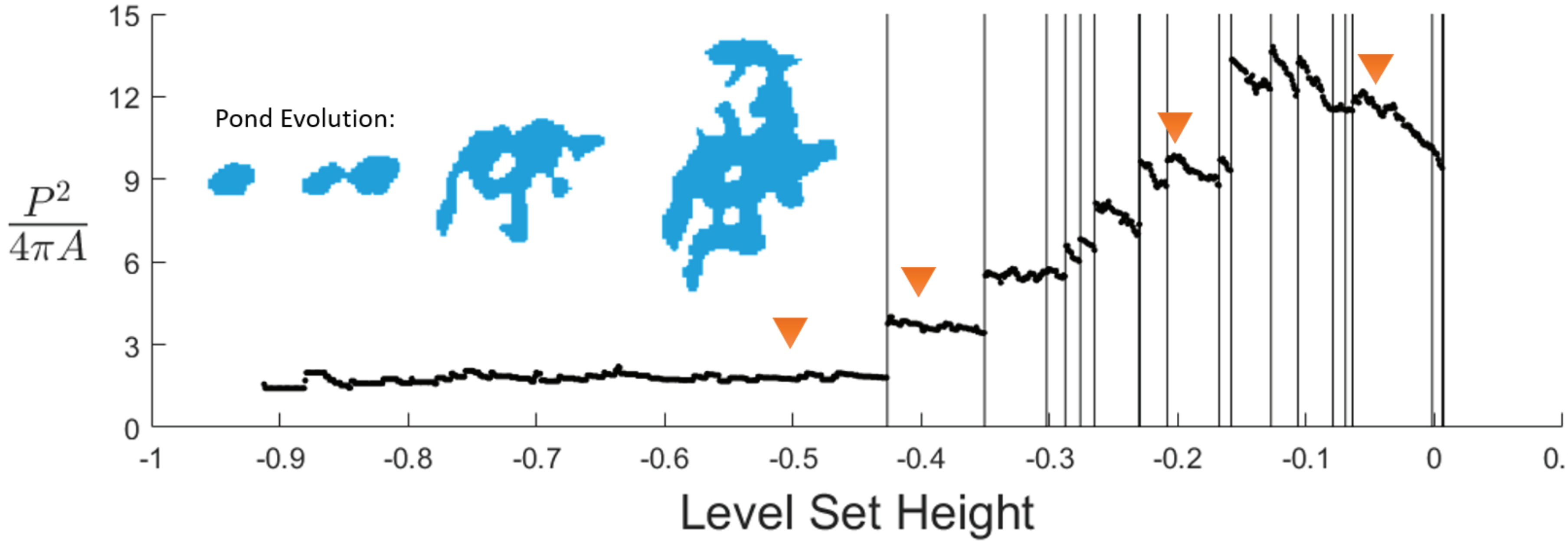
Morse theory



Morse theory tells us that changes in the topology of a surface occur at critical points of smooth functions on the surface: maxima, minima, and saddles.

Main results

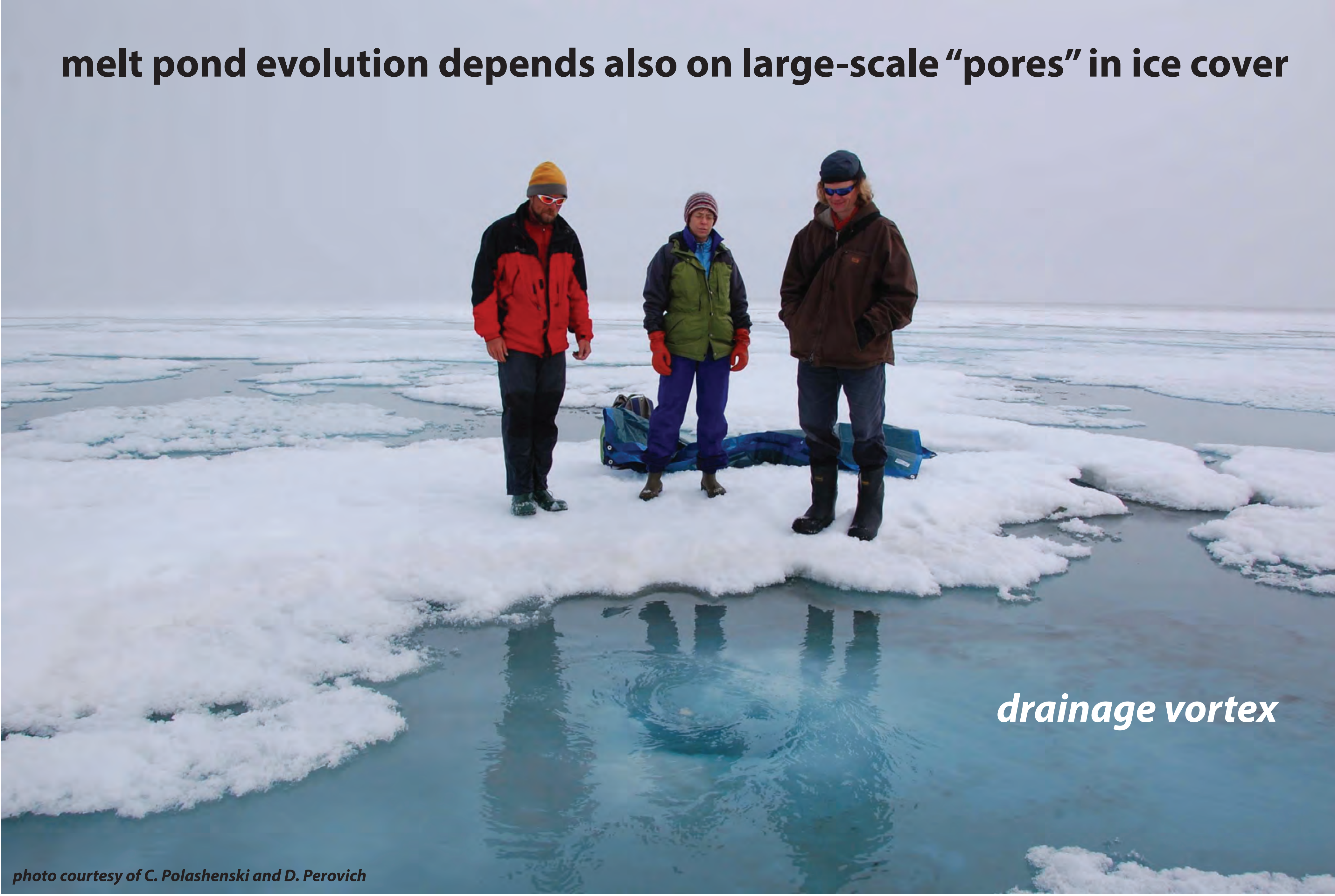
Isoperimetric quotient - as a proxy for fractal dimension - increases in discrete jumps when ponds coalesce at saddle points.



Horizontal fluid permeability "controlled" by saddles ~ electronic transport in 2D random potential.

drainage processes, seal holes

melt pond evolution depends also on large-scale “pores” in ice cover



Melt pond connectivity enables vast expanses of melt water to drain down seal holes, thaw holes, and leads in the ice.

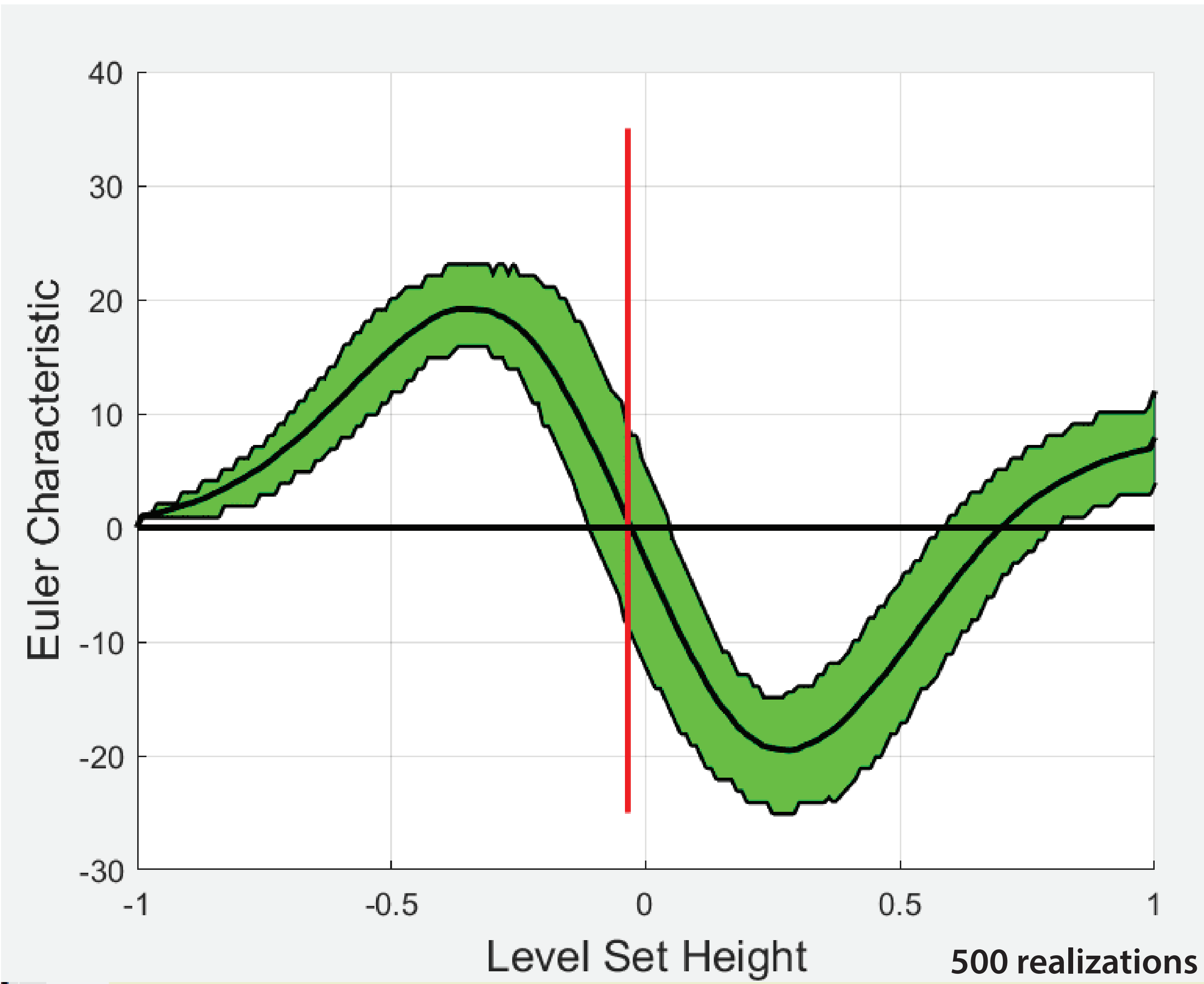
Topological Data Analysis

Euler characteristic = # maxima + # minima - # saddles

topological invariant

persistent homology

filtration - sequence of nested topological spaces, indexed by water level



Expected
Euler Characteristic Curve (ECC)

tracks the evolution of the EC of the flooded surface as water rises

zero of ECC ~ percolation

percolation on a torus
creates a giant cycle

Bobrowski & Skraba, 2020

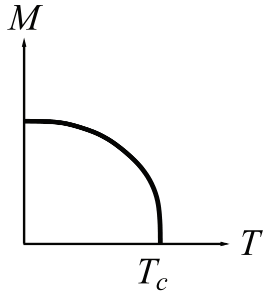
Carlsson, 2009

Vogel, 2002 GRF

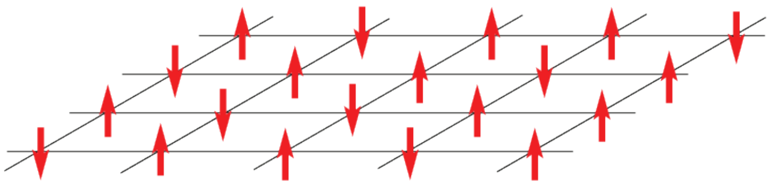
porous media
cosmology
brain activity

melt pond donuts

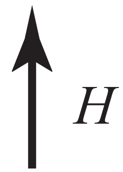
Ising Model for a Ferromagnet



Curie point
critical temperature



applied magnetic field



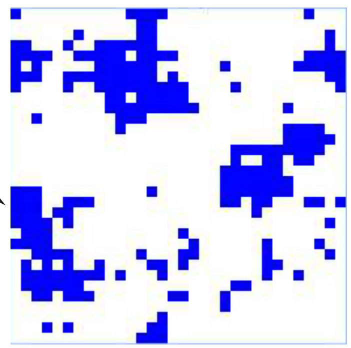
$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

blue
white

$$\mathcal{H} = -H \sum_i s_i - J \sum_{<i,j>} s_i s_j$$

nearest neighbor Ising Hamiltonian

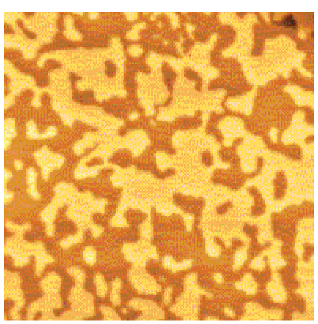
islands of like spins



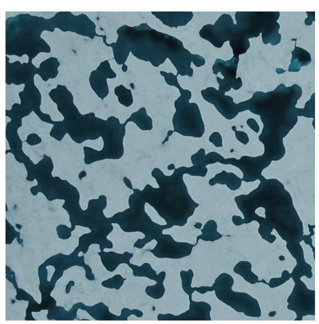
$$M(T, H) = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$$

effective magnetization

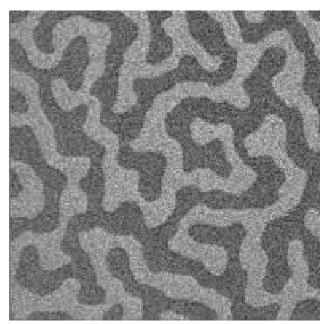
energy is lowered when nearby spins align with each other, forming **magnetic domains**



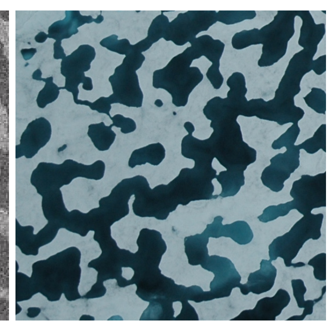
magnetic domains in cobalt



melt ponds (Perovich)



magnetic domains in cobalt-iron-boron



melt ponds (Perovich)

Ising model for ferromagnets ⟶ Ising model for melt ponds

Ma, Sudakov, Strong, Golden, *New J. Phys.*, 2019

$$\mathcal{H} = -\sum_{i}^{N} H_i\, s_i - J \sum_{<i,j>}^{N} s_i s_j \qquad s_i = \begin{cases} \uparrow \;\; +1 \;\; \text{water} \;\; \text{(spin up)} \\ \downarrow \;\; -1 \;\; \text{ice} \;\; \text{(spin down)} \end{cases}$$

random magnetic field represents snow topography

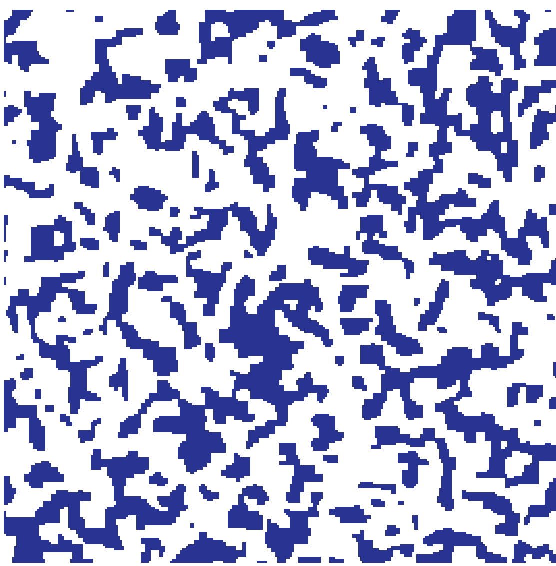
magnetization M

pond area fraction ***~ albedo*** $F = \dfrac{(M+1)}{2}$

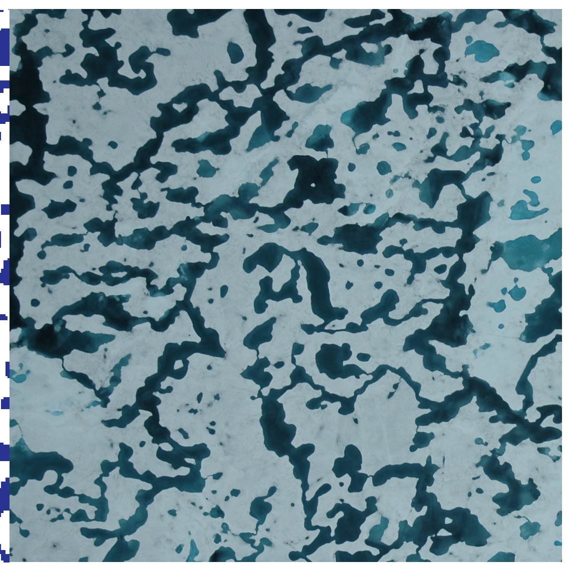
only nearest neighbor patches interact

Starting with random initial configurations, as Hamiltonian energy is minimized by Glauber spin flip dynamics, system "flows" toward metastable equilibria.

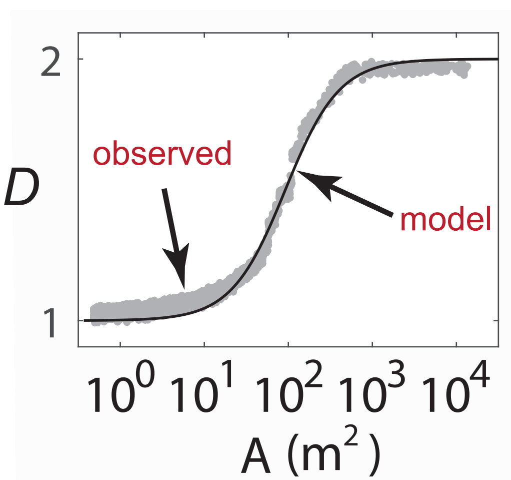
Order from Disorder



Ising model



melt pond photo **(Perovich)**



pond size distribution exponent

observed -1.5

(Perovich, *et al.* 2002)

model -1.58

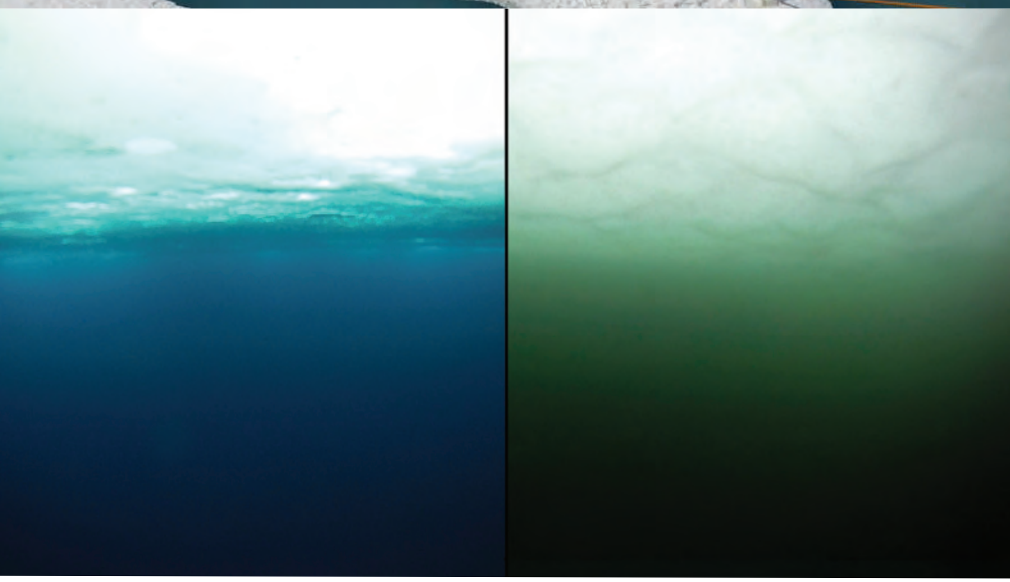
Scientific American
EOS, PhysicsWorld, ...

ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE) from snow topography data

Perovich

Melt ponds control transmittance of solar energy through sea ice, impacting upper ocean ecology.

WINDOWS



no bloom bloom

massive under-ice algal bloom

Arrigo et al., *Science* 2012

Have we crossed into a new ecological regime?

The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

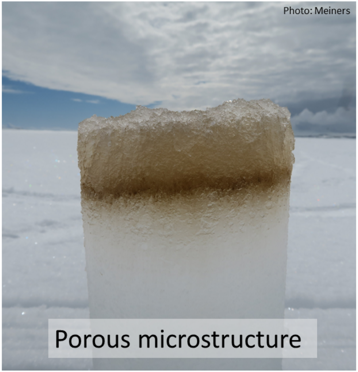
Horvat, Rees Jones, Iams, Schroeder, Flocco, Feltham, ***Science Advances*** 2017

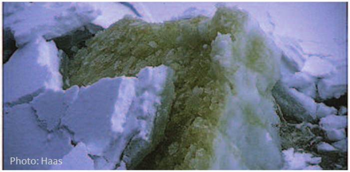
The effect of melt pond geometry on the distribution of solar energy under first year sea ice

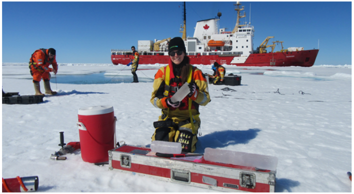
Horvat, Flocco, Rees Jones, Roach, Golden
Geophys. Res. Lett. 2019

(2015 AMS MRC)

Sea ice algae







80% of polar bear diet can be traced to ice algae*.

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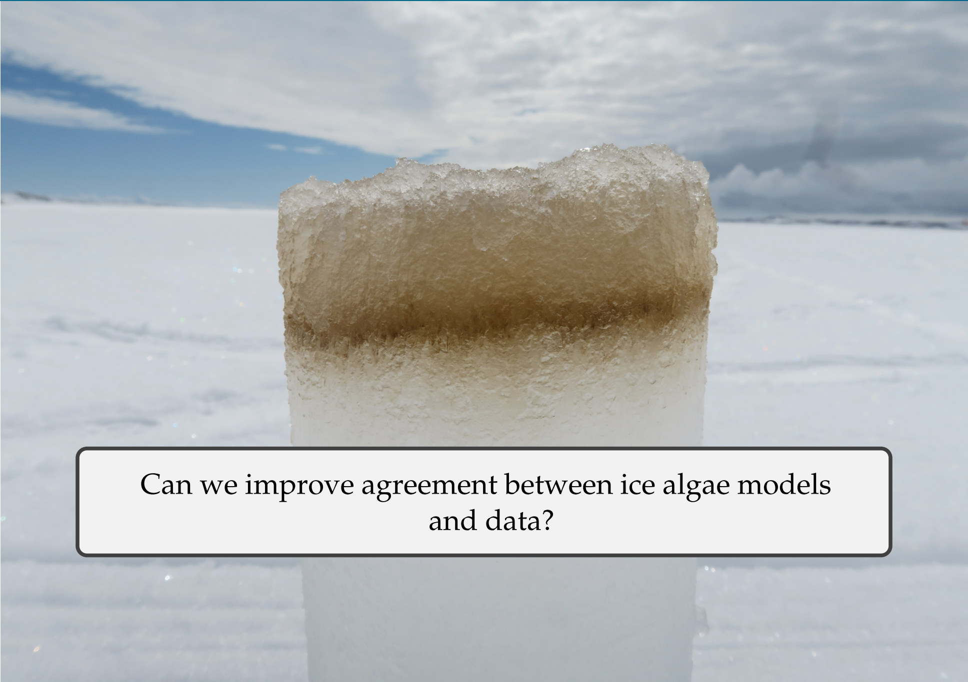
* Brown TA, et al. (2018). *PloS one*, 13(1), e0191631



Can we improve agreement between ice algae models and data?

Algal bloom model*

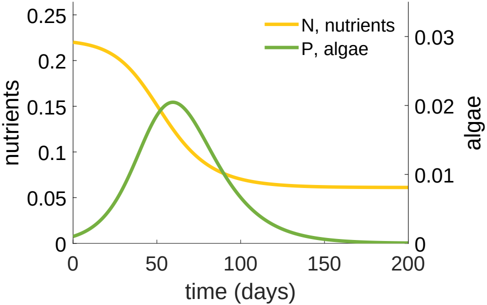
$$\text{nutrients:} \qquad \frac{dN}{dt} = \underbrace{\alpha}_{\text{input}} - \underbrace{\beta NP}_{\text{uptake}} - \underbrace{\eta N}_{\text{loss}}$$

$$\text{algae:} \qquad \frac{dP}{dt} = \underbrace{\gamma\beta NP}_{\text{growth}} - \underbrace{\delta P}_{\text{death}},$$

$$N(0) = n_0, \qquad P(0) = p_0$$

* Huppert, A., et al. (2002). *American Naturalist*, 159(2), 156-171

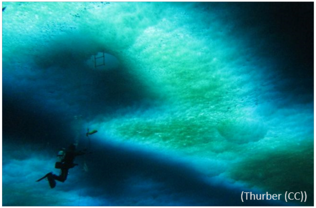
Algal bloom model

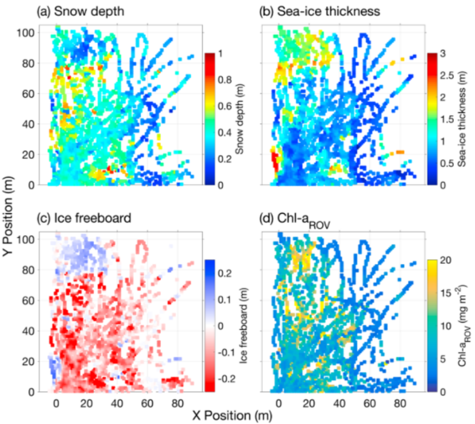


- poor agreement with data
- poor agreement between models

Steinacher, M., et al. (2010). *Biogeosciences*, 7(3), 979-1005

HETEROGENEITY





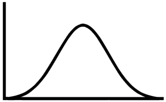
Meiners, K.M., et al. (2017). *Geophysical Research Letters*, 44(14), 7382-7390

Heterogeneity in initial conditions

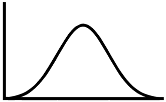
At each location within a larger region, we could consider

$$\frac{dN}{dt} = \alpha - BNP - \eta N$$
$$\frac{dP}{dt} = \gamma BNP - \delta P$$

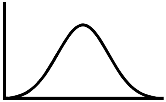
$$N(0) = N_0, \qquad P(0) = P_0$$



growth rate, B



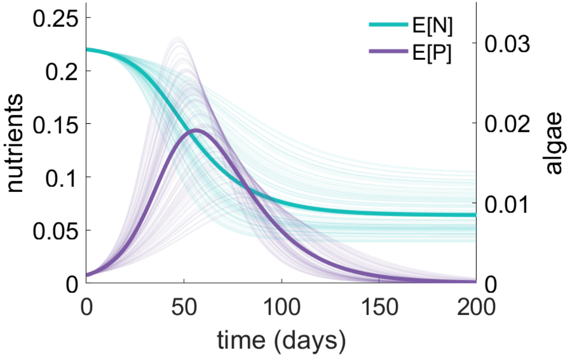
Initial nutrients, N_0



Initial algae, P_0

How do we analyze this model?

Monte Carlo simulations?



Too slow! Full algae model takes **8 hours** (cloud computing).

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DOI: 10.1111/ele.14095

ECOLOGY LETTERS cnrs WILEY

METHOD

Uncertainty quantification for ecological models with random parameters

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Abstract

There is often considerable uncertainty in parameters in ecological models. This uncertainty can be incorporated into models by treating parameters as random variables with distributions, rather than fixed quantities. Recent advances in uncertainty quantification methods, such as polynomial chaos approaches, allow for the analysis of models with random parameters. We introduce these methods with a motivating case study of sea ice algal blooms in heterogeneous environments. We compare Monte Carlo methods with polynomial chaos techniques to help understand the dynamics of an algal bloom model with random parameters.

Polynomial chaos expansions

$$N(t; B, P_0, N_0) \approx N_V(t; B, P_0, N_0) := \sum_{j=1}^{n} \widetilde{N}_j(t)\phi_j(B, P_0, N_0),$$

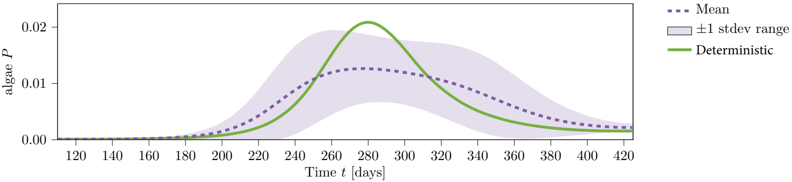
$$P(t; B, P_0, N_0) \approx P_V(t; B, P_0, N_0) := \sum_{j=1}^{n} \widetilde{P}_j(t)\phi_j(B, P_0, N_0),$$

where

- $V := \text{span}\{\phi_j\}_{j=1}^{n}$
- ϕ_j are orthogonal polynomials that form a basis for V
- $(\widetilde{N}_j, \widetilde{P}_j)$ need to be computed

Xiu, D. (2010). Numerical methods for stochastic computations. Princeton university press.

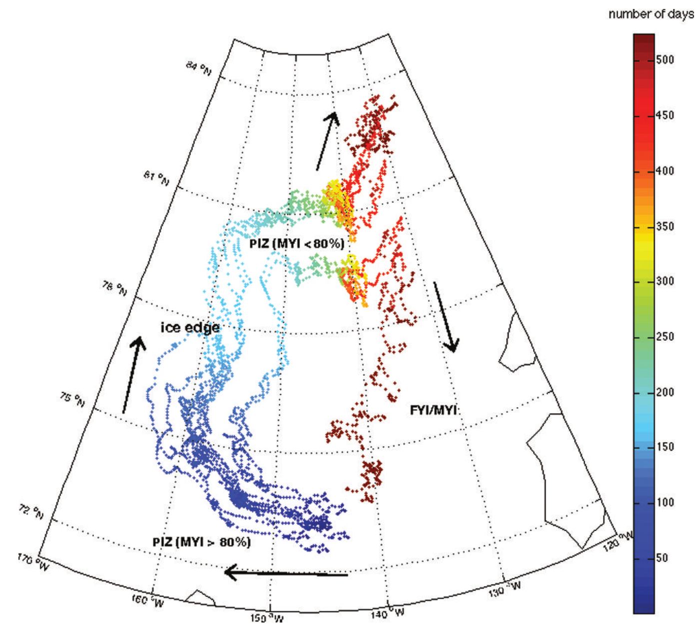
Ecological insights



- lower peak bloom intensity
- longer bloom duration
- able to compare variance to data

macroscale

Anomalous diffusion in sea ice dynamics



Ice floe diffusion in winds and currents

observations from GPS data:

Jennifer Lukovich, Jennifer Hutchings, David Barber, *Ann. Glac.* 2015

- **On short time scales floes observed (buoy data) to exhibit Brownian-like behavior, but they are also being advected by winds and currents.**

- **Effective behavior is purely diffusive, sub-diffusive or super-diffusive depending on ice pack and advective conditions - Hurst exponent.**

modeling:

Huy Dinh, Ben Murphy, Elena Cherkaev, Court Strong, Ken Golden 2022

floe scale model to analyze transport regimes in terms of ice pack crowding, advective conditions

Delaney Mosier, Jennifer Hutchings, Jennifer Lukovich, Marta D'Elia, George Karniadakis, Ken Golden 2022

learning fractional PDE governing diffusion from data

Floe Scale Model of Anomalous Diffusion in Sea Ice Dynamics

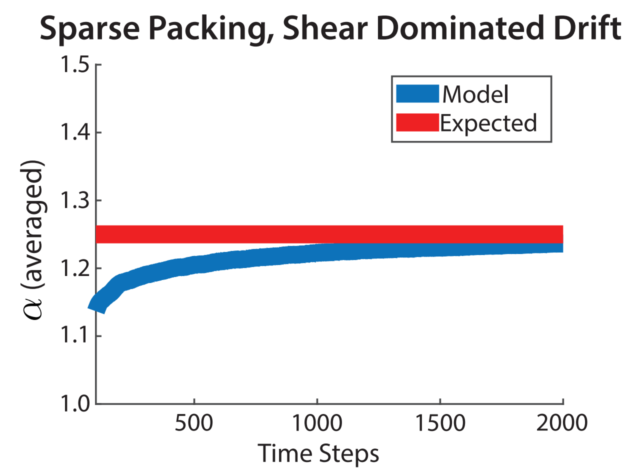
Huy Dinh, Ben Murphy, Elena Cherkaev, Court Strong, Ken Golden 2022

$$\left\langle \left| \mathbf{x}(t) - \mathbf{x}(0) - \langle \mathbf{x}(t) - \mathbf{x}(0) \rangle \right|^2 \right\rangle \sim t^\alpha \qquad \alpha = \text{Hurst exponent}$$

diffusive $\alpha = 1$

sub-diffusive $\alpha < 1$

super-diffusive $\alpha > 1$



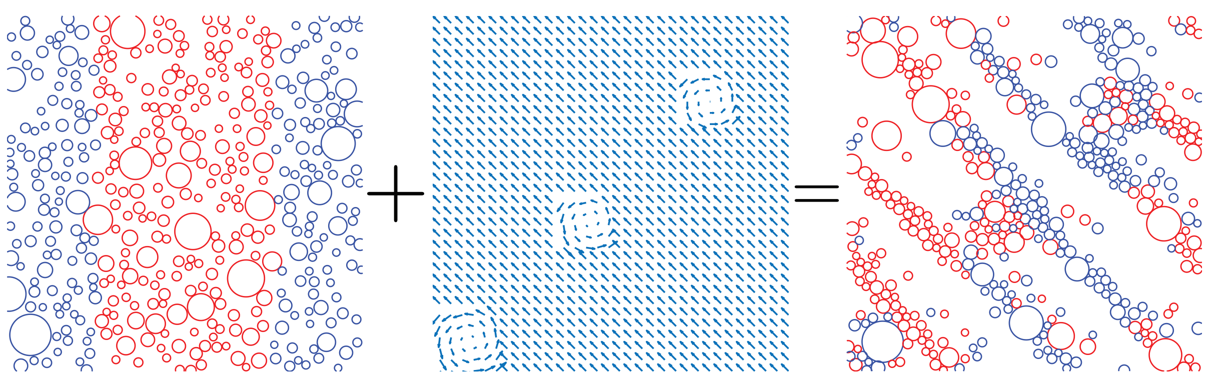
Model Approximations

Power Law Size Distribution: $N(D) \sim D^{-k}$

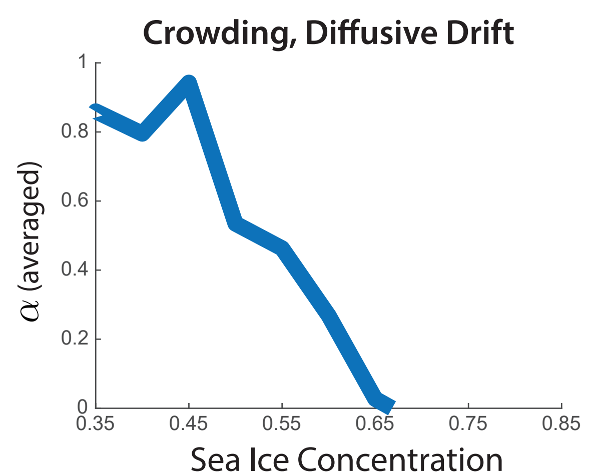
D. A. Rothrock and A. S. Thorndike Journal of Geophysical Research 1984

Floe-Floe Interactions: Linear Elastic Collisions

Advective Forcing: Passive, Linear Drag Law



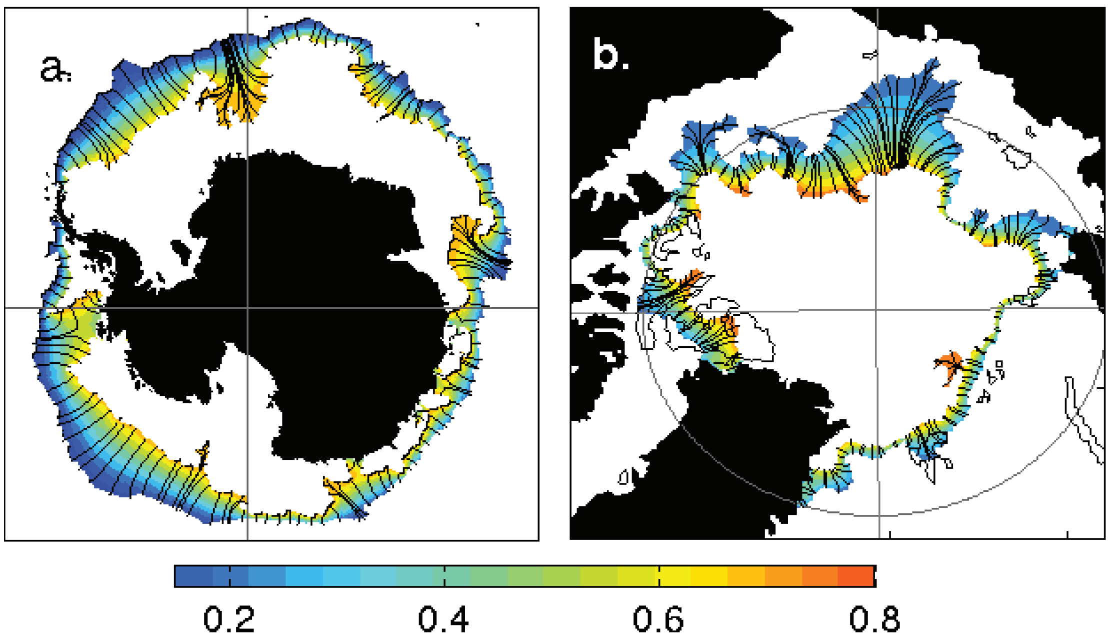
$\psi = 0.3$ **Vorticity Dominated Drift**



Marginal Ice Zone

MIZ

- biologically active region
- intense ocean-sea ice-atmosphere interactions
- region of significant wave-ice interactions



MIZ WIDTH
fundamental length scale of
ecological and climate dynamics

Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

transitional region between dense interior pack ($c > 80\%$) sparse outer fringes ($c < 15\%$)

How to objectively measure the "width" of this complex, non-convex region?

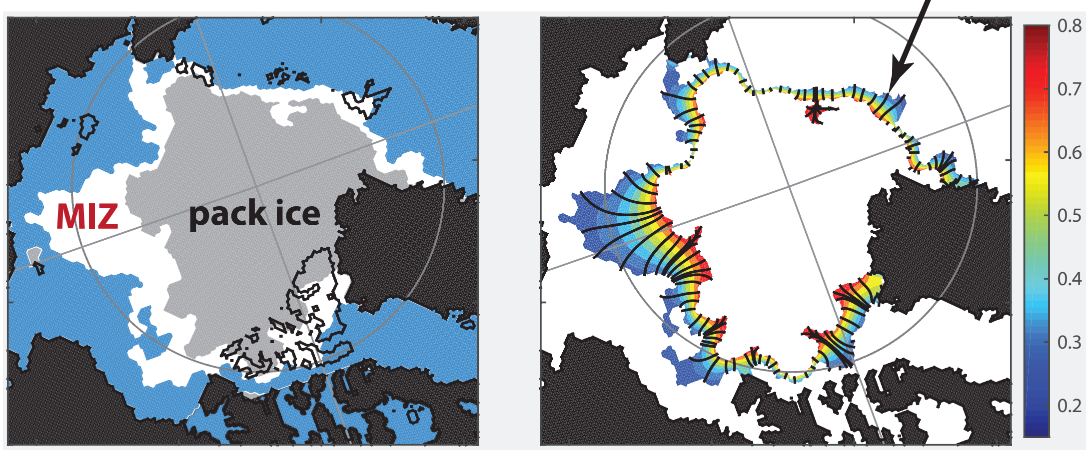
Objective method for measuring MIZ width motivated by medical imaging and diagnostics

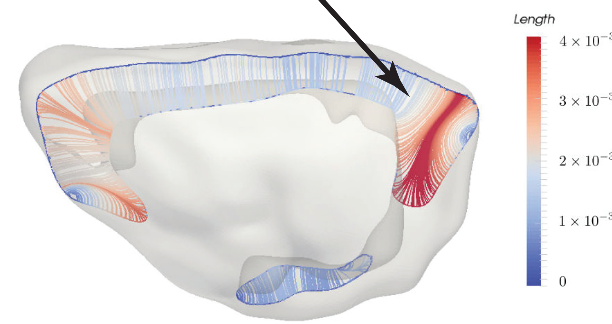
Strong, *Climate Dynamics* 2012
Strong and Rigor, *GRL* 2013

39% widening
1979 - 2012

streamlines of a solution to Laplace's equation

"average" lengths of streamlines





Arctic Marginal Ice Zone

crossection of the cerebral cortex of a rodent brain

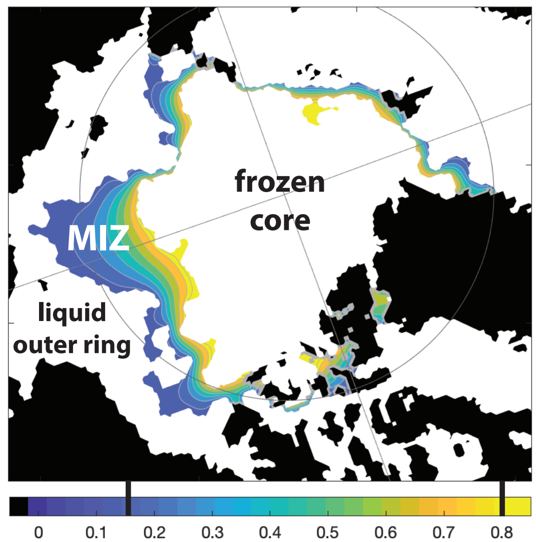
analysis of different MIZ WIDTH definitions

Strong, Foster, Cherkaev, Eisenman, Golden
J. Atmos. Oceanic Tech. 2017

Strong and Golden
Society for Industrial and Applied Mathematics News, April 2017

Model larger scale effective behavior with partial differential equations that *homogenize* complex local structure and dynamics.

Arctic MIZ



sea ice concentration ψ

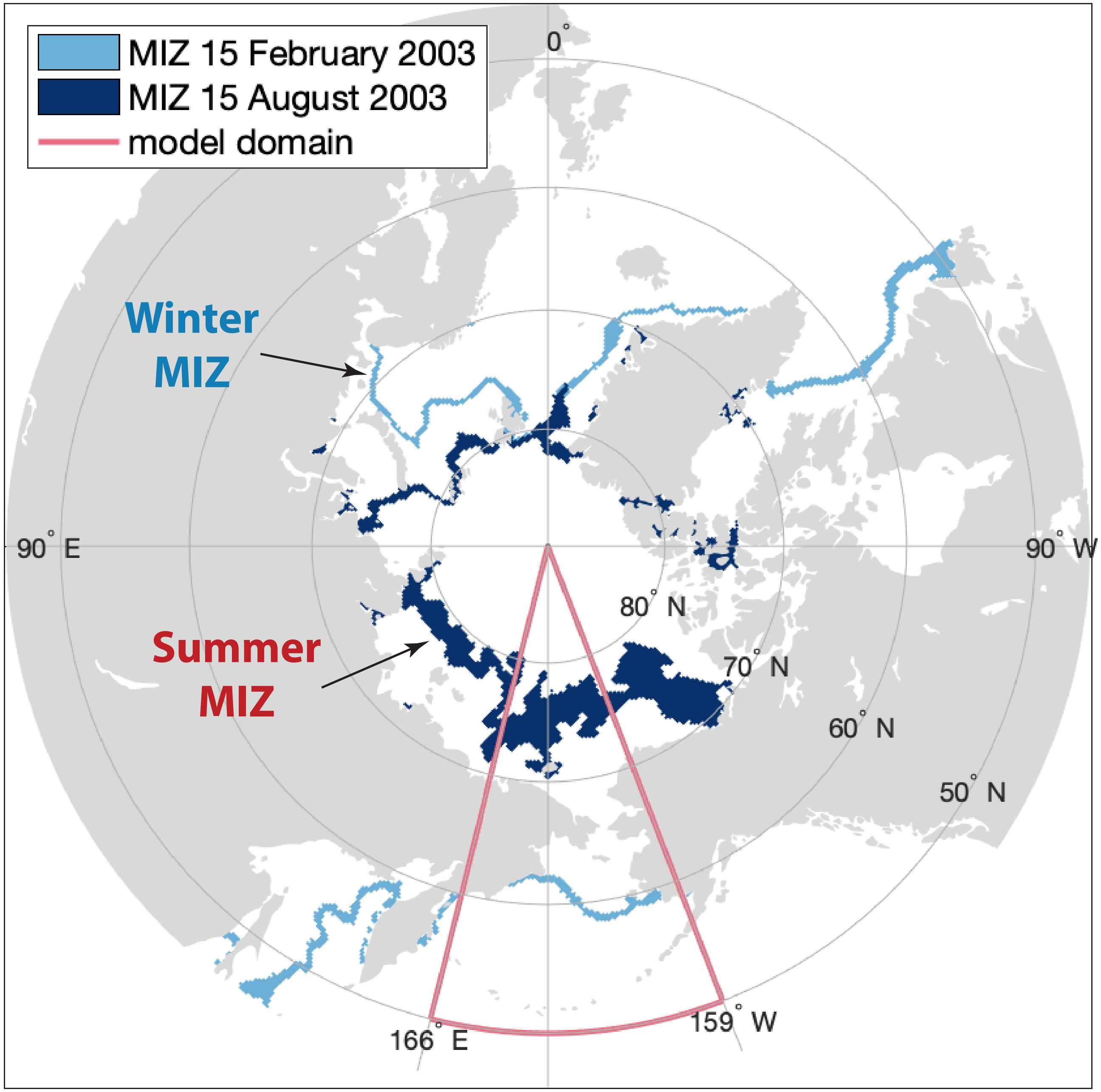
Predict MIZ width and location with basin-scale phase change model.

dynamic transitional region - mushy layer - separating two “pure” phases

seasonal and long term trends

C. Strong, E. Cherkaev, and K. M. Golden,
Annual cycle of Arctic marginal ice zone location
and width explained by dynamic phase transition model, 2022

Observed Arctic MIZ



MIZ as a moving phase transition region

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + S$$

$$S = [\rho(c_l - c_s)T + \rho L]\frac{\partial \psi}{\partial t}$$

$$\psi = 1 - \left(\frac{T - T_s}{T_l - T_s}\right)^{\alpha}$$

$$k_x = \left(\frac{\psi}{k_s} + \frac{1-\psi}{k_l}\right)^{-1}$$

homogenization

$$k_z = \psi k_s + (1-\psi)k_l$$

ρ effective density

T temperature

c specific heat

L latent heat of fusion

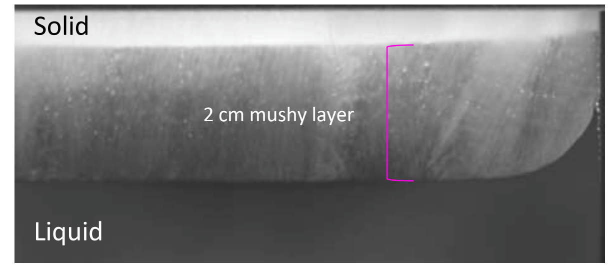
S models nonlinear phase change

ψ sea ice concentration

k effective diffusivity

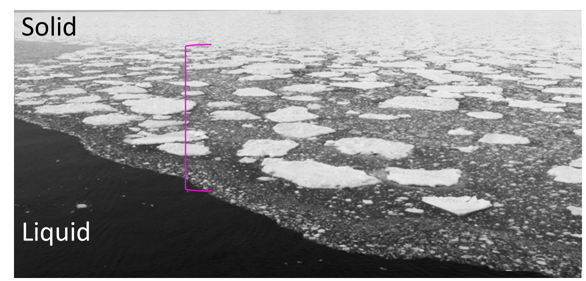
l liquid, s solid

Classical small-scale application



NaCl-H_2O in lab
(Peppin et al., 2007;, J. Fluid Mech.)

Macroscale application



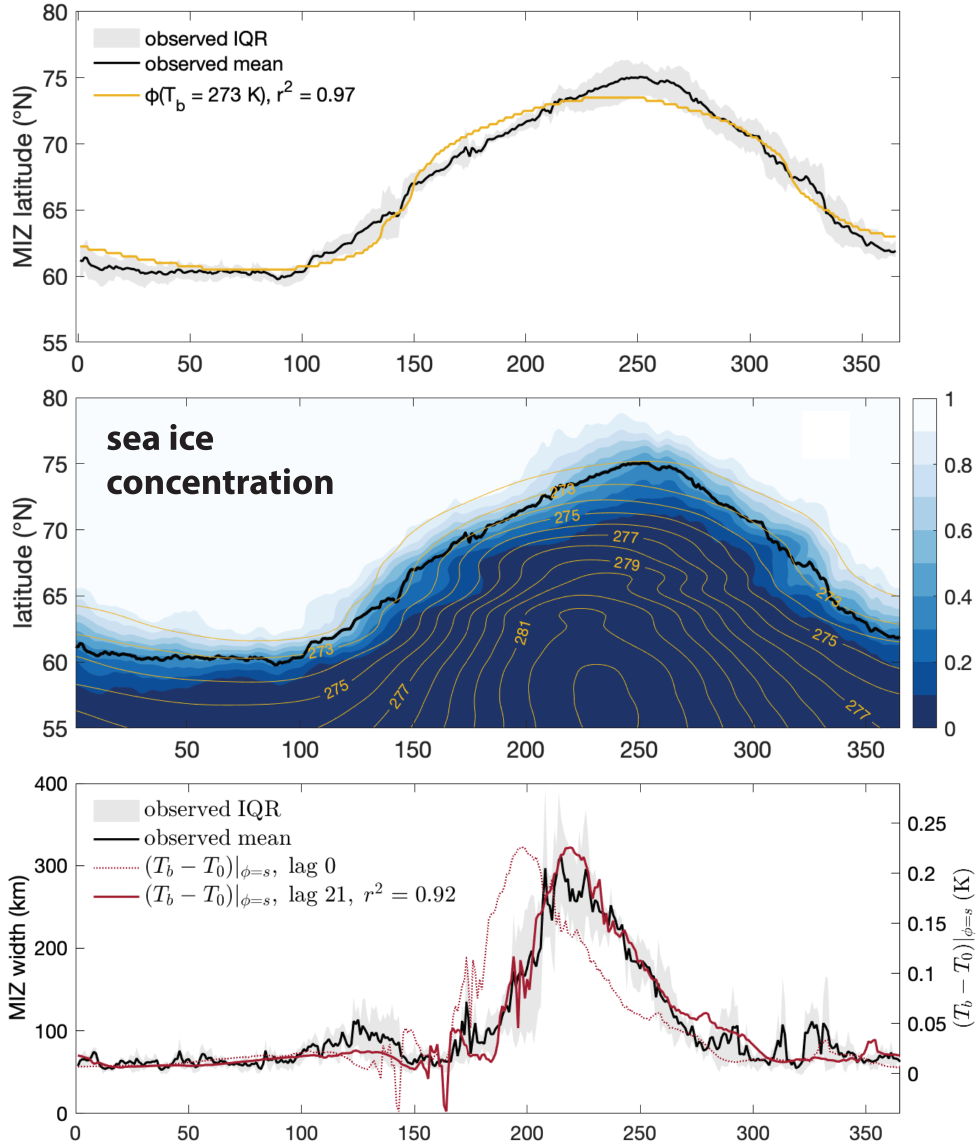
- Develop multiscale PDE model for simulating phase transition fronts to predict MIZ seasonal cycles and decadal trends
- Model simulates MIZ as a large-scale mushy layer with effective thermal conductivity derived from physics of composite materials

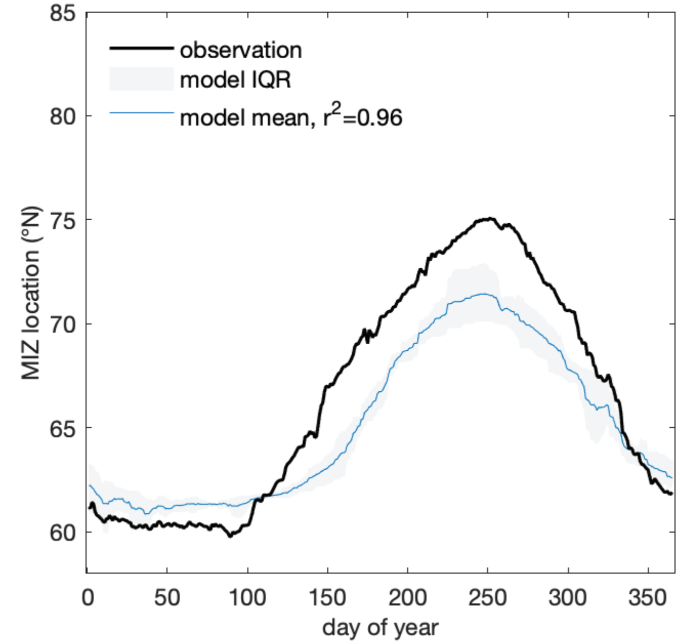
MIZ observations

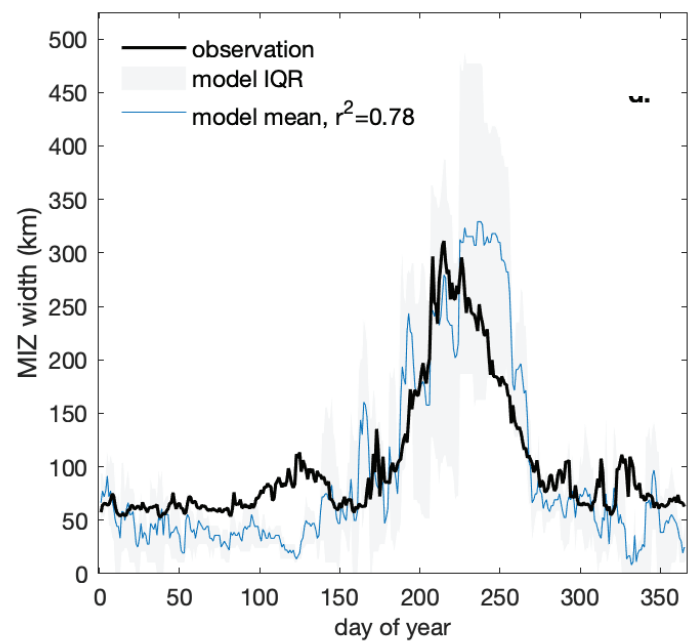
MIZ model vs. observations

location

width







Model captures basic physics of MIZ dynamics.

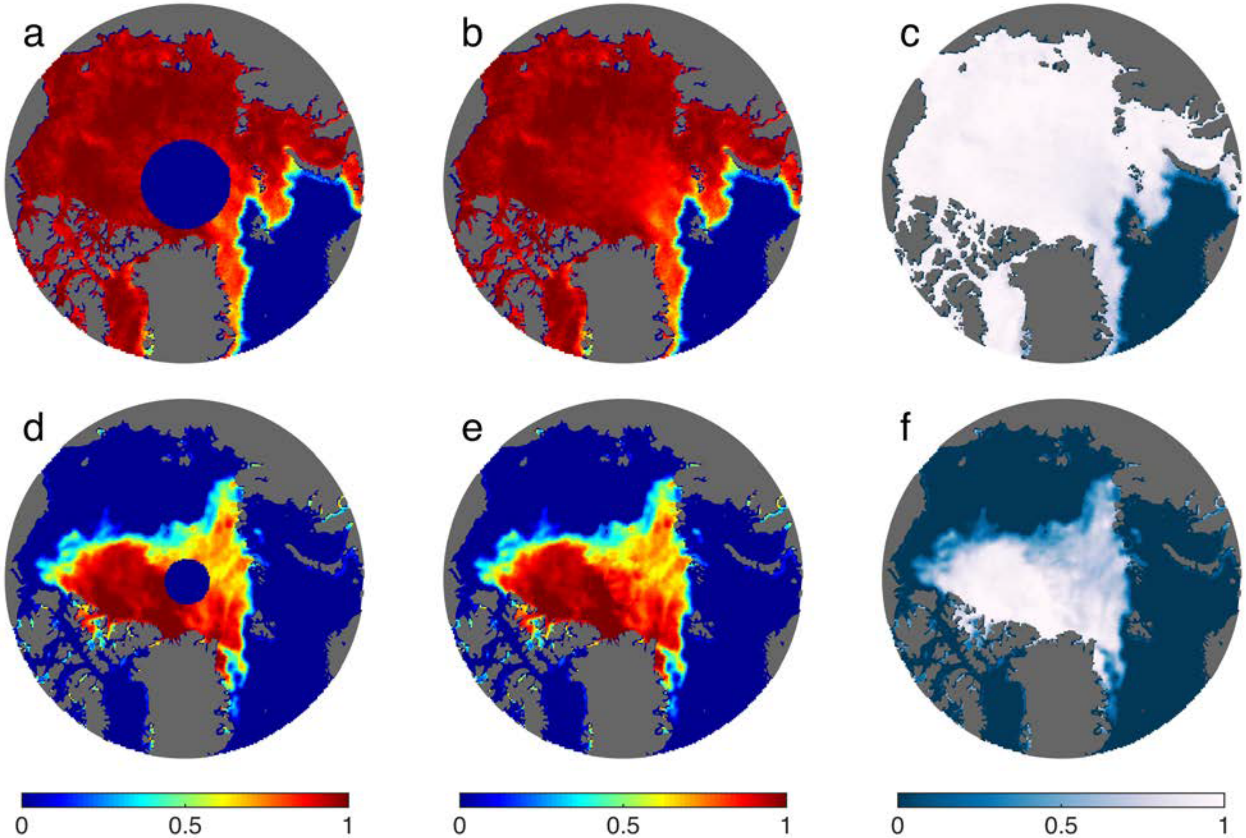
Filling the polar data gap with partial differential equations

hole in satellite coverage of sea ice concentration field

previously assumed ice covered

Gap radius: 611 km
06 January 1985

Gap radius: 311 km
30 August 2007



$$\Delta\psi=0$$

fill = harmonic function with learned stochastic term

Strong and Golden, *Remote Sensing* 2016
Strong and Golden, *SIAM News* 2017

NOAA/NSIDC Sea Ice Concentration CDR product update will use our PDE method.

Conclusions

1. Sea ice is a fascinating multiscale composite with structure similar to many other natural and man-made materials.

2. Mathematics developed for sea ice advances the theory of composites and other areas of science and engineering.

3. Homogenization and statistical physics help ***link scales in sea ice and composites***; provide rigorous methods for finding effective behavior; advance sea ice representations in climate models.

4. Fluid flow through sea ice mediates melt pond evolution and many processes important to climate change and polar ecosystems.

5. Field experiments are essential to developing relevant mathematics.

6. Our research is helping to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

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Advancing research. Creating connections.

The cover is based on "Modeling Sea Ice,"
page 1535.

THANK YOU

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Applied and Computational Analysis Program

Arctic and Global Prediction Program

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999

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Elena Cherkaev, Professor of Mathematics
Court Strong, Associate Professor of Atmospheric Sciences
Ben Murphy, Adjunct Assistant Professor of Mathematics

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Christian Sampson (now at UNC Chapel Hill with Chris Jones)
Huy Dinh (now a sea ice MURI Postdoc at NYU/Courant)
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Delaney Mosier
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Kitsel Lusted, Ruby Bowers, Kimball Johnston,
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High School Students: Jeremiah Chapman, Titus Quah, Dylan Webb

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