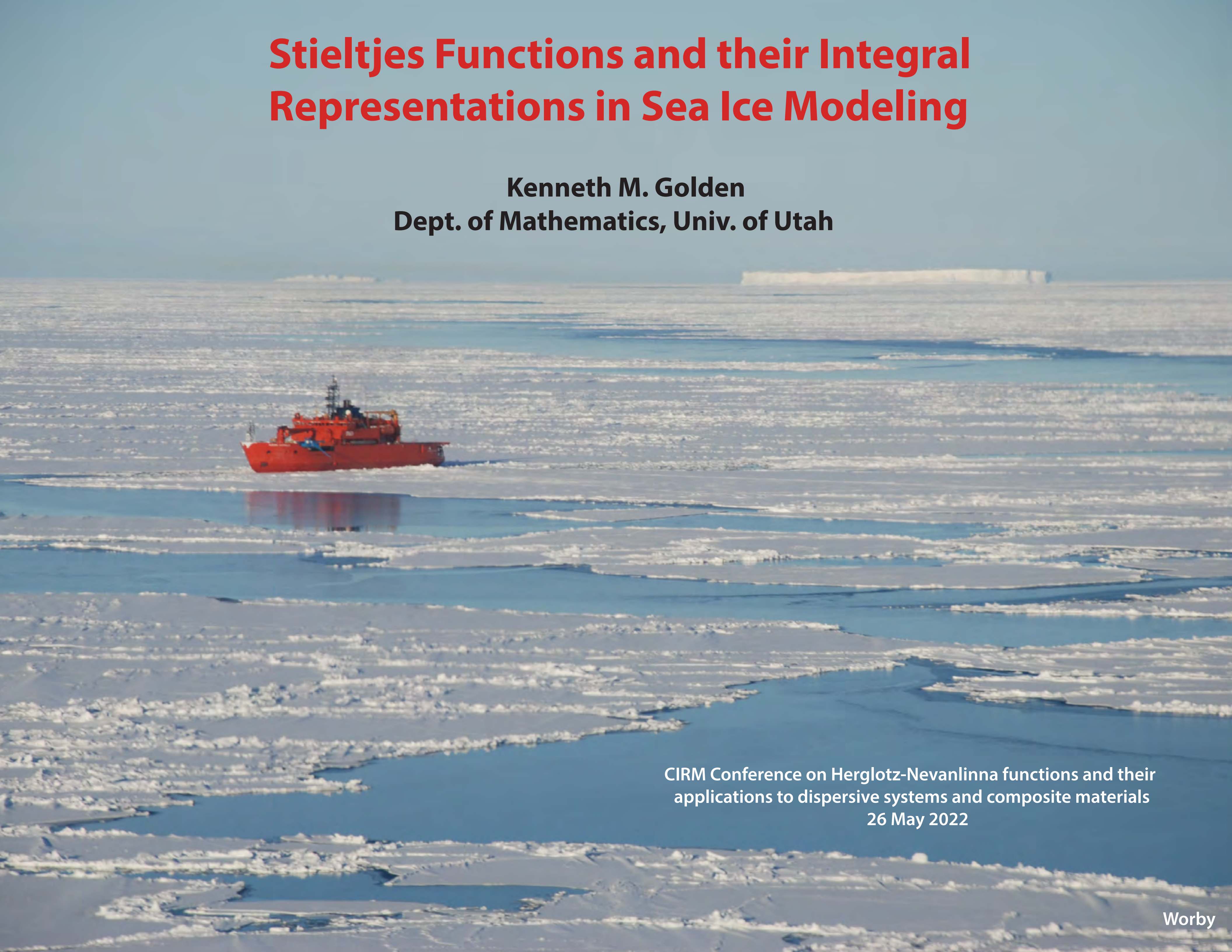
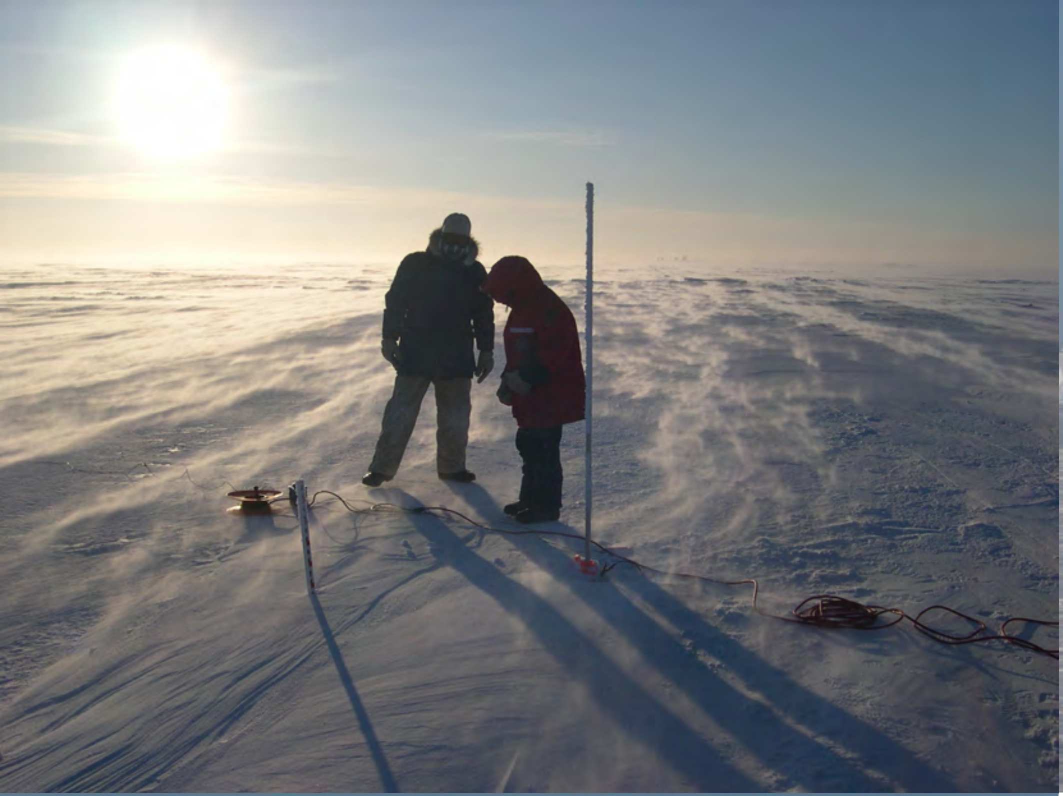
Stieltjes Functions and their Integral Representations in Sea Ice Modeling

Kenneth M. Golden
Dept. of Mathematics, Univ. of Utah



CIRM Conference on Herglotz-Nevanlinna functions and their applications to dispersive systems and composite materials
26 May 2022

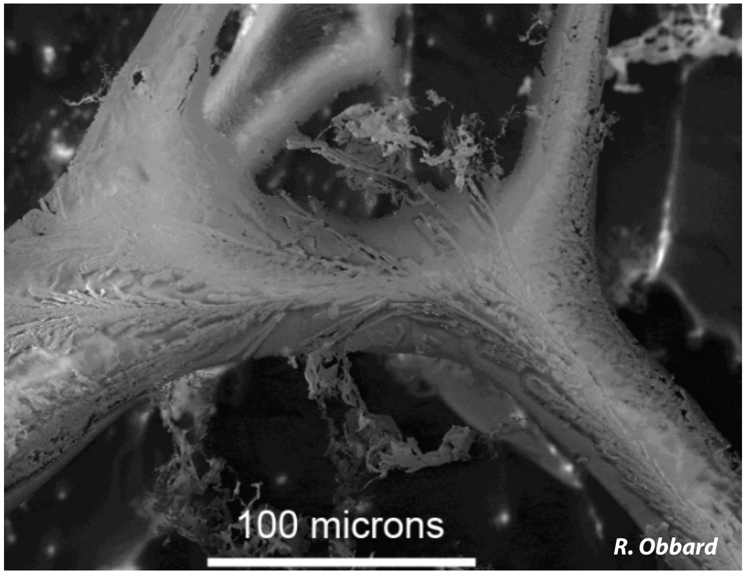
Worby



sea ice may appear to be a barren, impermeable cap ...

Golden

brine inclusions in sea ice (mm)

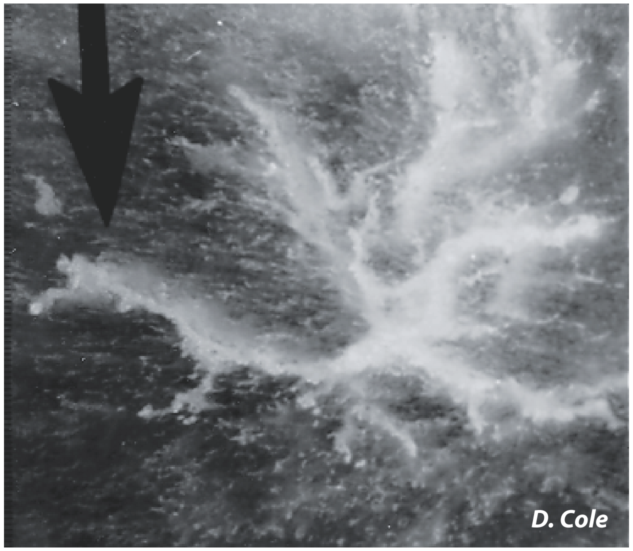


micro - brine channel (SEM)

sea ice is a porous composite

pure ice with brine, air, and salt inclusions

brine channels (cm)

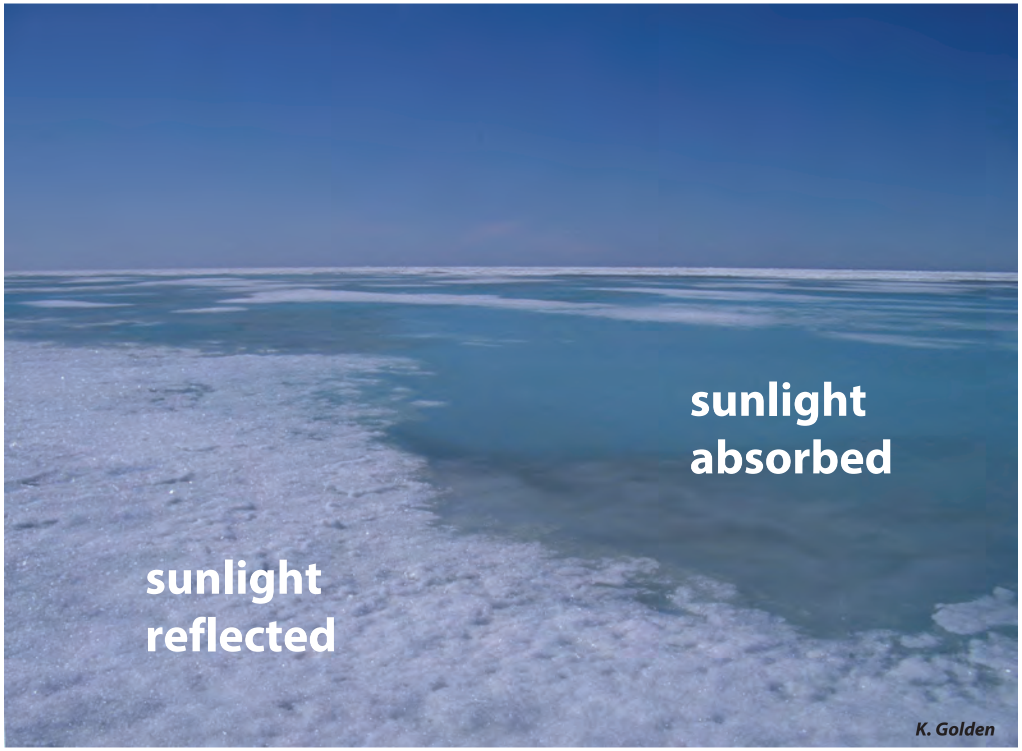


horizontal section

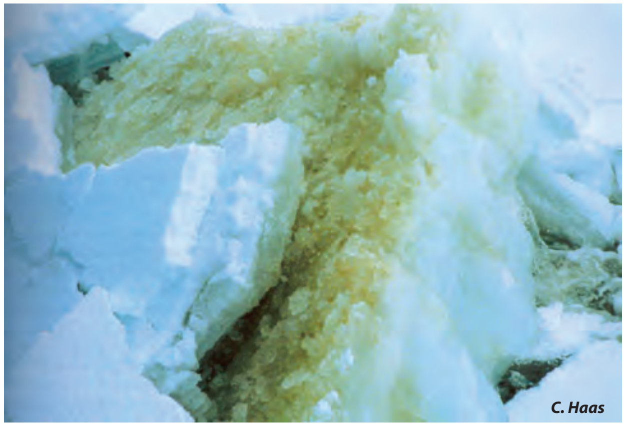
vertical section

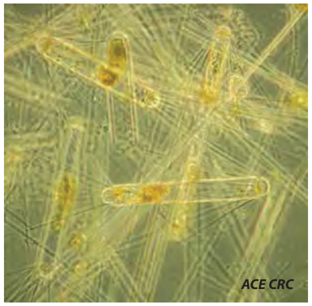
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

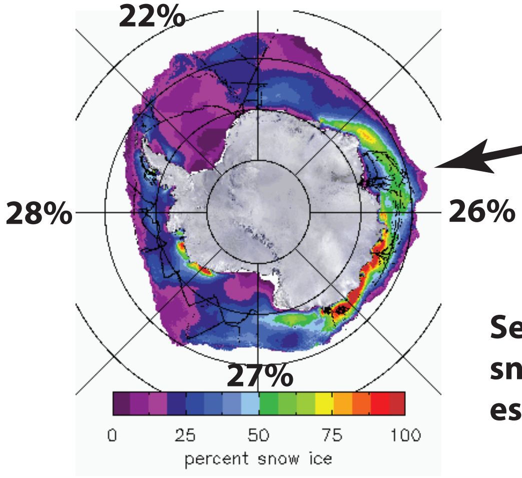
evolution of Arctic melt ponds and sea ice albedo



nutrient flux for algal communities







T. Maksym and T. Markus, 2008

September snow-ice estimates

Antarctic surface flooding and snow-ice formation

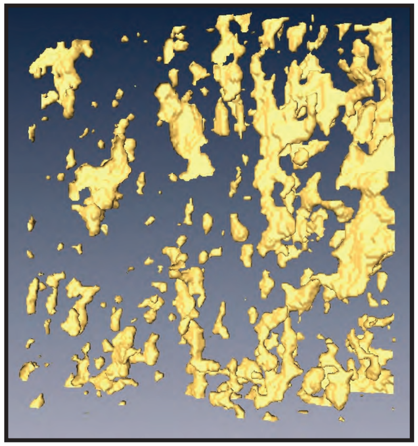
- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO_2*

Sea Ice is a Multiscale Composite Material

microscale

brine inclusions

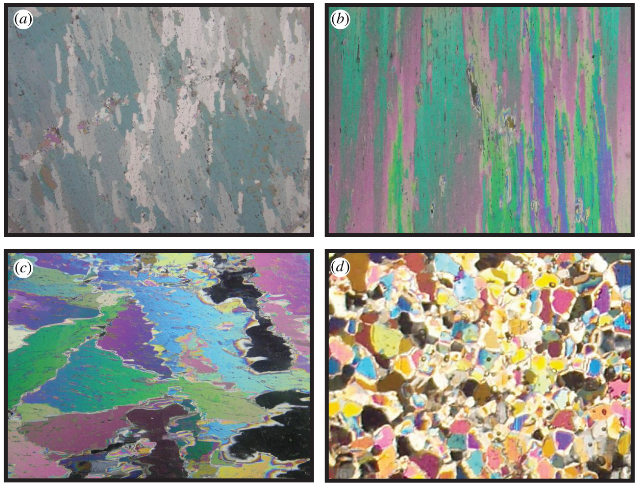
Weeks & Assur 1969



H. Eicken

Golden *et al. GRL* 2007

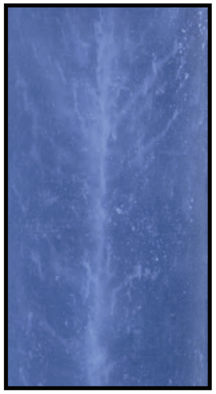
polycrystals



Gully *et al. Proc. Roy. Soc. A* 2015

brine channels

D. Cole



K. Golden

millimeters

centimeters

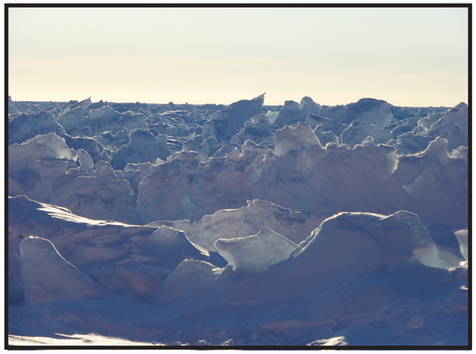
---

mesoscale

Arctic melt ponds

K. Frey

Antarctic pressure ridges



K. Golden

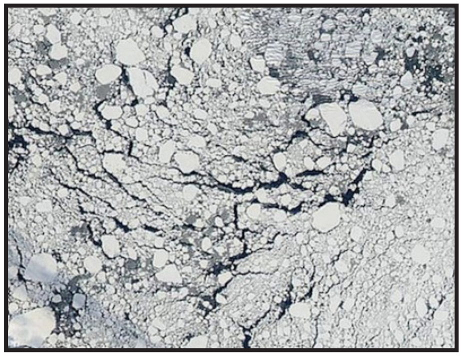
meters

macroscale

sea ice floes

J. Weller

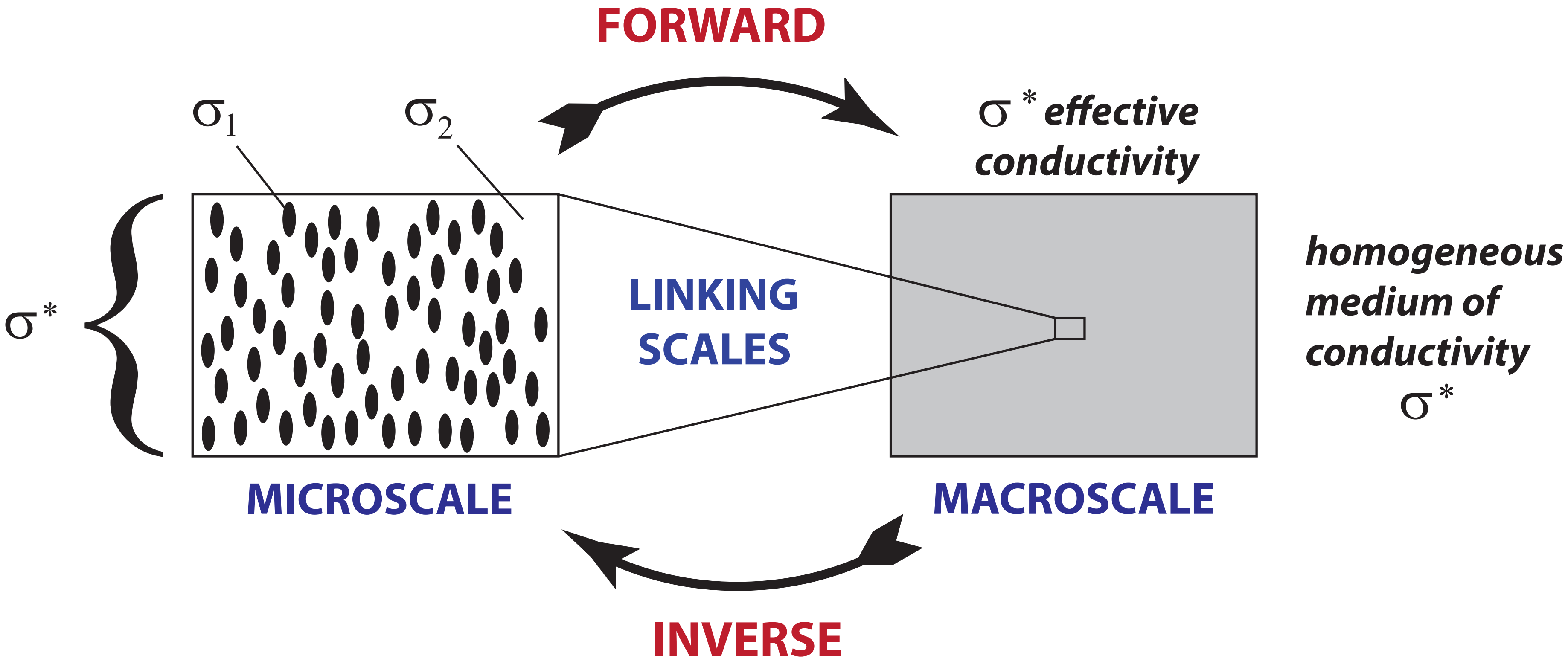
sea ice pack



NASA

kilometers

HOMOGENIZATION for Composite Materials



FORWARD

σ_1 σ_2

σ^*

LINKING SCALES

σ^* ***effective conductivity***

MICROSCALE

MACROSCALE

homogeneous medium of conductivity σ^*

INVERSE

Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

Wiener 1912 : arithmetic and harmonic mean bounds on effective conductivity
Hashin and Shtrikman 1962 : variational bounds on effective conductivity

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

What is this talk about?

the role of "microstructure" in determining sea ice effective properties

Using **homogenization and statistical physics** to compute effective behavior on scales relevant to coarse-grained sea ice and climate models, process studies, ...

MICROSCALE: brine + polycrystalline microstructure; EM, fluid transport

MESOSCALE: advection diffusion, thermal transport, ocean waves

A tour of Stieltjes functions in the study of sea ice and its role in climate.

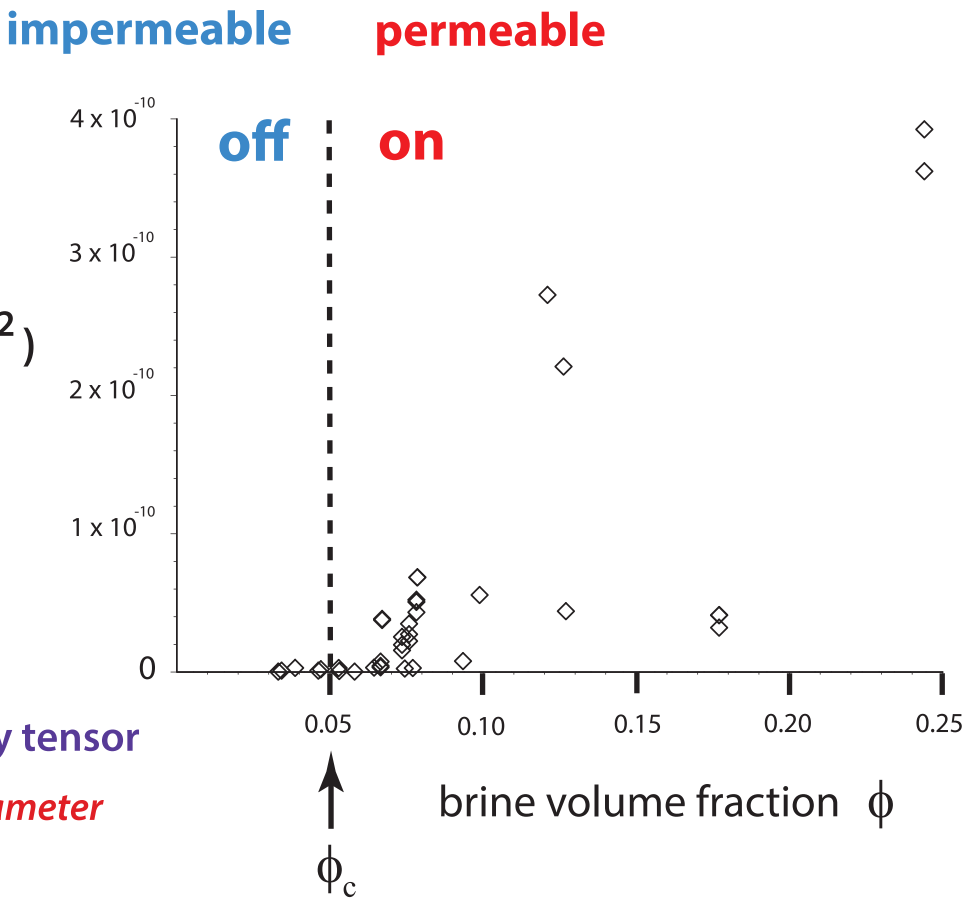
Solving problems in the physics of sea ice drives advances in theory of composite materials.

microscale

Critical behavior of fluid transport in sea ice

Arctic field data

vertical fluid permeability k (m^2)



"on - off" switch for fluid flow

Darcy's Law

$$\mathrm{v} = -\frac{\mathbf{k}}{\eta}\nabla p$$

$\mathbf{k}$ = fluid permeability tensor

homogenized parameter

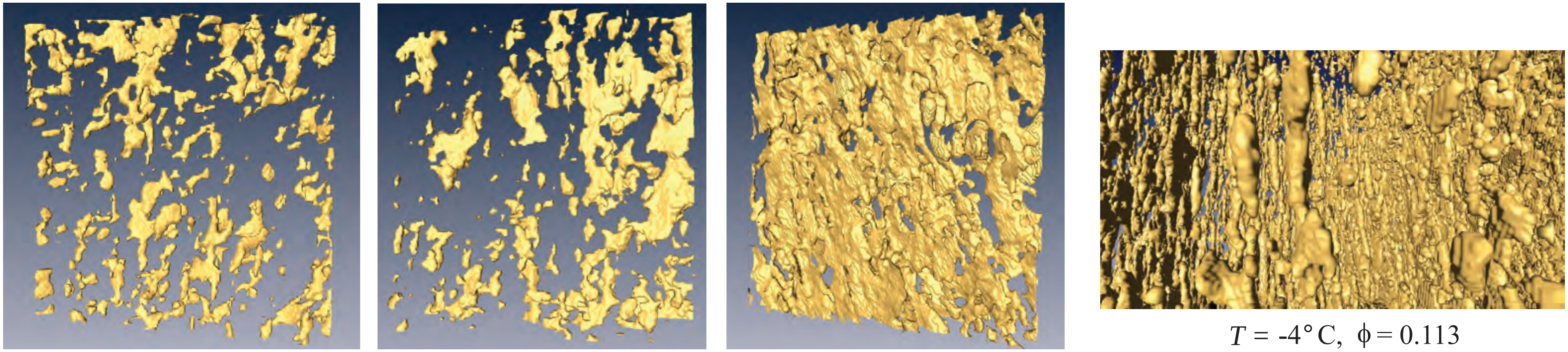
critical brine volume fraction $\phi_c \approx 5\%$ ⟷ $T_c \approx -5^\circ\mathrm{C},\ S \approx 5$ ppt

RULE OF FIVES

Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

sea ice ~ compressed powder in stealthy composites

brine volume fraction and ***connectivity*** increase with temperature



T = -15 °C, $\phi = 0.033$ T = -6 °C, $\phi = 0.075$ T = -3 °C, $\phi = 0.143$ T = -4 °C, $\phi = 0.113$

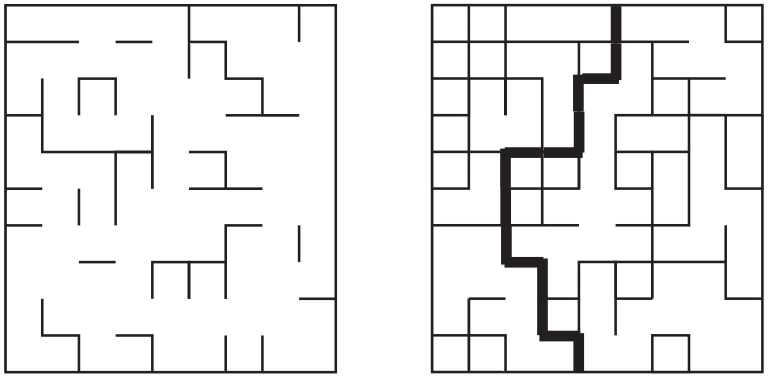
X-ray tomography for brine phase in sea ice

Golden, Eicken, *et al., Geophysical Research Letters* 2007

PERCOLATION THRESHOLD $\phi_c \approx 5\ \%$

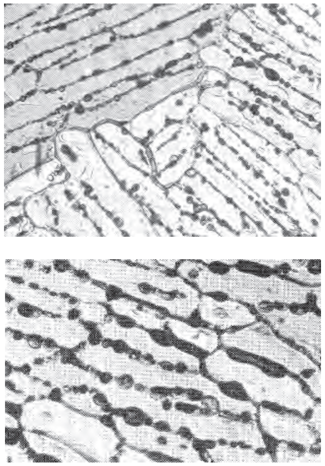
Golden, Ackley, Lytle, *Science* 1998

impermeable permeable

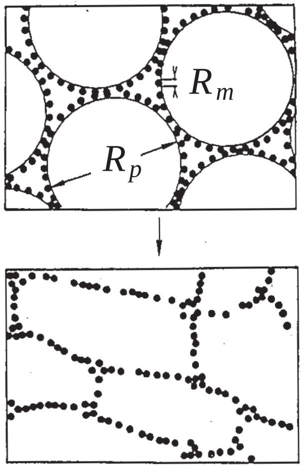


$p = 1/3$ $p = 2/3$

lattice percolation



sea ice



Kusy, Turner
Nature 1971

compressed powder

B-2 Stealth Bomber
F-117 Nighthawk
F-35

stealth

continuum percolation

Thermal evolution of permeability and microstructure in sea ice

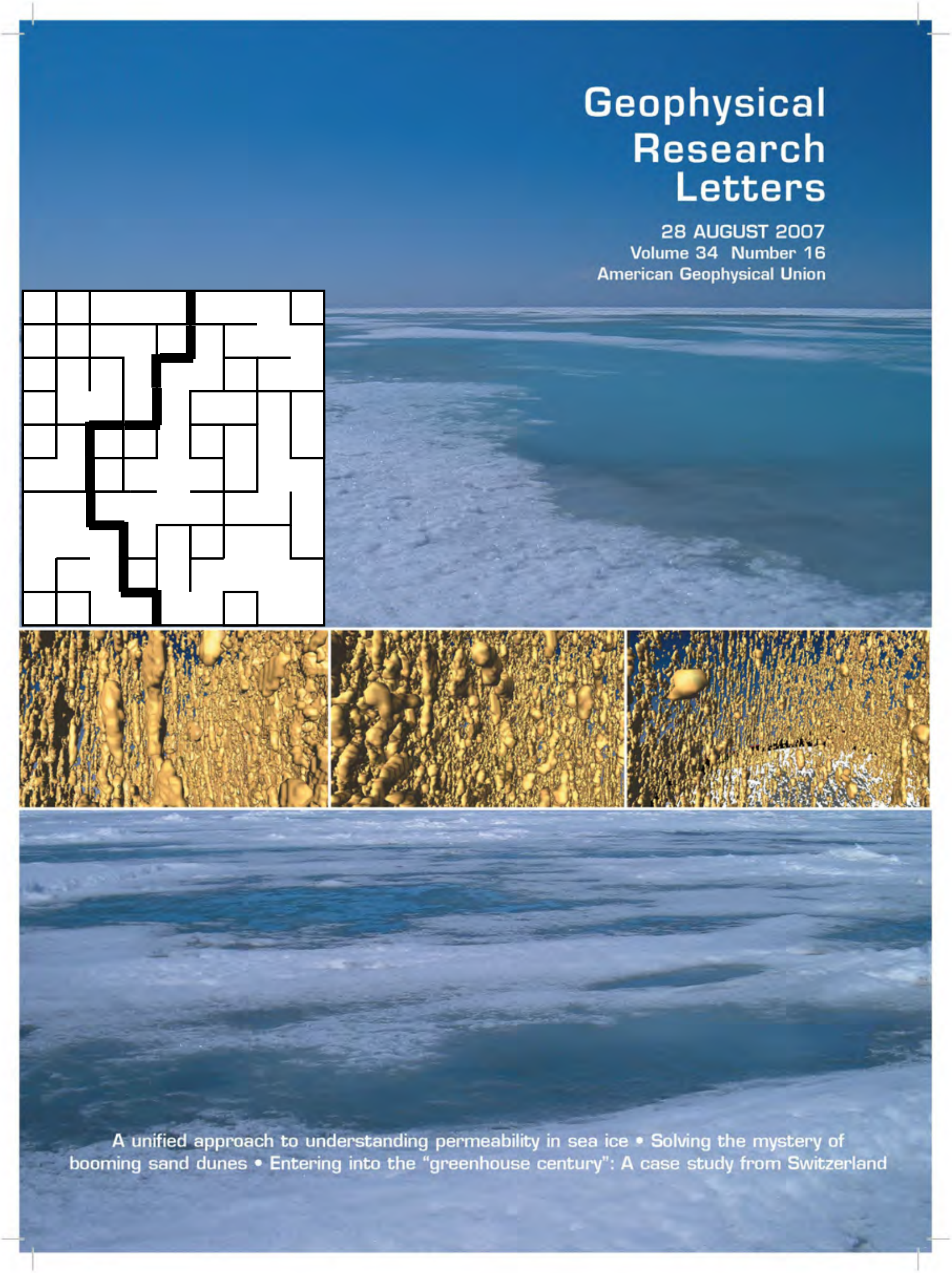
Golden, Eicken, Heaton*, Miner, Pringle, Zhu, Geophysical Research Letters 2007

microscale

governs

mesoscale
processes

***melt pond
evolution***



Geophysical
Research
Letters

28 AUGUST 2007
Volume 34 Number 16
American Geophysical Union

A unified approach to understanding permeability in sea ice • Solving the mystery of booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

**percolation theory
for fluid permeability**

$$k(\phi) = k_0 (\phi - 0.05)^2$$

critical
exponent
t

$$k_0 = 3 \text{ x } 10^{-8} \text{ m}^2$$

from critical path analysis
in **hopping conduction**

hierarchical model
rock physics
network model
rigorous bounds

X-ray tomography for
brine inclusions

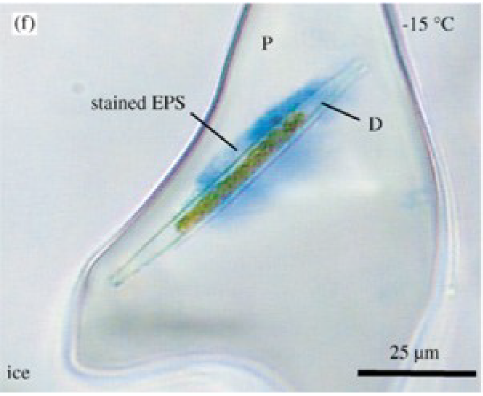
confirms rule of fives

Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009

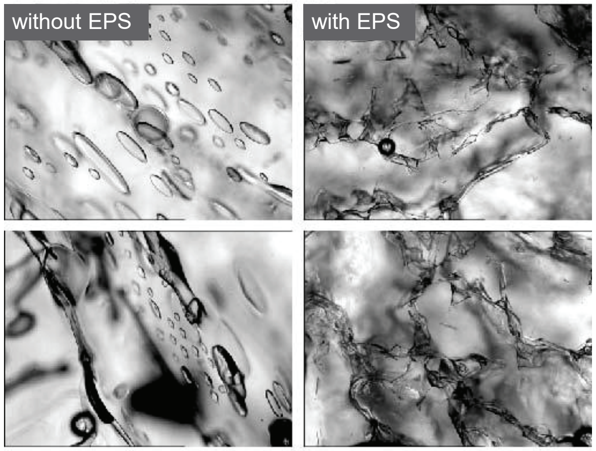
**theories agree closely
with field data**

Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

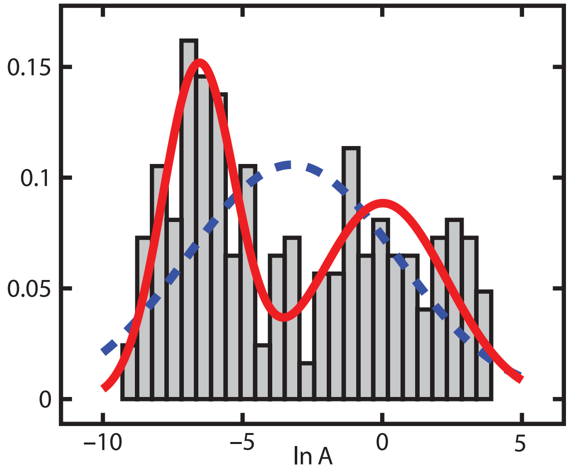
How does EPS affect fluid transport? **How does the biology affect the physics?**



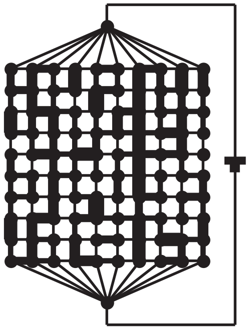
Krembs



Krembs, Eicken, Deming, PNAS 2011

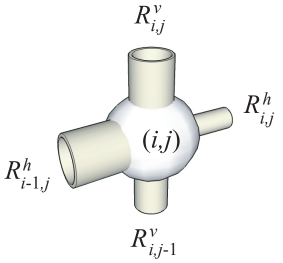


RANDOM PIPE MODEL



- 2D random pipe model with bimodal distribution of pipe radii

- Rigorous bound on permeability *k;* results predict observed drop in *k*

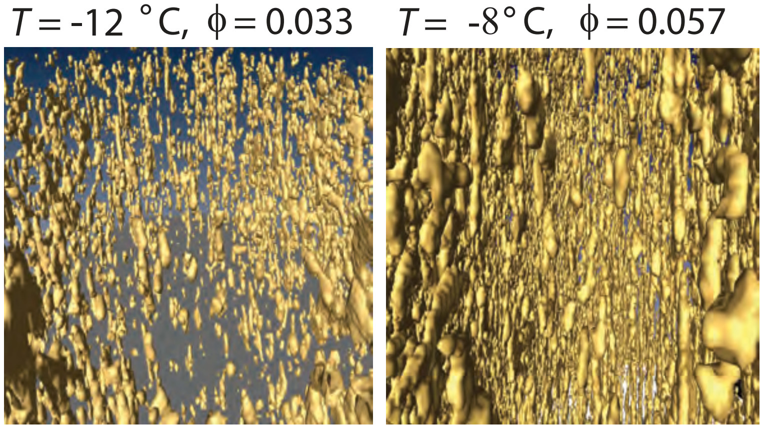


Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
***Multiscale Modeling and Simulation*, 2018**

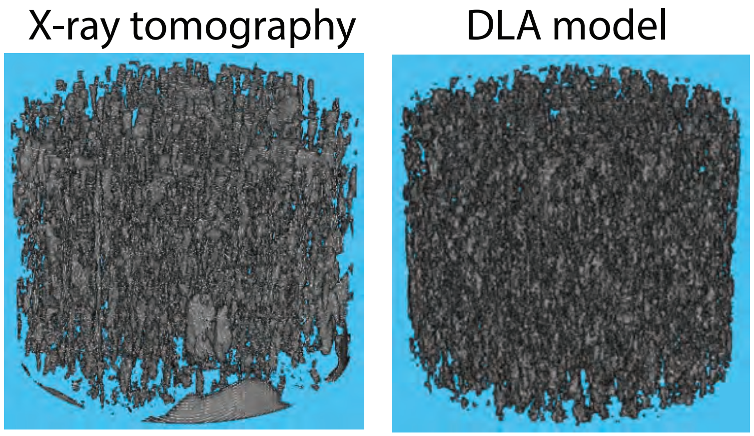
Zhu, Jabini, Golden,
Eicken, Morris
***Ann. Glac.* 2006**

Thermal evolution of the fractal geometry of the brine microstructure in sea ice

N. Ward, D. Hallman, J. Reimer, H. Eicken, M. Oggier and K. M. Golden, 2022



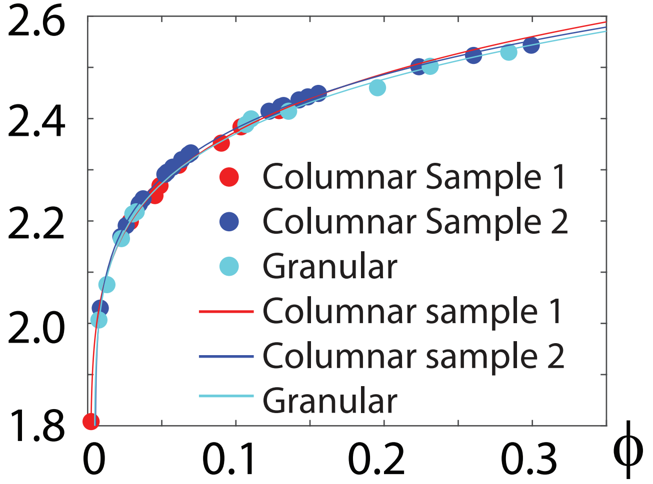
$T = -12\ ^\circ\mathrm{C},\ \phi = 0.033$ $T = -8^\circ\mathrm{C},\ \phi = 0.057$



X-ray tomography DLA model

+ implications for brine phase as a habitat

Fractal dimension



brine volume fraction (porosity)

theory of porosity as a function of fractal dimension

invert

excellent correspondence with data

Katz and Thompson, *PRL*, 1985

Arctic and Antarctic field experiments

develop electromagnetic methods of monitoring fluid transport and microstructural transitions

extensive measurements of fluid and electrical transport properties of sea ice:

2007	***Antarctic***	***SIPEX***
2010	***Antarctic***	***McMurdo Sound***
2011	***Arctic***	***Barrow AK***
2012	***Arctic***	***Barrow AK***
2012	***Antarctic***	***SIPEX II***
2013	***Arctic***	***Barrow AK***
2014	***Arctic***	***Chukchi Sea***

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Notices

of the American Mathematical Society

May 2009

Volume 56, Number 5

Climate Change and the Mathematics of Transport in Sea Ice

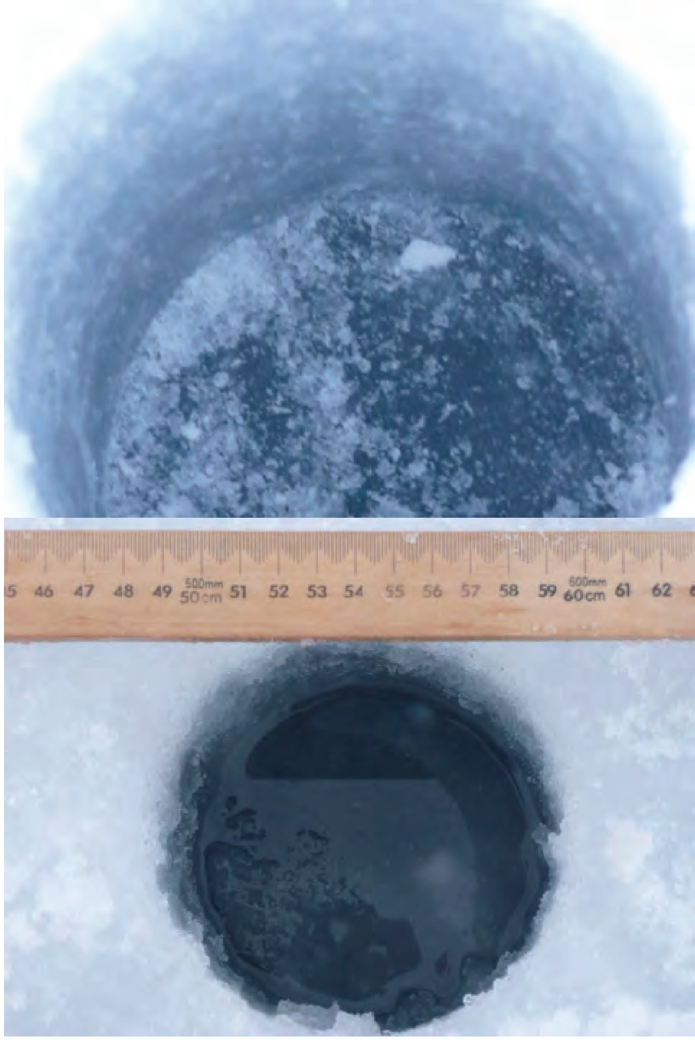
page 562

Mathematics and the Internet: A Source of Enormous Confusion and Great Potential

page 586

Real analysis in polar coordinates (see page 613)

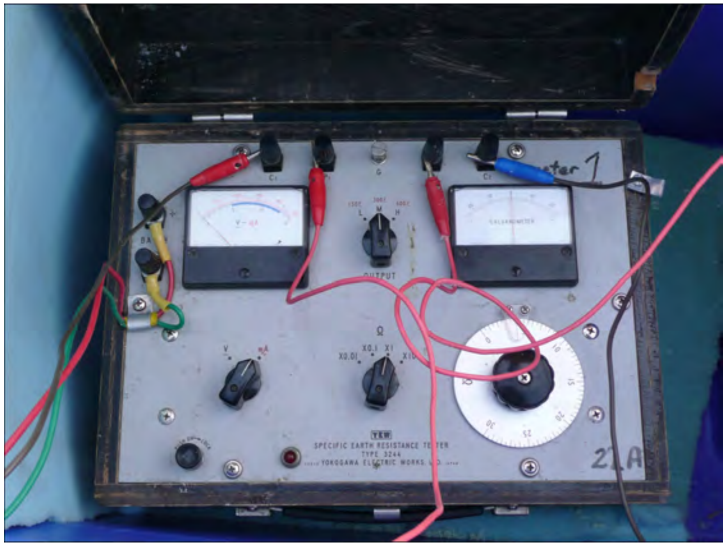
photo by Jan Lieser



measuring fluid permeability of Antarctic sea ice

SIPEX 2007

electrical measurements

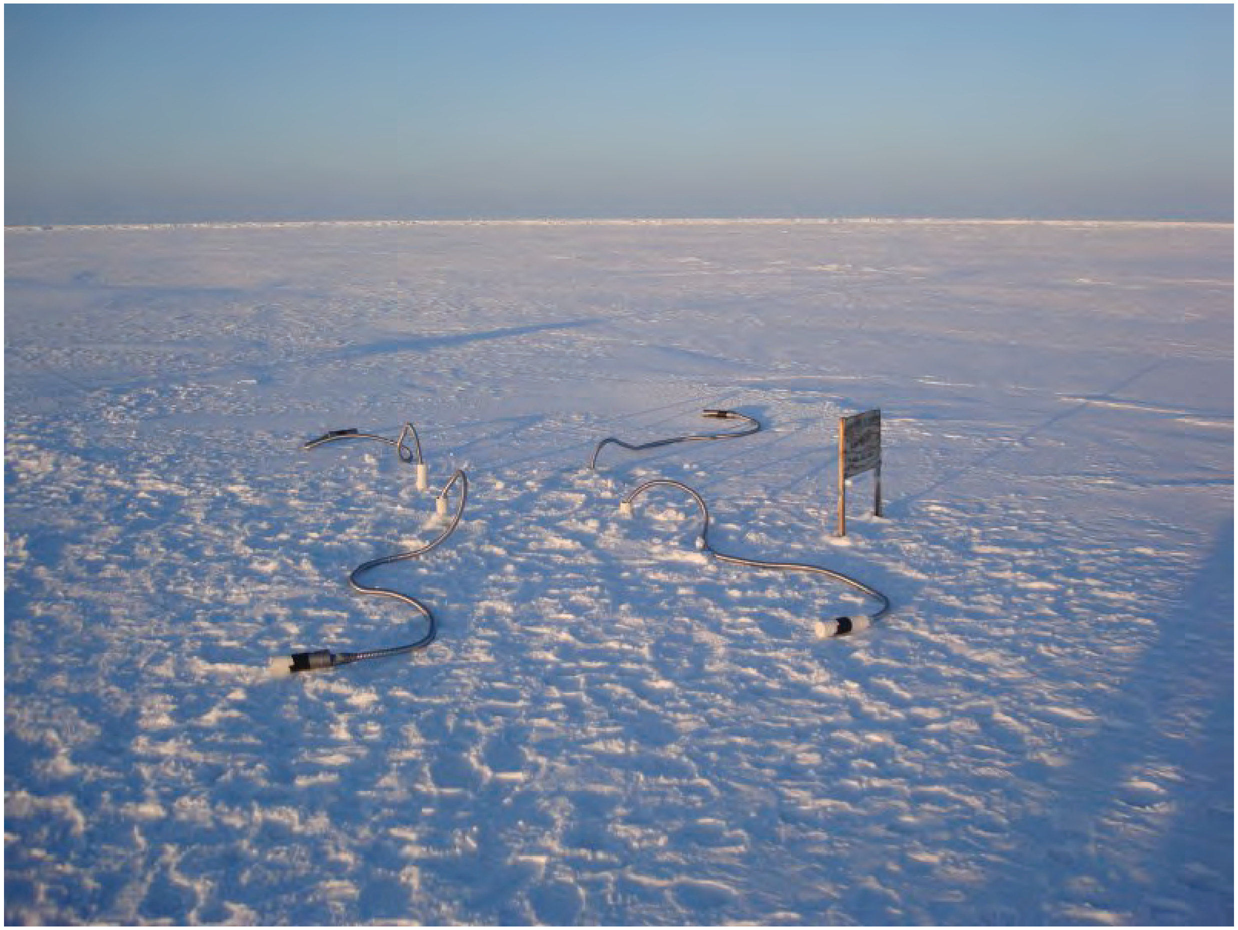


Wenner array

vertical conductivity

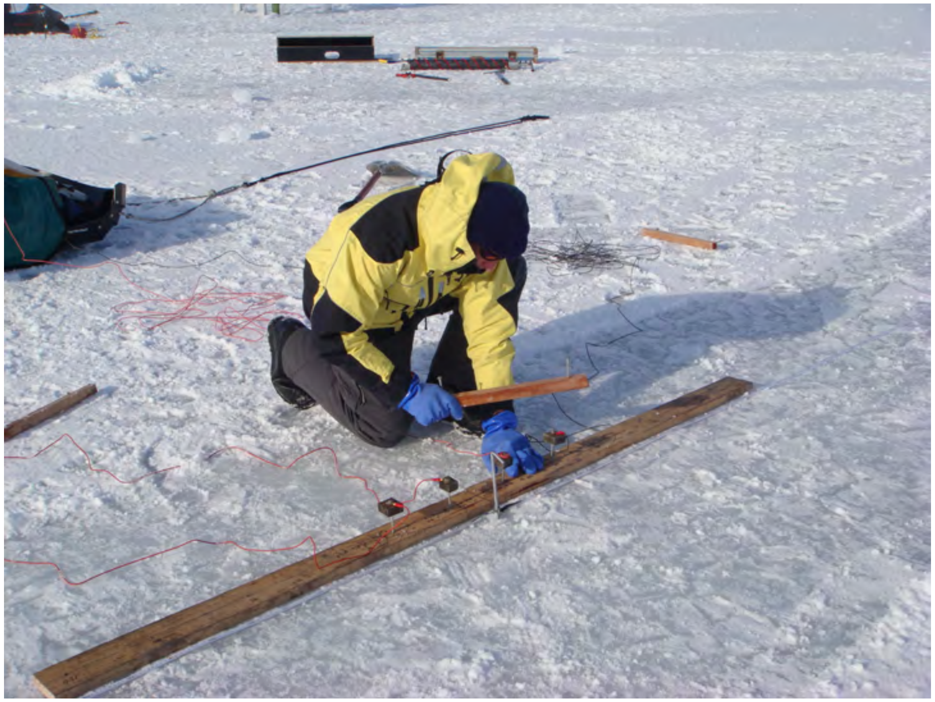
Zhu, Golden, Gully, Sampson ***Physica B*** 2010
Sampson, Golden, Gully, Worby ***Deep Sea Research*** 2011

cross borehole tomography



Ingham, Jones, Buchanan
Victoria University, Wellington, NZ

Measuring sea ice thickness



Golden

Remote sensing of sea ice

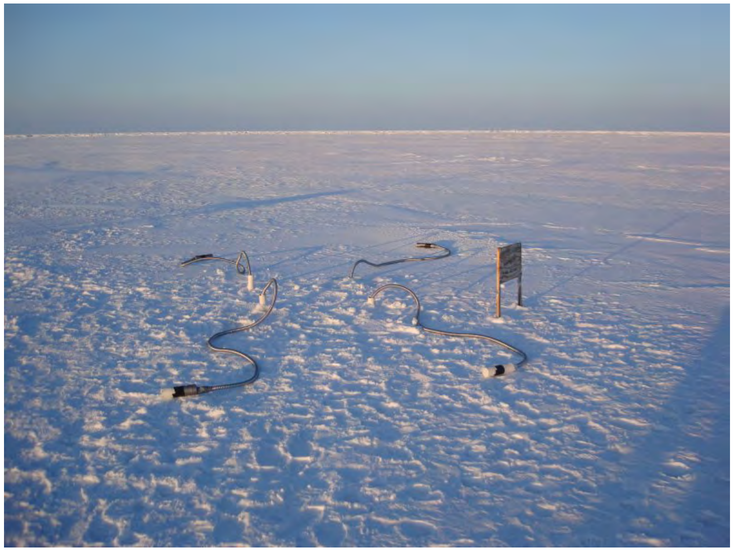
INVERSE PROBLEM

sea ice thickness
ice concentration

Recover sea ice properties from electromagnetic (EM) data

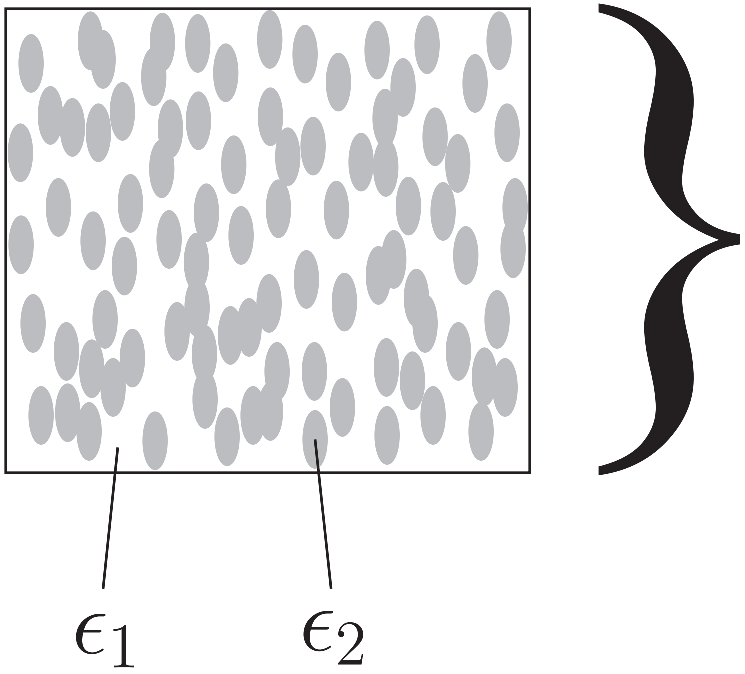
ε^*

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Effective complex permittivity of a two phase composite in the quasistatic (long wavelength) limit



$$\epsilon^*$$

$$D = \epsilon E$$

$$\nabla \cdot D = 0$$

$$\nabla \times E = 0$$

$$\langle D \rangle = \epsilon^* \langle E \rangle$$

p_1 , p_2 = volume fractions of the components

$$\epsilon^* = \epsilon^* \left(\frac{\epsilon_1}{\epsilon_2} \text{ , composite geometry} \right)$$

What are the effective propagation characteristics of an EM wave (radar, microwaves) in the medium?

Analytic Continuation Method for Homogenization

Bergman (1978), Milton (1979), Golden and Papanicolaou (1983), Theory of Composites, Milton (2002)

Stieltjes integral representation for homogenized parameter

separates geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s-z} \qquad s = \frac{1}{1-\epsilon_1/\epsilon_2}$$

material parameters

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

$$E = s\,(s + \Gamma\chi)^{-1} e_k$$

$\Gamma\chi$: microscale → macroscale

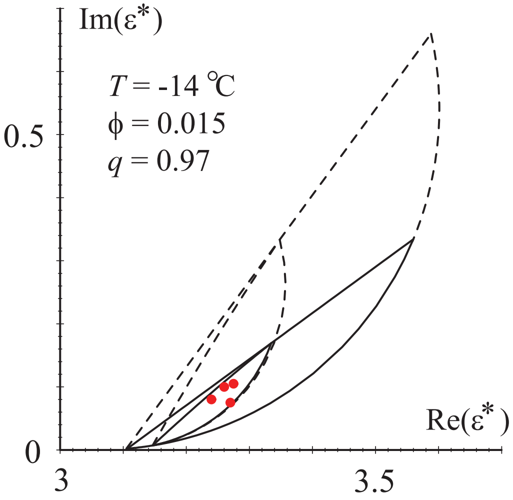
$\Gamma\chi$ *links scales*

Golden and Papanicolaou, Comm. Math. Phys. 1983

This representation distills the complexities of mixture geometry into the spectral properties of an operator like the Hamiltonian in physics.

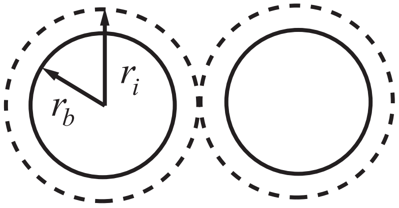
forward and inverse bounds on the complex permittivity of sea ice

forward bounds



4.75 GHz data

matrix particle

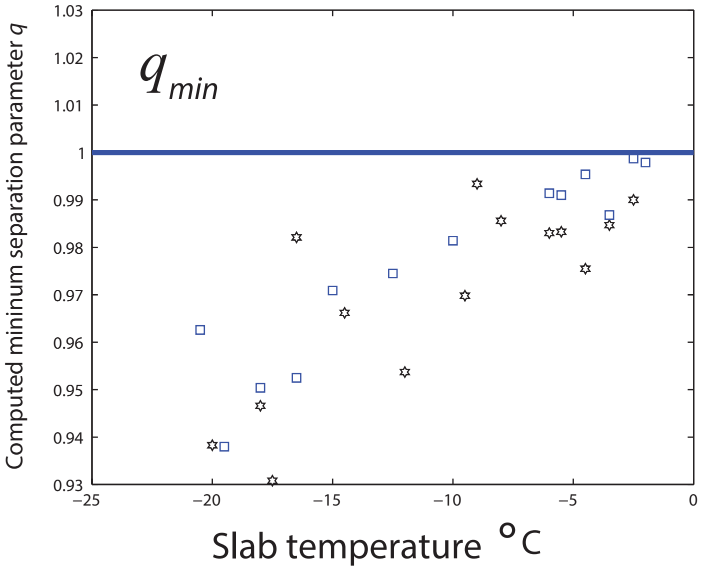


$$q = r_b / r_i$$

$$0 < q < 1$$

Golden 1995, 1997

inverse bounds



Inverse Homogenization

Cherkaev and Golden (1998), Day and Thorpe (1999), Cherkaev (2001), McPhedran, McKenzie, Milton (1982), *Theory of Composites*, Milton (2002)

ε^* ⟶ ***composite geometry***
(spectral measure μ)

inverse bounds and recovery of brine porosity

Gully, Backstrom, Eicken, Golden
Physica B, 2007

inversion for brine inclusion separations in sea ice from measurements of effective complex permittivity ε^*

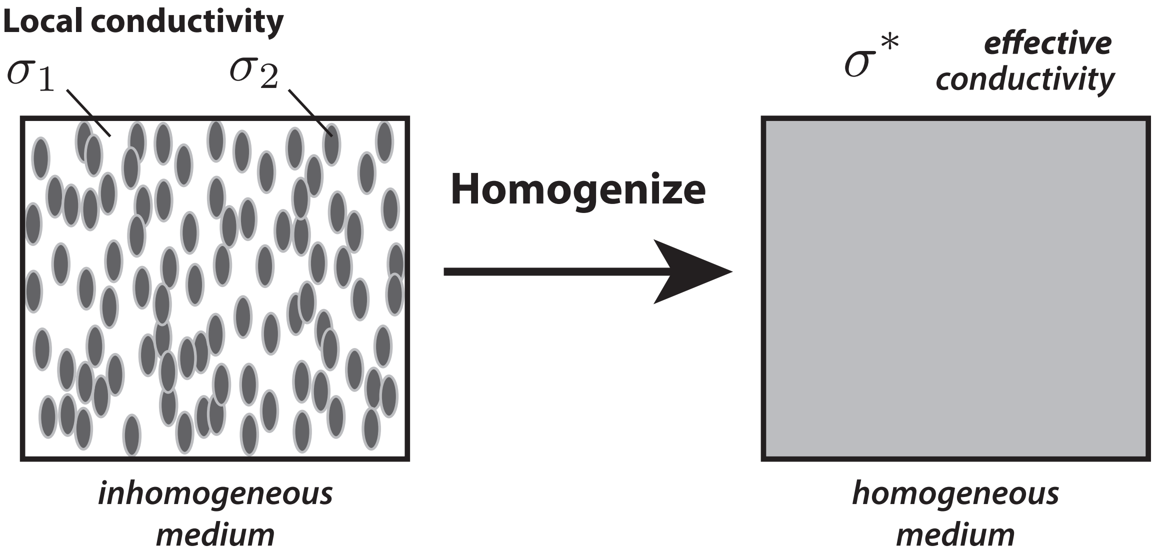
rigorous inverse bound
on spectral gap

construct algebraic curves which bound
admissible region in (p,q)-space

Orum, Cherkaev, Golden
Proc. Roy. Soc. A, 2012

Homogenization for polycrystalline materials

Two-component composites



Local conductivity

σ_1 σ_2

Homogenize

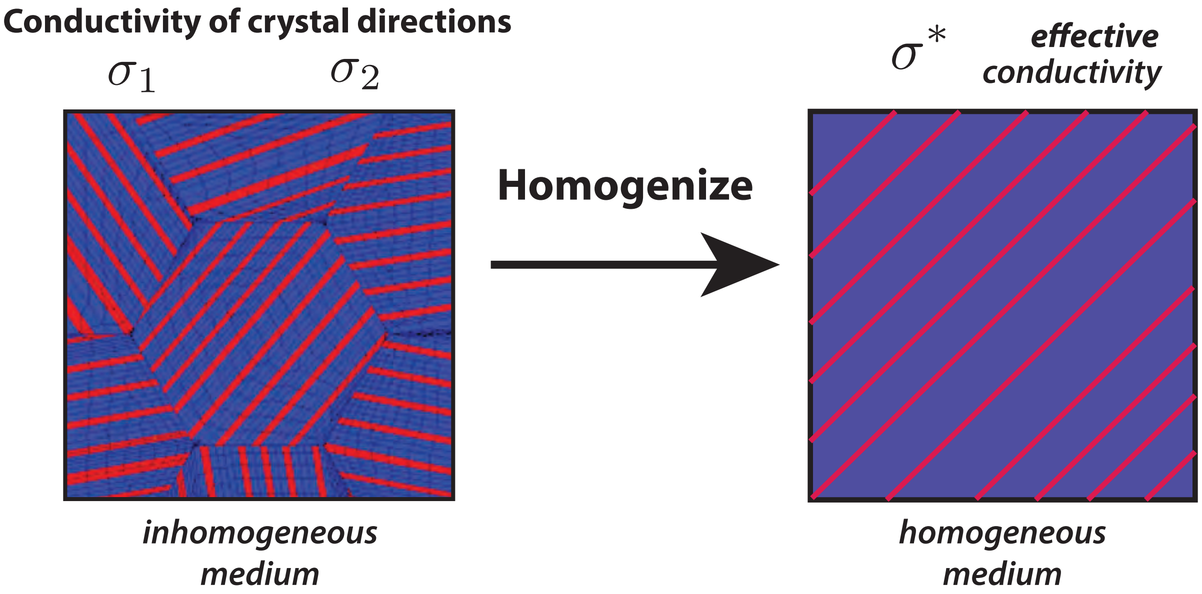
σ^* ***effective conductivity***

inhomogeneous medium

homogeneous medium

Find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Polycrystalline media



Conductivity of crystal directions

σ_1 σ_2

Homogenize

σ^* ***effective conductivity***

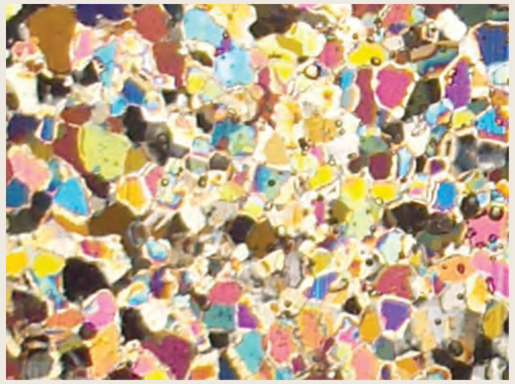
inhomogeneous medium

homogeneous medium

Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**
  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**
  orientation statistics
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A 8 Feb 2015

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS
OF SCIENTIFIC
PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

THE
ROYAL
SOCIETY
PUBLISHING

21/01/15 1:47 PM

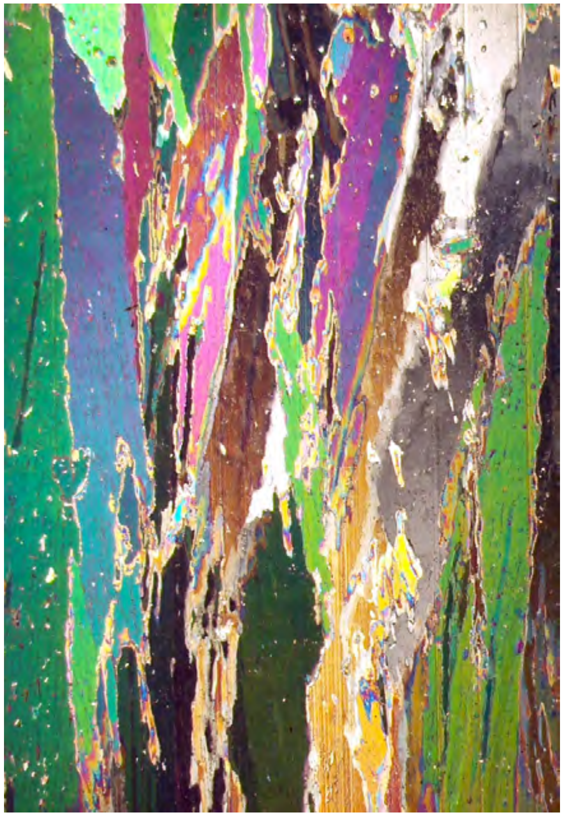
higher threshold for fluid flow in granular sea ice

microscale details impact "mesoscale" processes

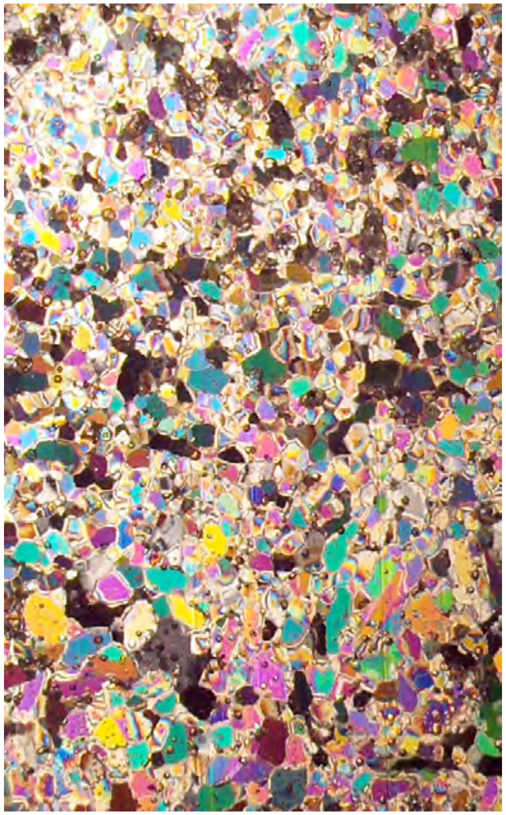
nutrient fluxes for microbes
melt pond drainage
snow-ice formation

columnar

5%



granular



10%

Golden, Sampson, Gully, Lubbers, Tison 2022

electromagnetically distinguishing ice types
Kitsel Lusted, Elena Cherkaev, Ken Golden

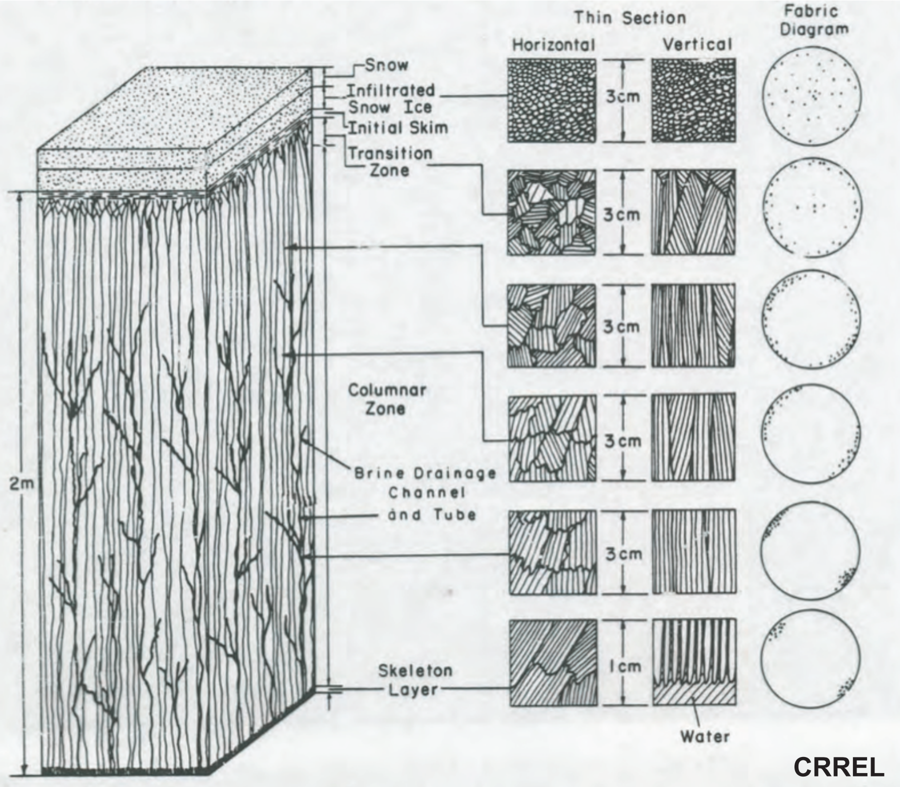
Rigorous bounds on the complex permittivity tensor of sea ice with polycrystalline anisotropy in the horizontal plane

Kenzie McLean, Elena Cherkaev, Ken Golden 2022

motivated by **Weeks and Gow, *JGR* 1979: c-axis alignment in Arctic fast ice off Barrow**

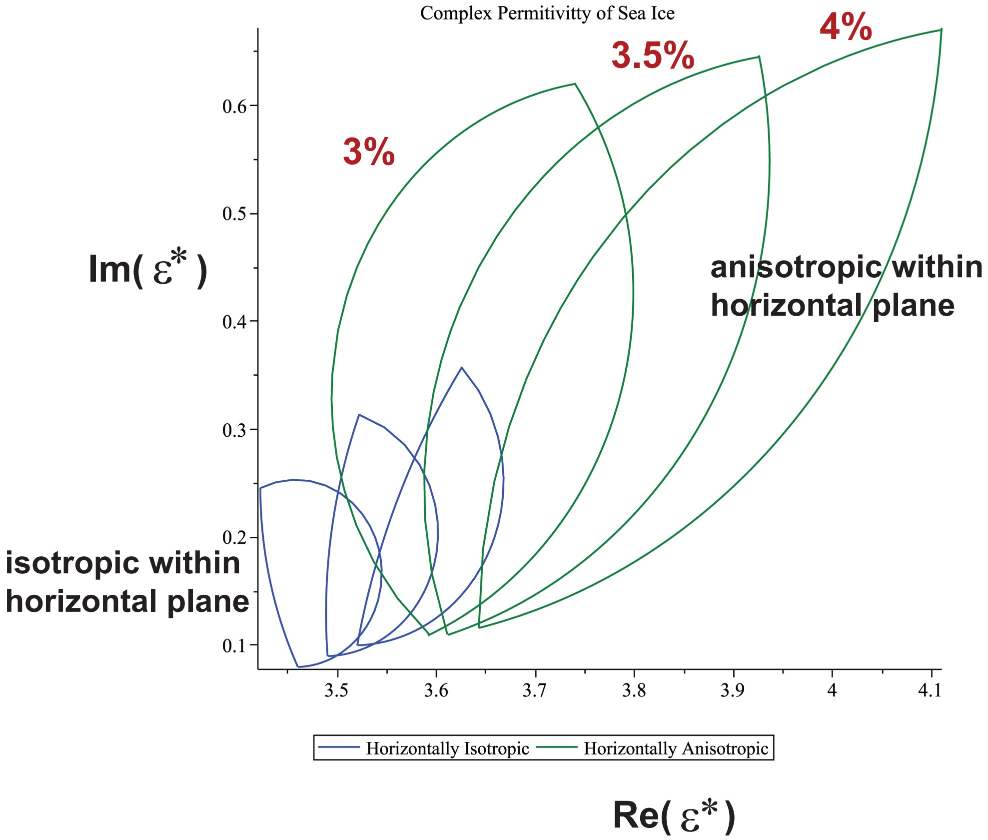
Golden and Ackley, *JGR* 1981: radar propagation model in aligned sea ice

input: orientation statistics



CRREL

output: bounds



mesoscale

wave propagation in the marginal ice zone (MIZ)

Stieltjes integral representation and bounds for the complex viscoelasticity of the ice - ocean layer

Sampson, Murphy, Cherkaev, Golden 2022

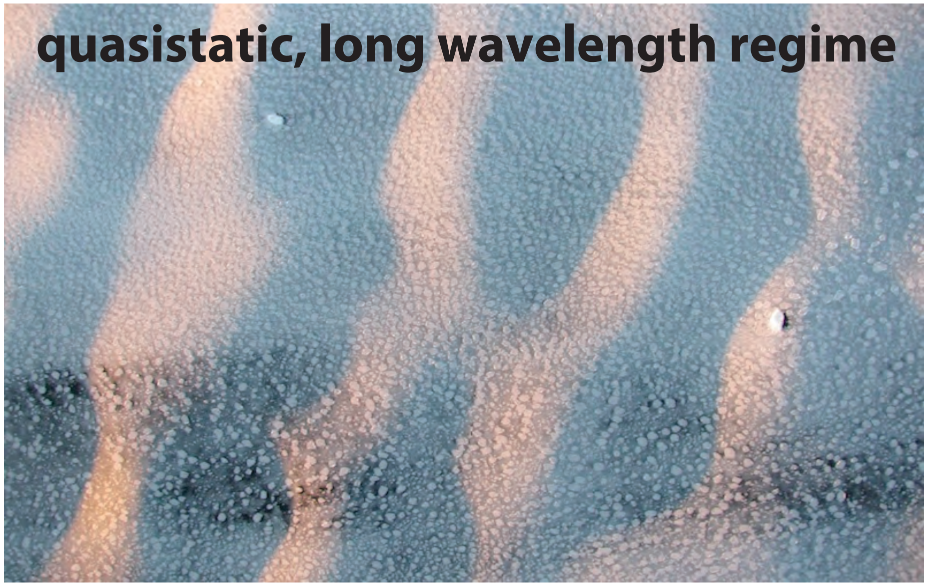
first theory of key parameter in wave-ice interactions only fitted to wave data before

Keller, 1998
Mosig, Montiel, Squire, 2015
Wang, Shen, 2012

Analytic Continuation Method

Bergman (78) - Milton (79)
integral representation for ϵ^*
Golden and Papanicolaou (83)

Milton, *Theory of Composites* (02)



quasistatic, long wavelength regime

homogenized parameter depends on sea ice concentration and ice floe geometry

like EM waves

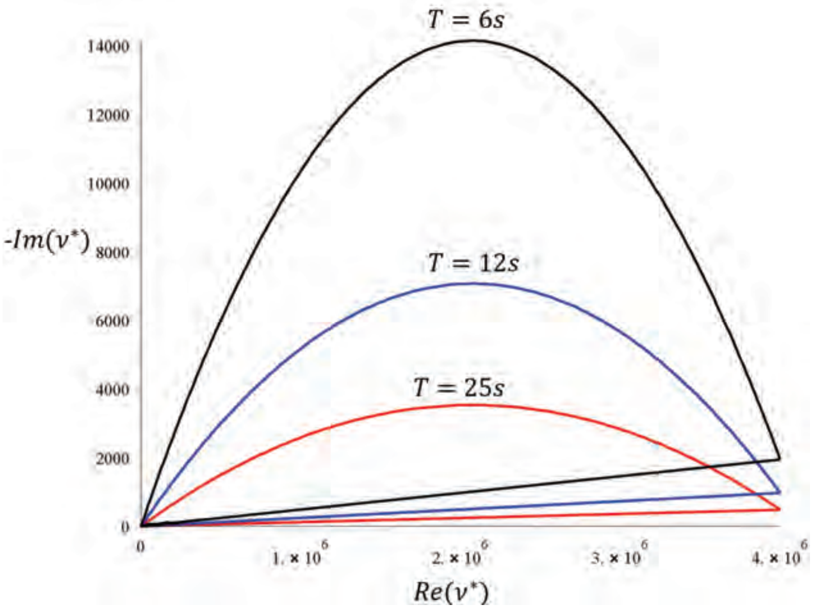
bounds on the effective complex viscoelasticity

$\nu_1 = 10^7 + i\,4875$ **pancake ice**

$\nu_2 = 5 + i\,0.0975$ **slush / frazil**

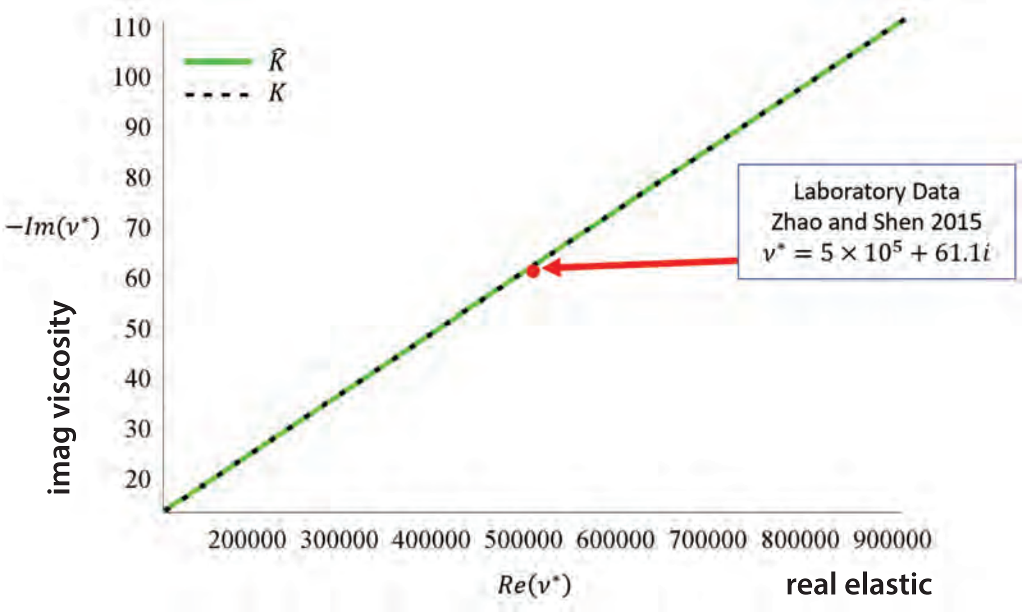
high contrast

complex elementary bounds (fixed area fraction of floes)



Elementary bounds for wave periods T.

matrix-particle bounds



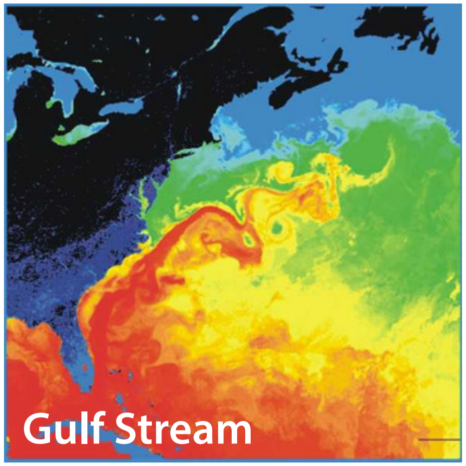
Laboratory Data
Zhao and Shen 2015
$\nu^* = 5 \times 10^5 + 61.1i$

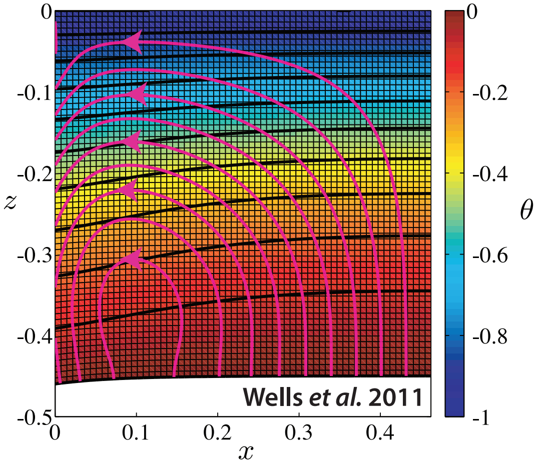
Golden

advection enhanced diffusion

effective diffusivity

nutrient and salt transport in sea ice
heat transport in sea ice with convection
sea ice floes in winds and ocean currents
tracers, buoys diffusing in ocean eddies
diffusion of pollutants in atmosphere





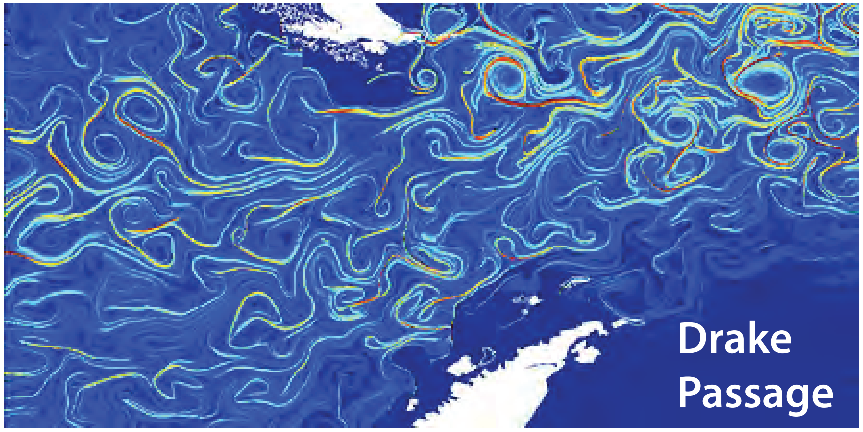
advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

↓ ***homogenize***

$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$



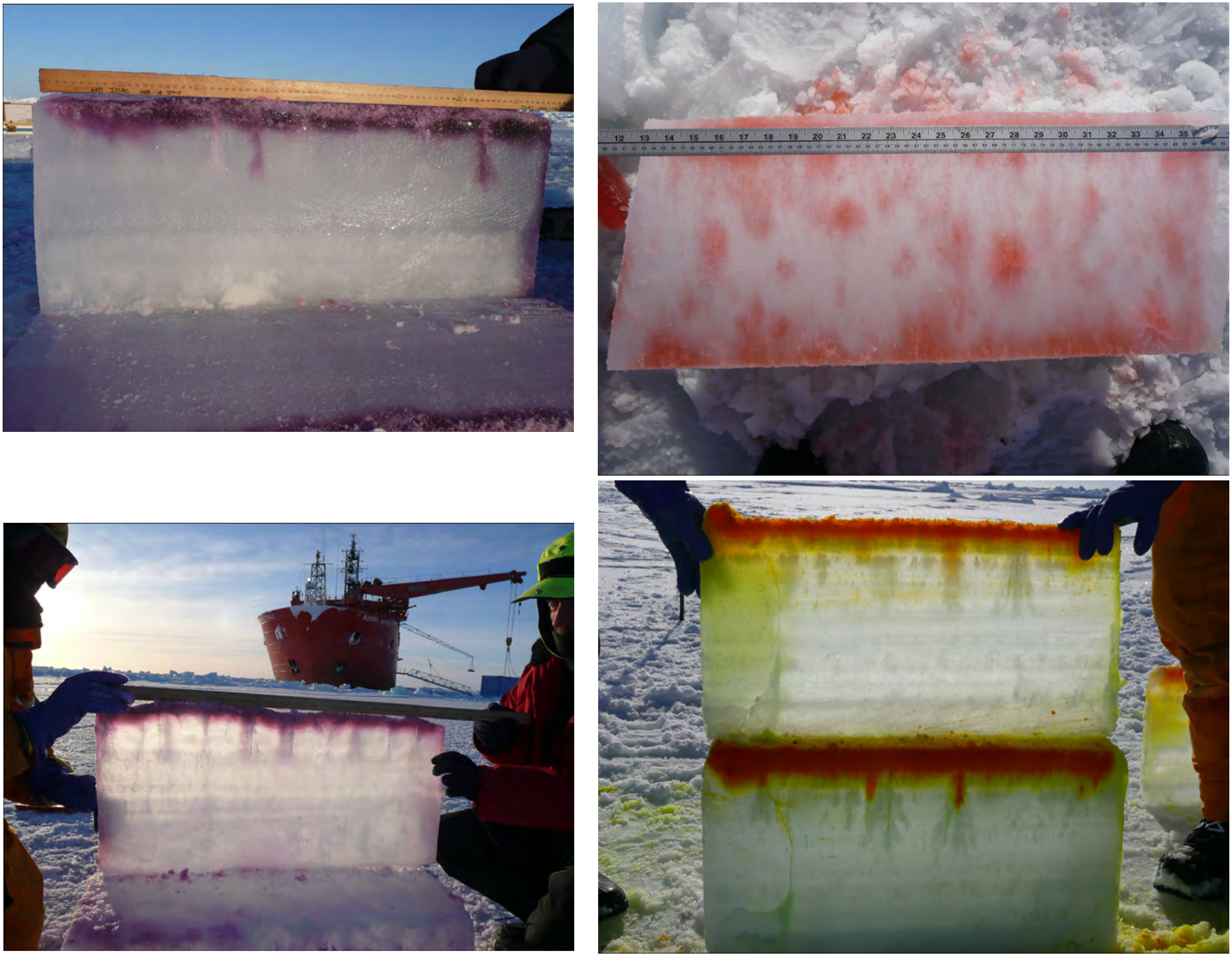
κ^* *effective diffusivity*

Stieltjes integral for κ^* with spectral measure

Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

tracers flowing through inverted sea ice blocks



Stieltjes Integral Representation for Advection Diffusion

Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math. Phys.* 2020

$$\kappa^* = \kappa\left(1 + \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2+\tau^2}\right), \quad F(\kappa) = \int_{-\infty}^{\infty} \frac{d\mu(\tau)}{\kappa^2+\tau^2}$$

- μ is a positive definite measure corresponding to the spectral resolution of the self-adjoint operator $i\Gamma H\Gamma$
- $H =$ stream matrix , $\kappa =$ local diffusivity
- $\Gamma := -\nabla(-\Delta)^{-1}\nabla\cdot$, Δ is the Laplace operator
- $i\Gamma H\Gamma$ is bounded for time independent flows
- $F(\kappa)$ is analytic off the spectral interval in the κ-plane

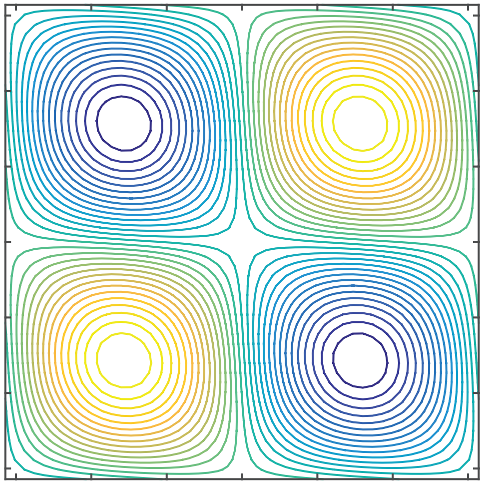
rigorous framework for numerical computations of spectral measures and effective diffusivity for model flows

new integral representations, theory of moment calculations

separation of material properties and flow field

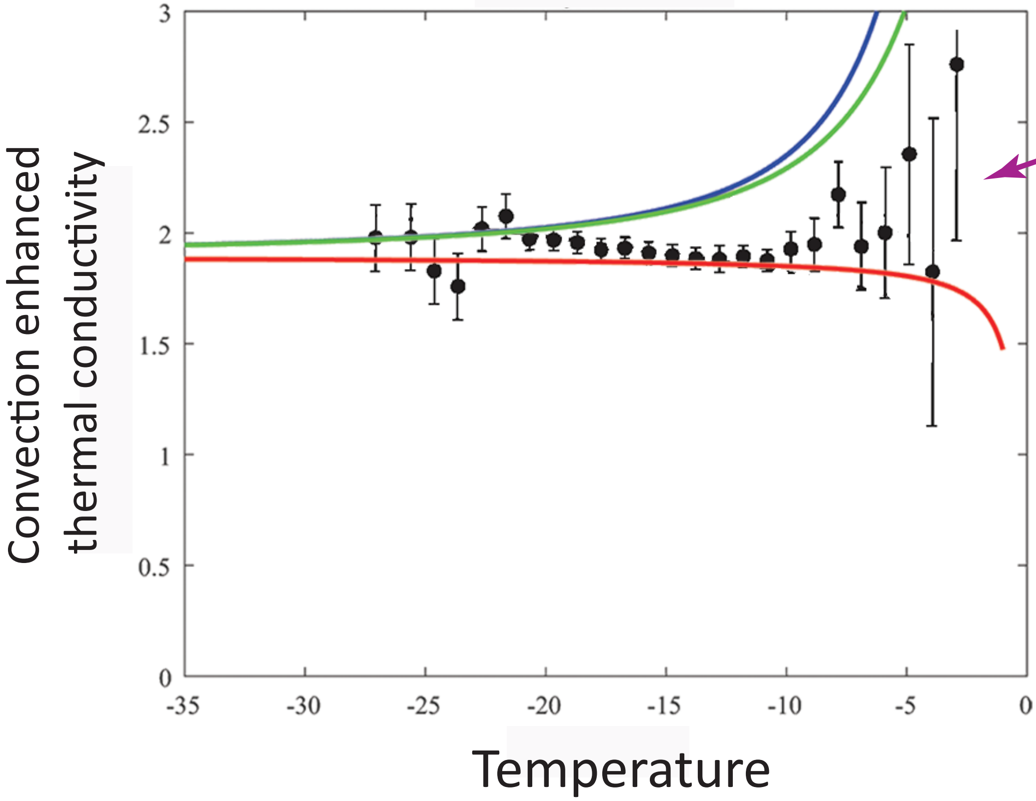
Rigorous bounds on convection enhanced thermal conductivity of sea ice

Kraitzman, Hardenbrook, Dinh, Murphy, Zhu, Cherkaev, Golden 2022



cat’s eye flow model for brine convection cells

similar bounds
for shear flows



data Trodahl et al. 2001

rigorous Padé bounds from Stieltjes integral +
analytical calculations of moments of measure

direct calculation of spectral measures

Murphy, Hohenegger, Cherkaev, Golden, *Comm. Math. Sci.* 2015

- depends only on the composite geometry
- discretization of microstructural image gives binary network
- fundamental operator becomes a random matrix
- spectral measure computed from eigenvalues and eigenvectors

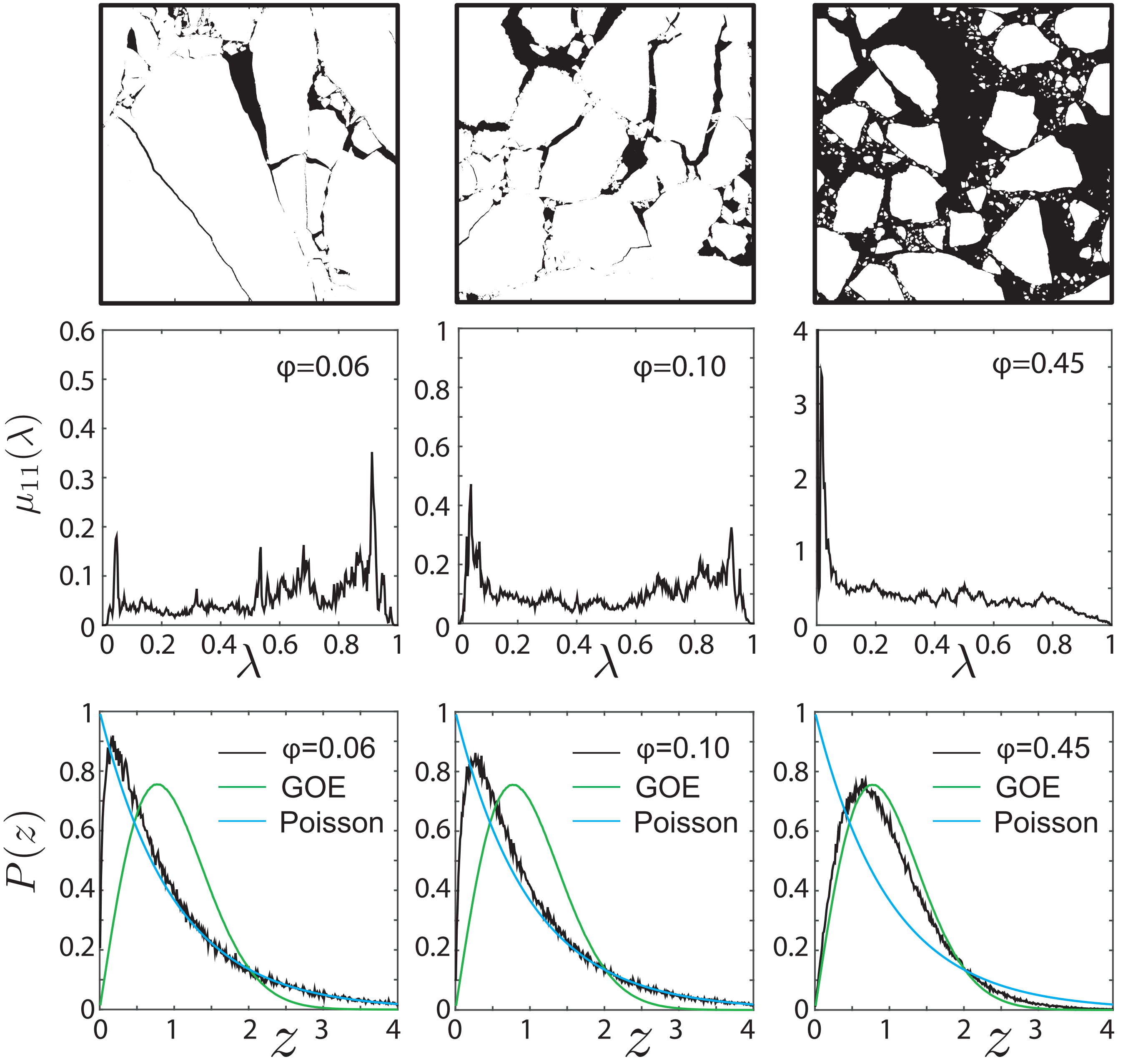
once we have the spectral measure μ it can be used in Stieltjes integrals for other transport coefficients:

electrical and thermal conductivity, complex permittivity, magnetic permeability, diffusion, fluid flow properties

earlier studies of spectral measures

Day and Thorpe 1996
Helsing, McPhedran, Milton 2011

Spectral computations for sea ice floe configurations



spectral measures

eigenvalue spacing distributions

UNIVERSAL Wigner-Dyson distribution

uncorrelated → **level repulsion**

Eigenvalue Statistics of Random Matrix Theory

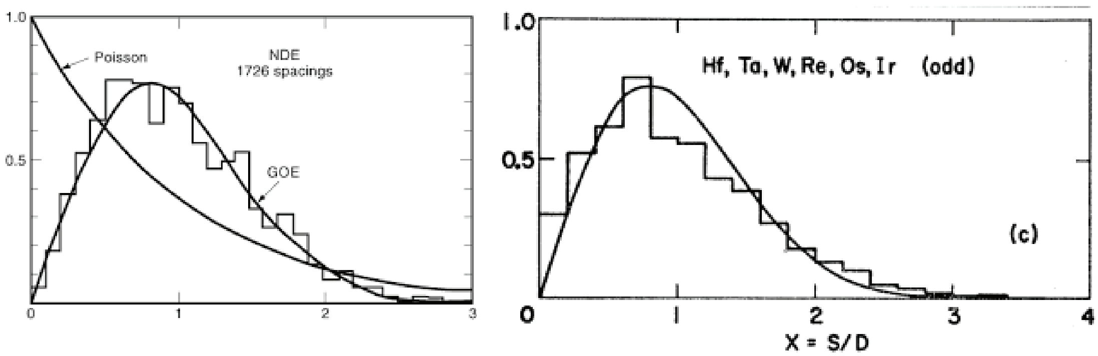
Wigner (1951) and Dyson (1953) first used random matrix theory (RMT) to describe quantized energy levels of heavy atomic nuclei.

$[\mathbf{N}]_{ij} \sim N(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^T)/2$ ***Gaussian orthogonal ensemble (GOE)***

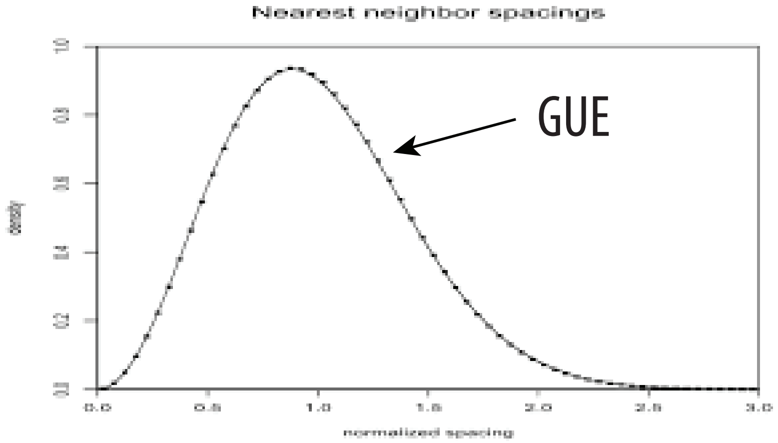
$[\mathbf{N}]_{ij} \sim N(0,1)+iN(0,1)$, $\mathbf{A} = (\mathbf{N}+\mathbf{N}^\dagger)/2$ ***Gaussian unitary ensemble (GUE)***

Short range and long range correlations of eigenvalues are measured by various eigenvalue statistics.

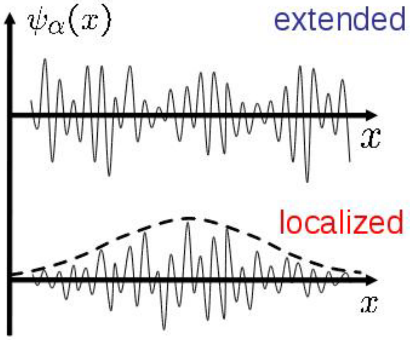
Spacing distributions of energy levels for heavy atomic nuclei



Spacing distributions of the first billion zeros of the Riemann zeta function



Universal eigenvalue statistics arise in a broad range of "unrelated" problems!



electronic transport in semiconductors

metal / insulator transition

localization

Anderson 1958
Mott 1949
Shklovshii et al 1993
Evangelou 1992

Anderson transition in wave physics:
quantum, optics, acoustics, water waves, ...

from analysis of spectral measures for brine, melt ponds, ice floes

we find percolation-driven

Anderson transition for classical transport in composites

Murphy, Cherkaev, Golden Phys. Rev. Lett. 2017

PERCOLATION TRANSITION → **universal eigenvalue statistics (GOE)**
extended states, mobility edges

-- but with NO wave interference or scattering effects ! --

local conductivity in 1D inhomogeneous material

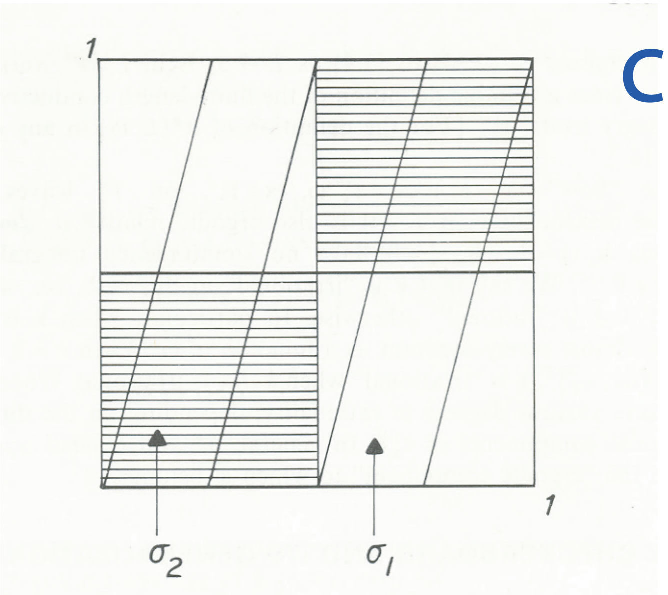
$$\sigma(x) = 3 + \cos x + \cos kx$$

effective conductivity

$$\sigma^*(k) = \begin{cases} \text{constant} & k \ \ \text{irrational} \qquad \text{quasiperiodic} \\ f(k) & k \ \ \text{rational} \qquad \text{periodic} \end{cases}$$

Golden, Goldstein, Lebowitz, Phys. Rev. Lett. 1985

Classical transport in quasiperiodic media



line of slope k through
an infinite checkerboard

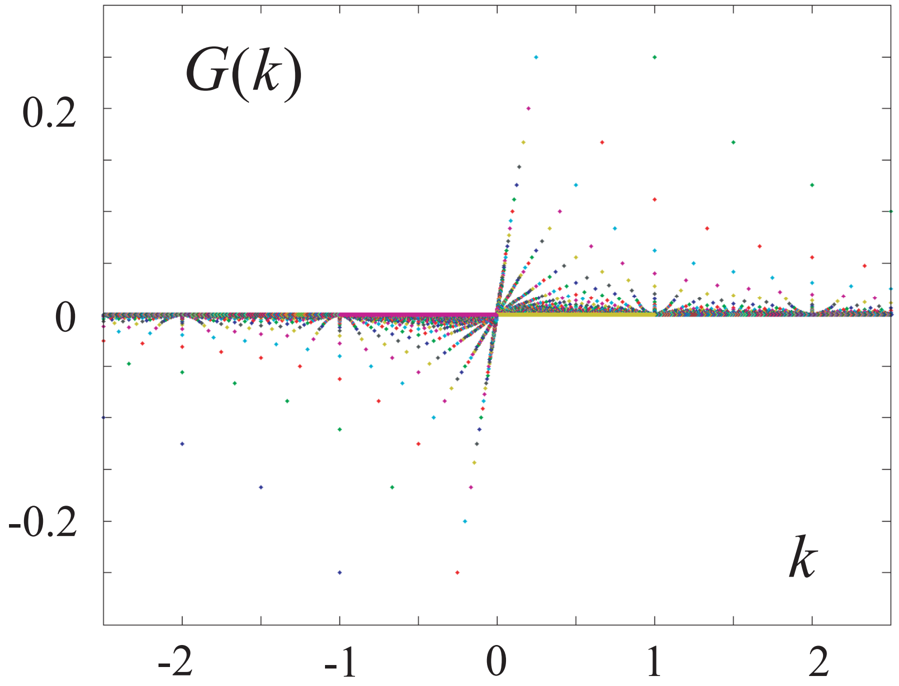
Golden, Goldstein, and Lebowitz
***Phys. Rev. Lett.* 1985**
***J. Stat. Phys.* 1990**

effective conductivity $\sigma^*(k)$

effective resistivity $1/\,\sigma^*(k) = 1 - G(k)$

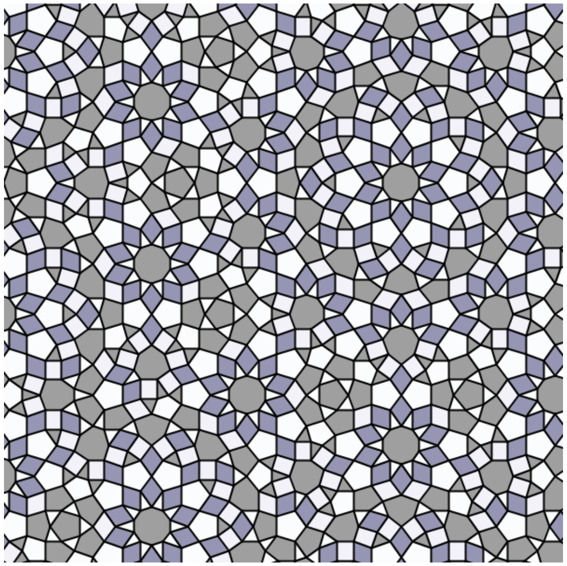
$$G(k) = \begin{cases} 0, & k \text{ irrational} \\ 1/pq, & k = p/q \text{ rational} \end{cases}$$

continuous at *k* irrational
discontinuous at *k* rational



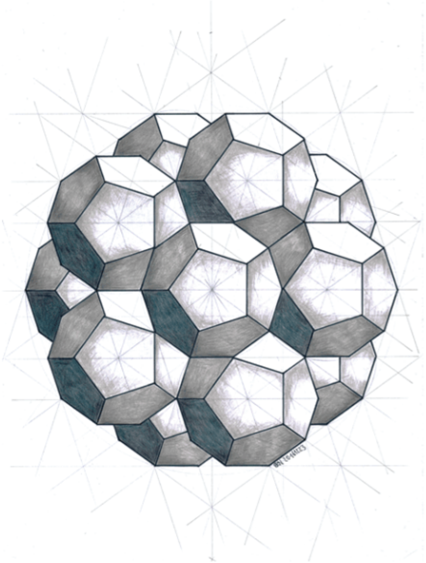
Order to Disorder in Quasiperiodic Composites

D. Morison (Physics), N. B. Murphy, E. Cherkaev, K. M. Golden, *Communications Physics* 2022



quasiperiodic checkerboard

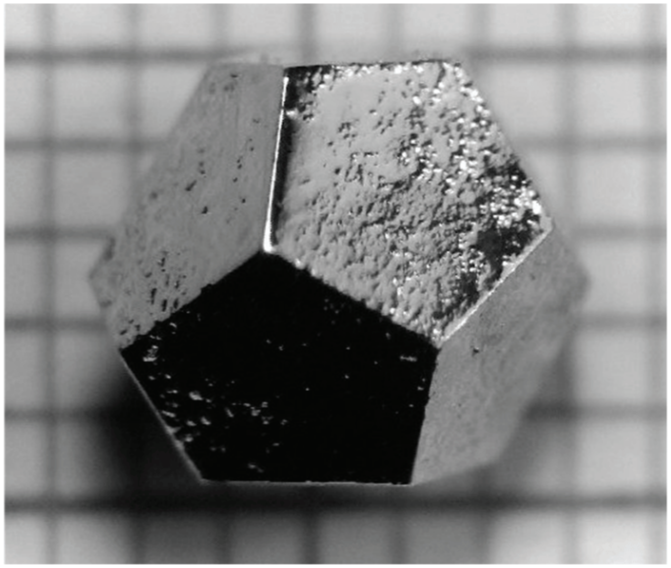
Stampfli, 2013



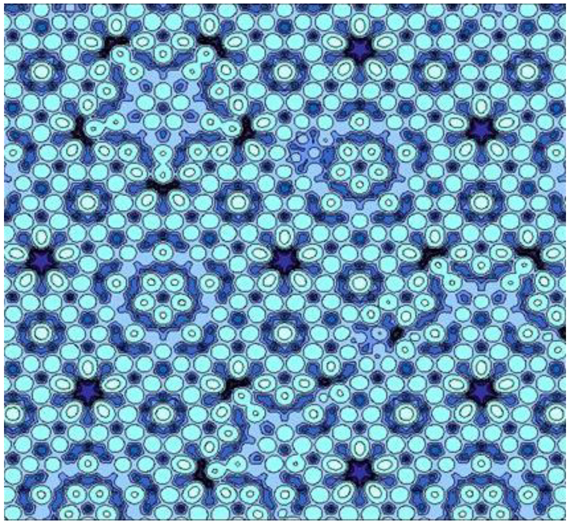
dense packing of dodecahedra

3D Penrose tiling

Tripkovic, 2019



Holmium–magnesium–zinc quasicrystal



energy surface Al-Pd-Mn quasicrystal

Unal et al., 2007

quasiperiodic crystal

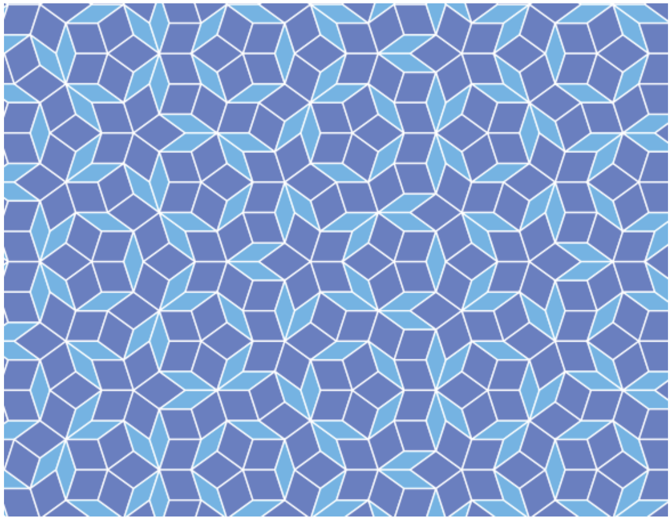
quasicrystal

ordered but aperiodic

lacks translational symmetry

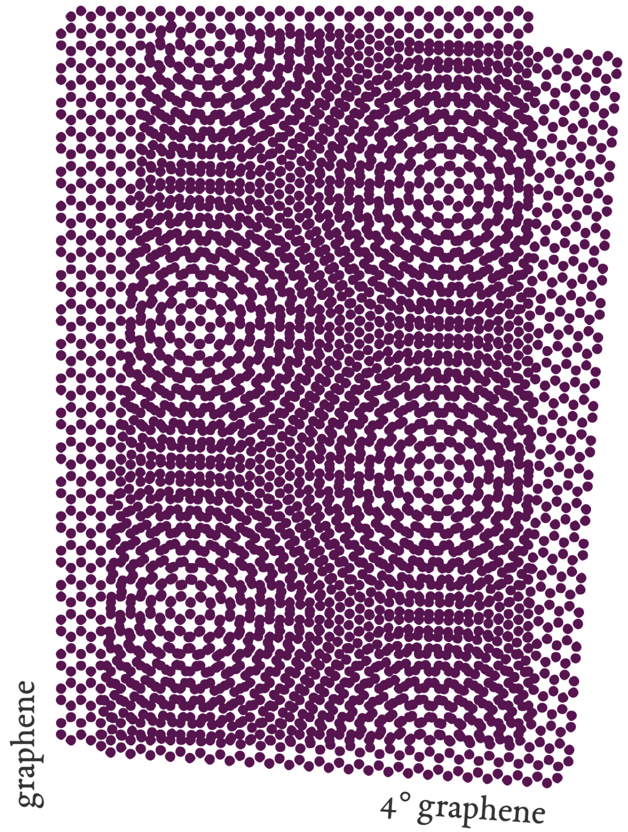
Schechtman et al., 1984

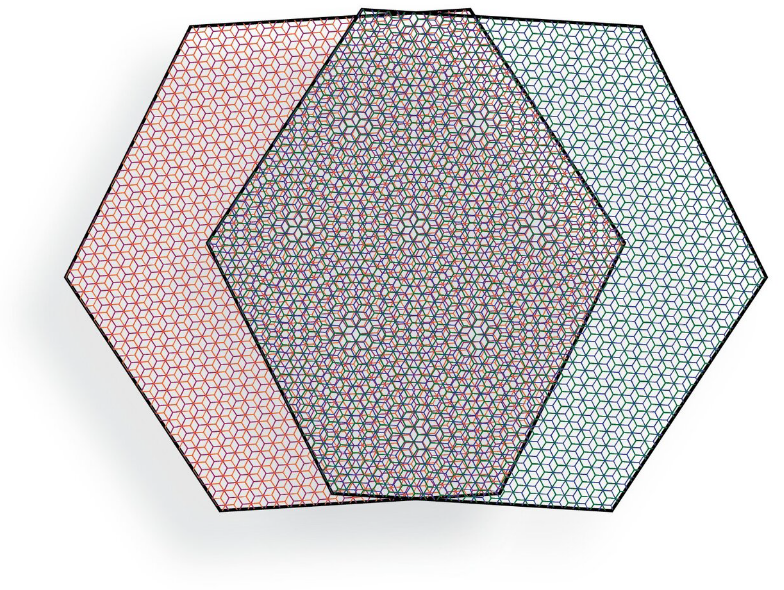
Levine & Steinhardt, 1984

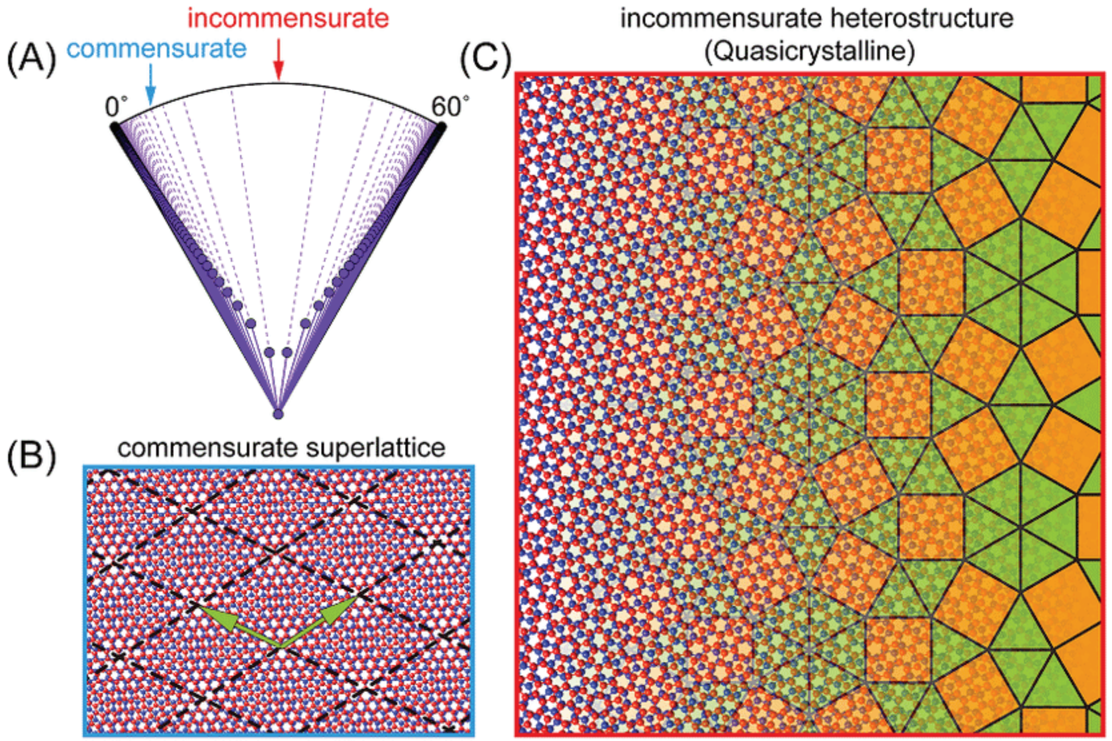


aperiodic tiling of the plane - R. Penrose 1970s

twisted bilayer graphene

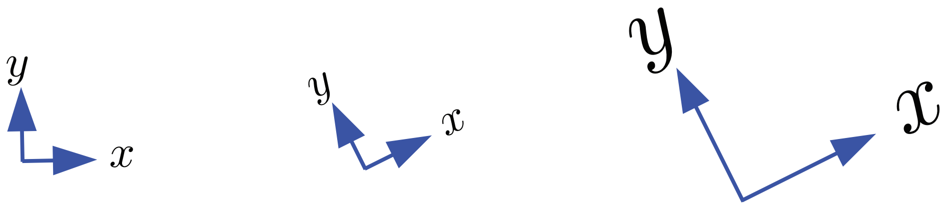


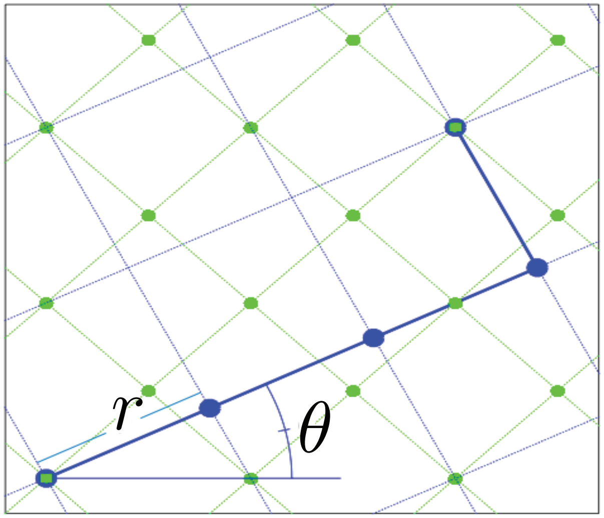




Yao et al., 2018

Moiré patterns generate two component composites





$$\begin{pmatrix} x' \\ y' \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\psi\left(x', y'\right) = \cos 2\pi x' \cos 2\pi y'$$

$$\chi = \begin{cases} 1, & \psi \geq 0 \\ 0, & \psi < 0 \end{cases}$$

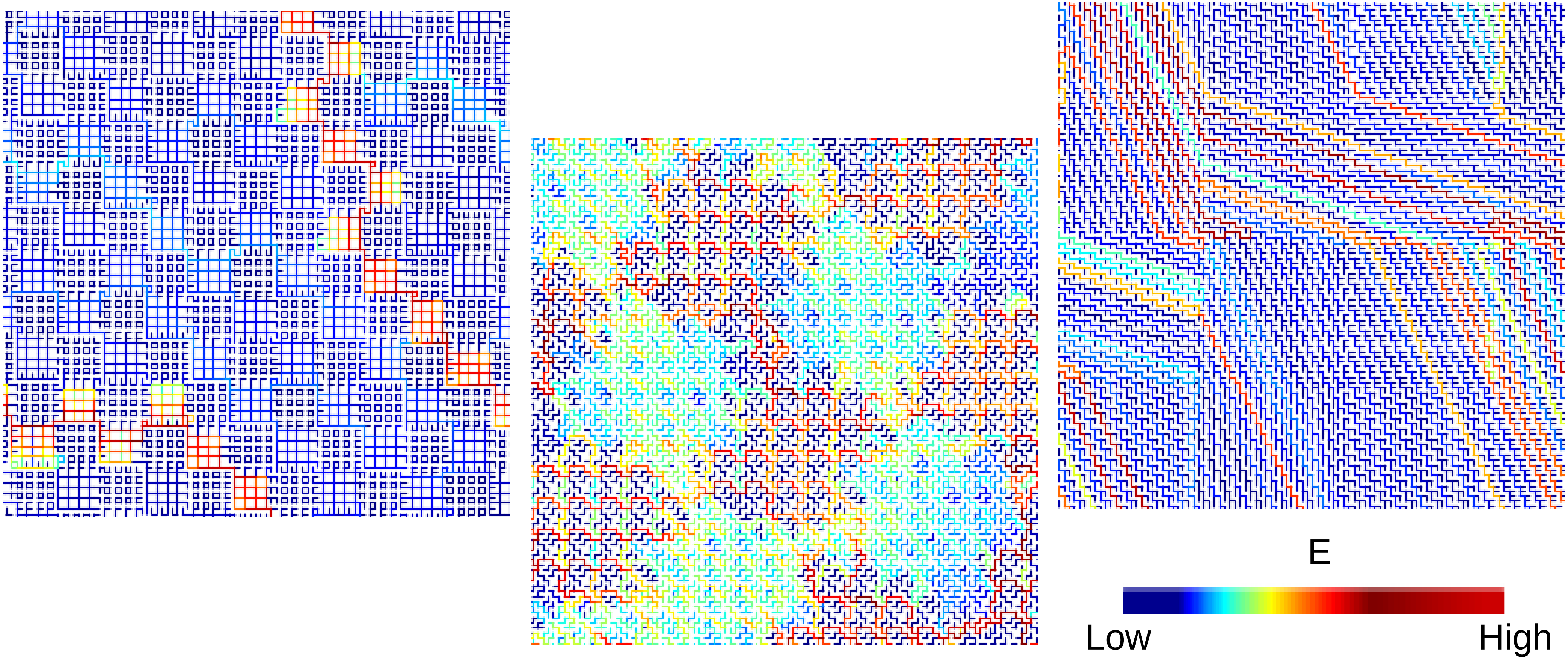
rotation and dilation

Small Difference in Moiré Parameters

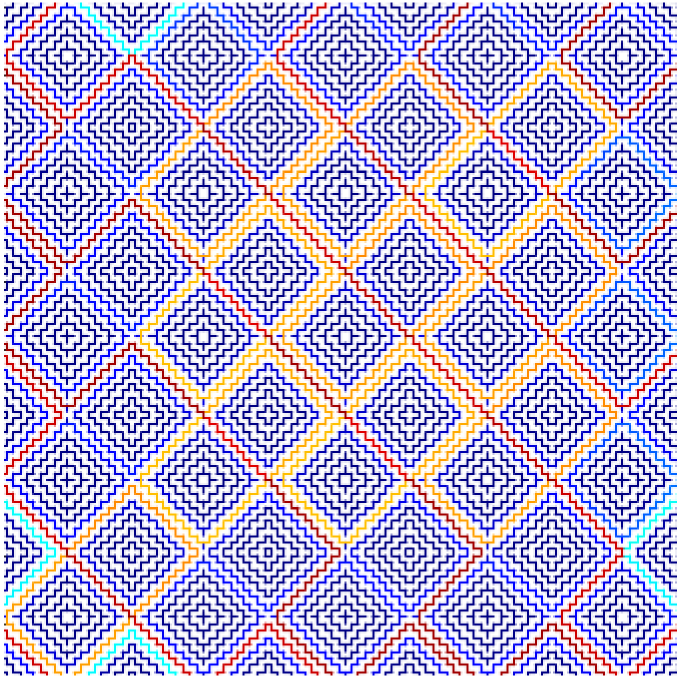
↓

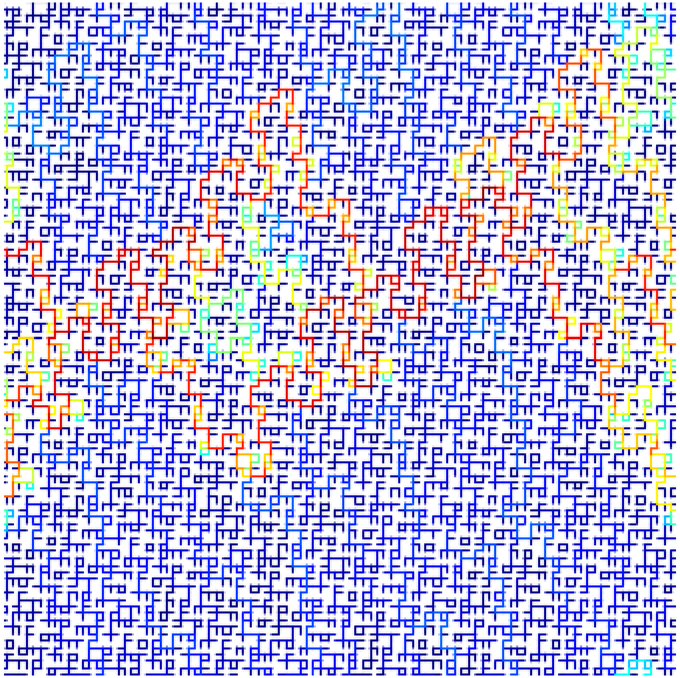
Big Difference in Material Properties

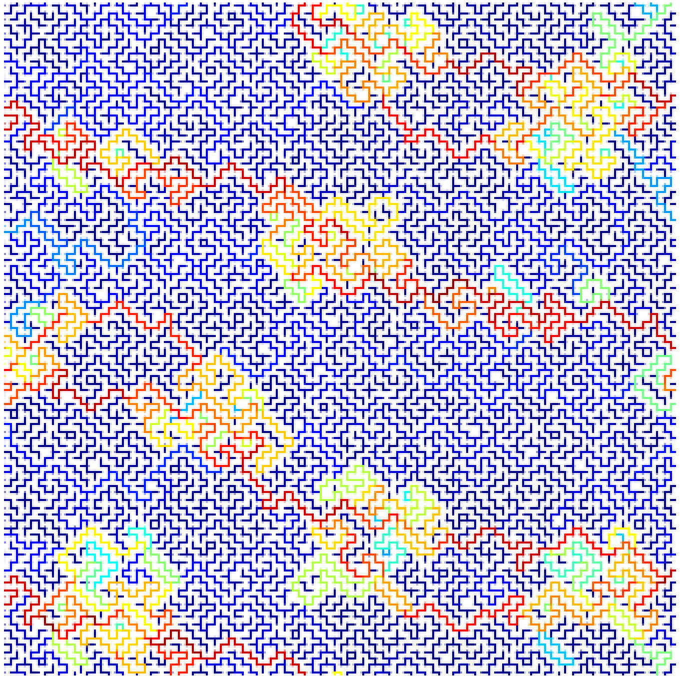
Wide Variety of Microgeometries



Wide Variety of Microgeometries



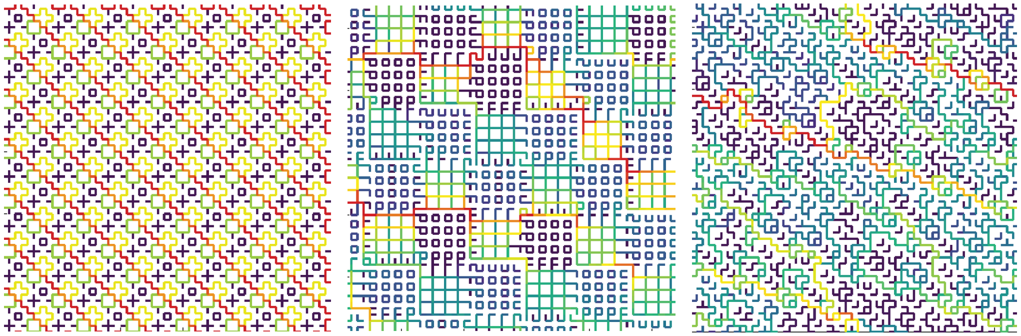




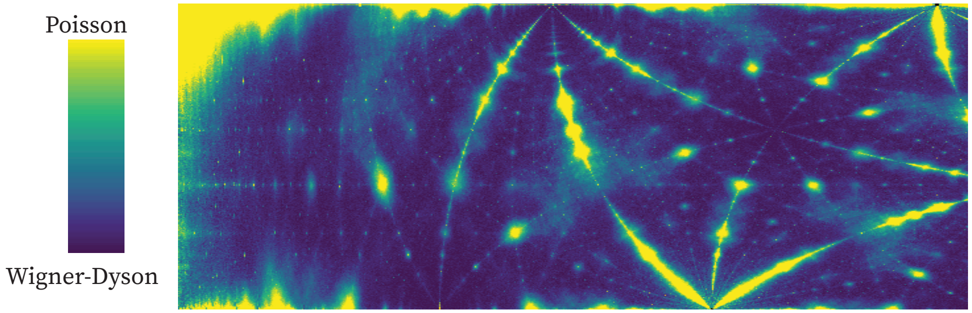
Order to disorder in quasiperiodic composites

Morison, Murphy, Cherkaev, Golden, *Commun. Phys.* 2022

Parameterized Moiré Pattern Creates Tunable Microgeometry



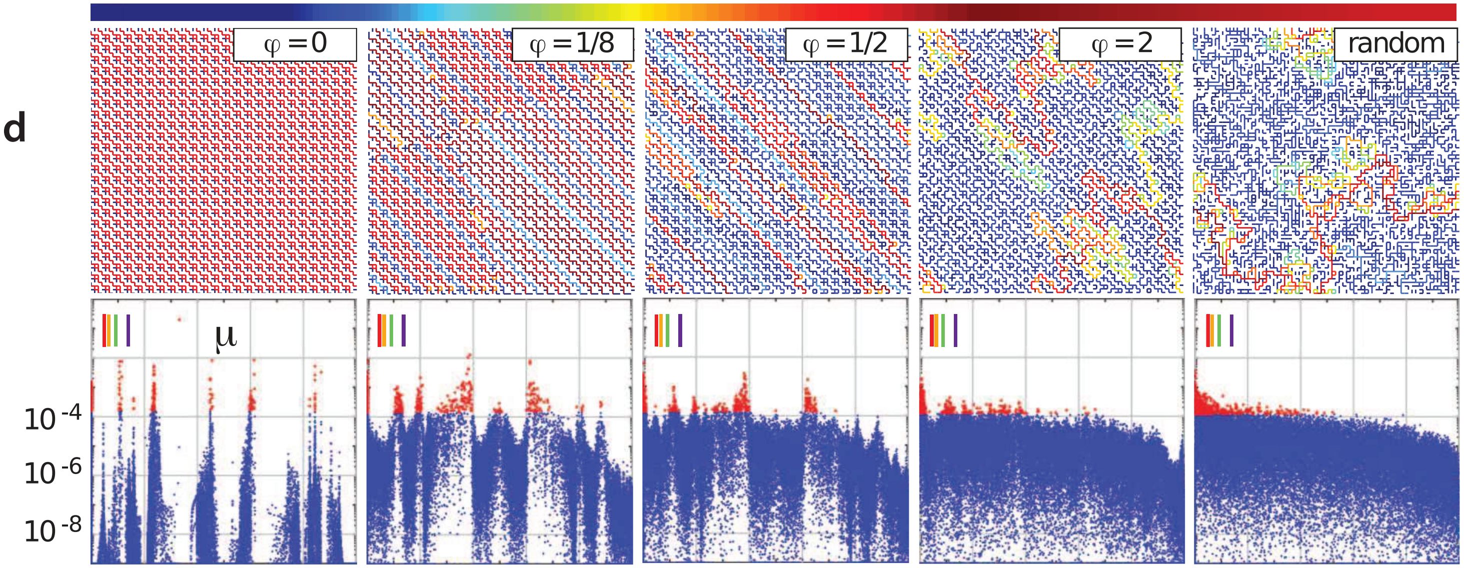
constellation of periodic systems in a sea of randomness



parameter space

periodic → **quasiperiodic**

electric field strength



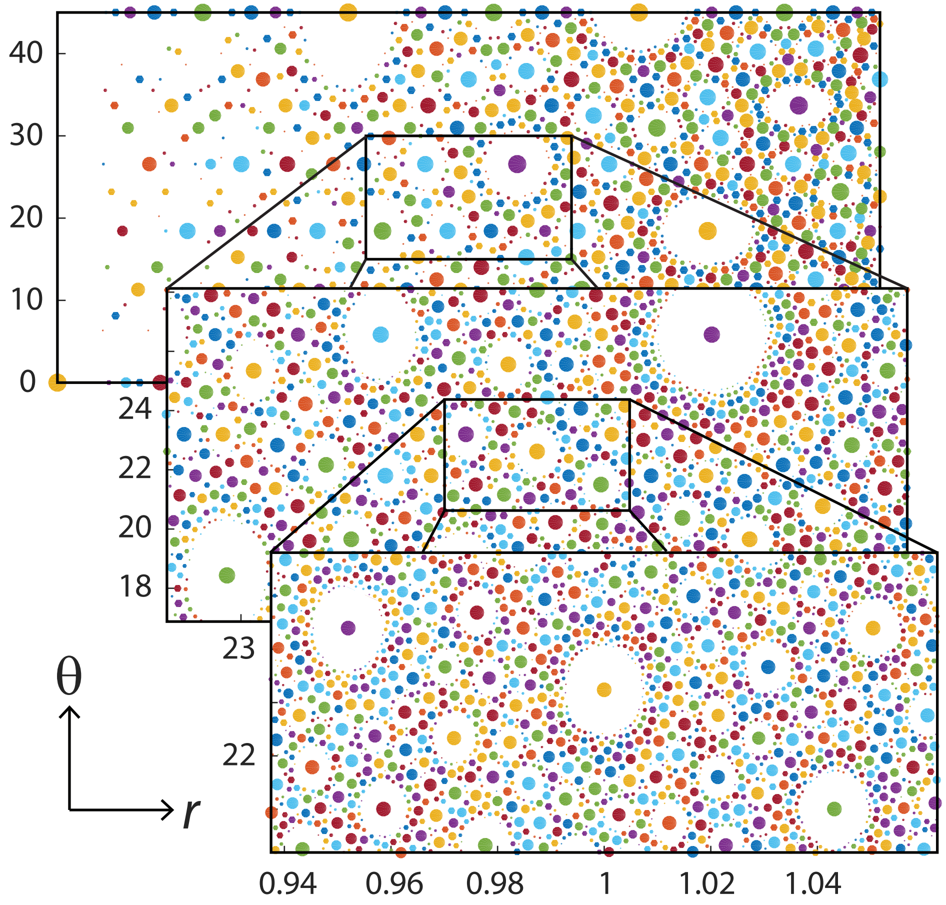
spectral measure

RRN at percolation threshold

we bring the framework of solid state physics of electronic transport and band gaps in semiconductors to classical transport in periodic and quasiperiodic composites

photonic crystals and quasicrystals

Fractal arrangement of periodic systems



Sequential insets zooming into smaller regions of parameter space.

size of the dots ~ length of period

(large dot ~ small period; small dot ~ large period; white space ~ “infinite” period)

Conclusions

1. Sea ice is a fascinating multiscale composite with structure similar to many other natural and man-made materials.

2. Homogenization and statistical physics help *link scales in the sea ice system*; provide rigorous methods for finding effective behavior; advance sea ice representations in climate models.

3. Stieltjes functions and their integral representations provide powerful methods of homogenization for EM waves, advection diffusion, polycrystalline media, and ocean surface waves in the ice cover.

4. Mathematical methods developed for sea ice advance the theory of composites, and quasiperiodic media in particular.

5. Our research is helping to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

University of Utah Sea Ice Modeling Group (2017-2021)

Senior Personnel: Ken Golden, Distinguished Professor of Mathematics
Elena Cherkaev, Professor of Mathematics
Court Strong, Associate Professor of Atmospheric Sciences
Ben Murphy, Adjunct Assistant Professor of Mathematics

Postdoctoral Researchers: Noa Kraitzman (now at ANU), Jody Reimer

Graduate Students: Kyle Steffen (now at UT Austin with Clint Dawson)
Christian Sampson (now at UNC Chapel Hill with Chris Jones)
Huy Dinh (now a sea ice MURI Postdoc at NYU/Courant)
Rebecca Hardenbrook
David Morison (Physics Department)
Ryleigh Moore
Delaney Mosier
Daniel Hallman

Undergraduate Students: Kenzie McLean, Jacqueline Cinella Rich,
Dane Gollero, Samir Suthar, Anna Hyde,
Kitsel Lusted, Ruby Bowers, Kimball Johnston,
Jerry Zhang, Nash Ward, David Gluckman

High School Students: Jeremiah Chapman, Titus Quah, Dylan Webb

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Sea Ice Ecology Group Postdoc Jody Reimer, Grad Student Julie Sherman,
Undergraduates Kayla Stewart, Nicole Forrester

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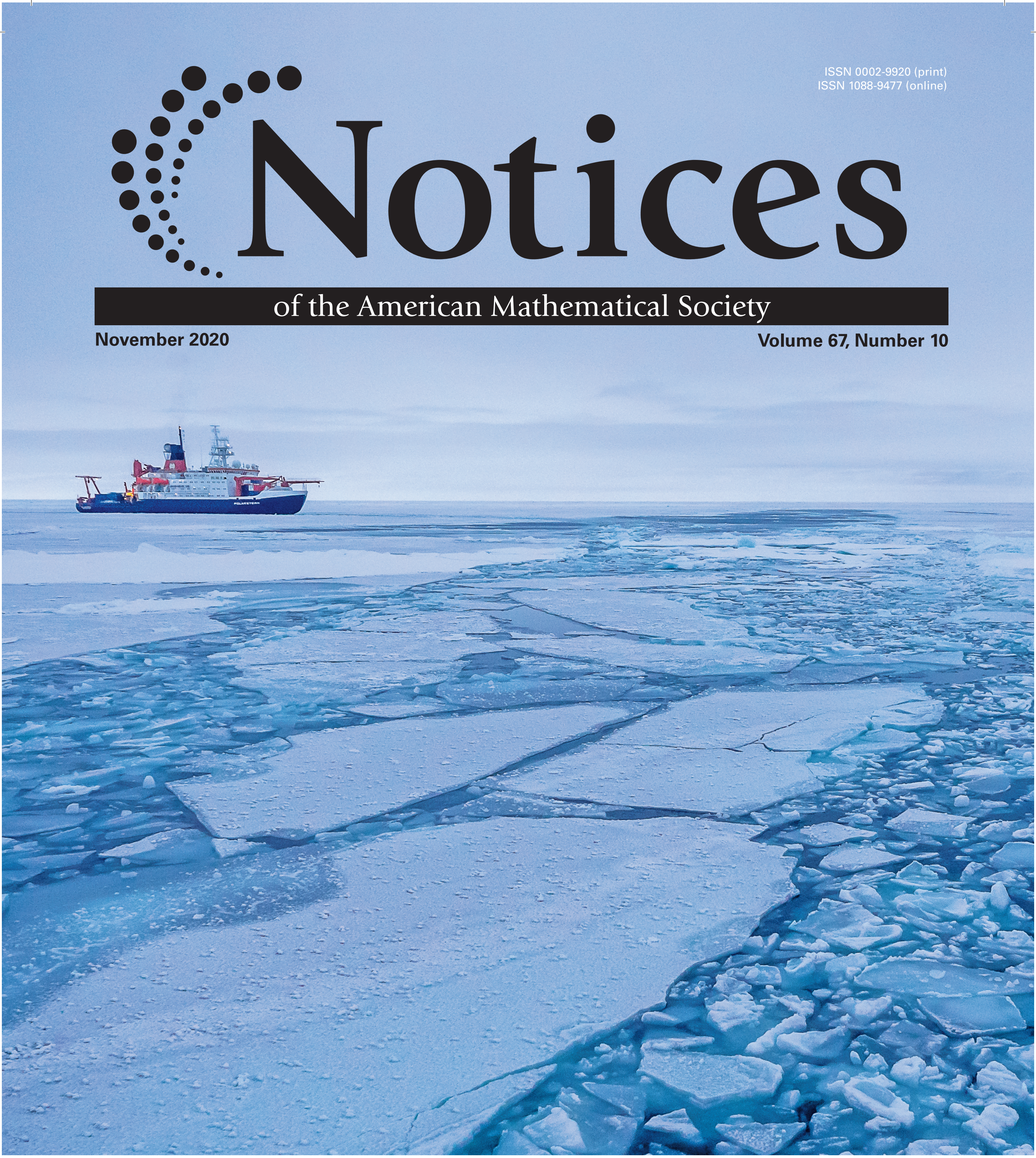

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The cover is based on "Modeling Sea Ice," page 1535.

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Buchanan Bay, Antarctica Mertz Glacier Polynya Experiment July 1999