SPATIAL BOUNDS ON THE EFFECTIVE COMPLEX PERMITTIVITY FOR TIME-HARMONIC WAVES IN RANDOM MEDIA*

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Abstract. We consider wave propagation in random cell materials when the wavelength is finite, so that scattering effects must be taken into account. An effective dielectric coefficient is introduced, which in general, is a spatially dependent function, yet reduces, under the infinite wavelength assumptions, to the constant effective parameter in the quasistatic limit. We present an upper bound on the effective permittivity and a bound on its spatial variations that depends on the maximum volume of the inhomogeneities and the contrast of the medium. Numerical experiments illustrate the rigorous results. The dependence of the effective dielectric coefficient on the contrast in the medium is also investigated and an approximation formula is derived.

Key words. Random media, effective properties, electromagnetic scattering.

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1. Background. Usually, when one considers the propagation of an electromagnetic wave in a random medium, two parameters are of importance. The first, δ/λ is the ratio of the length scales of the typical inhomogeneities in the medium to the wavelength of the electromagnetic wave probing the medium. The second one is the contrast of the medium. Considerable effort over many decades has been applied to building effective medium theories that are applicable to wave propagation when the wavelengths associated with the fields are much larger than the microstructural scale. This limit where the ratio δ/λ goes to zero is called the quasistatic or infinite wavelength limit. In this case the heterogeneous material is replaced by a homogeneous, fictitious medium whose macroscopic characteristics are good approximations of the initial ones. The solutions of a boundary value partial differential equation describing the propagation of waves converge to the solution of a limit boundary value problem which is explicitly described when the size of the heterogeneities goes to zero. Similarly, in the limit when the contrast goes to zero, convergence of the solution to the solution of a constant coefficient partial differential equation is obtained.

The problem of finding bounds on the effective properties of materials in the quasistatic limit has been investigated vigorously, and there have been significant advances not only in deriving optimal bounds, but also in describing the materials that attain these bounds. See [15] and references within. Wellander and Kristensson [20] and Conca and Vanninathan [4] have both recently analyzed the homogenization of time-harmonic wave problems in periodic media, using entirely different methods. Their results are each applicable to problems in which the wavelength of the incident field is much larger than the microstructure.

For waves in random media, Keller and Karal [13] and Papanicolaou [17] use averaging of random realizations of materials in order to describe the effective properties of the composites when interacting with electromagnetic waves. Both analyses

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assume that the random materials deviate slightly from a homogeneous material, i.e. the contrast of the random inclusions is small. Keller and Karal assume *a priori* that the effective dielectric coefficient is a constant. Using perturbation methods, they approximate the dielectric constant with a complex number, whose imaginary part accounts for the wave attenuation.

A comprehensive overview of the subject of wave propagation in random media is given in a book by Ishimaru [12]. Also, recent results in this field can be found in the AMS-IMS-SIAM proceedings edited by Kuchment [14].

The above methods that provide bounds and describe the behavior of the dielectric coefficients do not account for scattering effects which occur when the wavelength is no longer much larger than the inhomogeneities of the composite and when the contrast is large. Results for this problem are sparse. The problem is difficult and the techniques that come from the quasistatic regime cannot be applied directly to the scattering problem since the quasistatic methods utilize the condition that the size of the heterogeneities goes to zero.

Even the correct definition of "effective medium" is somewhat unclear outside the quasistatic regime. In this work, we assume that the purpose of the effective medium is to reproduce the average or expected wave field as the actual medium varies over a given set of random realizations.

For simplicity in this work we consider waves in two- or three-dimensional random cell materials (discussed in Section 2.2) governed by the Helmholtz equation

$$\triangle u + \omega^2 \varepsilon u = f,$$

where realizations of the random permittivity function $\varepsilon(x)$ belong to some probability space. We average over all the possible material realizations to obtain the equation

$$\triangle \langle u \rangle + \omega^2 \langle \varepsilon u \rangle = f,$$

where $\langle \cdot \rangle$ denotes expected value, i.e. averaging over the set of realizations, and not a spatial average. The source f is assumed to be independent of the material. We seek to find the dielectric coefficient ε^* that will solve the problem

$$\triangle \langle u \rangle + \omega^2 \varepsilon^* \langle u \rangle = f, \tag{1.1}$$

where $\langle u \rangle$ is the expected value of the solution u. From the above two equations, it is easy to see that the appropriate definition for ε^* is

$$\varepsilon^* = \frac{\langle \varepsilon u \rangle}{\langle u \rangle}. \tag{1.2}$$

Note that the definition of ε^* does not preclude spatial variations, $\varepsilon^* = \varepsilon^*(x)$.

The definition in 1.2 is similar to the definition of the effective dielectric coefficient of an isotropic medium in the quasistatic case. In this case, the effective permittivity ε^* is defined by

$$\varepsilon^* \langle E \rangle = \langle D \rangle = \langle \varepsilon E \rangle,$$

where the averaged electric field $\langle E \rangle = \bar{E}$ is a given constant, and the averaged dielectric displacement $\langle D \rangle$ is independent of x which ensures that ε^* in the quasistatic case is a constant.

We can calculte the quasistatic effective dielectric constant by letting the wavelength λ go to infinity, or equivalently, by letting the frequency ω approach zero. Let $\varepsilon = \varepsilon_0\chi - \varepsilon_1(1-\chi)$, where χ is a characteristic function of the material ε_0 and the expected value of χ when we sum over all possible material realizations is p, i.e. $\langle\chi\rangle = p$. Let G_{ω,ε_1} be the free-space Green's function for the operator $Lv = \triangle v + \omega^2 v$ (with the outgoing wave condition). Our problem can be rewritten to yield the Lippmann-Schwinger equation

$$u(x) = \omega^2(\varepsilon_1 - \varepsilon_0)\int_\Omega G_{\omega,\varepsilon_1}(|x-y|)\chi(y)u(y)dy + q(x), \tag{1.3}$$

where $q = G_{\omega,\varepsilon_1} \star f$. Define the operator $A_\omega : L^2(\Omega) \to L^2(\Omega)$ by

$$(A_{\omega,\varepsilon_1}v)(x) = \int_\Omega G_{\omega,\varepsilon_1}(|x-y|)v(y)dy, \quad x \in \Omega. \tag{1.4}$$

In the case when $\omega^2|\varepsilon_1 - \varepsilon_0|\,\|A_{\omega,\varepsilon_1}\| < 1$,

$$u = (I - \omega^2(\varepsilon_1 - \varepsilon_0)A_{\omega,\varepsilon_1}\chi)^{-1}q, \tag{1.5}$$

and the Neumann series

$$u = q + \omega^2(\varepsilon_1 - \varepsilon_0)A_{\omega,\varepsilon_1}\chi q + \ldots \tag{1.6}$$

converges absolutely. Take the average over all realizations to obtain

$$\begin{aligned}\langle u\rangle &= q + \omega^2(\varepsilon_1 - \varepsilon_0)A_{\omega,\varepsilon_1}\langle\chi\rangle q + \ldots\\ &= q + \omega^2(\varepsilon_1 - \varepsilon_0)pA_{\omega,\varepsilon_1}q + \ldots\end{aligned}$$

and

$$\langle\varepsilon u\rangle = \langle\varepsilon\rangle q + \omega^2(\varepsilon_1 - \varepsilon_0)\langle\varepsilon A_{\omega,\varepsilon_1}\chi\rangle q + \ldots$$

Thus, the quasistatic effective dielectric coefficient is

$$\lim_{\omega->0}\varepsilon^* = \frac{\lim_{\omega->0}\langle\varepsilon u\rangle}{\lim_{\omega->0}\langle u\rangle} = \frac{\langle\varepsilon\rangle q}{q} = \varepsilon_0 p + \varepsilon_1(1-p).$$

Wave localization and cancellation must be accounted for when the wavelength is on the same order as the size of the heterogeneities, which means that the effective coefficients are no longer necessarily constants as in the quasistatic case, but functions of the spatial variable. We have illustrated in section 4 that as ω increases (which will decrease the wavelength), we begin to see spatial variations in the effective dielectric coefficient due to the presence of scattering effects. Nevertheless ε^* as defined in (1.2) is a "correct" definition of the effective dielectric coefficient, in that it reproduces the average field response through equation (1.1).

Since ε^* cannot be calculated explicitly in general, to be useful in applications it is important that we can bound both ε^* itself, and some measure of the spatial variations in ε^*. The main result of this paper, presented in Theorem 3.1, is a bound on the magnitude of ε^* and a local bound on the total variation, $\|\varepsilon^*\|_{BV}$. The estimates hold for any fixed frequency $\omega > 0$ and show an explicit dependence on the feature size and contrast of the random medium.

The paper is organized as follows. We pose the model problem of electromagnetic wave propagation in a composite material in subsection 2.1. The two-component composite material is random, and its structure is defined in subsection 2.2 using random variables which describe its geometry and component dependence. In subsection 2.3 we obtain existence and uniqueness of solutions and uniform bounds on the solutions, as well as Lipschitz bounds with respect to the dielectric coefficients of the materials.

Both the uniform and Lipschitz bounds are instrumental in obtaining the results of the paper. Spatial variations due to scattering effects are allowed. Bounds on the effective dielectric coefficient and its spatial variations are obtained when certain conditions are satisfied. These results are stated in the theorem in section 3, which is proved using methods that incorporate both PDE analysis and probability arguments.

We note that while the paper is focused on results in two- and three-dimensional spaces, simple modifications also provide one-dimensional results.

2. Model Problem.

2.1. Electromagnetic wave propagation. Consider time-harmonic electromagnetic wave propagation through nonmagnetic ($\mu = 1$) heterogeneous media. Assuming that the electric field vector $E = (0, 0, u)$ and ε is independent of x_3, Maxwell's equations reduce to the Helmholtz equation

$$\triangle u + \omega^2 \varepsilon u = 0, \tag{2.1}$$

where ω represents the frequency, and $\varepsilon \in L^\infty(\mathbb{R}^n)$ is the dielectric coefficient. In media with heterogeneities in all three dimensions, each field component satisfies (2.1).

Let our bounded spatial domain be $\Omega \subseteq \mathbb{R}^n$, where $n = 2, 3$. The region outside Ω is filled with a homogeneous material. In particular, assume for $x \notin \Omega$, we have $\varepsilon(x) = 1$. Let S_0 be the sphere of radius R_0, i.e. $S_0 = \{r = R_0\}$, and let $\Omega_0 = \{|x| < R_0\}$.

Outside the ball Ω_0, we separate the solution u to (2.1) into the incident and scattered field: $u = u_i + u_s$. The scattered field u_s can also be separated. Wellposedness of the problem requires imposing Sommerfeld's radiation condition as a boundary condition at infinity, i.e.

$$\lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial}{\partial r} - i\omega \right) u = 0,$$

uniformly in all directions, where $n = 2, 3$ is the spatial dimension. Here, it is assumed that the time-harmonic field is $e^{-i\omega t}u$.

The linear operator $T\colon H^{\frac{1}{2}}(S_0) \to H^{-\frac{1}{2}}(S_0)$ (Dirichlet-to-Neumann map) defines the relationship between the traces $u_s|_{\{r=R_0\}}$ and $\partial_r u_s|_{\{r=R_0\}}$, i.e. $T(u_s|_{\{r=R_0\}}) = (\partial_r u_s)|_{\{r=R_0\}}$. The Dirichlet-to-Neumann operator defines an exact nonreflecting boundary condition on the artificial boundary S_0, i.e. there are no spurious reflections of the scattered solution introduced at S_0. We write T explicitly for the two- and three-dimensional cases in the Appendix. On the boundary $S_0 = \{r = R_0\}$, the solution $u = u_i + u_s$ should then satisfy

$$\partial_r u - Tu = \partial_r u_i - Tu_i + \partial_r u_s - Tu_s = \partial_r u_i - Tu_i \equiv c.$$

In this way the problem on $\mathbb{R}^n$ is equivalently replaced by

$$\begin{aligned} &\triangle u + \omega^2 \varepsilon u = 0 \quad \text{in } \Omega_0 \supset \Omega, \\ &(\partial_r u - Tu) = c \quad \text{on } S_0. \end{aligned}$$

2.2. Random structure. We are interested in computing expected values of wave fields as the underlying medium ranges over some class of random materials. In this section, we define the probability space characterizing these materials.

We fill our bounded domain Ω by random cell materials (see e.g. Milton [15]). Our two-phase random materials are constructed as follows. The first step is to divide Ω into a finite number of cells. The cells may vary in size and shape, but their volume is bounded by a parameter.

The second step is to randomly assign to each cell a material of permittivity ϵ_0 with probability p or ϵ_1 with probability $1-p$ in a way that is uncorrelated both with the shape of the cell and with the phases assigned to the surrounding cells. We then have a probability space $(\Psi_\delta, \mathcal{J}_\delta, P_\delta)$, where Ψ_δ is a set of material realizations with a σ-algebra $\mathcal{J}_\delta$ of subsets of Ψ_δ, and a probability measure P_δ on $\mathcal{J}_\delta$ with $P_\delta(\Psi_\delta) = 1$. The parameter δ bounds the volume of each cell and its precise definition is given later in the section.

Elements $\psi \in \Psi_\delta$ are characterized by two random variables, $\psi = (m, g)$, where the variable m depends on the random variable g. The variable g describes the geometry of the material by partitioning the domain Ω into N_g parts, each of which is filled either with material ε_0 or material ε_1, which is done by the random variable m. Thus, g describes the subdivision of our domain into subdomains; once the geometry g is fixed, the random variable m distributes the material in the subdomains. Denoting some set of partitions of Ω by Γ_δ, the variable $g \in \Gamma_\delta$, partitions the spatial domain Ω into N_g disjoint subdomains $\{\Omega_j\}_{j=1}^{N_g}$ such that $\cup\Omega_j = \Omega$. The variable $m_g = \{m_1, \ldots, m_{N_g}\}$ assigns zero for material ε_0 with probability p or one for material ε_1 with probability $1-p$ in each spatial subdomain. The real part of the dielectric constant in the composite material is defined by

$$\varepsilon_{m,g}(x) = \begin{cases} \varepsilon_0 & \text{if } m_j = 0 \text{ and } x \in \Omega_j; \\ \varepsilon_1 & \text{if } m_j = 1 \text{ and } x \in \Omega_j. \end{cases}$$

We assume without loss of generality that $\varepsilon_1 > \varepsilon_0$.

Fix a geometry g. Denote the set of realizations for geometry g by R_g:

$$R_g = \{m_g = (m_1, \ldots, m_{N_g}) : m_j = 0 \text{ or } m_j = 1,\ j = 1, \ldots, N_g\}$$

The set R_g has 2^{N_g} elements. Thus the set of material realizations, Ψ_δ is described as follows,

$$\Psi_\delta = \{(g, m_g) : g \in \Gamma_\delta,\ m_g \in R_g\}.$$

The probability measure is

$$P = \sum_{m_g \in R_g} \prod_{j=1}^{N_g} p^{1-m_j} (1-p)^{m_j} G_\delta, \tag{2.2}$$

where G_δ is the probability measure on the space of all geometries, Γ_δ. The product describes the multiplication of the probabilities of the materials in each subdomain Ω_j, which is summed over the set of all realizations for a particular geometry g.

$(\Psi_\delta, \mathcal{J}_\delta, P_\delta)$ depends on a parameter $\delta > 0$. Let k be a whole number, independent of δ. We make the following assumptions on the subdomain partitions in Γ_δ:

A1: The volume of each subdomain $\{\Omega_j\}_{j=1}^{N_g}$ is bounded by δ, i.e., $|\Omega_j| \le \delta$. Note that since the volume of Ω is fixed, as δ decreases, the set of realizations Ψ_δ must change.

A2: For each δ, there exists η with $0 < \eta \leq \delta$ such that a ball with volume η, $B_r(x)$, intersects at most k subdomains Ω_j for all $x \in \Omega$. This condition excludes from consideration materials with infinitely many subdomains interfacing at any $x \in \Omega$. Here $B_r(x)$ denotes the ball of radius $r = \sqrt{\eta/\pi}$ in two dimensions and radius $r = \left(\frac{3\eta}{4\pi}\right)^{1/3}$ in three dimensions, centered at x.

A3: Using $B_r(x_0)$ from A2, define the set

$$S_{x_0,r} = \left(\bigcup \partial\Omega_j\right) \bigcap B_r(x_0).$$

There exists a constant C_p (independent of δ) such that the Lebesgue measure of the set $S_{x_0,r}$ satisfies

$$\mathcal{L}^{n-1}(S_{x_0,r}) \leq C_p r^{n-1},$$

for every $x_0 \in \Omega$. This condition excludes from consideration materials containing subdomains with boundaries with infinite perimeter in $B_r(x)$.

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2.3. Existence and uniqueness of solutions and Lipschitz bounds. For a fixed dissipation constant $\epsilon_i > 0$, define a set

$$\mathcal{A} := \{\varepsilon = \varepsilon_r + i\,\varepsilon_i : \varepsilon_r = \varepsilon_{m,g} \text{ for some } (m,g) \in \Psi_\delta\}.$$

Given an incident field u_i, we must solve the following problem

$$\triangle u + \omega^2 \varepsilon_r u + i\omega^2 \varepsilon_i u = 0 \quad \text{in } \Omega_0 \tag{2.3}$$

$$\left(\frac{\partial u}{\partial r} - Tu\right) = c \quad \text{on } S_0. \tag{2.4}$$

Existence and uniqueness of weak solutions, with a uniform bound, may be obtained for materials with a little bit of absorption, i.e. $\varepsilon_i > 0$.

Throughout the remainder of the paper, in order to simplify estimates within proofs, C will denote a constant which is independent of (ε, u), whose value may change from line to line.

LEMMA 2.1. *For each $\varepsilon \in \mathcal{A}$, problem (2.3)-(2.4) admits a unique weak solution $u \in H^2(\Omega)$. Furthermore, there exists a constant C depending on $\mathcal{A}$, such that $\|u\|_{H^2(\Omega)} \leq C$, independent of $\varepsilon \in \mathcal{A}$.*

Proof. The ideas for the proof of the lemma come from the proof of a similar lemma in [6]. Define for $u, v \in H^1(\Omega)$

$$a(u,v) = \int_\Omega \nabla u \cdot \overline{\nabla v} - \omega^2 \int_\Omega \varepsilon u \overline{v} - \int_{S_0} (Tu)\overline{v},$$

and

$$b(v) = c \int_{S_0} \overline{v}.$$

Using bounds (7.3) and (7.5) in the Appendix for the two- and three-dimensional problems respectively, it is straightforward to show that $a(u,v)$ defines a bounded sesquilinear form over $H^1(\Omega) \times H^1(\Omega)$, and that $b(v)$ is a bounded linear functional on $H^1(\Omega)$. Weak solutions $u \in H^1(\Omega)$ of (2.3) solve the variational problem

$$a(u,v) = b(v) \quad \text{for all } v \in H^1(\Omega). \tag{2.5}$$

The sesquilinear form a uniquely defines a linear operator $A : H^1(\Omega) \to H^1(\Omega)$ such that $a(u, v) = \langle Au, v\rangle_{H^1(\Omega)}$, and the functional $b(v)$ is uniquely identified with an element $b \in H^1(\Omega)$ such that $b(v) = \langle b, v\rangle$. By reflexivity, problem (2.5) is then equivalently stated as

$$Au = b. \tag{2.6}$$

We intend to show that a is coercive by establishing a bound $|a(u, u)| \geq c > 0$ for all $u \in H^1(\Omega)$ with $\|u\|_{H^1(\Omega)} = 1$. We have

$$\begin{aligned} a(u,u) = \int_\Omega |\nabla u|^2 - \omega^2 \int_\Omega \varepsilon_r |u|^2 - \Re\left(\int_{S_0} (Tu)\overline{u}\right) \\ -i\,\Im\left(\int_{S_0} (Tu)\overline{u}\right) - i\omega^2\varepsilon_i \int_\Omega |u|^2. \end{aligned} \tag{2.7}$$

For the two-dimensional problem we have

$$\int_{S_0} (Tu)\overline{u} = \int_{S_0} \sum_{m=1}^{\infty} \gamma_m \hat{u}_m e^{im\theta}\overline{u} = \sum_{m=1}^{\infty} \gamma_m |\hat{u}_m|^2,$$

where $\hat{u}_m$ are the Fourier coefficients of the trace $u|_{S_0}$ (see Appendix). $\Re(\gamma_m) < 0$ and $\Im(\gamma_m) > 0$ for every m. Thus,

$$\Re\left(\int_{S_0} (Tu)\overline{u}\right) < 0 \quad \text{and} \quad \Im\left(\int_{S_0} (Tu)\overline{u}\right) > 0.$$

Similarly, for the three-dimensional case

$$\int_{S_0} (Tu)\overline{u} = \int_{S_0} \sum_{l=0}^{\infty} \gamma_l \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm}\overline{u} = \sum_{l=0}^{\infty} \gamma_l \sum_{m=-l}^{l} |\hat{u}_{lm}|^2,$$

where $\hat{u}_{lm}$ are the coefficients in the spherical harmonics expansion of the trace $u|_{S_0}$ (see Appendix). $\Re(\gamma_l) < 0$ and $\Im(\gamma_l) > 0$ for every l. Thus,

$$\Re\left(\int_{S_0} (Tu)\overline{u}\right) < 0 \quad \text{and} \quad \Im\left(\int_{S_0} (Tu)\overline{u}\right) > 0.$$

Assuming $\|u\|^2_{H^1(\Omega)} = \int_\Omega |\nabla u|^2 + \int_\Omega |u|^2 = 1$, and noticing that the first three terms on the right-hand side of (2.7) are purely real and the last two terms are purely imaginary, we find

$$\begin{aligned} 2|a(u,u)| \geq \left|1 - \int_\Omega (1 + \omega^2\varepsilon_r|u|^2 - \Re\left(\int_{S_0} (Tu)\overline{u}\right)\right| \\ + \left|-\omega^2\varepsilon_i \int_\Omega |u|^2 - \Im\left(\int_{S_0} (Tu)\overline{u}\right)\right|. \end{aligned}$$

For convenience, write $r = \int_\Omega (1 + \omega^2\varepsilon_r)|u|^2$, $s = \int_\Omega |u|^2$, and

$$t = \begin{cases} -\sum_{m=1}^{\infty} \Re(\gamma_m)|\hat{u}_m|^2 & \text{in two dimensions;} \\ -\sum_{l=0}^{\infty} \Re(\gamma_l)\sum_{m=-l}^{l} |\hat{u}_{lm}|^2 & \text{in three dimensions.} \end{cases}$$

Obviously t, r, and s are nonnegative real numbers which depend on u (and ε in the case of r). Although t and s are essentially independent, r must satisfy

$$(1+\omega^2\varepsilon_0)s \le r \le (1+\omega^2\varepsilon_1)s. \tag{2.8}$$

With this notation,

$$2|a(u,u)| \ge |1+t-r| + \omega^2\varepsilon_i s.$$

Note that in the case $s \ge \frac{1}{2(1+\omega^2\varepsilon_1)}$, we have $|a(u,u)| \ge \frac{1}{2}\omega^2\varepsilon_i s \ge \frac{\omega^2\varepsilon_i}{4(1+\omega^2\varepsilon_1)}$. Otherwise, $s < \frac{1}{2(1+\omega^2\varepsilon_1)}$ so that $r < \frac{1}{2}$, and $|a(u,u)| \ge \frac{1}{2}|1+t-r| > \frac{1}{4}$. Hence, for all $s, t \ge 0$, and all r satisfying (2.8),

$$|a(u,u)| \ge c = \min\left\{\frac{\omega^2\varepsilon_i}{4(1+\omega^2\varepsilon_1)}, \frac{1}{4}\right\}.$$

The bound thus holds for every u with $\|u\|_{H^1(\Omega)} = 1$ and for every $\varepsilon \in \mathcal{A}$ with $\varepsilon_i > 0$. Given this coercivity bound, direct application of the Lax-Milgram Theorem yields existence of the bounded solution operator A^{-1} for problem (2.6) such that $\|A^{-1}\| \le 1/c$. Thus $\|u\|_{H^1(\Omega)} \le \|b\|_{H^1(\Omega)}/c$.

Given the bound on $\|u\|_{H^1(\Omega)}$, a uniform $H^2(\Omega)$ bound follows easily, since $\triangle u = -\omega^2\varepsilon u$ is uniformly bounded in $L^2(\Omega)$. ☐

LEMMA 2.2. *There exists a constant K such that for every $\varepsilon_s, \varepsilon_t \in \mathcal{A}$, if $u_s(\varepsilon_s)$, $u_t(\varepsilon_t)$ are the corresponding solutions of the Helmholtz equation (2.3)-(2.4), then u_s and u_t satisfy the Lipschitz condition:*

$$\|u_t - u_s\|_{H^2} \le K\|\varepsilon_t - \varepsilon_s\|_{L^2}. \tag{2.9}$$

Moreover, there exists a constant C such that,

$$\|u_t - u_s\|_{W^{1,\infty}} \le CK\|\varepsilon_t - \varepsilon_s\|_{L^2} \tag{2.10}$$

Proof. We subtract one of the Helmholtz equations from the other to obtain:

$$\triangle u_t - \triangle u_s + \omega^2\varepsilon_t u_t - \omega^2\varepsilon_s u_s = 0.$$

Subtract $\omega^2\varepsilon_t u_s$ on both sides:

$$\triangle(u_t - u_s) + \omega^2\varepsilon_t(u_t - u_s) = -\omega^2(\varepsilon_t - \varepsilon_s)u_s.$$

Let $w = u_t - u_s$. Thus the above equation is written as:

$$\triangle w + \omega^2\varepsilon_t w = -\omega^2(\varepsilon_t - \varepsilon_s)u_s \tag{2.11}$$

The function $-\omega^2(\varepsilon_t - \varepsilon_s)u_s \in L^2(\Omega)$ and thus Lemma 2.1 applies and w is a solution to our equation (2.11). Let us rewrite (2.11) using the operator L_{ε_t}:

$$L_{\varepsilon_t} w := \triangle w + \omega^2\varepsilon_t w = -\omega^2(\varepsilon_t - \varepsilon_s)u_s.$$

Lemma 2.1 ensures that the inverse operator $L_{\varepsilon_t}^{-1} \colon L^2(\Omega) \to H^2(\Omega)$ exists and is uniformly bounded with respect to $\varepsilon_t \in \mathcal{A}$. Thus,

$$w = -\omega^2 L_{\varepsilon_t}^{-1}(\varepsilon_t - \varepsilon_s)u_s.$$

For both two- and three- dimensional materials, the Sobolev Imbedding Theorem implies that $H^2(\Omega) \subset C^0_B(\Omega)$ [1] and hence $\|u_s\|_{L^\infty}$ is bounded, so

$$\|w\|_{H^2} \leq \|L^{-1}_{\varepsilon_t}\|_{L^2(\Omega),H^2(\Omega)} \|\varepsilon_t - \varepsilon_s\|_{L^2} \|u_s\|_{L^\infty} \leq K \|\varepsilon_t - \varepsilon_s\|_{L^2}.$$

To prove the second part of the Lemma, we use the Sobolev Imbedding Theorem and interpolation inequalities. We prove that $w \in W^{2,q}$ for any q such that $3 < q < \infty$. Using the interpolation inequalities in [1] we see that for any solution u of (2.3)-(2.4)

$$\|\triangle u\|_{L^q} \leq \|\triangle u\|^{2/q}_{L^2} \|\triangle u\|^{1-2/q}_{L^\infty} \leq \omega^2 \|u\|^{2/q}_{H^2} \|\varepsilon u\|^{1-2/q}_{L^\infty} \leq \omega^2 \varepsilon_1^{1-2/q} \|u\|_{H^2}.$$

Thus $u \in W^{2,q}$. However, the Sobolev Imbedding Theorem [1] implies that $W^{2,q}(\Omega) \subset C^1_B(\Omega)$, i.e., there exists a constant C such that

$$\|u_t - u_s\|_{1,\infty} \leq C \|u_t - u_s\|_{W^{2,q}} \leq CK \|\varepsilon_t - \varepsilon_s\|_{L^2}, \tag{2.12}$$

where

$$\|u\|_{1,\infty} := \max_{0 \leq |\alpha| \leq 1} \sup_{x \in \Omega} |D^\alpha u(x)|.$$

We deduce the Lipschitz condition (2.10) from (2.12). ▯

We also obtain a Lipshitz-type bound that estimates the proximity of solutions u of the Helmholtz equation (2.3)-(2.4) and the solution $\tilde{u}$ of the constant coefficient Helmholtz equation, where the constant coefficient is the expected value of ε, i.e. $\tilde{\varepsilon} \equiv \langle \varepsilon \rangle = \varepsilon_0 p + \varepsilon_1 (1-p)$. The bound is in terms of the local proximity of the random medium ε and the homogeneous medium $\tilde{\varepsilon}$.

LEMMA 2.3. *Let $\tilde{u}$ be the solution to the Helmholtz equation with constant coefficient $\tilde{\varepsilon} = \varepsilon_0 p + \varepsilon_1(1-p)$, still satisfying the boundary condition (2.4):*

$$\triangle \tilde{u} + \omega^2 \tilde{\varepsilon} \tilde{u} = 0. \tag{2.13}$$

Let $\nu > 0$ and $3 < q < \infty$ be fixed. For any subdomain $\tilde{\Omega} \subset \Omega$ we define the diameter

$$d(\tilde{\Omega}) = \sup_{x,y \in \tilde{\Omega}} |x - y|.$$

There exist constants K^* and K^*_∞, and $\gamma > 0$ such that if Ω is divided into N' non-overlapping subdomains O_i such that $d(O_i) \leq \gamma$ for all $i = 1, \ldots, N'$, then

$$\|u - \tilde{u}\|_{L^2} \leq K^* \left(\sum_{i=1}^{N'} \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| \right) + \nu \tag{2.14}$$

and

$$\|u - \tilde{u}\|_{L^\infty} \leq K^*_\infty(q) \left(K^* \left(\sum_{i=1}^{N'} \left| \int_{O_i} (\tilde{\varepsilon} - \varepsilon)\, dx \right| \right) + C\nu \right)^{\frac{1}{q}}, \tag{2.15}$$

for all realizations (u, ε), and $3 < q < \infty$.

Proof. In the following proof, the difference between the solutions of the two equations (2.1) and (2.13) is written in terms of the solution operator $L^{-1}_{\tilde{\varepsilon}}$. This compact solution operator is approximated by a sequence of finite-rank operators L_n^{-1}, written in their canonical form in terms of orthonormal basis functions. These measurable functions are approximated outside of a set of small measure by continuous functions. The domain Ω is divided into N' non-overlapping subdomains O_i of diameter at most γ, such that the uniformly continuous functions are approximated by a sequence of step functions with characteristic functions of O_i. Hölder continuity of u is proven, and the difference between the solution u for every x in O_i and the maximum of u over the set O_i is bounded in terms of the diameter γ. All of these are combined to give the desired inequalities. The details of the proof follow.

Subtract the two equations (2.1) and (2.13) and manipulate them to get the equation:

$$\triangle(u-\tilde{u})+\omega^2\tilde{\varepsilon}(u-\tilde{u})=\omega^2(\tilde{\varepsilon}-\varepsilon)u$$

for any realization (ε, u). Thus, we can apply the solution operator $L_{\tilde{\varepsilon}}^{-1}$ to obtain:

$$u-\tilde{u}=\omega^2 L_{\tilde{\varepsilon}}^{-1}((\tilde{\varepsilon}-\varepsilon)u).$$

Now, $L_{\tilde{\varepsilon}}^{-1}$ is a bounded operator $L_{\tilde{\varepsilon}}^{-1}: L^2 \to H^2$ and a compact operator $L_{\tilde{\varepsilon}}^{-1}: L^2 \to L^2$. Since $L_{\tilde{\varepsilon}}^{-1}: L^2 \to L^2$ is compact, it can be approximated by a sequence of finite-rank operators L_n^{-1}, and for every given error $\nu_1 > 0$, there exists M_1 such that $\|L_{\tilde{\varepsilon}}^{-1} - L_n^{-1}\|_{L^2(\Omega),L^2(\Omega)} \leq \nu_1$ for $n \geq M_1$ [5]. We apply the triangle inequality to obtain:

$$\begin{aligned}
&\|u-\tilde{u}\|_{L^2} = \omega^2\|L_{\tilde{\varepsilon}}^{-1}(\tilde{\varepsilon}-\varepsilon)u\|_{L^2}\\
&\leq \omega^2\|L_{\tilde{\varepsilon}}^{-1}-L_n^{-1}\|_{L^2(\Omega),L^2(\Omega)}\|\tilde{\varepsilon}-\varepsilon\|_{L^\infty}\|u\|_{L^2}+\omega^2\|L_n^{-1}(\tilde{\varepsilon}-\varepsilon)u\|_{L^2}\\
&\leq C\nu_1+\omega^2\|L_n^{-1}(\tilde{\varepsilon}-\varepsilon)u\|_{L^2},
\end{aligned}$$

where C is independent of the material ε. Finite-rank operators can be decomposed

$$L_n^{-1}(\tilde{\varepsilon}-\varepsilon)u=\sum_{i=1}^{N} w_i^n\langle(\tilde{\varepsilon}-\varepsilon)u, g_i^n\rangle_{L^2}$$

where $g_i^n \in L^2(\Omega)$ and $w_i^n \in \mathrm{Range}(L_n^{-1})$. Thus,

$$\|L_n^{-1}(\tilde{\varepsilon}-\varepsilon)u\|_{L^2}=\|\sum_{i=1}^{N} w_i^n\int_\Omega(\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\|_{L^2}\leq\sum_{i=1}^{N}\|w_i^n\|_{L^2}\left|\int_\Omega(\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right|.$$

Fix $n \geq M_1$; g_i^n is a measurable function on Ω. Given $\nu_2 \geq 0$, there exist continuous functions v_i^n on Ω such that $|S_{\nu_2}| = m\{x : g_i^n(x) \neq v_i^n(x)\} \leq \nu_2$, for each $i = 1, \ldots, N$ [18]. Decompose the integral

$$\int_\Omega(\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx=\int_{\Omega\setminus S_{\nu_2}}(\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx+\int_{S_{\nu_2}}(\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx.$$

Using this we obtain the following bound for each $i = 1, \ldots, N$

$$\begin{aligned}
&\left|\int_\Omega(\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right|\leq\left|\int_{\Omega\setminus S_{\nu_2}}(\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right|+\left|\int_{S_{\nu_2}}(\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right|\\
&\leq\left|\int_{\Omega\setminus S_{\nu_2}}(\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right|+\|\tilde{\varepsilon}-\varepsilon\|_{L^\infty}|S_{\nu_2}|^{\frac{1}{2}}\|u\|_{L^\infty}\|g_i^n\|_{L^2}\leq\left|\int_{\Omega\setminus S_{\nu_2}}(\tilde{\varepsilon}-\varepsilon)ug_i^n\,dx\right|+C_2\nu_2.
\end{aligned}$$

The function v_i^n is continuous on the compact domain Ω and thus it is uniformly continuous and can be approximated by a sequence of step functions $\psi_{N'}$. Divide Ω into N' non-overlapping subdomains O_i such that $d(O_i) \le \gamma$. Define $\psi_{N'} = \sum_{i=1}^{N'} a_i^{N'} \chi_{O_i}$, where χ_{O_i} is a characteristic function of the subdomain O_i. For every given error $\nu_3 > 0$, there exists $\gamma > 0$ such that $\|v_i^n - \psi_{N'}\|_{L^\infty} \le \nu_3$. Thus,

$$\begin{aligned}
&\left|\int_{\Omega\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon) u g_i^n \, dx\right| = \left|\int_{\Omega\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon) u v_i^n \, dx\right| = \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u v_i^n \, dx - \int_{S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon) u v_i^n \, dx\right| \\
&\le \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u v_i^n \, dx\right| + \|\tilde{\varepsilon}-\varepsilon\|_{L^\infty} |S_{\nu_2}| \, \|u\|_{L^\infty} \|v_i^n\|_{L^\infty} \\
&\le \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u (v_i^n - \psi_{n'}) \, dx\right| + \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u \psi_{N'} \, dx\right| + C_2\nu_2 \\
&\le \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u \psi_{N'} \, dx\right| + \|v_i^n - \psi_{N'}\|_{L^\infty} \|\tilde{\varepsilon}-\varepsilon\|_{L^1} \|u\|_{L^\infty} + C_2\nu_2 \\
&\le \left|\int_{\Omega} (\tilde{\varepsilon}-\varepsilon) u \sum_{i=1}^{N'} a_i^{N'} \chi_{O_i} \, dx\right| + C_3\nu_3 + C_2\nu_2 \le \sum_{i=1}^{N'} |a_i^{N'}| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon) u \, dx\right| + C_3\nu_3 + C_2\nu_2.
\end{aligned}$$

Lemma 2.2 implies there exists a constant K such that $\|u\|_{H^2} \le K$ for every realization u. Since H^2 imbeds in $C^{0,1/2}$, there exists a constant K_L such that

$$|u(x) - u(y)| \le K_L |x-y|^{1/2},$$

for all u and for all $x, y \in \Omega$. Let

$$u_\gamma^i = \max_{x \in O_i} u(x)$$

and we have

$$|u(x) - u_\gamma^i| \le K_L \gamma^{1/2}$$

for all $x \in O_i$. Thus,

$$\begin{aligned}
&\left|\int_{\Omega\setminus S_{\nu_2}} (\tilde{\varepsilon}-\varepsilon) u g_i^n \, dx\right| \\
&\le \sum_{i=1}^{N'} |a_i^{N'}| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon)(u - u_\gamma^i) \, dx\right| + \sum_{i=1}^{N'} |a_i^{N'}| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon)(u_\gamma^i) \, dx\right| + C_3\nu_3 + C_2\nu_2 \\
&\le K_L \gamma^{1/2} \sum_{i=1}^{N'} |a_i^{N'}| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon) \, dx\right| + \sum_{i=1}^{N'} |a_i^{N'}| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon)(u_\gamma^i) \, dx\right| + C_3\nu_3 + C_2\nu_2 \\
&\le C\gamma^{1/2} + \sum_{i=1}^{N'} |a_i^{N'}| |u_\gamma^i| \left|\int_{O_i} (\tilde{\varepsilon}-\varepsilon) \, dx\right| + C_3\nu_3 + C_2\nu_2.
\end{aligned}$$

We obtain the desired bound by taking γ, ν_2, and ν_3 sufficiently small. Let $C\gamma^{1/2} + C_2\nu_2 + C_3\nu_3 < \nu$; hence

$$\|u - \tilde{u}\|_{L^2} \le K^* \left(\sum_{i=1}^{N'} \left| \int_{O_i} (\tilde{\varepsilon}-\varepsilon) \, dx \right| \right) + \nu. \tag{2.16}$$

The interpolation inequality [1] states that there exists a constant K_I such that

$$\|u\|_{W^{1,q}} \leq K_I \|u\|_{W^{2,q}}^{\frac{1}{2}} \|u\|_{L^q}^{\frac{1}{2}}.$$

Since $W^{1,q}$ imbeds in C_B for $3 < q < \infty$ [1], there exists a constant C such that

$$\|u - \tilde{u}\|_{L^\infty} \leq C\|u - \tilde{u}\|_{W^{1,q}}.$$

Also, the interpolation inequality for L^p-spaces [8] states that when $3 < q < \infty$

$$\|u\|_{L^q} \leq \|u\|_{L^2}^{\frac{2}{q}} \|u\|_{L^\infty}^{\frac{q-2}{q}}.$$

Combining the above inequalities and the bound (2.16), we prove the second bound in the statement of the Lemma:

$$\|u-\tilde{u}\|_{L^\infty} \leq CK_I\|u-\tilde{u}\|_{W^{2,q}}^{\frac{1}{2}}\|u-\tilde{u}\|_{L^q}^{\frac{1}{2}} \leq CK_I\|u-\tilde{u}\|_{W^{2,q}}^{\frac{1}{2}}\|u-\tilde{u}\|_{L^2}^{\frac{1}{q}}\|u-\tilde{u}\|_{L^\infty}^{\frac{q-2}{2q}}$$
$$\leq K_\infty^*(q)\left(K^*\left(\sum_{i=1}^{N'}\left|\int_{O_i}(\tilde{\varepsilon}-\varepsilon)\,dx\right|\right)+\nu\right)^{\frac{1}{q}}.$$

□

3. Effective Dielectric Coefficient. The expected value $\langle u\rangle$ of the solution u of the Helmholtz equation (2.3)-(2.4), that depends on the random variables through its dependence on the composite material, is defined, recalling (2.2), as follows

$$\langle u\rangle = \int_{\Psi_\delta} u\,dP = \int_{\Gamma_\delta}\sum_{m_g\in R_g}\prod_{j=1}^{N_g} p^{1-m_j}\,(1-p)^{m_j}\,u(\varepsilon_{m,g},x)\,dG_\delta. \tag{3.1}$$

Note that $\langle\rangle$ is an expectation over material realizations, not the spatial variables, so that $\langle u\rangle$ is in general still a function of x. Thus, the effective dielectric coefficient, defined in (1.2) as

$$\varepsilon^* = \frac{\langle\varepsilon u\rangle}{\langle u\rangle},$$

is a function of the spatial variable x.

Our main theorem gives a bound on the effective dielectric coefficient and its spatial variations provided we have a lower bound on the expected value of u. Such a bound is proven to exist for sufficiently small δ. The theorem shows that as the maximum volume δ of the subdomains decreases, so does the magnitude of the spatial variations, and as $\delta \to 0$, the effective coefficient equals the constant predicted by the quasistatic case.

THEOREM 3.1. *Let $\varepsilon^*(x)$ be the effective dielectric coefficient of the medium defined by (1.2). There exists $\delta_0 > 0$ and a constant C^* such that for all $0 < \delta < \delta_0$ and any $x_0 \in \Omega$, the local total variation of ε^* satisfies*

$$\int_{B_r(x_0)} |\nabla\varepsilon^*|\,dx \leq C^*|\varepsilon_1 - \varepsilon_0|\delta,$$

where r is determined as in Assumption A2. As the size of the inhomogeneities goes to 0, the spatial variations decrease in magnitude, and $\varepsilon^*(x) \to p\varepsilon_0 + (1-p)\varepsilon_1$.

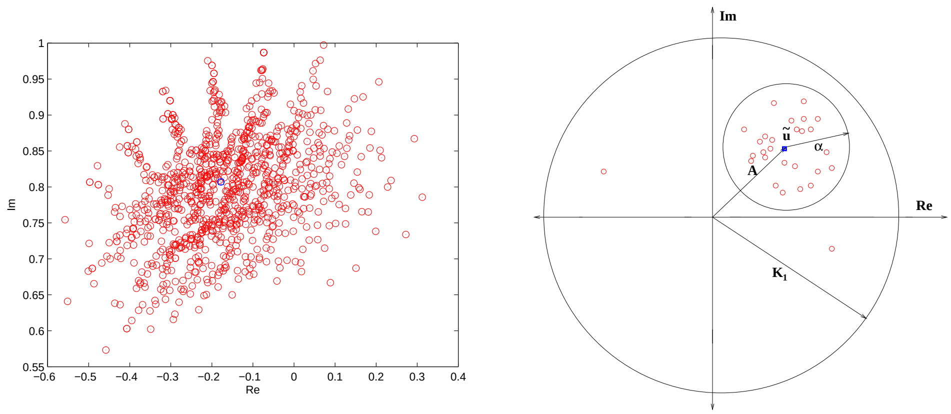


FIG. 3.1. *Proximity to the constant coefficient solution. Left: From numerical experiments, solutions u for a medium with 10 layers at $x = 0.5$ (red dots) and the solution to the constant coefficient problem $\tilde{u}(0.5)$ (blue square); Right: For an appropriate parameter δ, the probability that solutions u cluster within a circle with center $\tilde{u}$ and radius α is $1-\beta$. The probability β that solutions lie outside this circle depends on δ, and $\beta \to 0$ as $\delta \to 0$. All solutions are contained in the circle with radius K_1, since $\|u\|_{L^\infty} \leq K_1$.*

Thus, $|\varepsilon^*(x)|$ is uniformly bounded above for all x, and the spatial variations of ε^* are bounded in terms of the size of the inhomogeneities δ and the contrast of the medium $|\varepsilon_1 - \varepsilon_0|$.

Proof. The proof applies to one-, two-, and three-dimensional random media. In order to obtain a bound on $|\varepsilon^*| = \frac{|\langle \varepsilon u\rangle|}{|\langle u\rangle|}$, we must obtain a lower bound on the denominator $|\langle u\rangle|$. We show that a uniform bound exists provided δ is chosen sufficiently small, i.e. $|\langle u\rangle| \geq c > 0$ for all $x \in \Omega$. The proof is based on a probability argument that shows that the probability that the solutions u will be within a certain radius α from the solution of the constant boundary value problem with dielectric constant $\tilde{\varepsilon} = p\varepsilon_0 + (1-p)\varepsilon_1$ goes to one as the maximum volume δ or the contrast $|\varepsilon_1 - \varepsilon_0|$ goes to zero. The probability β that a solution u lies outside the circle with radius α depends on the parameter δ, and $\beta \to 0$ as $\delta \to 0$. This prevents $\langle u\rangle$ from equaling 0 and gives a lower bound on $|\langle u\rangle| \geq c > 0$. The numerical experiment in Figure 3.1 illustrates this argument, and the proof follows.

We let α and β be arbitrary constants such that $\beta \leq 1$ and $\alpha \leq K_1$. We want to prove that for every such α and β, one can find $\delta > 0$ such that

$$|\langle u\rangle| \geq (1-\beta)(A-\alpha) - \beta K_1,$$

where $\|\tilde{u}\|_{L^\infty} = A$ and $\|u\|_{L^\infty} \leq K_1$.

We use Lemma 2.3. There our domain Ω was divided into N' non-overlapping subdomains O_i such that $d(O_i) \leq \gamma$ for all $i = 1, \ldots, N'$. Each O_i contains at most $\tilde{N}$ subdomains Ω_j and subdomains $\Omega_j \cap O_i$. We are guaranteed that any subdomain Ω_j coming from material realizations has volume less than or equal to δ, hence $|\Omega_j \cap O_i| \leq \delta$. Denote by χ the indicator function assigning 1 if we have material ε_0 or 0 if we have material ε_1 in a given domain. Given the radius α and using Chebyshev's inequality

[7] and estimate (2.15), we obtain

$$
\begin{aligned}
P(\|u-\tilde{u}\|_{L^\infty} \le \alpha) &\ge P\left(K_\infty^*\left(K^*\left(\sum_{i=1}^{N'}\left|\int_{O_i}(\tilde{\varepsilon}-\varepsilon)\,dx\right|\right)+\nu\right)^{\frac{1}{q}} \le \alpha\right)\\
&\ge P\left(\max_i\left|\int_{O_i}(\tilde{\varepsilon}-\varepsilon)\,dx\right| \le \frac{\left(\frac{\alpha}{K_\infty^*}\right)^q-\nu}{K^*N'}\right)\\
&\ge P\left(\left|\sum_{j=1}^{\tilde{N}}\chi_j|O_j^M|-p|O_M|\right| \le \frac{\left(\frac{\alpha}{K_\infty^*}\right)^q-\nu}{K^*N'(\varepsilon_1-\varepsilon_0)}\right)\\
&= 1-P\left(\left|\sum_{j=1}^{\tilde{N}}\chi_j|O_j^M|-p|O_M|\right| \ge \frac{\left(\frac{\alpha}{K_\infty^*}\right)^q-\nu}{K^*N'(\varepsilon_1-\varepsilon_0)}\right)\\
&\ge 1-\left(\frac{(K_\infty^*)^qK^*N'(\varepsilon_1-\varepsilon_0)}{\alpha^q-\nu(K_\infty^*)^q}\right)^2\mathrm{Var}\left(\sum_{j=1}^{\tilde{N}}\chi_j|O_j^M|\right)\\
&\equiv 1-\beta \qquad (3.2)
\end{aligned}
$$

Here O^M is the set over which the quantity $\left|\int_{O_i}(\tilde{\varepsilon}-\varepsilon)\,dx\right|$ is maximized and the sets $\emptyset_j^M \equiv \Omega_j \cap O^M$. We have also used the fact that $\left\langle\sum_{j=1}^{\tilde{N}}\chi_j|O_j^M|\right\rangle = p|O^M|$. We notice that the random variables χ_j are independent and calculate the variance

$$
\mathrm{Var}\left(\sum_{j=1}^{\tilde{N}}\chi_j|O_j^M|\right) = \sum_{j=1}^{\tilde{N}}|O_j^M|^2\mathrm{Var}(\chi_j) = p(1-p)\sum_{j=1}^{\tilde{N}}|O_j^M|^2 \le p(1-p)\tilde{N}\delta^2.
$$

Thus,

$$
\begin{aligned}
\beta &\equiv \left(\frac{(K_\infty^*)^qK^*N'(\varepsilon_1-\varepsilon_0)}{\alpha^q-\nu(K_\infty^*)^q}\right)^2\mathrm{Var}\left(\sum_{j=1}^{\tilde{N}}\chi_j|O_j^M|\right)\\
&\le \left(\frac{(K_\infty^*)^qK^*N'(\varepsilon_1-\varepsilon_0)}{\alpha^q-\nu(K_\infty^*)^q}\right)^2 p(1-p)\tilde{N}\delta^2.
\end{aligned}
$$

We have shown that the probability that solutions u are within a radius α of the constant coefficient solution $\tilde{u}$ goes to one as either δ or the contrast in the medium $|\varepsilon_1-\varepsilon_0|$ goes to 0.

Let us call $\|u-\tilde{u}\|_{L^\infty} \le \alpha$ condition L and the complement - condition L^c. Define the conditional expectations

$$
\langle u|L\rangle \equiv \frac{\int_{\Psi_\delta(L)} u\,dP}{P(L)} \quad \text{and} \quad \langle u|L^c\rangle \equiv \frac{\int_{\Psi_\delta(L^c)} u\,dP}{P(L^c)},
$$

and note that $P(L) \ge 1-\beta$ and $P(L^c) \le \beta$. The expected value $\langle u\rangle$ is given by

$$
\langle u\rangle = P(L)\langle u|L\rangle + P(L^c)\langle u|L^c\rangle,
$$

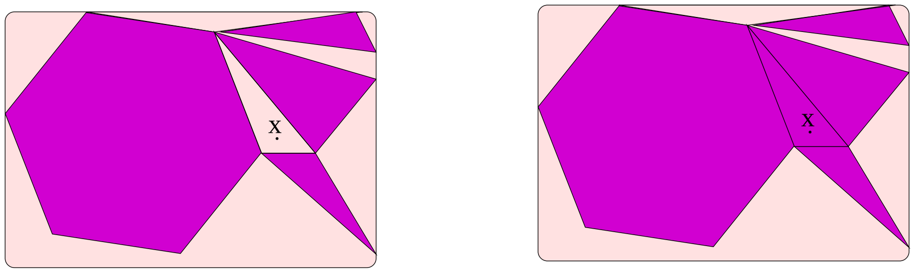


FIG. 3.2. *Sample materials in* Ψ_δ^0 *and* Ψ_δ^1 *for fixed* x. *Left: Material realization* ψ_0*; Right: Corresponding material realization* ψ_1 *obtained by switching material* ε_0 *with material* ε_1 *in the domain containing* x.

and using estimate (3.2) we obtain

$$|\langle u\rangle| \geq (1-\beta)|\langle u|L\rangle| - \beta|\langle u|L^c\rangle|.$$

If u satisfies condition L, then u satisfies the inequality

$$\|u\|_{L^\infty} \geq \|\tilde{u}\|_{L^\infty} - \alpha \geq A - \alpha.$$

Now using the uniform upper bound $\|u\|_{L^\infty} \leq K_1$, we obtain the desired result:

$$|\langle u\rangle| \geq (1-\beta)(A-\alpha) - \beta K_1,$$

where the constant β depends on δ, the maximum volume of the subdomains, and on the contrast $|\varepsilon_1 - \varepsilon_0|$, and $\beta \to 0$ as δ or $|\varepsilon_1 - \varepsilon_0| \to 0$. Thus by picking the appropriate α and β, where β is controlled by the parameter δ, we obtain the lower bound $|\langle u\rangle| \geq c > 0$ for all $x \in \Omega$. This provides a bound on the effective dielectric coefficient:

$$|\varepsilon^*| \leq \frac{\tilde{\varepsilon} K_1}{c}.$$

The uniform lower bound on $|\langle u\rangle|$ is utilized in proving that $\|\varepsilon^*\|_{BV} \leq C^*|\varepsilon_1 - \varepsilon_0|\delta$, as follows. Formally, the gradient $\nabla\varepsilon^*$ is given by:

$$\nabla\varepsilon^* = \frac{\langle u\rangle\langle(\nabla\varepsilon)u\rangle + \langle u\rangle\langle\varepsilon\nabla u\rangle - \langle\nabla u\rangle\langle\varepsilon u\rangle}{\langle u\rangle^2}, \tag{3.3}$$

where $\nabla\varepsilon$ is understood in the sense of a distribution. Now choose δ such that $|\langle u\rangle| \geq c > 0$. We want to bound the numerator in terms of this δ and the contrast $|\varepsilon_1 - \varepsilon_0|$. First we bound

$$|\langle u\rangle\langle\varepsilon\nabla u\rangle - \langle\nabla u\rangle\langle\varepsilon u\rangle| \leq C_1\delta|\varepsilon_1 - \varepsilon_0| \tag{3.4}$$

pointwise, where C_1 is a constant. In the proof we use the Lipschitz bound (2.10) from Lemma 2.2.

The bound (3.4) is obtained by looking at material realizations that differ only in one particular subdomain Ω_j and realizing that the pointwise difference in solutions propagating through two such material realizations can be bounded in terms of the L^2-norm of the difference in the two materials, where the two materials differ only on the subdomain Ω_j with $|\Omega_j| \leq \delta$.

Fix x. Divide the set of material realizations Ψ_δ into two subsets $\Psi_\delta = \Psi_\delta^0 \cup \Psi_\delta^1$, where Ψ_δ^0 is the subset of realizations such that $\varepsilon(x) = \varepsilon_0$ and Ψ_δ^1 is the subset of realizations such that $\varepsilon(x) = \varepsilon_1$. Representative elements of the subsets Ψ_δ^0 and Ψ_δ^1 are shown in Figure 3.2. For each geometry g, let R_g^0 and R_g^1 be subsets of the set of material assignments R_g such that

$$R_g^0 = \{m_g = \left(m_1, \ldots, m_{N_g}\right) : m_j = 0 \text{ for } x \in \Omega_j\},$$

and

$$R_g^1 = \{m_g = \left(m_1, \ldots, m_{N_g}\right) : m_j = 1 \text{ for } x \in \Omega_j\}.$$

Thus, $R_g = R_g^0 \cup R_g^1$. The expected value of u is given by:

$$\langle u\rangle(x) = \int_{\Psi_\delta} u\, dP = \int_{\Gamma_\delta} \sum_{m_g \in R_g} \prod_{l=1}^{N_g} p^{1-m_l} \left(1-p\right)^{m_l} u(\varepsilon_{m,g}, x)\, dG_\delta$$

$$= p \int_{\Gamma_\delta} \sum_{m_g \in R_g^0} \prod_{\substack{l=1 \\ l \neq j}}^{N_g} p^{1-m_l} \left(1-p\right)^{m_l} u\, dG_\delta + (1-p) \int_{\Gamma_\delta} \sum_{m_g \in R_g^1} \prod_{\substack{l=1 \\ l \neq j}}^{N_g} p^{1-m_l} \left(1-p\right)^{m_l} u\, dG_\delta$$

$$= = p\langle u\rangle_{\Psi_\delta^0} + (1-p)\langle u\rangle_{\Psi_\delta^1},$$

where $\langle u\rangle_{\Psi_\delta^0} = \langle u | \varepsilon(x) = \varepsilon_0\rangle$ and $\langle u\rangle_{\Psi_\delta^1} = \langle u | \varepsilon(x) = \varepsilon_1\rangle$. Using this notation we can rewrite

$$\begin{aligned} &\langle u\rangle\langle \varepsilon \nabla u\rangle - \langle \nabla u\rangle\langle \varepsilon u\rangle \\ &= \varepsilon_1 p(1-p) \left(\langle u\rangle_{\Psi_\delta^0}\langle \nabla u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^1}\langle \nabla u\rangle_{\Psi_\delta^0}\right) \\ &+ \varepsilon_0 p(1-p) \left(\langle u\rangle_{\Psi_\delta^1}\langle \nabla u\rangle_{\Psi_\delta^0} - \langle u\rangle_{\Psi_\delta^0}\langle \nabla u\rangle_{\Psi_\delta^1}\right). \end{aligned}$$

For every material described by Ψ_δ^0, there exists a material described by Ψ_δ^1 such that the two materials differ only in a subdomain $\Omega_j \ni x$. Let us call u_{ψ_0} the solution of the Helmholtz equation when the material realization belongs to Ψ_δ^0 and u_{ψ_1} the corresponding solution of the Helmholtz equation when the material realization, differing only in m_j, belongs to Ψ_δ^1. We have

$$\left| \int_{\Psi_\delta^1} u_{\psi_1}(x)\, dP - \int_{\Psi_\delta^0} u_{\psi_0}(x)\, dP \right| \leq \int_{\Gamma_\delta} \sum_{i=1}^{2^{N_g - 1}} \prod_{\substack{l=1 \\ l \neq j}}^{N_g} p^{1-m_l^i} \left(1-p\right)^{m_l^i} |u_{\psi_1} - u_{\psi_0}|(x)\, dG_\delta$$

$$\leq \sup_{\substack{g \in \Gamma_\delta \\ m_1 \in R_g^1 \\ m_0 \in R_g^0}} \|u_{\psi_1}(m_1, g) - u_{\psi_0}(m_0, g)\|_{L^\infty}$$

$$\leq CK \sup_{\substack{g \in \Gamma_\delta \\ m_1 \in R_g^1 \\ m_0 \in R_g^0}} \|\varepsilon_{\psi_1}(m_1, g) - \varepsilon_{\psi_0}(m_0, g)\|_{L^2} \leq CK\delta|\varepsilon_1 - \varepsilon_0|.$$

The preceding comes from the fact that for any material realization in Ψ_δ^1, there exists a material realization in Ψ_δ^0. The application of Lemma 2.2 yields the second-to-last inequality. Thus, we have that $\left|\langle u\rangle_{\Psi_\delta^1} - \langle u\rangle_{\Psi_\delta^0}\right| \to 0$ pointwise as $\delta \to 0$. By a similar

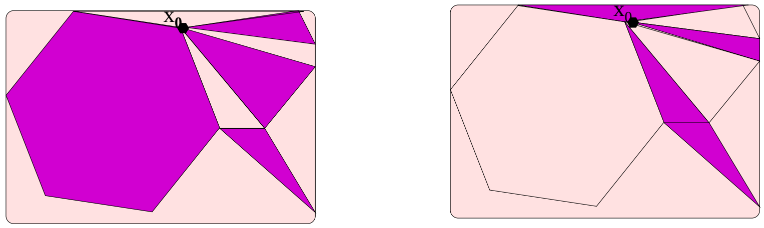


FIG. 3.3. *Sample materials in Ψ^α_δ and Ψ^β_δ for fixed x on the boundary between several materials. Left: Material realization ψ_α; Right: Corresponding material realization ψ_β obtained by interchanging the materials at domains interfacing at x.*

argument, $\left|\langle\nabla u\rangle_{\Psi^1_\delta} - \langle\nabla u\rangle_{\Psi^0_\delta}\right| \le CK\delta|\varepsilon_1-\varepsilon_0|$, and $\left|\langle\nabla u\rangle_{\Psi^1_\delta} - \langle\nabla u\rangle_{\Psi^0_\delta}\right| \to 0$ pointwise as $\delta\to 0$. Now,

$$\begin{aligned}&\left|\langle u\rangle_{\Psi^0_\delta}\langle\nabla u\rangle_{\Psi^1_\delta} - \langle u\rangle_{\Psi^1_\delta}\langle\nabla u\rangle_{\Psi^0_\delta}\right| \\ &\le \left|\langle u\rangle_{\Psi^0_\delta}\right|\left|\langle\nabla u\rangle_{\Psi^1_\delta} - \langle\nabla u\rangle_{\Psi^0_\delta}\right| + \left|\langle\nabla u\rangle_{\Psi^0_\delta}\right|\left|\langle u\rangle_{\Psi^1_\delta} - \langle u\rangle_{\Psi^0_\delta}\right|.\end{aligned} \tag{3.5}$$

Referring to Lemmas 2.2 and 2.1, we know that $u \in C^1_B(\Omega)$, and that there exist constants K_1 and K_2 such that $\|u\|_{L^\infty} \le K_1$ and $\|\nabla u\|_{L^\infty} \le K_2$ for every u. Then

$$\left|\langle u\rangle_{\Psi^0_\delta}\langle\nabla u\rangle_{\Psi^1_\delta} - \langle u\rangle_{\Psi^1_\delta}\langle\nabla u\rangle_{\Psi^0_\delta}\right| \le KC|\varepsilon_1-\varepsilon_0|\delta(K_1+K_2) \to 0 \quad \text{as} \quad \delta\to 0$$

and similarly for the second term in (3.5). Thus, we obtain the following bound

$$\begin{aligned}&|\langle u\rangle\langle\varepsilon\nabla u\rangle - \langle\nabla u\rangle\langle\varepsilon u\rangle| \\ &\le \varepsilon_1 p(1-p)\left|\langle u\rangle_{\Psi^0_\delta}\langle\nabla u\rangle_{\Psi^1_\delta} - \langle u\rangle_{\Psi^1_\delta}\langle\nabla u\rangle_{\Psi^0_\delta}\right| + \varepsilon_0 p(1-p)\left|\langle u\rangle_{\Psi^1_\delta}\langle\nabla u\rangle_{\Psi^0_\delta} - \langle u\rangle_{\Psi^0_\delta}\langle\nabla u\rangle_{\Psi^1_\delta}\right| \\ &\le KCp(1-p)(\varepsilon_1+\varepsilon_0)|\varepsilon_1-\varepsilon_0|(K_1+K_2)\delta.\end{aligned} \tag{3.6}$$

Looking back at (3.3) to get an upper bound on $|\nabla\varepsilon^*|$, we now want to prove that $|\langle(\nabla\varepsilon)u\rangle| \le C_2\delta|\varepsilon_1-\varepsilon_0|$ in the distributional sense.

Since $\varepsilon(x)$ equals a constant in every subdomain Ω_j, $\nabla\varepsilon = 0$ there, and the only problem occurs at the interface between two or more subdomains with different materials, where ε is discontinuous and $\nabla\varepsilon$ is defined only in the distributional sense.

Fix a realization ψ_α such that x_0 is at the interface between k subdomains Ω_j, $j = 1...k$ with alternating materials ε_0 and ε_1 in them. This assumption will pose no loss of generality since the other cases are attained at material realizations satisfying our assumptions. Call ψ_β the realization that has the same geometry as realization ψ_α, but with the materials in the k subdomains interfacing at x_0 switched, e.g. Figure 3.3. Without loss of generality let realization ψ_α have material ε_0 in Ω_1; thus realization ψ_β has material ε_1 in the same subdomain Ω_1. Let ϕ be a test function $\phi\in C^\infty_0(\Omega,\mathbb{R}^n)$ such that $\operatorname{supp}\phi\subset B_r(x_0)$. We can find $\nabla(\varepsilon_\alpha)u_\alpha$ at x_0 in the generalized sense:

$$\begin{aligned}&\int_{B_r(x_0)} u_\alpha\nabla(\varepsilon_\alpha)\phi\,dx \\ &= (\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_1\cap\Omega_2)} u_\alpha\phi\nu_{\partial(\Omega_1\cap\Omega_2)}\,dx + (\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_2\cap\Omega_3)} u_\alpha\phi\nu_{\partial(\Omega_2\cap\Omega_3)}\,dx + \cdots \\ &+(\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_{k-1}\cap\Omega_k)} u_\alpha\phi\nu_{\partial(\Omega_{k-1}\cap\Omega_k)}\,dx + (\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_1\cap\Omega_k)} u_\alpha\phi\nu_{\partial(\Omega_1\cap\Omega_k)}\,dx,\end{aligned}$$

where $\partial(\Omega_1 \cap \Omega_2)$ is the interface between subdomains Ω_1 and Ω_2 and $\nu_{\partial(\Omega_1\cap\Omega_2)}$ is the unit normal vector to Ω_1 on the interface with Ω_2. Note that $\nu_{\partial(\Omega_1\cap\Omega_2)} = -\nu_{\partial(\Omega_2\cap\Omega_1)}$.

Similarly, we find that $\nabla(\varepsilon_\beta)u_\beta$ at x_0 in the generalized sense is

$$\begin{aligned}
&\int_{B_r(x_0)} u_\beta \nabla(\varepsilon_\beta)\phi\, dx \\
&= -(\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_1\cap\Omega_2)} u_\beta\phi\nu_{\partial(\Omega_1\cap\Omega_2)}\, dx - (\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_2\cap\Omega_3)} u_\beta\phi\nu_{\partial(\Omega_2\cap\Omega_3)}\, dx - \cdots \\
&-(\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_{k-1}\cap\Omega_k)} u_\beta\phi\nu_{\partial(\Omega_{k-1}\cap\Omega_k)}\, dx - (\varepsilon_1-\varepsilon_0)\int_{\partial(\Omega_1\cap\Omega_k)} u_\beta\phi\nu_{\partial(\Omega_1\cap\Omega_k)}\, dx
\end{aligned}$$

Divide again Ψ_δ into three subsets $\Psi_\delta = \Psi_\delta^c \cup \Psi_\delta^\alpha \cup \Psi_\delta^\beta$: Ψ_δ^c is the subset of realizations such that x_0 is inside some subdomain; Ψ_δ^α is the subset of realizations such that x_0 is at the interface between k subdomains Ω_j, $j = 1\dots k$ for any integer k with alternating materials ε_0 and ε_1 in them and material ε_0 in Ω_1; Ψ_δ^β is the subset of realizations such that x_0 is at the interface between k subdomains Ω_j, $j = 1\dots k$ for any integer k with alternating materials ε_1 and ε_0 in them and material ε_1 in Ω_1. Note that $\langle\nabla\varepsilon\rangle_{\Psi_\delta^c} = 0$. Utilizing assumptions A2 and A3, we obtain

$$\begin{aligned}
&\left|\langle\int_{B_r(x_0)} (u\nabla\varepsilon)\phi dx\rangle\right| \qquad (3.7)\\
&\le |\varepsilon_1-\varepsilon_0|\,\|\phi\|_{L^\infty}\sum_{j=1}^{k}\|\chi_{\Omega_j}\|_{BV}\int_{G_\delta}\sum_{i=1}^{2^{N_g-1}} p^{\frac{k}{2}}(1-p)^{\frac{k}{2}}\prod_{\substack{l=1\\ l\neq j+1\dots\\ j+k}}^{N_g} p^{1-m_l^i}(1-p)^{m_l^i}\,\|u_\alpha-u_\beta\|_{L^\infty}\, dG_\delta \\
&\le kKCC_p p(1-p)\|\phi\|_{L^\infty}\,|\varepsilon_1-\varepsilon_0|^2\,\delta.
\end{aligned}$$

Note that the inequality

$$\|u_\alpha - u_\beta\|_{L^\infty} \le kKC\,|\varepsilon_1-\varepsilon_0|\,\delta$$

comes from Lemma 2.2 and the fact that for any material in Ψ_δ^α one can find a material in Ψ_δ^β, which differs only on the subdomains Ω_j through Ω_{j+k} each with volume less than or equal to δ.

Choose δ small enough that $|\langle u\rangle| \ge c > 0$. Using the lower bound $|\langle u\rangle| \ge c > 0$, (3.6), and (3.7), we obtain

$$\int_{B_r(x_0)} |\nabla\varepsilon^*|\, dx \le \frac{C|\varepsilon_1-\varepsilon_0|\delta\|\phi\|_{L^\infty}}{c^2} \le C^*|\varepsilon_1-\varepsilon_0|\delta, \qquad (3.8)$$

where $\nabla\varepsilon^*$ is defined in the generalized sense. This will ensure that $\varepsilon^* \in BV(\Omega)$, and thus, we can bound the spatial variations of ε^*

$$\begin{aligned}
&V(\varepsilon^*,\Omega) := \sup\left\{\int_\Omega \varepsilon^*\mathrm{div}\phi\, dx : \phi\in C_0^1(\Omega,\mathbb{R}^n),\ \|\phi\|_{L^\infty(\Omega)}\le 1\right\}\\
&\le C\int_\Omega |\nabla\varepsilon^*|\, dx \to 0 \quad \text{ as } \delta \text{ or } |\varepsilon_1-\varepsilon_0| \to 0.
\end{aligned}$$

The formula that prescribes the appropriate δ takes into account the contrast $|\varepsilon_1-\varepsilon_0|$ in the medium (Theorem 3.1, (3.6) and (3.8)).

Note that

$$\varepsilon^* = \frac{\langle \varepsilon u \rangle}{\langle u \rangle} = \frac{p\varepsilon_0 \langle u \rangle_{\Psi_\delta^0} + (1-p)\varepsilon_1 \langle u \rangle_{\Psi_\delta^1}}{p\langle u \rangle_{\Psi_\delta^0} + (1-p)\langle u \rangle_{\Psi_\delta^1}}.$$

Since $\left| \langle u \rangle_{\Psi_\delta^1} - \langle u \rangle_{\Psi_\delta^0} \right| \to 0$ pointwise as $\delta \to 0$, we obtain that $\varepsilon^* \to p\varepsilon_0 + (1-p)\varepsilon_1$ as $\delta \to 0$, which is consistent with the quasistatic case since by letting $\delta \to 0$, we are effectively operating in the quasistatic limit.
□

We can obtain an estimate of how much ε^* differs from the expected value $\tilde{\varepsilon}$:

$$\begin{aligned}
|\varepsilon^* - \tilde{\varepsilon}| &= \frac{|\langle \varepsilon u \rangle - \tilde{\varepsilon}\langle u \rangle|}{|\langle u \rangle|} \\
&\leq \frac{|p\varepsilon_0 \langle u \rangle_{\Psi_0} + (1-p)\varepsilon_1 \langle u \rangle_{\Psi_1} - (p\varepsilon_0 + (1-p)\varepsilon_1)(p\langle u \rangle_{\Psi_0} + (1-p)\langle u \rangle_{\Psi_1})|}{c} \\
&\leq \frac{p(1-p)|\varepsilon_1 - \varepsilon_0||\langle u \rangle_{\Psi_1} - \langle u \rangle_{\Psi_0}|}{c} \\
&\leq p(1-p)C|\varepsilon_1 - \varepsilon_0|\delta.
\end{aligned}$$

4. Numerical Experiments. We observe the spatial dependence of the effective dielectric coefficient by numerically calculating ε^* and graphing it as a function of x. In these numerical experiments, ε^* is calculated by dividing the interval (0, 1) into the corresponding number of intervals m, each layer of length $\frac{1}{m}$, and going through all possible realizations by assigning in each layer either material of type one or material of type two, both with probability $\frac{1}{2}$. The solution u for each particular layered material is computed by the transfer matrix method [19]. Sample realizations in the case of a six-layer medium are given in Figure 4.1. In these numerical experiments $\omega = 53$. The graph on the left shows the sample six-layer medium, composed of material of type one ($\varepsilon_0 = 1$) in the first, second, and fifth layers, and material of type two ($\varepsilon_1 = 2$) in the third, forth, and sixth layers (above), and the solution u and the product εu (below). The graph on the right shows a six-layer sample medium, composed of material of type one ($\varepsilon_0 = 1$) in the first, second, and sixth layers, and material of type two ($\varepsilon_1 = 2$) in the third, forth, and fifth layers (above), and the solution u and the product εu.

The expected $\langle u \rangle$ is obtained by evaluation the solution u for each realization and multiplying it by the probability of the particular realization, i.e.

$$\langle u \rangle = \sum_{m_g \in R_g} u(x, m_g) \prod_{j=1}^{N_g} p^{1-m_j} (1-p)^{m_j}.$$

In the case when both materials are assigned according to probability $\frac{1}{2}$, each solution u is multiplied by $\left(\frac{1}{2}\right)^m$. The expected $\langle \varepsilon u \rangle$ is computed similarly. We observe that when the length of the layers is 1/6, the spatial variations of ε^* are more pronounced than in the case when the length of the layer is 1/16 (Figure 4.2).

Without loss of generality, assume that the dielectric coefficient of the medium is

$$\varepsilon(x) = 1 + z\chi(x, \psi),$$

where the function $\chi(x, \psi)$ is a random characteristic function in x. The main theorem 3.1 showed that the spatial variations in the effective coefficient are bounded by the contrast in the medium z (or as appears in the theorem, $|\varepsilon_1 - \varepsilon_0|$).

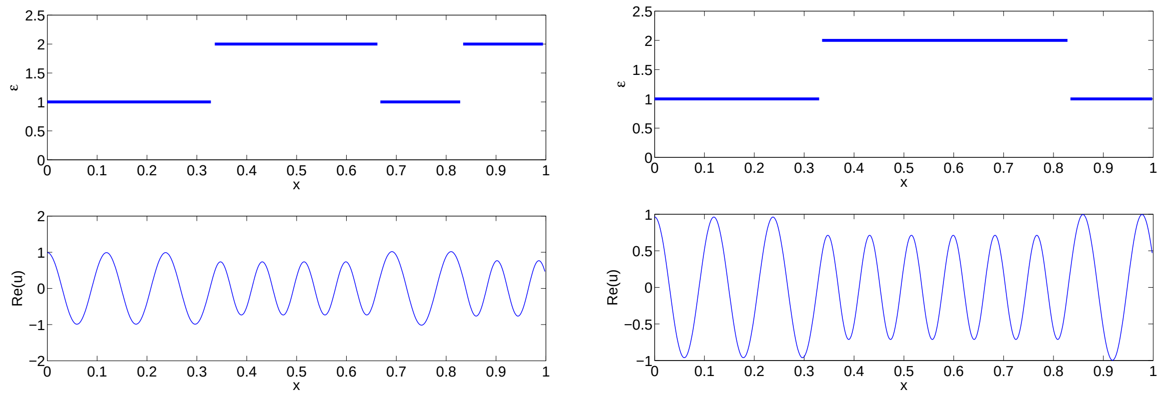


FIG. 4.1. *Sample realizations in a six layer medium: ε (top) and corresponding real part of u (bottom).*

Numerical experiments also show that the spatial variations decrease in magnitude when the contrast z between the two materials is small (Figure 4.3). In these experiments we are looking at a ten-layer medium and $\omega = 53$. We vary the contrast. In the first experiment, we assign material of type one ($\varepsilon_0 = 1$) or material of type two ($\varepsilon_1 = 1.5$), both with probability $\frac{1}{2}$. In the second experiment, we assign material of type one ($\varepsilon_0 = 1$) or material of type two ($\varepsilon_1 = 13$), both with probability $\frac{1}{2}$. The dependence of the magnitude of the spatial variations on the contrast in the medium is obvious.

Numerical experiments are performed in a two dimensional random medium, that is periodic in the x direction. The medium is obtained by randomly picking points in a square cell with sides equal to 2π and drawing circles of random radii around the randomly selected points. The coordinates of the points and the values of the radii are drawn from a normal distribution. After the cell is divided into subdomains either material ε_0 or material ε_1 is assigned to each subdomain, both with probability $1/2$. The variational problem (2.6) was discretized with a first-order finite element method, using piecewise bilinear elements on a uniform, rectangular grid. The design variable ε was approximated by a piecewise constant function on the same uniform grid. The nonlocal boundary operators T defined by (7.1) in the Appendix were approximated by explicitly calculating the Fourier coefficients of the traces of the finite element basis, then truncating the sum in (7.1). The resulting finite element scheme can be shown to converge and to conserve energy, provided all the propagating terms are included in the sum [2]. This discretization leads to a large, sparse (except for the boundary terms), non-Hermitian matrix problem, which for simplicity is solved using the direct sparse solver in *Matlab*. Despite the convenience of imposing a positive lower bound on the imaginary part of ε in Lemma 2.1 for obtaining a uniform upper bound on solutions, we found that the numerical experiments were quite insensitive to small dissipations. Thus in all of the examples below, we set $\varepsilon_i = 0$.

In all two-dimensional numerical experiments, the frequency $\omega = 1.2$. In Figure 4.4 a single material realization (top), the real part of the corresponding solution u (middle) and the real part of the product εu (bottom) for a medium with contrast $z = 0.5$ are displayed. In Figure 4.5 another material realization (top), the real part of the corresponding solution u (middle) and the real part of the product εu (bottom) for

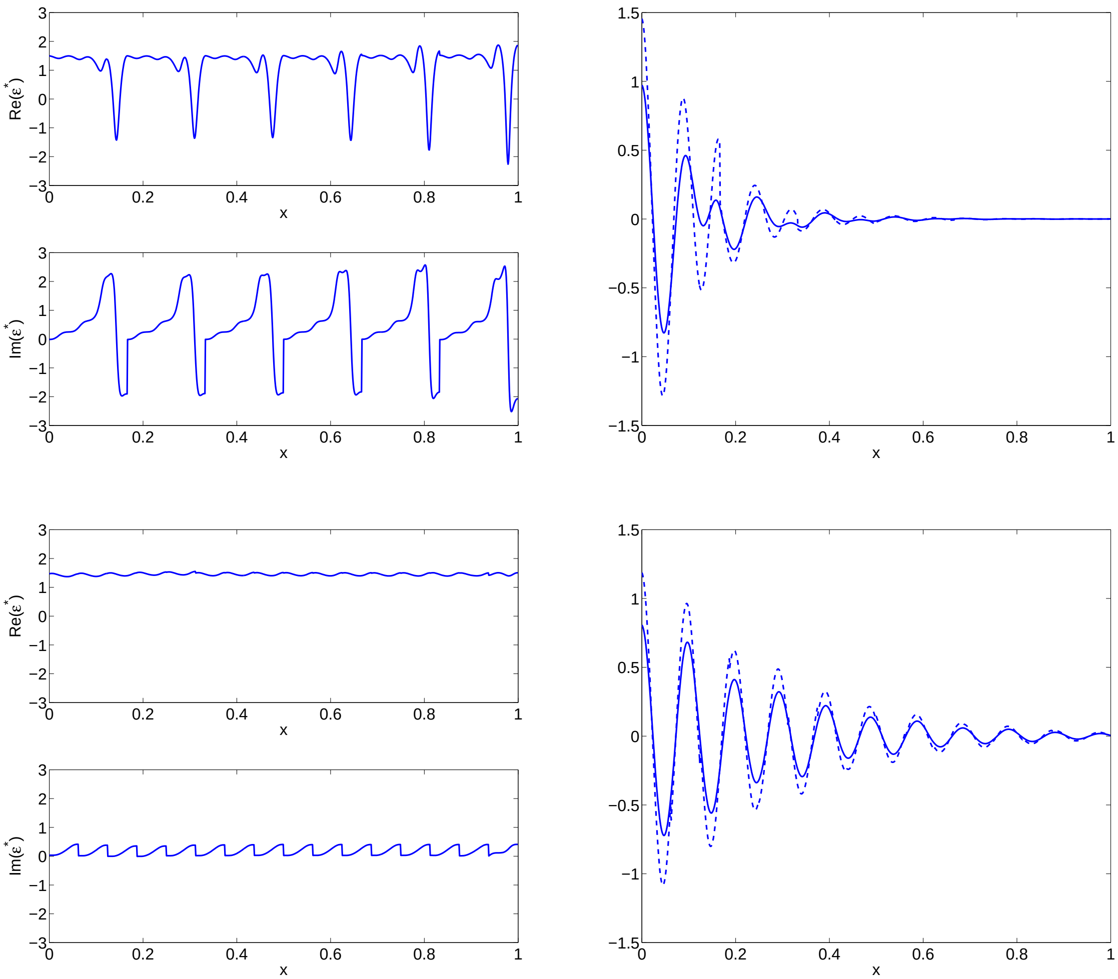


FIG. 4.2. *Spatial Variations. Upper left: Real and imaginary ε^* in a medium of six layers; upper right: Real part of $\langle \varepsilon u \rangle$ (dashed line) and $\langle u \rangle$ (solid line) in a medium of six layers; lower left: Real and imaginary ε^* in a medium of sixteen layers; lower right: Real part of $\langle \varepsilon u \rangle$ (dashed line) and $\langle u \rangle$ (solid line) in a medium of sixteen layers.*

a medium with contrast $z = 3$ are shown. The average $\langle \varepsilon u \rangle$ is obtained by calculating εu for each material realization, summing up over realizations, and dividing the sum by the number of realizations. In our experiments the number of material realizations is 75000. The expectation $\langle u \rangle$ is calculated similarly. The effective coefficient ε^* is the quotient of these quantities: $\varepsilon^* = \frac{\langle \varepsilon u \rangle}{\langle u \rangle}$.

In Figure 4.6 the expectations $\langle u \rangle$ and $\langle \varepsilon u \rangle$ are shown. The effective dielectric coefficient for a random medium with contrast $z = 0.5$ is displayed in Figure 4.6. Let us investigate the effect of increasing the contrast z in the medium on the magnitude of the spatial variation in ε^*. In Figure 4.7 we have shown the averaged quantities $\langle u \rangle$ and $\langle \varepsilon u \rangle$ for a random medium with contrast $z = 3$. The spatial variations of the effective coefficient (Figure 4.7) are much larger in magnitude for the medium with

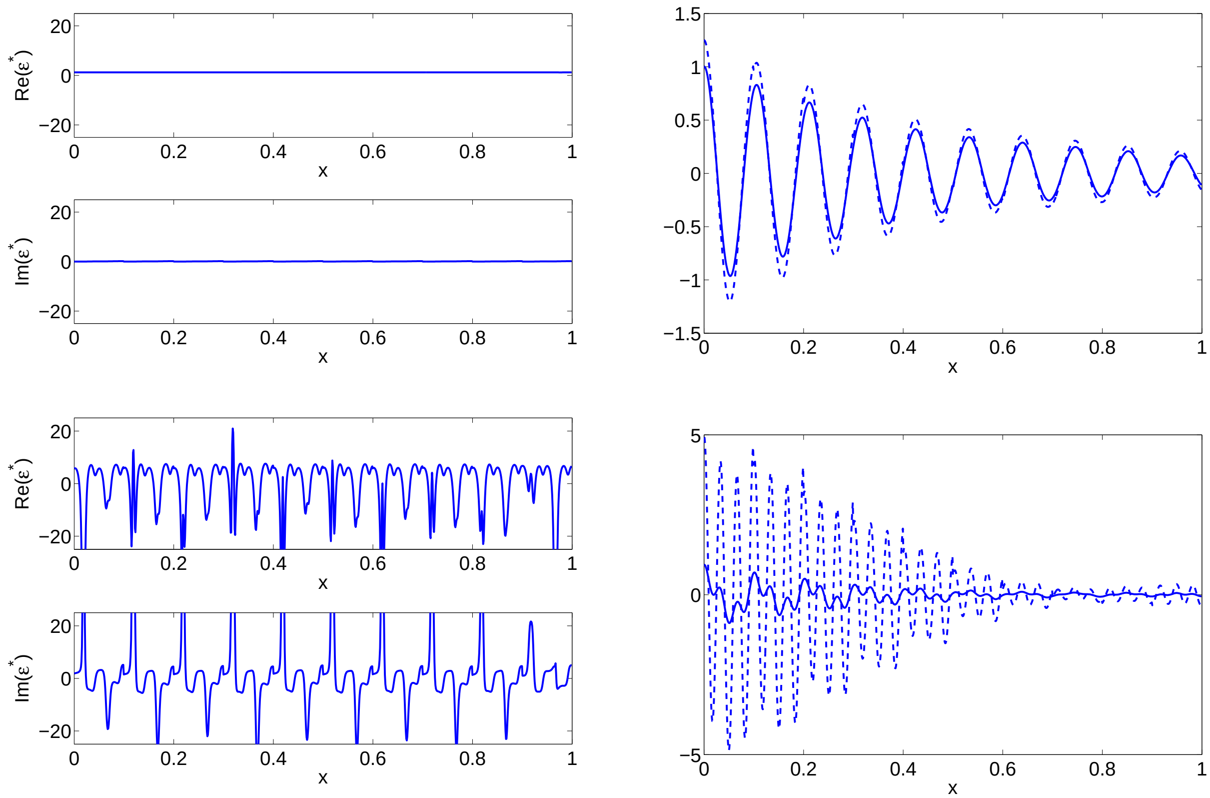


FIG. 4.3. *Spatial variations. Upper left: Real and imaginary ε^* in a medium of ten layers and contrast $z = 0.5$; upper right: real part of $\langle \varepsilon u \rangle$ (dashed line) and $\langle u \rangle$ (solid line) in a medium of ten layers and contrast $z = 0.5$; lower left: Real and imaginary ε^* in a medium of ten layers and contrast $z = 12$; lower right: real part of $\langle \varepsilon u \rangle$ (dashed line) and $\langle u \rangle$ (solid line) in a medium of ten layers and contrast $z = 12$.*

the greater contrast.

5. Approximation formulas. In general, it is difficult to calculate exactly the effective dielectric coefficient ε^*. Thus, finding good approximation formulas is important. These approximation formulas are derived assuming the contrast z is small. They take into account the geometry of the material through the material distribution and correlation functions.

Keller and Karal analyzed the propagation of waves in a random medium assuming that the medium differs slightly from a homogeneous medium. Assuming that the homogenized medium has a constant dielectric coefficient, and thus the averaged wave is a plane wave, the authors derive an equation satisfied by the average wave that is correct through terms of order ν^2, where ν measures the deviation of the medium from homogeneity. From this equation, they determine the effective dielectric constant of the medium. The propagation constant for the average or coherent wave is complex even for a nondissipative medium, because the coherent wave is continually scattered by the inhomogeneities and converted into the incoherent wave. The propagation velocity of the average wave is also diminished by the inhomogeneities. The effective dielectric constant depends upon certain trigonometric integrals of the auto-

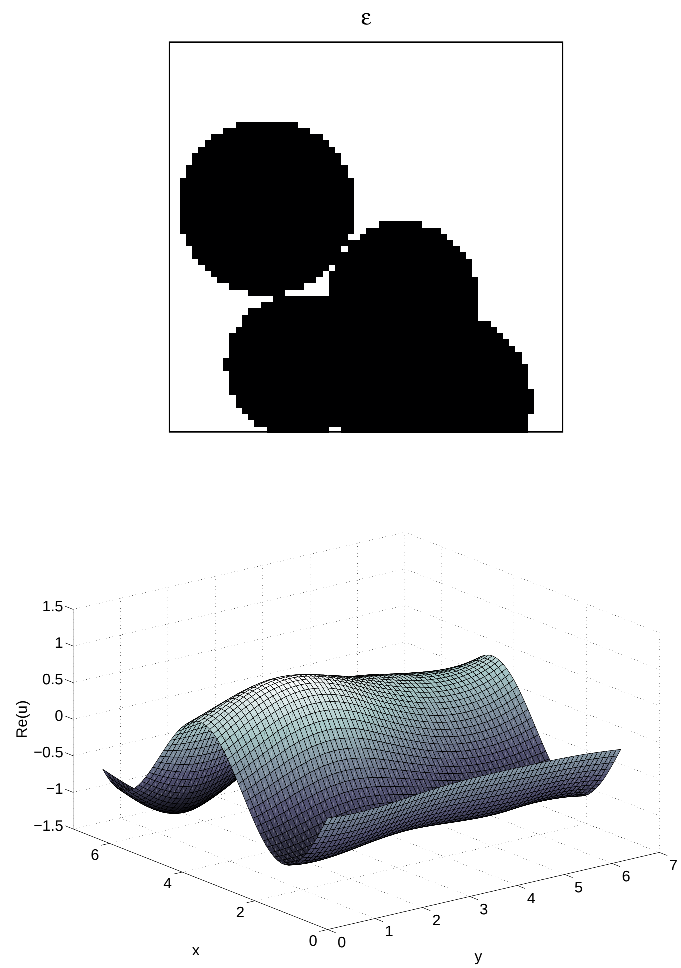


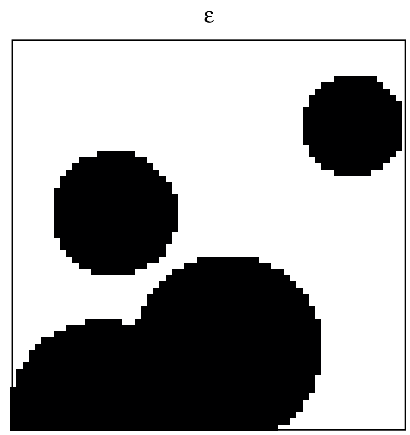
FIG. 4.4. *Sample Material I: constitutive materials $\varepsilon_0 = 1$ and $\varepsilon_1 = 1.5$ (top). Contributions from Sample Material I to the real part of solution u (middle) and real part of εu (bottom).*

and cross-correlation functions of the coefficients in the original equations, i.e., of the various coefficients characterizing the medium [13].

In our analysis, we also calculate the effective dielectric coefficient using integrals of the the auto- and cross-correlation functions of the coefficients in the original equations as suggested by Keller and Karal. However, our analysis does not assume that the effective dielectric coefficient is a constant, but allows for spatial variations. We illustrate below by considering media with particular random variations so that our approximation formulas capture the spatial variations.

Let g_ω be the free-space Green's function for the operator $Lv = v'' + \omega^2 v$ (with the outgoing wave condition). Our problem can be rewritten to yield the Lippmann-Schwinger equation

$$u(x) = -z\omega^2 \int_\Omega g_\omega(x-y)\chi(y)u(y)dy + q(x), \tag{5.1}$$



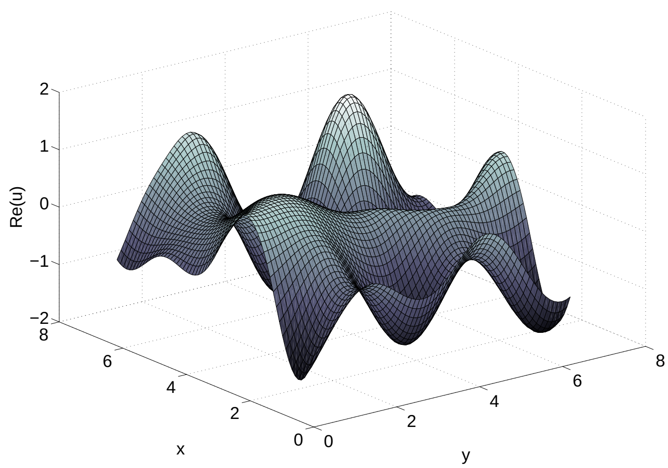


FIG. 4.5. *Sample Material II: constitutive materials $\varepsilon_0 = 1$ and $\varepsilon_1 = 4$ (top). Contributions from Sample Material II to the real part of solution u (middle) and real part of εu (bottom).*

where $q = g_\omega \star f$. As in (1.4), define the operator $A_\omega : L^2(\Omega) \to L^2(\Omega)$ by

$$(A_\omega v)(x) = \int_\Omega g_\omega(x-y)v(y)dy, \quad x \in \Omega. \tag{5.2}$$

In the case when $|z\omega^2|\|A_\omega\| < 1$,

$$u = (I + z\omega^2 A_\omega \chi)^{-1} q, \tag{5.3}$$

and the Neumann series

$$u = q - z\omega^2 A_\omega \chi q + z^2\omega^4 (A_\omega \chi)^2 q - \dots \tag{5.4}$$

converges absolutely. Take the average over all realizations to obtain

$$\begin{aligned} \langle u \rangle &= q - z\omega^2 A_\omega \langle \chi \rangle q + z^2\omega^4 A_\omega \langle \chi A_\omega \chi \rangle q - \dots \\ &= q - zp\omega^2 A_\omega q + z^2\omega^4 A_\omega \langle \chi A_\omega \chi \rangle q - \dots \end{aligned}$$

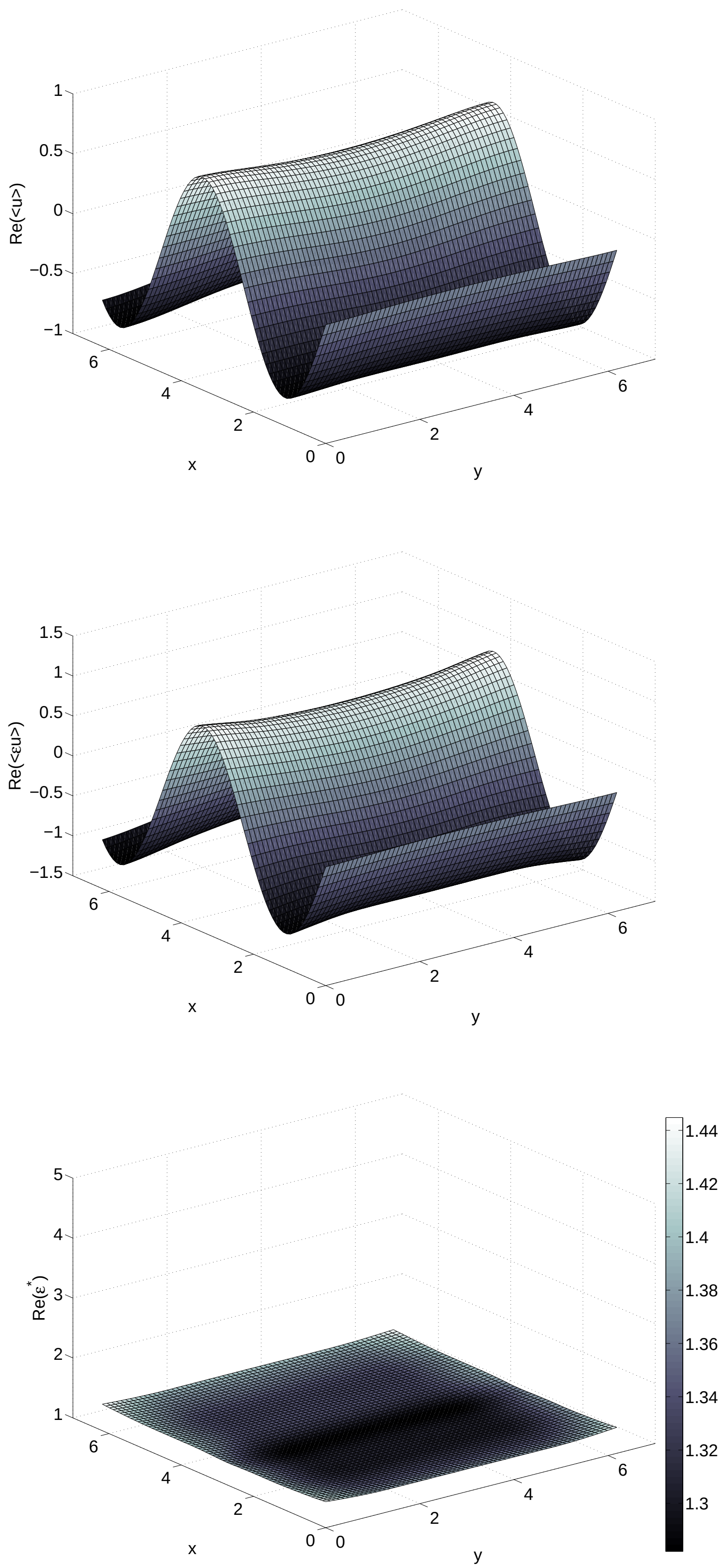


FIG. 4.6. *Averaged quantities of a medium with contrast* $z = 0.5$*: real part of* $\langle u \rangle$ *(top); real part of* $\langle \varepsilon u \rangle$ *(middle); real part of* ε^* *(bottom).*

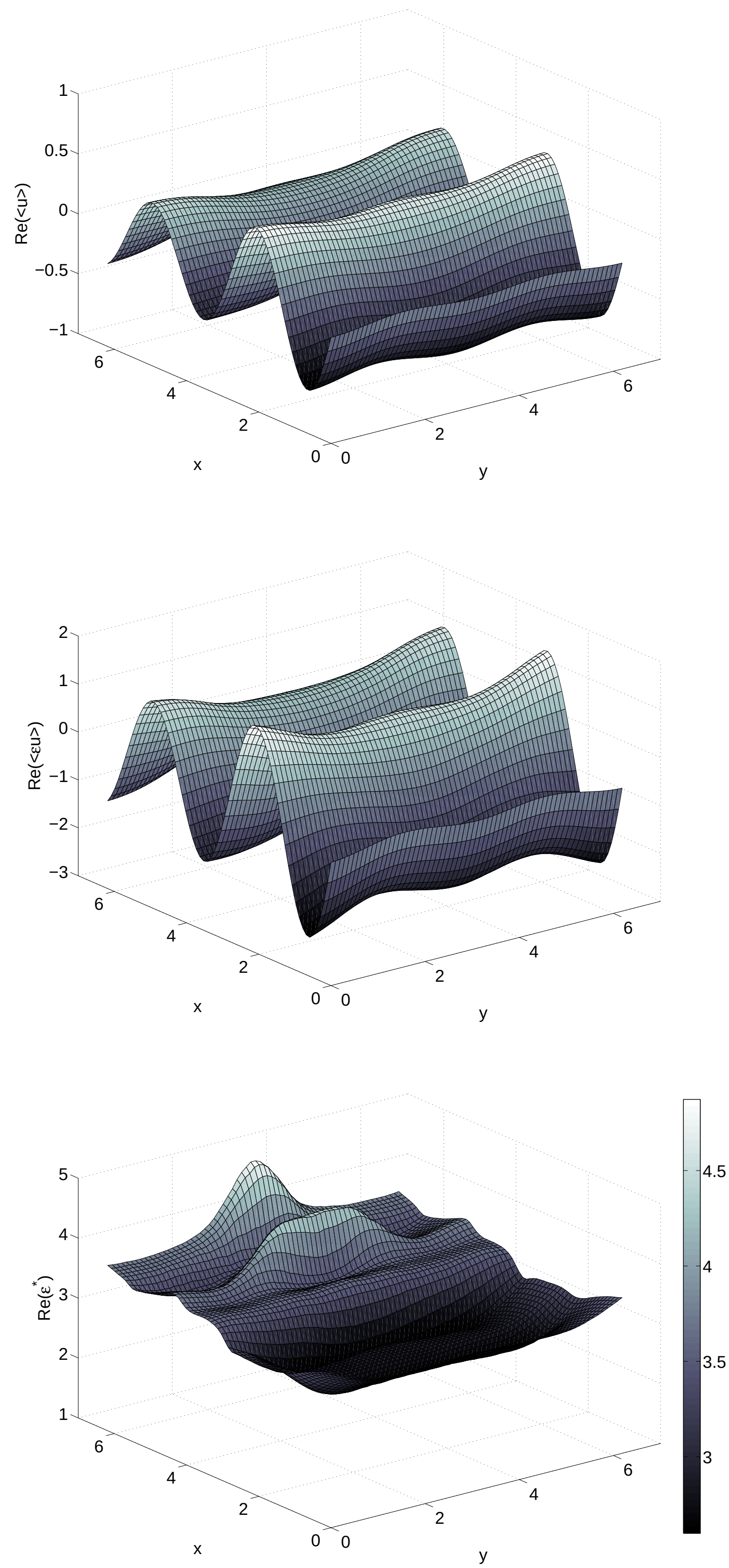


FIG. 4.7. *Averaged quantities of a medium with contrast* $z = 3$*: real part of* $\langle u \rangle$ *(top); real part of* $\langle \varepsilon u \rangle$ *(middle); real part of* ε^* *(bottom)*

and

$$\begin{aligned}\langle \chi u\rangle &= \langle \chi\rangle q - z\omega^2\langle \chi A_\omega \chi\rangle q + z^2\omega^4\langle \chi A_\omega \chi A_\omega \chi\rangle q - \dots \\ &= pq - z\omega^2\langle \chi A_\omega \chi\rangle q + z^2\omega^4\langle \chi A_\omega \chi A_\omega \chi\rangle q - \dots\end{aligned}$$

Thus, the effective dielectric coefficient can be represented with a Taylor series expansion in z

$$\begin{aligned}\varepsilon^*(x) = 1 + zp - 2z^2\omega^2 \frac{\langle \chi A_\omega \chi\rangle q - p^2 A_\omega q}{q} \\ + 6z^3\omega^4\left(\frac{\langle \chi A_\omega \chi A_\omega \chi\rangle q - p\langle \chi A_\omega \chi\rangle q}{q} - \frac{pA_\omega q(\langle \chi A_\omega \chi\rangle q - p^2 A_\omega q)}{q^2}\right)\dots\end{aligned} \quad (5.5)$$

We have also proven that the complex-valued $\langle u\rangle$ is a holomorphic function of z provided that $\frac{1}{z}$ belongs in the resolvent of the operator $\omega^2 A_\omega \chi$. In this case, we can approximate the effective dielectric coefficient by the Taylor series expansion in z. In the case of small z every term in the series is a constant provided the medium is stationary. In such media, the correlation functions depend only on the distance between the points, and not their positions. In this case all the correlation functions depend on the distance between all of the points, i.e., the three-point correlation function $N(x, y, s) = N(|x-y|, |x-s|, |s-y|)$. An example of such a medium is one constructed by varying the length of the first layer (and thus the last to compensate), and leaving the length of the middle layers constant. The material in each layer is assigned with a chosen probability. Knowing the N-point correlation function allows us to approximate ε^*. In order to do that we must develop a method to evaluate the terms of form $\langle \chi A_\omega \chi\rangle q$, $\langle \chi A_\omega \chi A_\omega \chi\rangle q$, etc. Keller and Karal suggest how this can be done when calculating $\langle \chi A_\omega \chi\rangle q$. The mean value theorem for the solution $q = g_\omega \star f$ of the constant coefficient problem

$$u'' + \omega^2 u = f$$

is applied [13], where f is the delta function and q is a plane wave solution. The Green's function for this problem is

$$g_\omega = \frac{ie^{i\omega|x|}}{2\omega}.$$

Knowing the correlation function $N(x, y) = N(|x-y|) = N(r)$ and using the mean-value theorem in one dimension, we can calculate

$$\begin{aligned}\langle \chi A_\omega \chi\rangle q &= (p-p^2)\int g_\omega(|x-y|)N(|x-y|)q(y)\,dy + p^2\int g_\omega(|x-y|)q(y)\,dy \\ &= 2(p-p^2)\int_0^\infty g_\omega(r)N(r)cos(\omega r)\,dr\,q(x) + 2p^2\int_0^\infty g_\omega(r)cos(\omega r)\,dr\,q(x)\end{aligned}$$

In a medium where the length of the first layer varies from 0 to d, the correlation function is

$$N(x, y) = N(|x-y|) = \begin{cases} 1 - \frac{|x-y|}{d} & \text{when } 0 \le |x-y| \le d; \\ 0 & \text{otherwise.}\end{cases}$$

Calculating the integrals for $r \in supp(N)$, we obtain

$$\langle \chi A_\omega \chi\rangle q = \left(-\frac{(p-p^2)i}{8\omega^3 d}(2d^2\omega^2 - 1 + e^{2i\omega d}) + \frac{p}{4\omega^2}(e^{2i\omega d} + 2i\omega d - 1)\right)q(x).$$

The mean value theorem is applied repeatedly to calculate the multiple integrals of the form $\langle \chi A_\omega \dots \chi A_\omega \chi \rangle q$, when the N-point correlation is given. Once these are calculated our formula (5.5) gives the approximation to the needed order, e.g.

$$\varepsilon^* \approx 1 + zp$$
$$- 2z\omega^2 \left(-\frac{(p-p^2)i}{8\omega^3 d}(2d^2\omega^2 - 1 + e^{2i\omega d}) + \frac{p}{4\omega^2}(e^{2i\omega d} + 2i\omega d - 1) - p^2 \left(\frac{id}{2\omega} + \frac{e^{2i\omega d}}{4\omega^2} - \frac{1}{4\omega^2} \right) \right).$$

Numerical experiments show that in a medium with a correlation function depending on position, the best approximation may be a function of the space variable. In the numerical experiments illustrated in Figures 5.1 and 5.2, we have graphed the real and imaginary parts of ε^* and its second order spatially dependent approximation, calculated using (5.5). The appropriate correlation function for the medium is spatially dependent and assigns 1 (or fully correlated), if the two points are in the same interval and 0 (no correlation), otherwise. Since the expansion is done around $z = 0$, it gives a better approximation for small z's and ω's. In the experiment, depicted in Figure 5.1, we use media of four layers and contrast $z = 0.1$ and $z = 0.5$, when the frequency is $\omega = 10$. We see that our second order approximations (thick line) give a very good approximation of both the real and imaginary parts of ε^* (thin line), capturing the spatial variations. In Figure 5.2, we observe the real and imaginary parts of ε^* (thin line), and its second order spatially dependent approximation (thick line) in a medium of four layers, contrast $z = 0.5$, and $\omega = 2$. The approximation is very good in the case when z and ω are small even if we have only four, relatively long, layers. In the last numerical experiment depicted in Figure 5.2, $\omega = 53$. For large frequencies, we expect the approximation to fail, but nevertheless, we see that our second order spatially dependent approximation (thick line) captures some of the behavior of the real and imaginary parts of ε^* (thin line). In this experiment, we are looking at a medium of four layers and contrast $z = 0.5$.

6. Conclusions. When we consider wave propagation in a medium for which the size of the inhomogeneities is of the same order as the wave length, scattering effects must be accounted for and the effective dielectric coefficient is no longer a constant, but a spatially dependent function. In this paper we use novel approaches to bound the spatial variations of the effective permittivity. Numerical experiments confirm the presence of spatial variations and their dependence on the size of the inhomogeneities and the magnitude of the contrast. Related optimization problems that seek the class of materials, described by the probability density function of the geometry of the medium, that optimize certain properties of the effective permittivity will be considered in the future.

7. APPENDIX. In two dimensions using polar coordinates (r, θ) and assuming no incoming waves, the exterior scattered solution is

$$u_{ex}(r, \theta) = \sum_{m=1}^{\infty} A_m H_m^1(\omega r) e^{im\theta},$$

where $H_m^1(\omega r)$ are Hankel functions of first kind. Suppose that the Dirichlet data u_{in} is given on the circle. The interior solution $u_{in} \in L^2(S_0)$, and thus it has a Fourier series representation

$$u_{in}(\theta) = \sum_{m=1}^{\infty} \hat{u}_m e^{im\theta},$$

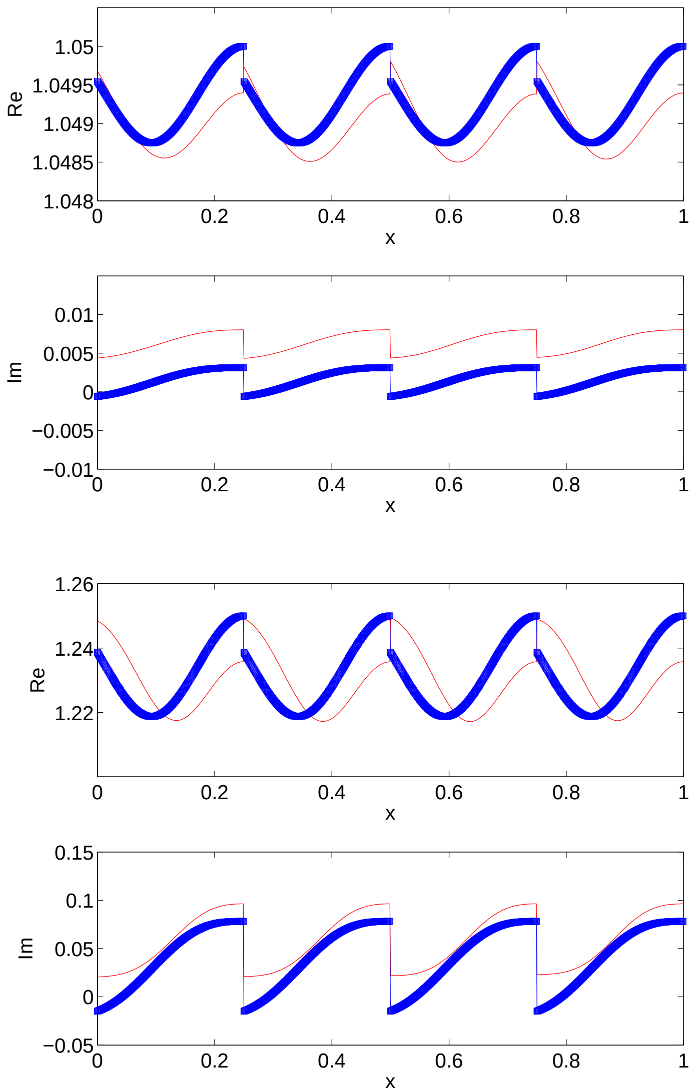


FIG. 5.1. *Second order approximation I. Above: Real and imaginary ε^* (thin line), and its second order spatially dependent approximation (thick line) in a medium of four layers, contrast $z = 0.1$; below: Real and imaginary ε^* (thin line), and its second order spatially dependent approximation (thick line) in a medium of four layers, contrast $z = 0.5$. The frequency $\omega = 10$.*

where

$$\hat{u}_m = \frac{1}{2\pi}\int_0^{2\pi} u(\omega R_0, \theta')e^{-im\theta'}\, d\theta'.$$

The constants A_m are found from the Dirichlet condition to be

$$A_m = \frac{\hat{u}_m}{H_m^1(\omega R_0)}.$$

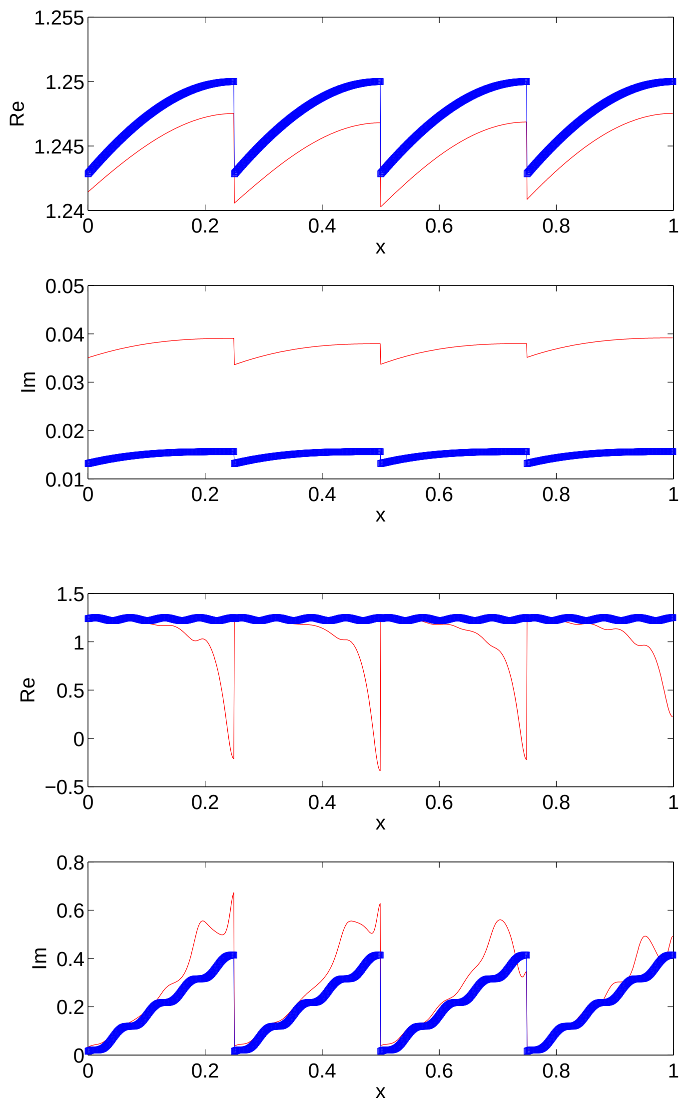


FIG. 5.2. *Second order approximation II. Above: Real and imaginary ε^* (thin line), and its second order spatially dependent approximation (thick line) in a medium of four layers, contrast $z = 0.5$, $\omega = 2$; Below: Real and imaginary ε^* (thin line) and its second order spatially dependent approximation (thick line) in a medium of four layers, contrast $z = 0.5$, $\omega = 53$.*

Thus the radiating solution is given by

$$u_s(r,\theta) = \sum_{m=1}^{\infty} \frac{H_m^1(\omega r)}{H_m^1(\omega R_0)} \hat{u}_m e^{im\theta}.$$

Differentiating in the radial direction and setting $r = R_0$ leads to

$$\frac{\partial u_s}{\partial r}(R_0,\theta) = \omega \sum_{m=1}^{\infty} \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \hat{u}_m e^{im\theta} \equiv (Tu_s)(\theta).$$

Thus, we see that

$$(Tv)(\theta) = \omega \sum_{m=1}^{\infty} \left(\frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \right) \hat{v}_m e^{im\theta}, \tag{7.1}$$

where $\hat{v}_m$ are the Fourier coefficients of v, where v satisfies the Helmholtz equation (2.1).

Let

$$\gamma_m \equiv \omega \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)}. \tag{7.2}$$

By using the properties and identities of Hankel functions, it can be shown that $\Im(\gamma_m) > 0$ and $\Re(\gamma_m) < 0$ for all m.

For $m \geq 0$ and r in compact subsets of $(0, \infty)$, we have [3]

$$\left| H_m^1(\omega r) \right| \leq C \frac{2^m m!}{(\omega r)^m}.$$

The derivative of the Hankel function is

$$\frac{\partial H_m^1}{\partial r}(\omega r) = \frac{m H_m^1(\omega r)}{r} - \omega H_{m+1}^1(\omega r).$$

This way we can bound the ratio

$$\left| \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \right| \leq Cm.$$

We obtain the bound

$$\begin{aligned} \|Tv\|^2_{H^{-\frac{1}{2}}(S_0)} &\leq \sum_{m=1}^{\infty} (1+m^2)^{-\frac{1}{2}} \left| \frac{\frac{\partial H_m^1}{\partial r}(\omega R_0)}{H_m^1(\omega R_0)} \right|^2 |\hat{v}_m|^2 \\ &\leq \sum_{m=1}^{\infty} C(1+m^2)^{-\frac{1}{2}} m^2 |\hat{v}_m|^2 \\ &\leq \sum_{m=1}^{\infty} C(1+m^2)^{\frac{1}{2}} |\hat{v}_m|^2 \leq C\|v\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \leq C\|v\|^2_{H^1(\Omega_0)}, \end{aligned}$$

where we have used the trace imbedding theorem [1].

In three dimensions using spherical coordinates (r, θ, ϕ) assuming $\varepsilon(x) = 1$ and no incoming waves, the scattered solution is

$$u_{ex}(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{lm} h_l^1(\omega r) Y_{lm}(\theta, \phi),$$

where $h_l^1(\omega r)$ are spherical Hankel functions of first kind and $Y_{lm}(\theta, \phi)$ are the normalized spherical harmonics. The latter form an orthonormal complete set of $L^2(S_0)$

[16]. Suppose that the Dirichlet data is given on the sphere. Since $u_{in} \in L^2(S_0)$, it can be expanded into spherical harmonics as

$$u_{in}(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm}(\theta,\phi)$$

with

$$\hat{u}_{lm} = \int_{S_0} u(R_0,\theta',\phi')\,\overline{Y_{lm}}(\theta',\phi')\,dS'.$$

The constants B_{lm} are found from the Dirichlet condition to be

$$B_{lm} = \frac{\hat{u}_{lm}}{h_l(wR_0)}.$$

Thus,

$$u_s(r,\theta,\phi) = \sum_{l=0}^{\infty} \frac{h_l^1(\omega r)}{h_l(\omega R_0)} \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm}(\theta,\phi).$$

Differentiating in the radial direction and setting $r = R_0$ gives

$$\frac{\partial u_s}{\partial r}(R_0,\theta,\phi) = \sum_{l=0}^{\infty} \omega \frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)} \sum_{m=-l}^{l} \hat{u}_{lm} Y_{lm}(\theta,\phi) \equiv (Tu_s)(\theta,\phi).$$

We see that

$$(Tv)(\theta,\phi) = \sum_{l=0}^{\infty} \omega \left(\frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)} \right) \sum_{m=-l}^{l} \hat{v}_{lm} Y_{lm}(\theta,\phi), \tag{7.3}$$

where $\hat{v}_{lm}$ are the coefficients in the spherical harmonics expansion of v, where v satisfies the Helmholtz equation (2.1).

Let

$$\gamma_l \equiv \omega \frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)}. \tag{7.4}$$

The following is obtained by very slight modification of the analysis of the exterior scattering problem discussed in [11]: for all l, $\Im \gamma_l > 0$ and $\Re \gamma_l < 0$.

The Sobolev space $H^s(S_0)$ with real parameter s consists of all distributions f such that

$$\|f\|^2_{H^s(S_0)} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (1+\lambda_l)^s |\hat{f}_{lm}|^2 < \infty,$$

where $\hat{f}_{lm}$ are the spherical harmonics Fourier coefficients and $\lambda_l = l(l+1)$, $l \geq 0$ is the eigenvalue of the Laplace-Beltrami operator on S_0. For $l \geq 0$ and r in compact subsets of $(0, \infty)$, we have

$$\left| h_l^1(\omega r) \right| \leq C \frac{2^l l!}{(\omega r)^{l+1}}.$$

The derivative of the spherical Hankel function is

$$\frac{\partial h_l^1}{\partial r}(\omega r) = \frac{1}{2}\left(\omega h_{l-1}^1(\omega r) - \frac{h_l^1(\omega r) + \omega r h_{l+1}^1(\omega r)}{r}\right).$$

This way we can bound the ratio

$$\left|\frac{\frac{\partial h_l^1}{\partial r}(\omega R_0)}{h_l^1(\omega R_0)}\right| \leq Cl.$$

We then obtain the bound

$$\begin{aligned}\|Tv\|^2_{H^{-\frac{1}{2}}(\Gamma_0)} &\leq \omega \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(1+l(l+1))^{-\frac{1}{2}}\left|\frac{\frac{\partial H_l^1}{\partial r}(\omega R_0)}{H_l^1(\omega R_0)}\right|^2 |\hat{v}_{l,m}|^2 \\ &\leq \omega \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C(1+l(l+1))^{\frac{1}{2}}|\hat{v}_{l,m}|^2 \leq C\|v\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \leq C\|v\|^2_{H^1(\Omega_0)},\end{aligned} \tag{7.5}$$

where we have used the trace imbedding theorem [1].

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