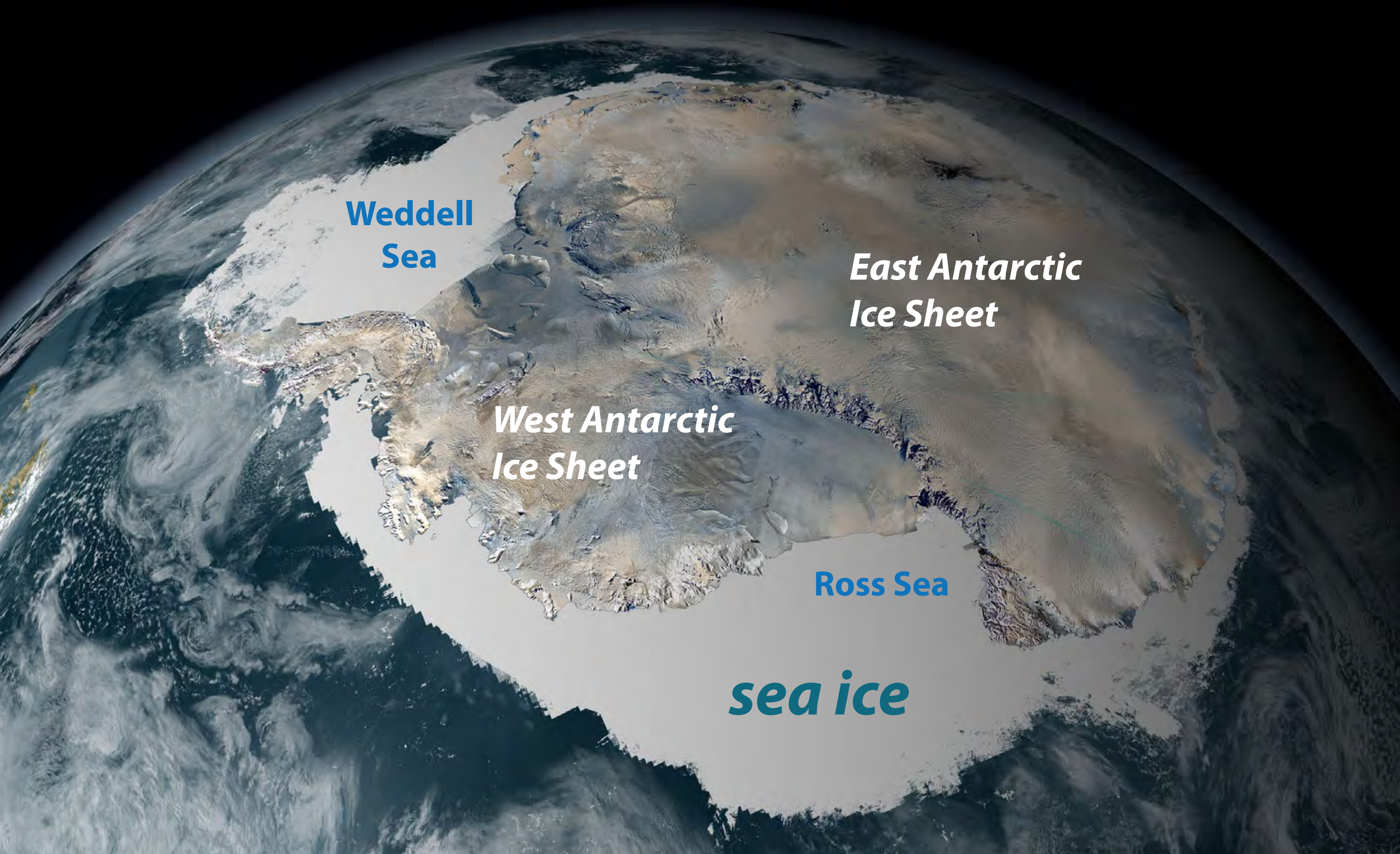
MODELING the MELT:
what math tells us about disappearing polar ice

Kenneth M. Golden
Department of Mathematics
University of Utah

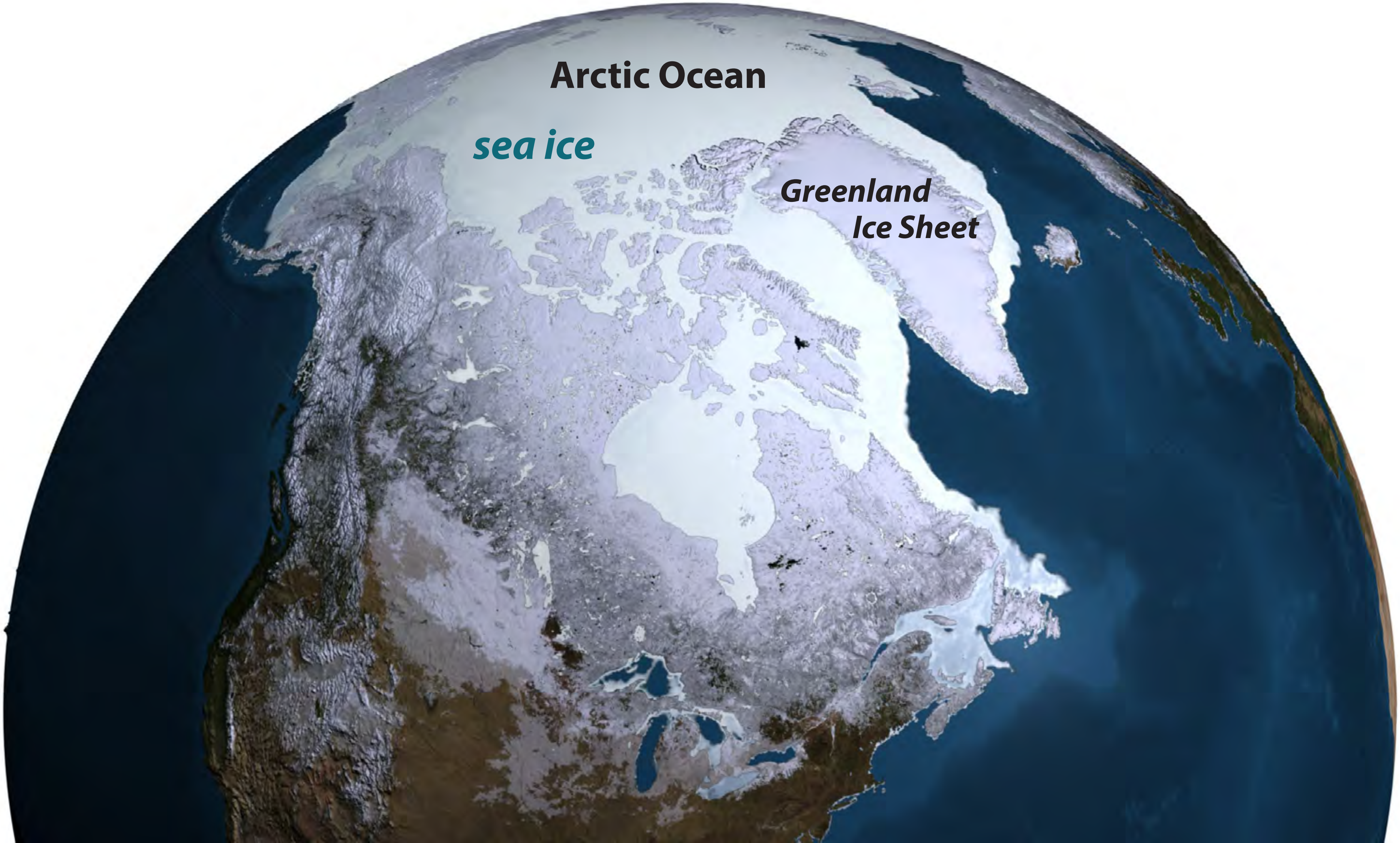
Frey

ANTARCTICA

southern cryosphere



THE ARCTIC
northern cryosphere

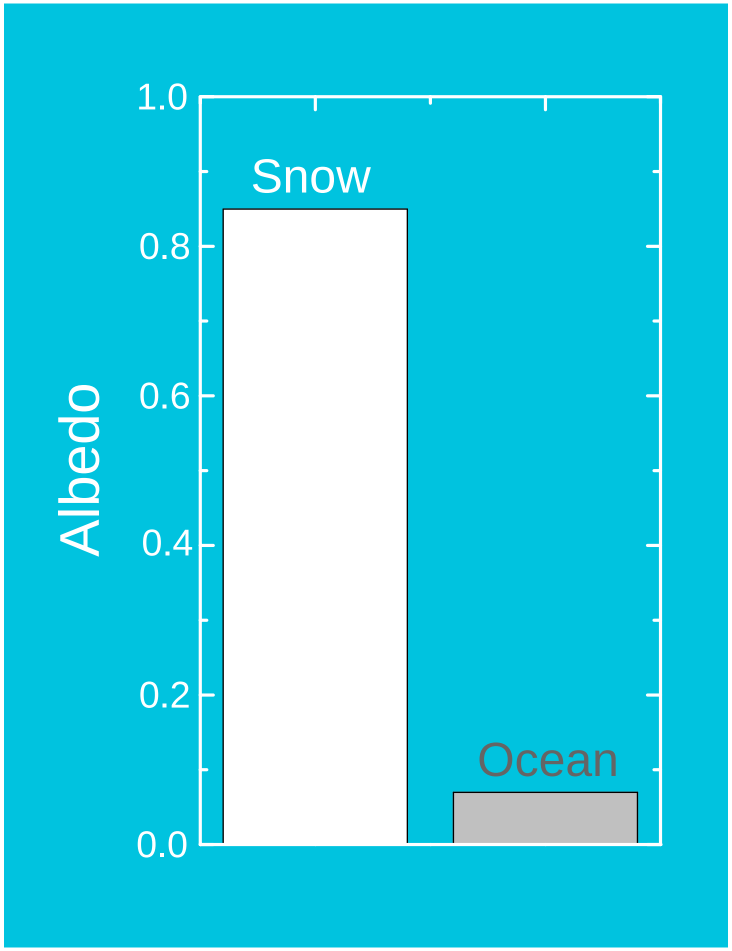


SEA ICE covers ~12% of Earth's ocean surface

- boundary between ocean and atmosphere
- mediates exchange of heat, gases, momentum
- global ocean circulation
- indicator and agent of climate change

polar ice caps critical to global climate in reflecting incoming solar radiation

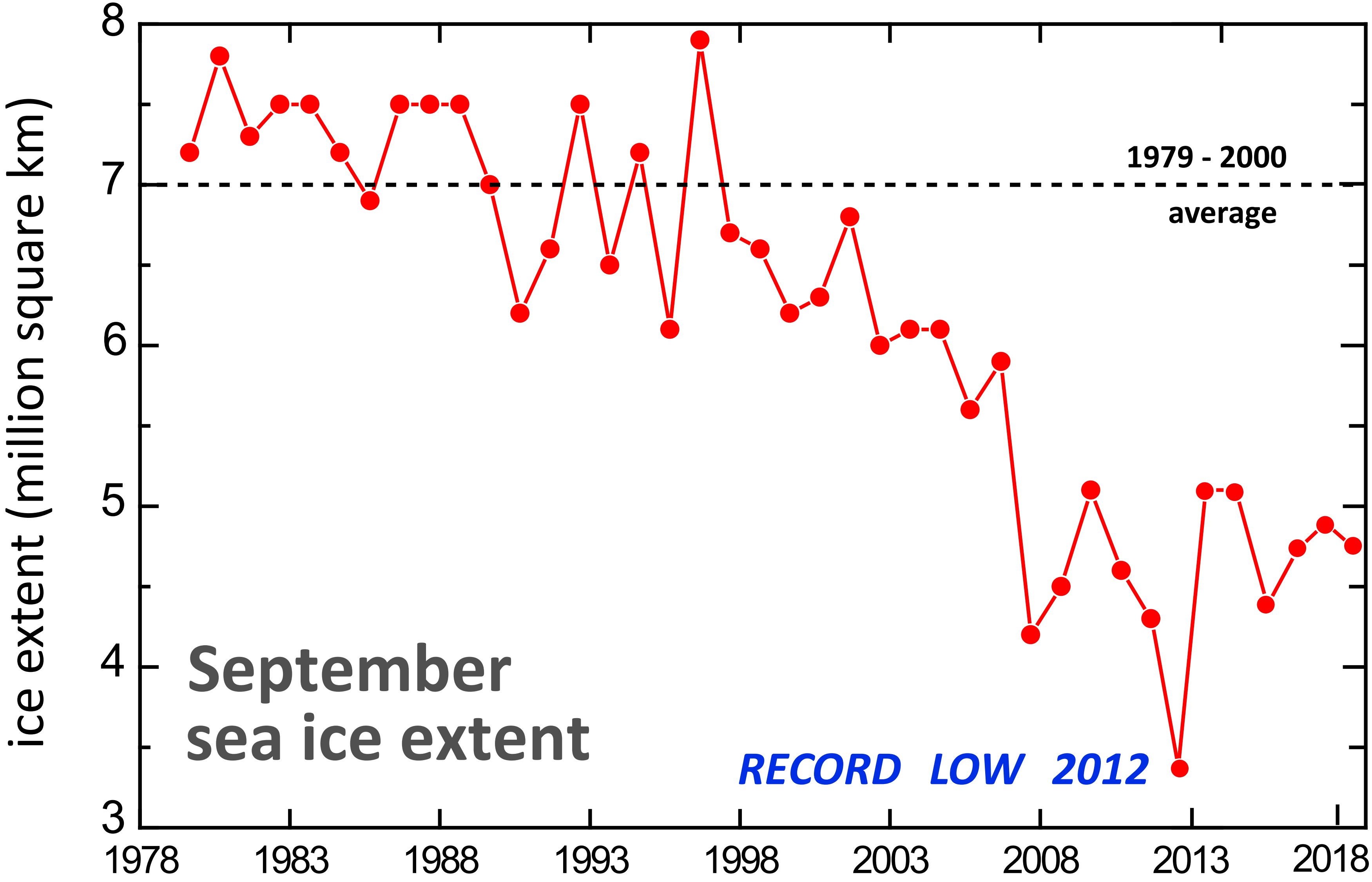
white snow and ice reflect



dark water and land absorb

$$\textbf{\textit{albedo}}\ \alpha = \frac{\text{reflected sunlight}}{\text{incident sunlight}}$$

the summer Arctic sea ice pack is melting

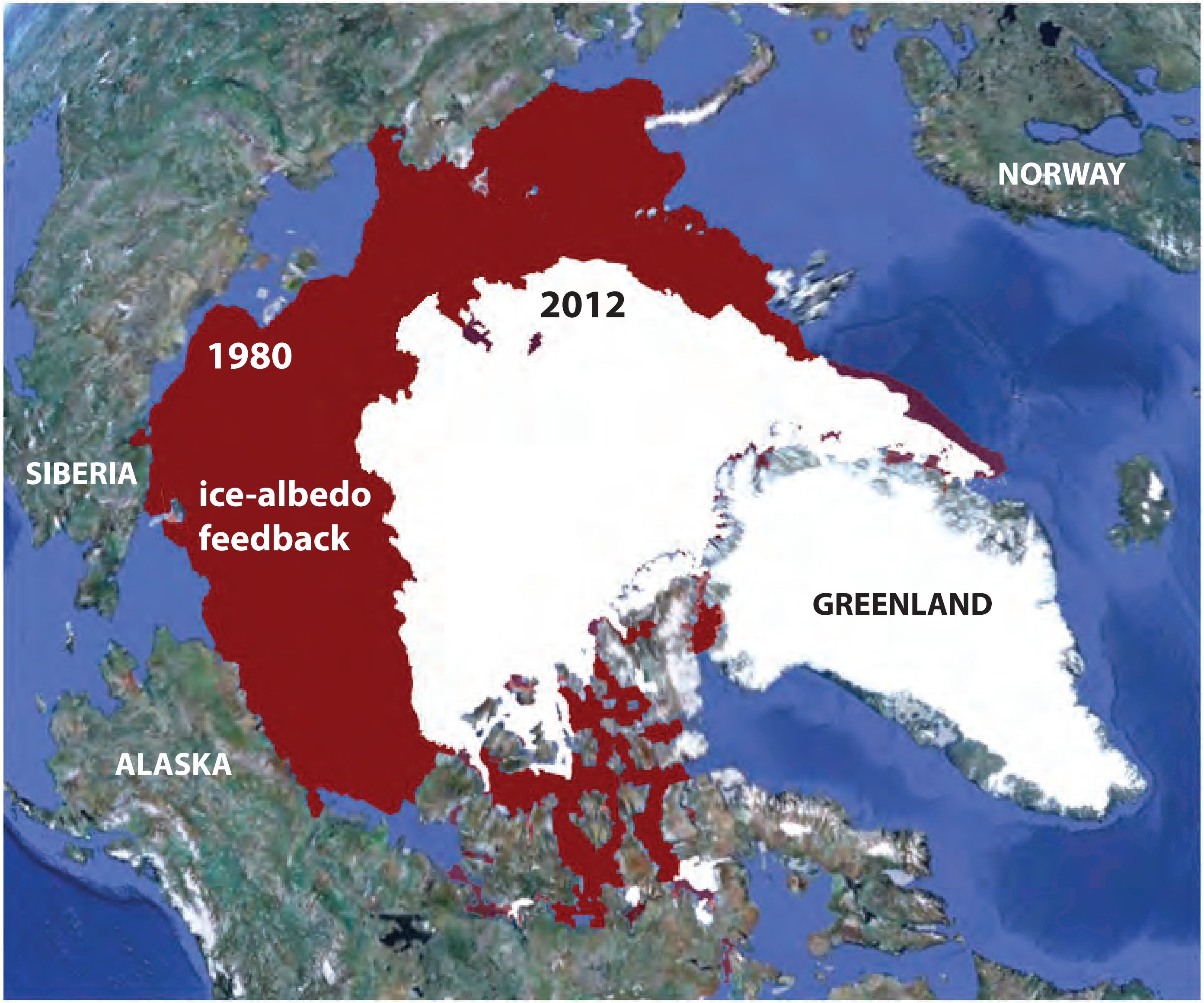


National Snow and Ice Data Center

Change in Arctic Sea Ice Extent

September 1980 -- 7.8 million square kilometers

September 2012 -- 3.4 million square kilometers



Perovich

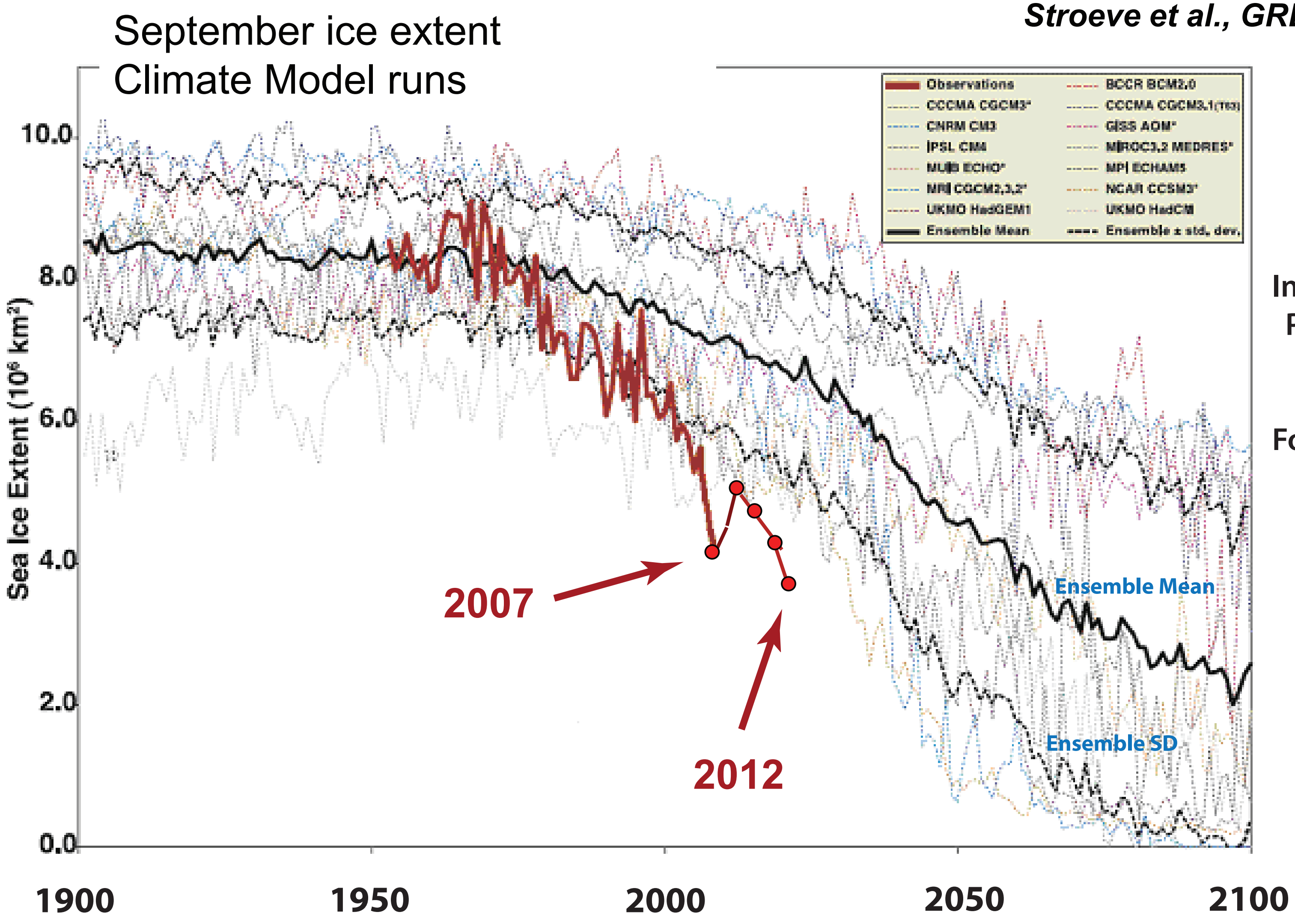
recent losses
in comparison to
the United States

Perovich

Arctic sea ice decline: faster than predicted by climate models

Stroeve et al., GRL, 2007

Stroeve et al., GRL, 2012



September ice extent
Climate Model runs

IPCC AR4 Models

Intergovernmental Panel on Climate Change (IPCC)

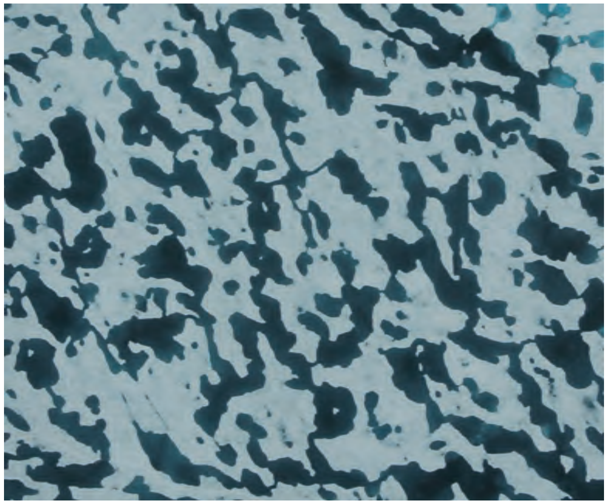
Fourth Assessment AR4, 2007

challenge

represent sea ice more rigorously in climate models

account for key processes

such as melt pond evolution



Impact of melt ponds on Arctic sea ice simulations from 1990 to 2007

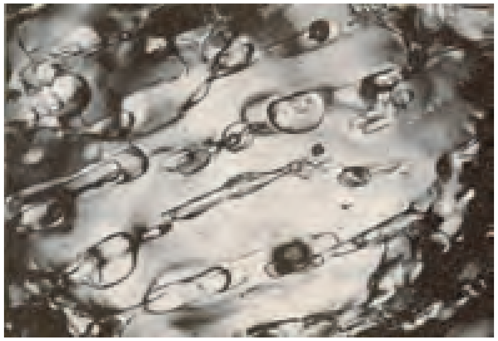
Flocco, Schroeder, Feltham, Hunke, *JGR Oceans* 2012

For simulations with ponds September ice volume is nearly 40% lower.

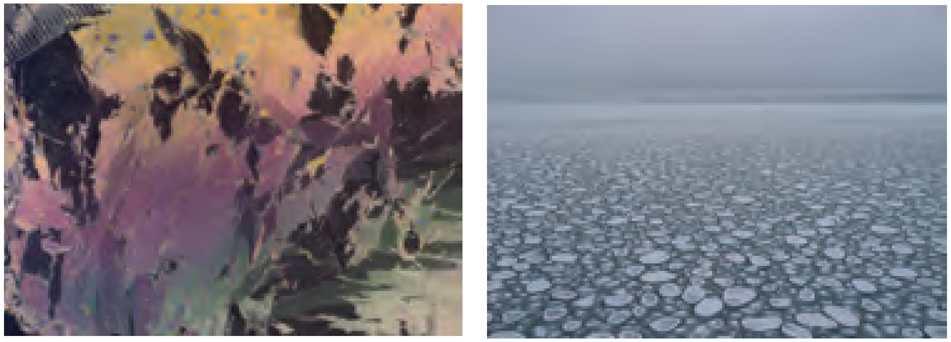
... and other sub-grid scale structures and processes

linkage of scales

sea ice is a multiscale composite

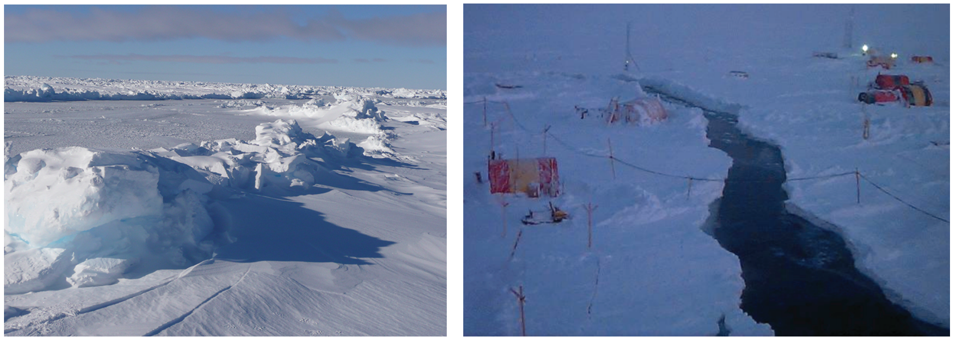


millimeters

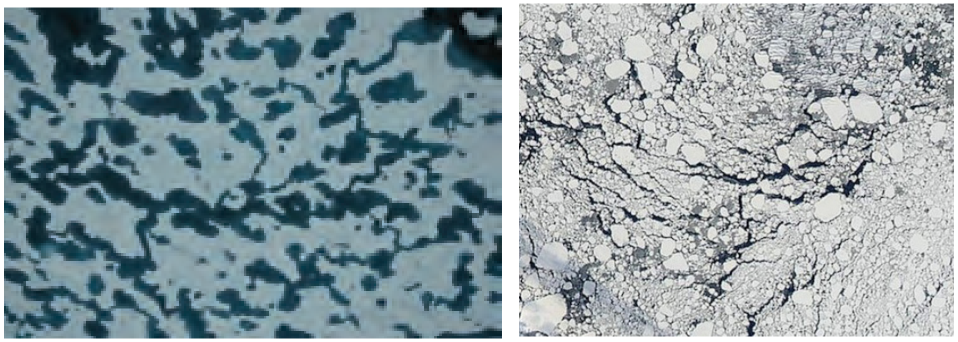


centimeters

meters



meters



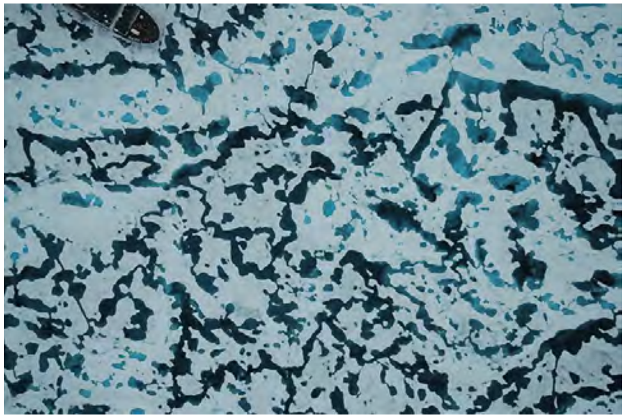
kilometers

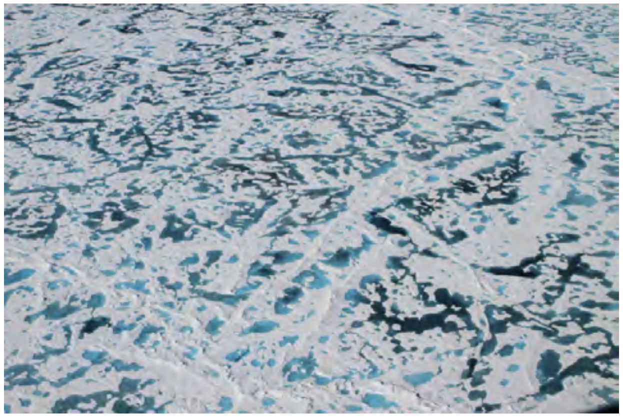
How do scales interact in the sea ice system?

basin scale -
grid scale
albedo

Linking Scales

km
scale
melt
ponds





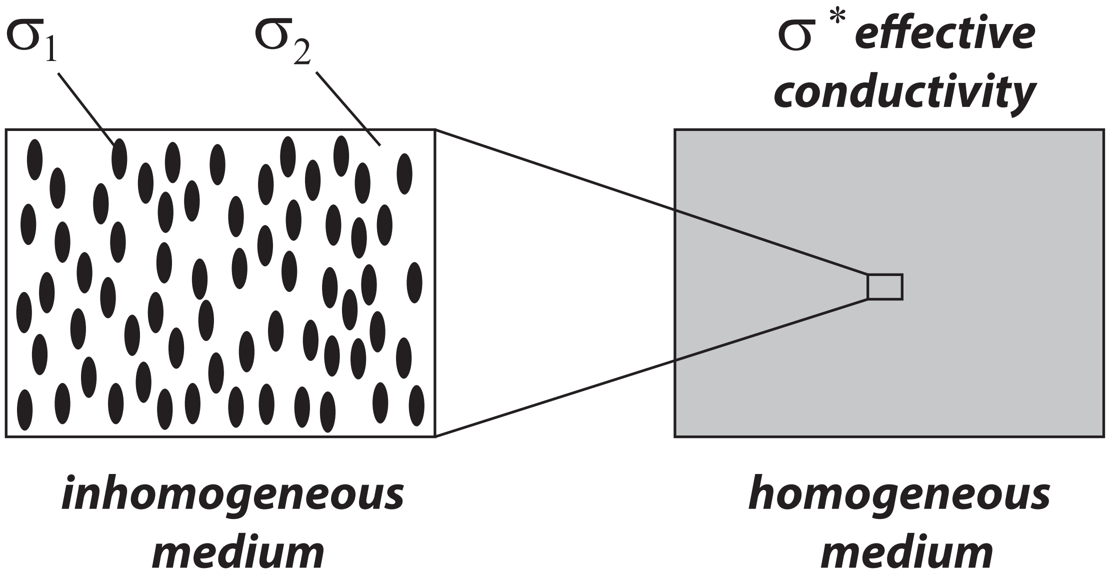
km
scale
melt
ponds

Linking Scales

mm
scale
brine
inclusions

meter
scale
snow
topography

HOMOGENIZATION - Linking Scales in Composites



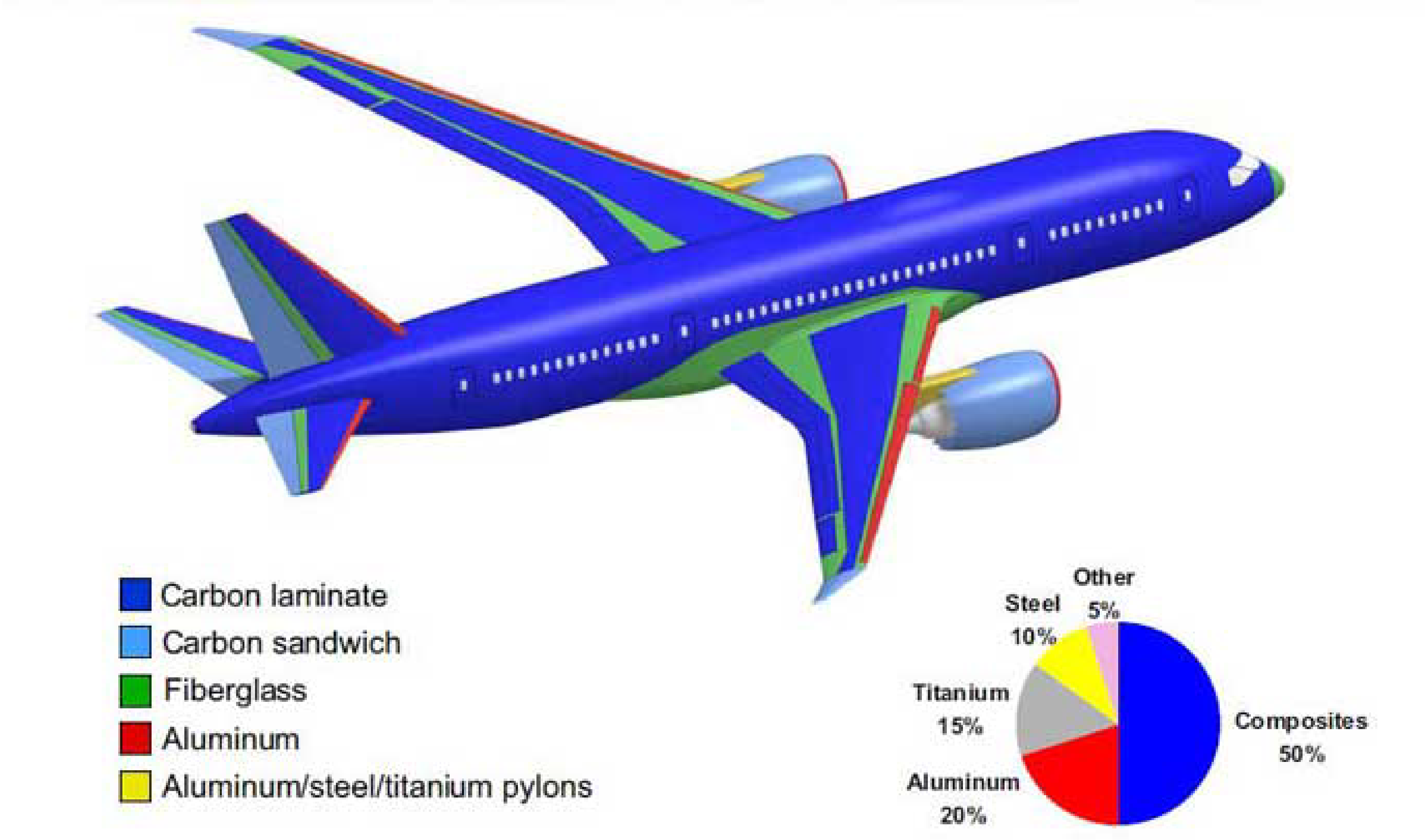
find the homogeneous medium which behaves macroscopically the same as the inhomogeneous medium

Maxwell 1873 : effective conductivity of a dilute suspension of spheres
Einstein 1906 : effective viscosity of a dilute suspension of rigid spheres in a fluid

*Wiener 1912 : arithmetic and harmonic mean **bounds** on effective conductivity*
*Hashin and Shtrikman 1962 : variational **bounds** on effective conductivity*

widespread use of composites in late 20th century due in large part to advances in mathematically predicting their effective properties

Composite materials in the Boeing 787 Dreamliner



sea ice microphysics

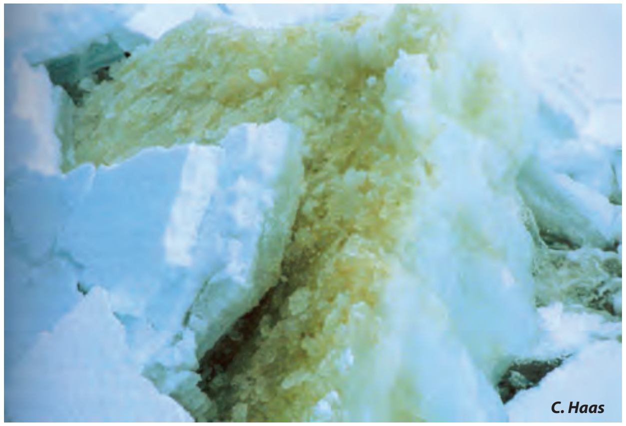
fluid transport

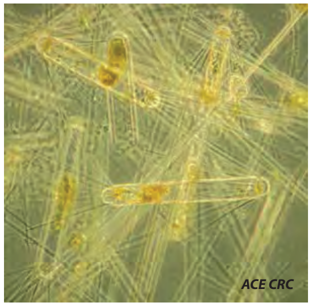
fluid flow through the porous microstructure of sea ice governs key processes in polar climate and ecosystems

evolution of Arctic melt ponds and sea ice albedo

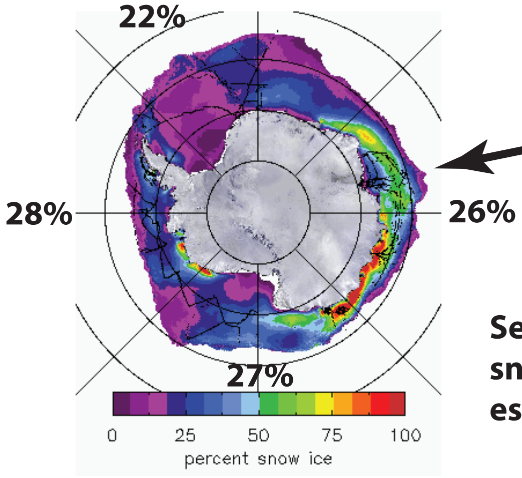

K. Golden

nutrient flux for algal communities


C. Haas


ACE CRC


C. Krembs



T. Maksym and T. Markus, 2008

September snow-ice estimates

Antarctic surface flooding and snow-ice formation

- *evolution of salinity profiles*
- *ocean-ice-air exchanges of heat, CO_2*

sea ice ecosystem

sea ice algae
support life in the polar oceans

fluid permeability *k* of a porous medium

porous
concrete

how much water gets through the sample per unit time?

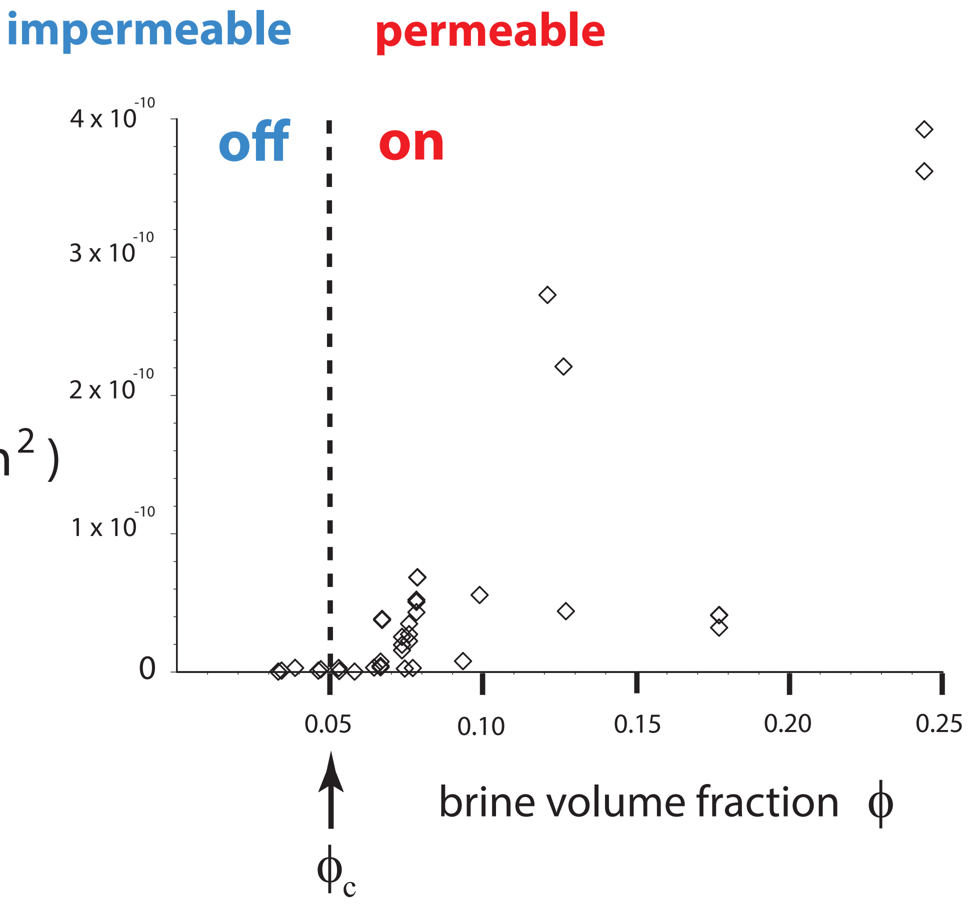
HOMOGENIZATION

mathematics for analyzing effective behavior of heterogeneous systems

Critical behavior of fluid transport in sea ice

Arctic field data

vertical fluid permeability k (m^2)



"on - off" switch for fluid flow

critical brine volume fraction $\phi_c \approx 5\%$ $\longleftrightarrow$ $T_c \approx -5^\circ\mathrm{C}$, $S \approx 5$ ppt

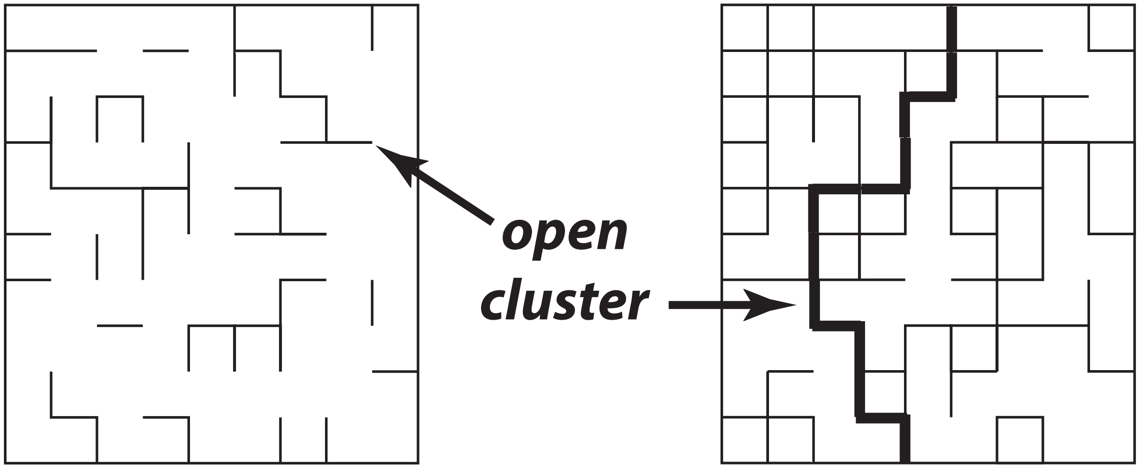
RULE OF FIVES

Golden, Ackley, Lytle Science 1998
Golden, Eicken, Heaton, Miner, Pringle, Zhu GRL 2007
Pringle, Miner, Eicken, Golden J. Geophys. Res. 2009

Why is the rule of fives true?

percolation theory

probabilistic theory of connectedness



impermeable

permeable

open cluster

$p = 1/3$

$p = 2/3$

bond → ***open*** *with probability* p
closed *with probability* $1\text{-}p$

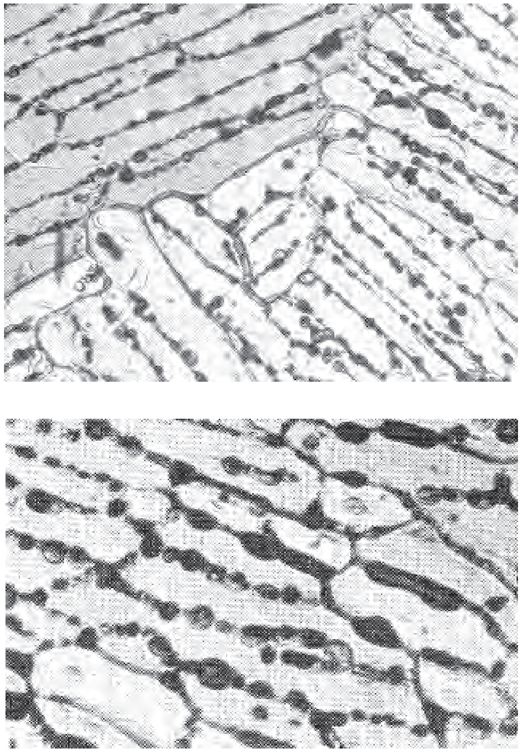
percolation threshold

$$p_c = 1/2 \quad \text{for} \quad d = 2$$

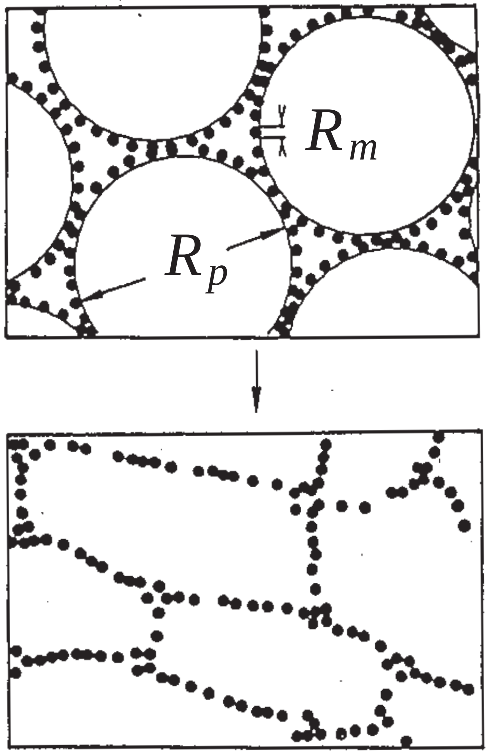
smallest p for which there is an infinite open cluster

Continuum percolation model for ***stealthy*** materials applied to sea ice microstructure explains **Rule of Fives** and Antarctic data on ice production and algal growth

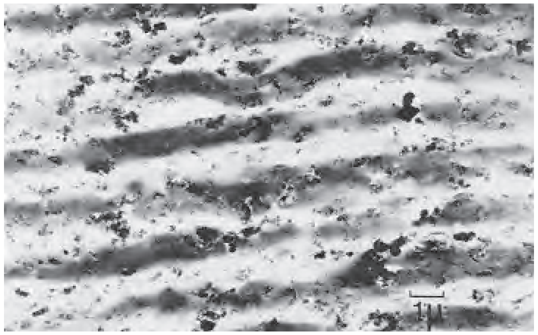
$\phi_c \approx 5\,\%$ Golden, Ackley, Lytle, *Science*, 1998



sea ice



compressed powder

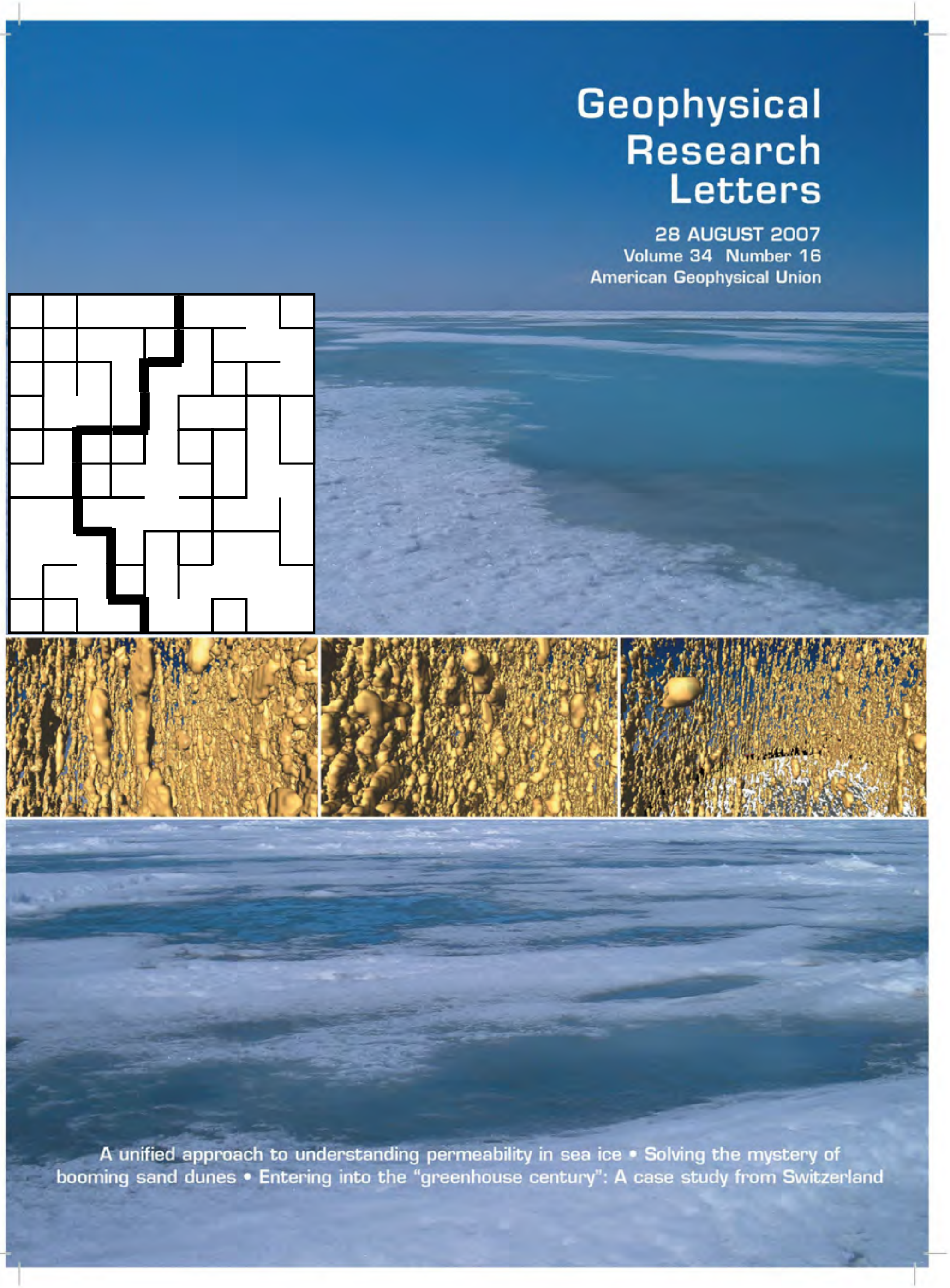


radar absorbing composite

sea ice is radar absorbing

Thermal evolution of permeability and microstructure in sea ice

Golden, Eicken, Heaton, Miner, Pringle, Zhu, Geophysical Research Letters 2007



Geophysical
Research
Letters

28 AUGUST 2007
Volume 34 Number 16
American Geophysical Union

A unified approach to understanding permeability in sea ice • Solving the mystery of booming sand dunes • Entering into the "greenhouse century": A case study from Switzerland

micro-scale

controls

macro-scale

processes

percolation theory

$$k(\phi) = k_0(\phi - 0.05)^2 \longleftarrow$$

critical exponent
t

$$k_0 = 3 \text{ x } 10^{-8} \text{ m}^2$$

hierarchical model

network model

rigorous bounds

agree closely
with field data

X-ray tomography for
brine inclusions

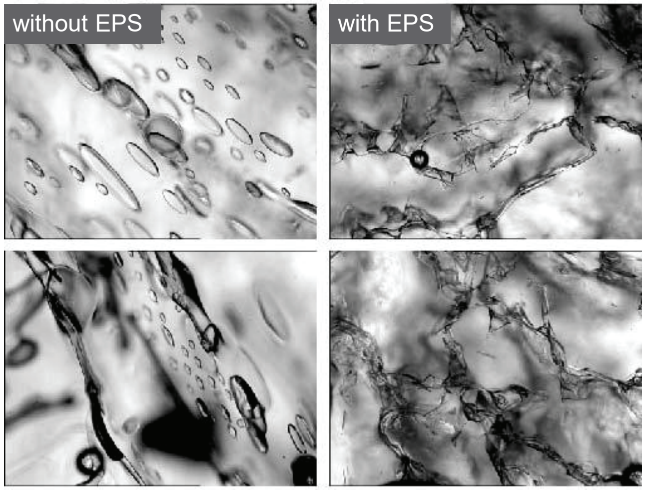
unprecedented look
at thermal evolution
of brine phase and
its connectivity

confirms rule of fives

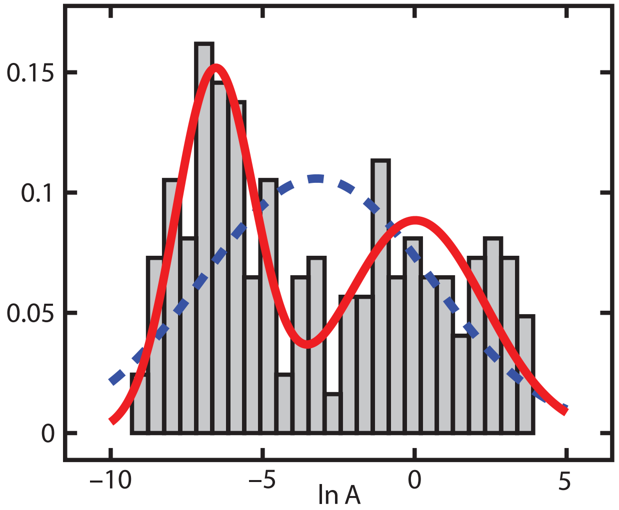
Pringle, Miner, Eicken, Golden
J. Geophys. Res. 2009

Sea ice algae secrete extracellular polymeric substances (EPS) affecting evolution of brine microstructure.

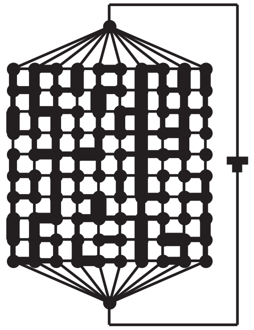
How does EPS affect fluid transport?

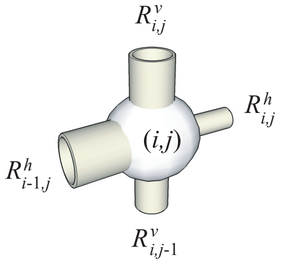


Krembs, Eicken, Deming, PNAS 2011



RANDOM PIPE MODEL





Zhu, Jabini, Golden,
Eicken, Morris
***Ann. Glac.* 2006**

- **Bimodal** lognormal distribution for brine inclusions
- Develop random pipe network model with bimodal distribution; Use numerical methods that can handle larger variances in sizes.
- Results predict observed drop in fluid permeability *k*.
- **Rigorous bound on *k* for bimodal distribution of pore sizes**

Steffen, Epshteyn, Zhu, Bowler, Deming, Golden
***Multiscale Modeling and Simulation*, 2018**

How does the biology affect the physics?

Remote sensing of sea ice

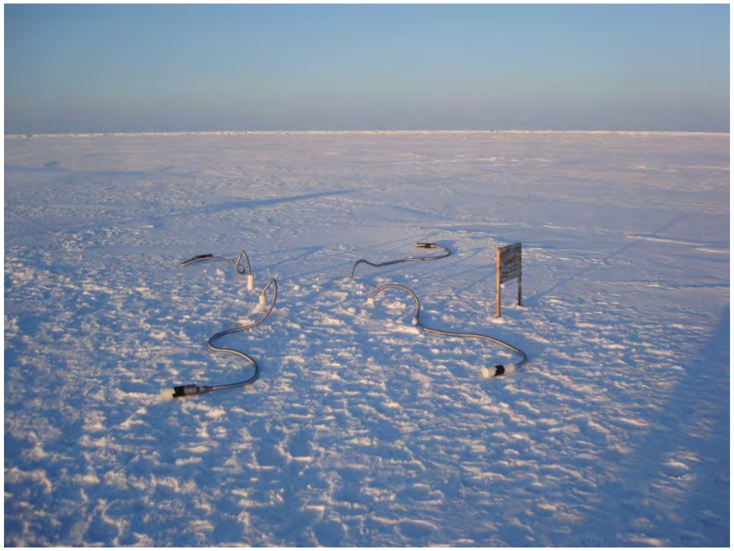
sea ice thickness
ice concentration

INVERSE PROBLEM

Recover sea ice
properties from
electromagnetic
(EM) data

ε^*

effective complex permittivity
(dielectric constant, conductivity)



brine volume fraction
brine inclusion connectivity

Stieltjes integral representation

separates geometry from parameters

geometry

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{d\mu(z)}{s - z}$$

material parameters

μ

- spectral measure of self adjoint operator $\Gamma\chi$
- mass = p_1
- higher moments depend on n-point correlations

$$\Gamma = \nabla(-\Delta)^{-1}\nabla\cdot$$

$\chi =$ characteristic function of the brine phase

$$E = (s + \Gamma\chi)^{-1} e_k$$

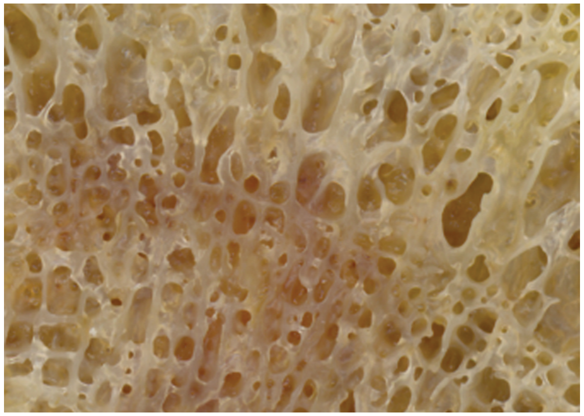
$\Gamma\chi$ **: microscale ⟶ macroscale**

$\Gamma\chi$ ***links scales***

Golden and Papanicolaou, Comm. Math. Phys. 1983

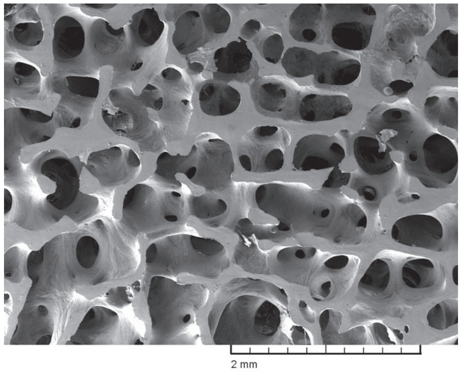
SEA ICE

HUMAN BONE

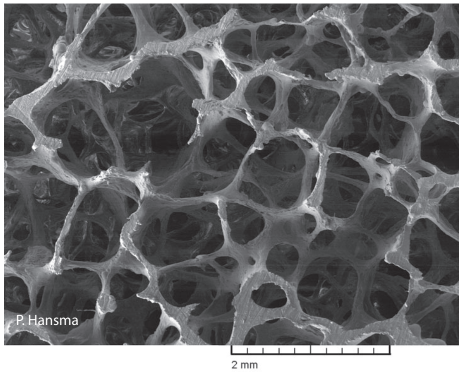


spectral characterization of porous microstructures in human bone

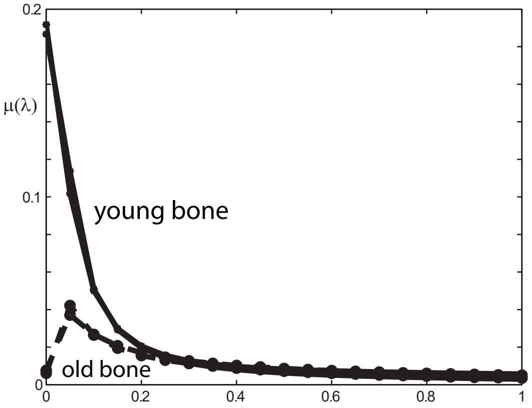
young healthy trabecular bone



old osteoporotic trabecular bone



reconstruct spectral measures from complex permittivity data



use regularized inversion scheme

apply spectral measure analysis of brine connectivity and spectral inversion to electromagnetic monitoring of osteoporosis

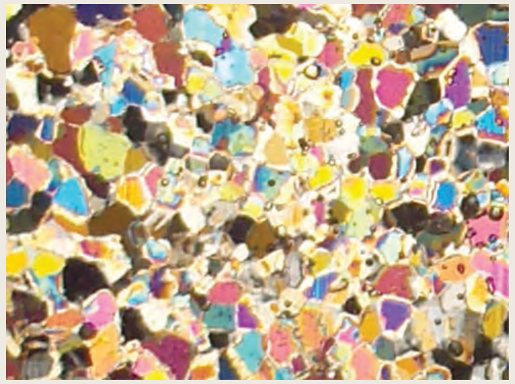
Golden, Murphy, Cherkaev, J. Biomechanics 2011

the math doesn't care if it's sea ice or bone!

Bounds on the complex permittivity of polycrystalline materials by analytic continuation

Adam Gully, Joyce Lin,
Elena Cherkaev, Ken Golden

- **Stieltjes integral representation for effective complex permittivity**
  Milton (1981, 2002), Barabash and Stroud (1999), ...
- **Forward and inverse bounds**
- **Applied to sea ice using two-scale homogenization**
- **Inverse bounds give method for distinguishing ice types using remote sensing techniques**



Proc. Roy. Soc. A 8 Feb 2015

Proc. R. Soc. A | Volume 471 | Issue 2174 | 8 February 2015

ISSN 1364-5021 | Volume 471 | Issue 2174 | 8 February 2015

PROCEEDINGS A

350 YEARS
OF SCIENTIFIC
PUBLISHING

An invited review commemorating 350 years of scientific publishing at the Royal Society

A method to distinguish between different types of sea ice using remote sensing techniques

A computer model to determine how a human should walk so as to expend the least energy

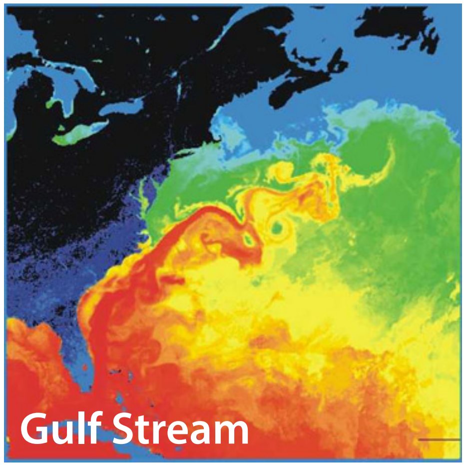
THE
ROYAL
SOCIETY
PUBLISHING

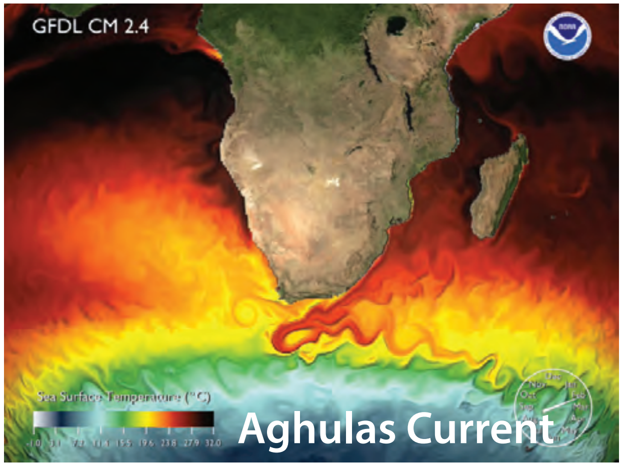
21/01/15 1:47 PM

advection enhanced diffusion

effective diffusivity

sea ice floes diffusing in ocean currents
diffusion of pollutants in atmosphere
salt and heat transport in ocean
heat transport in sea ice with convection





advection diffusion equation with a velocity field $\vec{u}$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \vec{\nabla} T = \kappa_0 \Delta T$$

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$\Big\downarrow \quad \textit{homogenize}$$

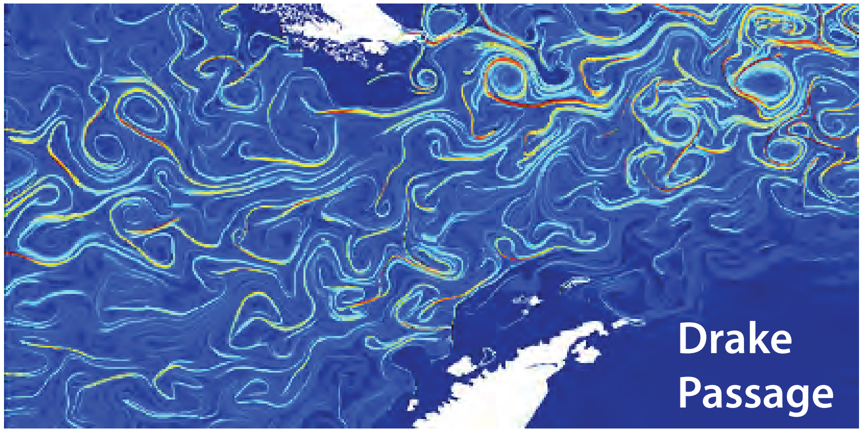
$$\frac{\partial \overline{T}}{\partial t} = \kappa^* \Delta \overline{T}$$

κ^* ***effective diffusivity***

Stieltjes integral for κ^* with spectral measure

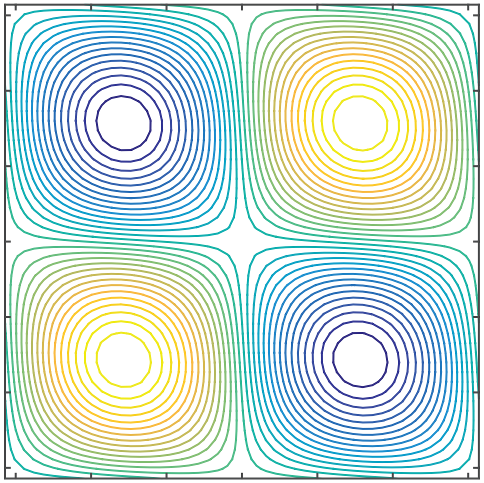
Avellaneda and Majda, PRL 89, CMP 91

Murphy, Cherkaev, Xin, Zhu, Golden, *Ann. Math. Sci. Appl.* 2017
Murphy, Cherkaev, Zhu, Xin, Golden, *J. Math Phys.* 2018

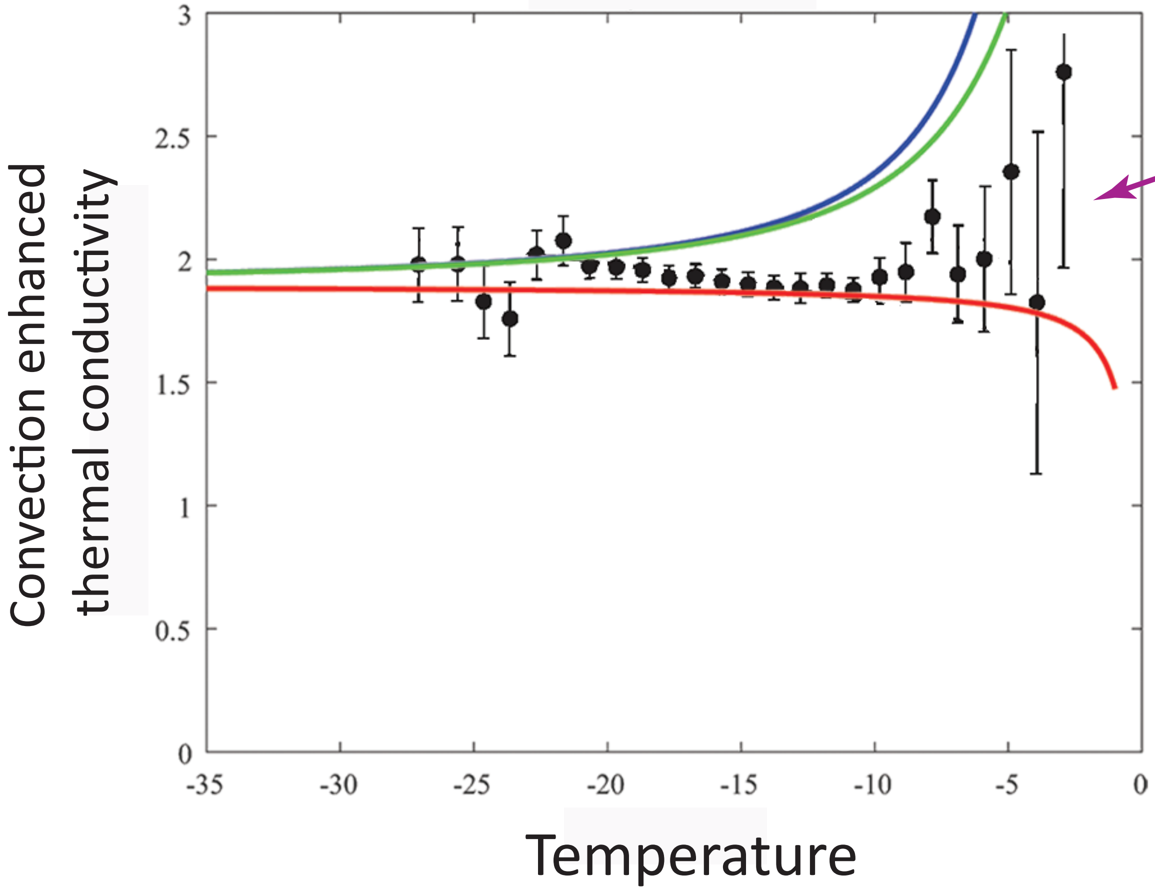


Rigorous bounds on convection enhanced thermal conductivity of sea ice

Kraitzman, Hardenbrook, Dinh, Murphy, Zhu, Cherkaev, Golden 2018



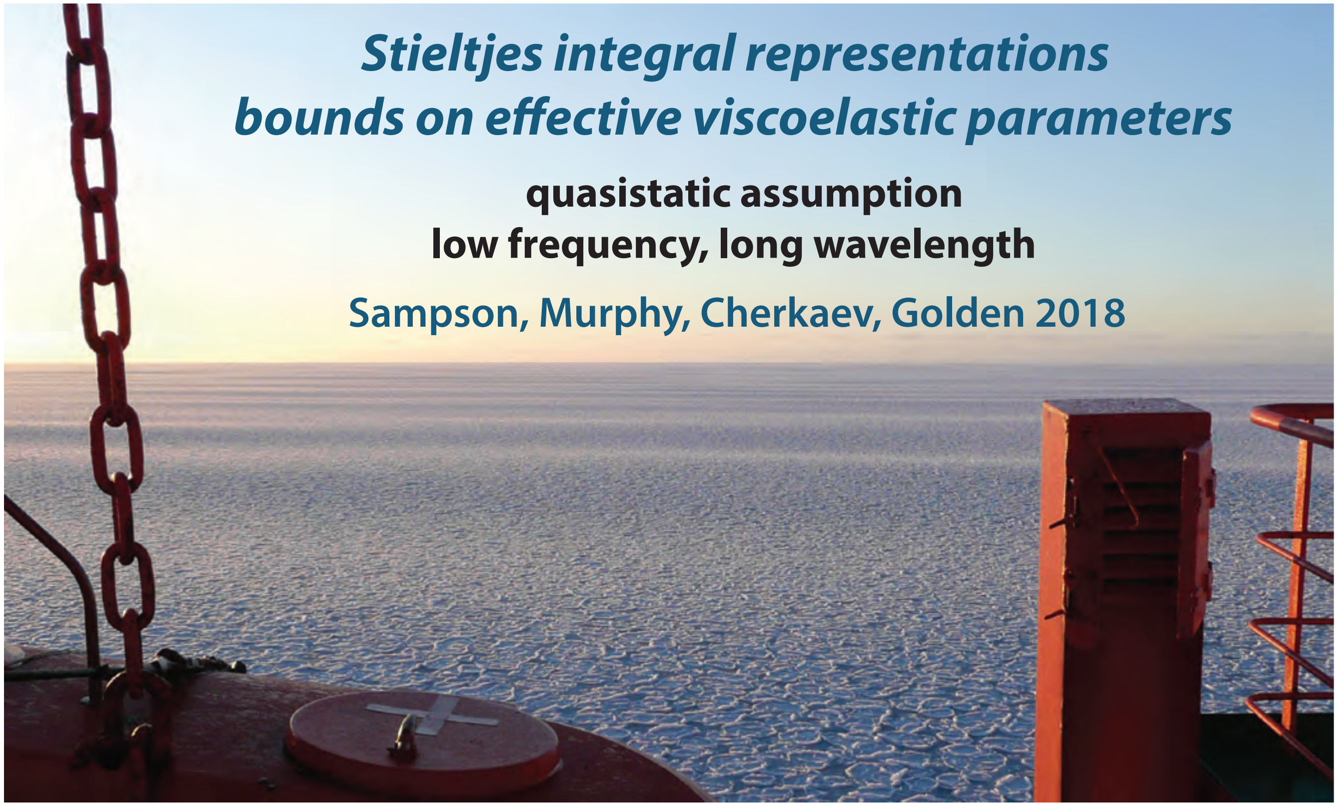
cat’s eye flow model for
brine convection cells



data
Trodahl
et al. 2001

rigorous Padé bounds from Stieltjes integral +
analytical calculations of moments of measure

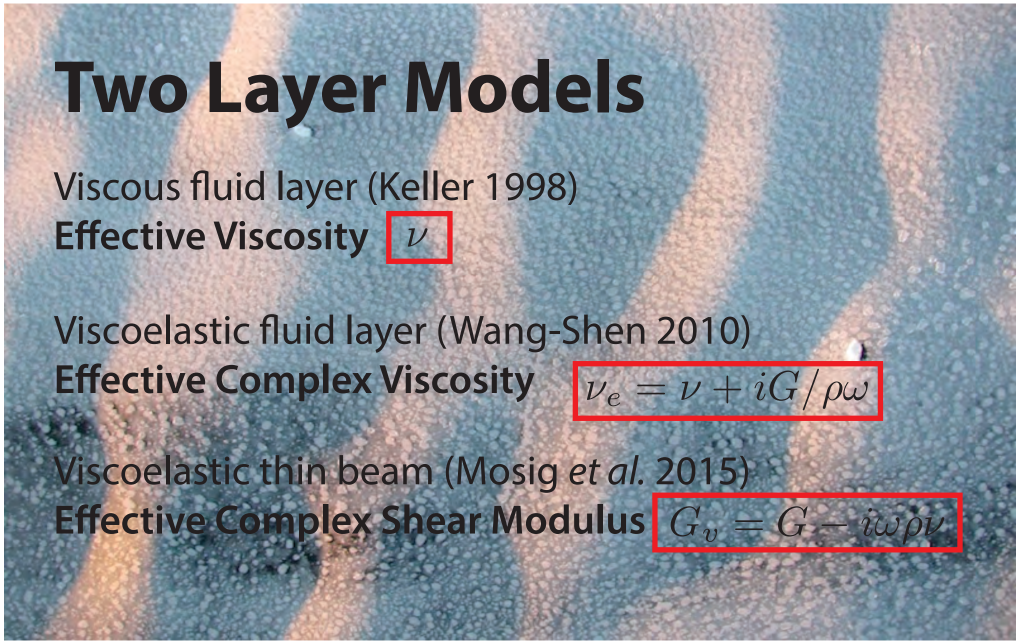
wave propagation in the marginal ice zone



Stieltjes integral representations
bounds on effective viscoelastic parameters

quasistatic assumption
low frequency, long wavelength

Sampson, Murphy, Cherkaev, Golden 2018



Two Layer Models

Viscous fluid layer (Keller 1998)
Effective Viscosity $\boxed{\nu}$

Viscoelastic fluid layer (Wang-Shen 2010)
Effective Complex Viscosity $\boxed{\nu_e = \nu + iG/\rho\omega}$

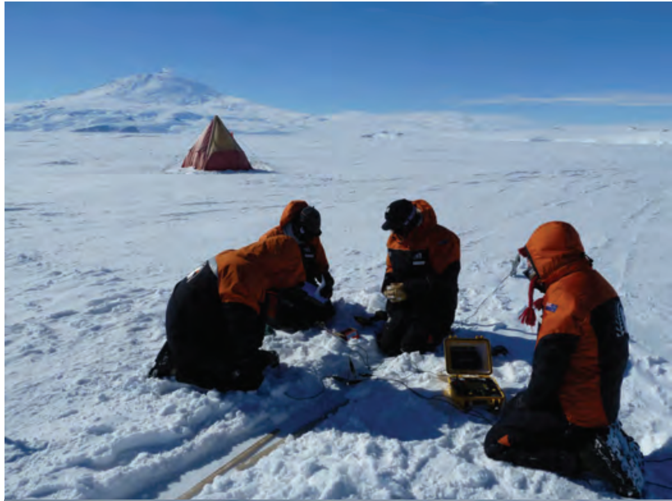
Viscoelastic thin beam (Mosig *et al.* 2015)
Effective Complex Shear Modulus $\boxed{G_v = G - i\omega\rho\nu}$

Arctic and Antarctic field experiments

develop electromagnetic methods of monitoring fluid transport and microstructural transitions

extensive measurements of fluid and electrical transport properties of sea ice:

2007	*Antarctic*	*SIPEX*
2010	*Antarctic*	*McMurdo Sound*
2011	*Arctic*	*Barrow AK*
2012	*Arctic*	*Barrow AK*
2012	*Antarctic*	*SIPEX II*
2013	*Arctic*	*Barrow AK*
2014	*Arctic*	*Chukchi Sea*



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Notices

of the American Mathematical Society

May 2009 Volume 56, Number 5

Climate Change and the Mathematics of Transport in Sea Ice

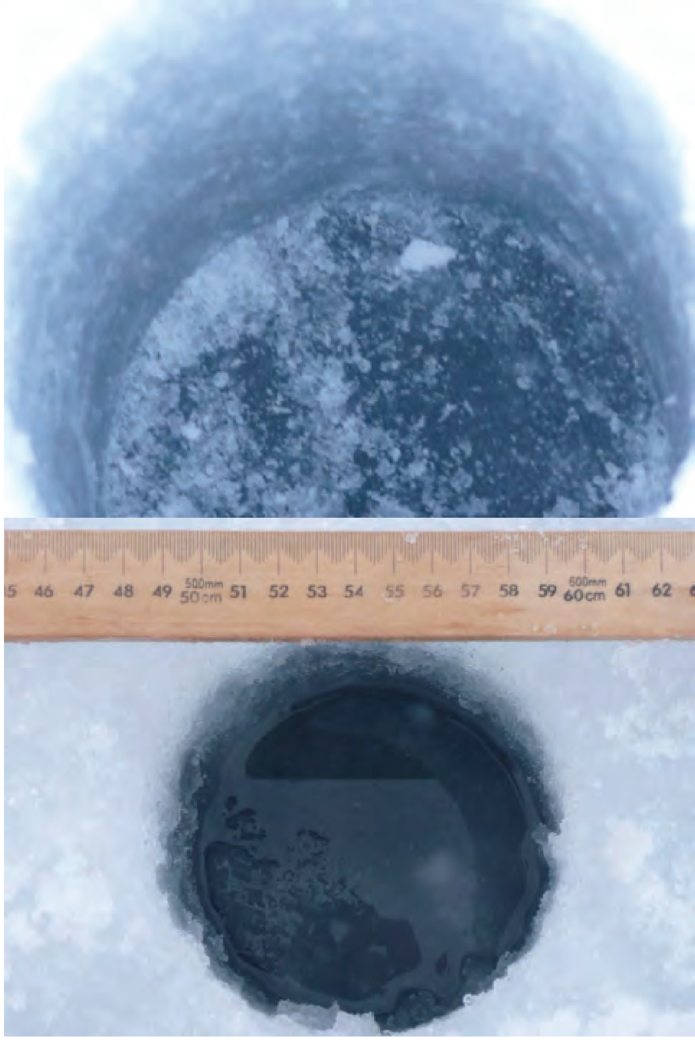
page 562

Mathematics and the Internet: A Source of Enormous Confusion and Great Potential

page 586

Real analysis in polar coordinates (see page 613)

photo by Jan Lieser

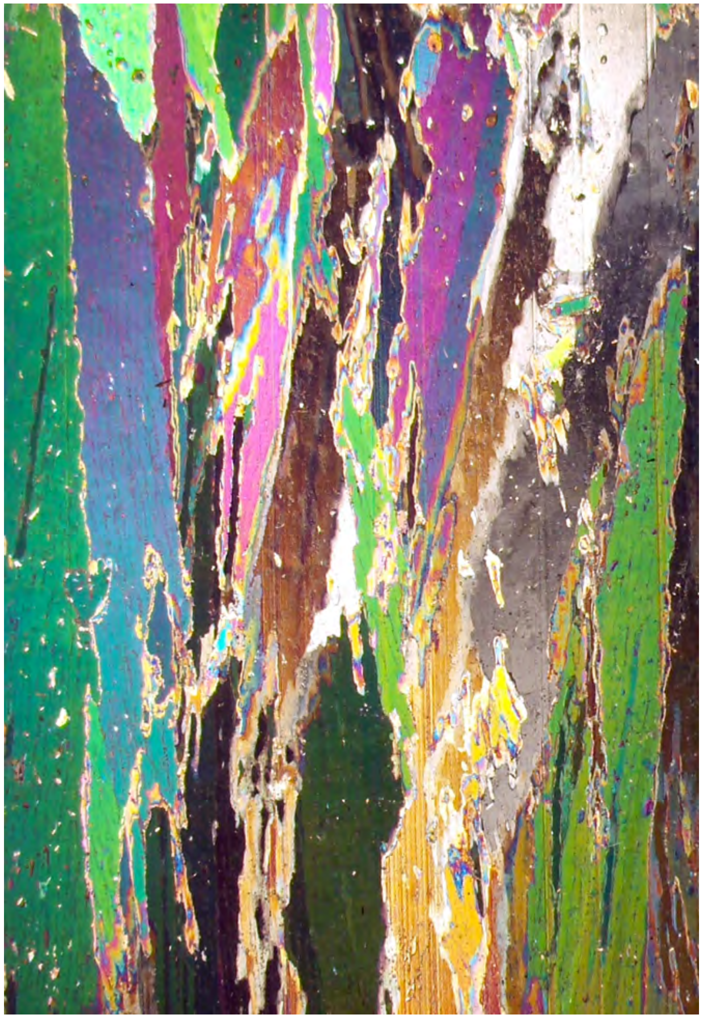


measuring fluid permeability of Antarctic sea ice

SIPEX 2007

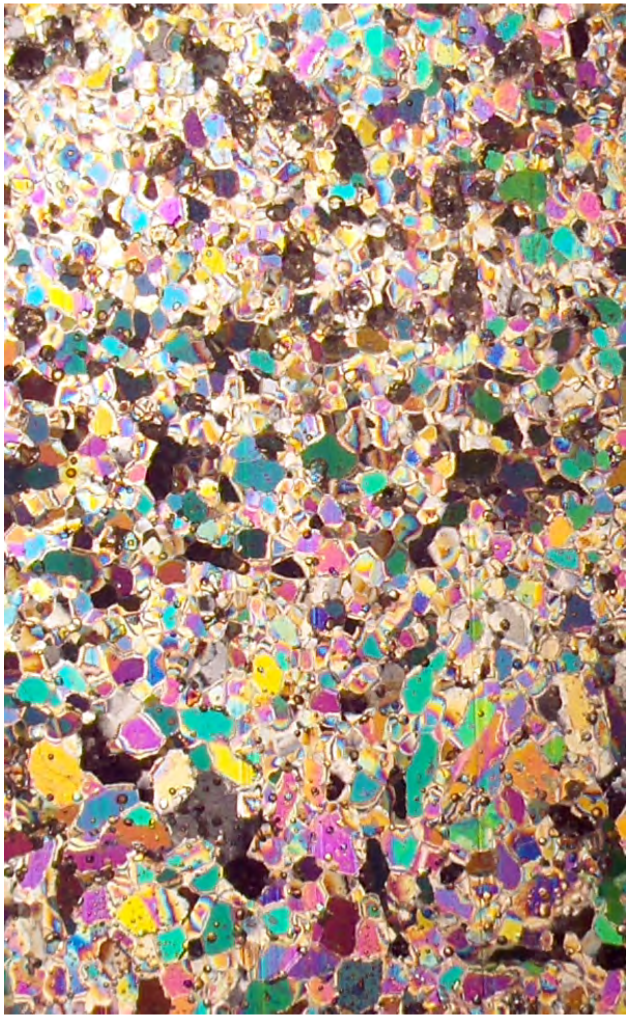
higher threshold for fluid flow in Antarctic granular sea ice

columnar



5%

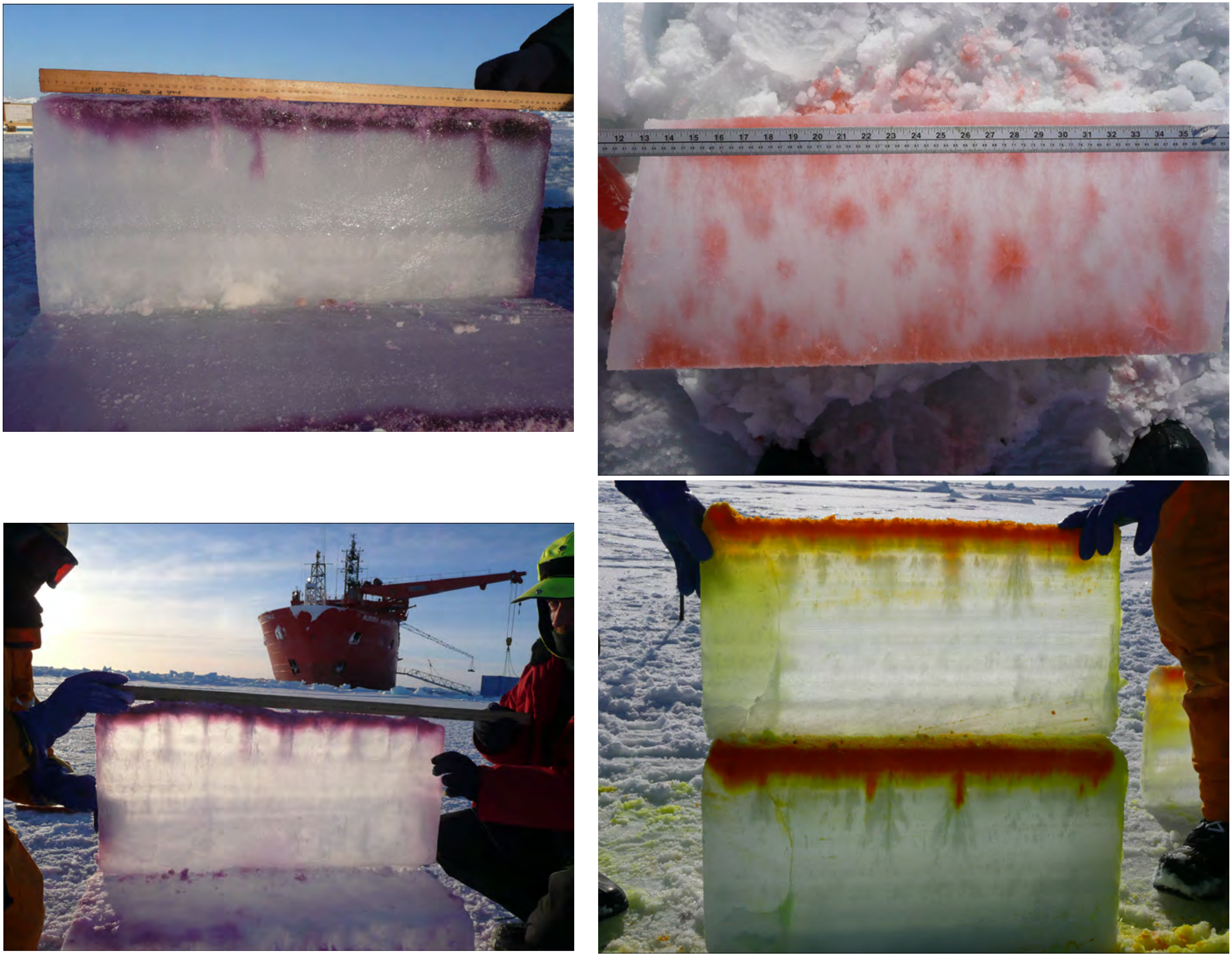
granular



10%

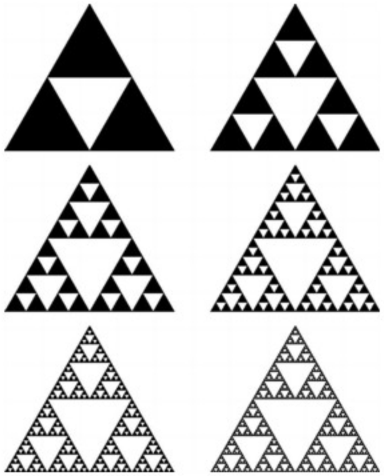
Golden, Sampson, Gully, Lubbers, Tison 2018

tracers flowing through inverted sea ice blocks



fractals and multiscale structure

© 2004 Douglas R. Cootey www.cootey.com/fractals



melt pond formation and albedo evolution:

- ***major drivers in polar climate***
- ***key challenge for global climate models***

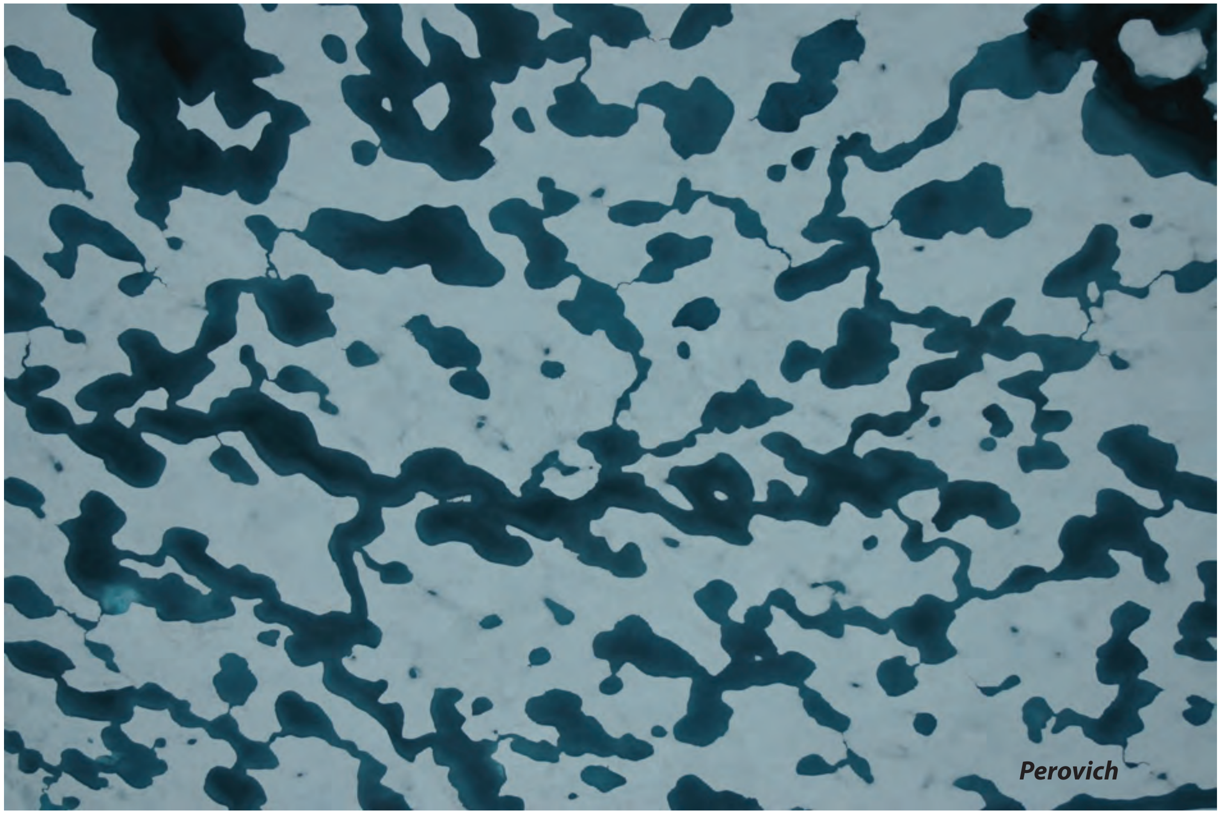
numerical models of melt pond evolution, including topography, drainage (permeability), etc.

Lüthje, Feltham, Taylor, Worster 2006

Flocco, Feltham 2007

Skyllingstad, Paulson, Perovich 2009

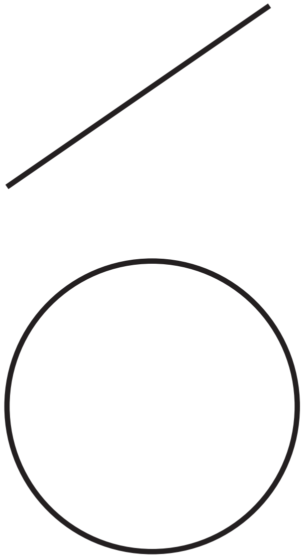
Flocco, Feltham, Hunke 2012



Are there universal features of the evolution similar to phase transitions in statistical physics?

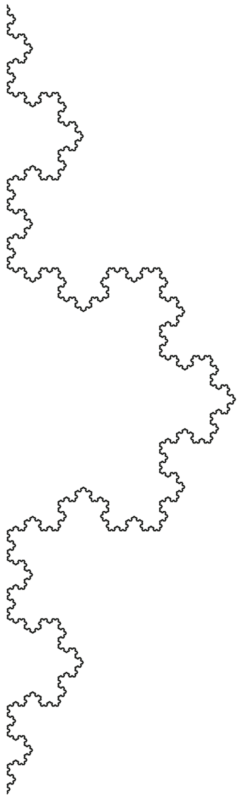
fractal curves in the plane

they wiggle so much that their dimension is >1



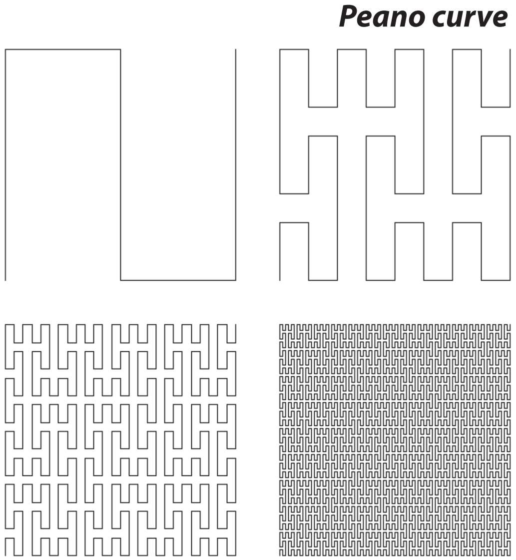
simple curves

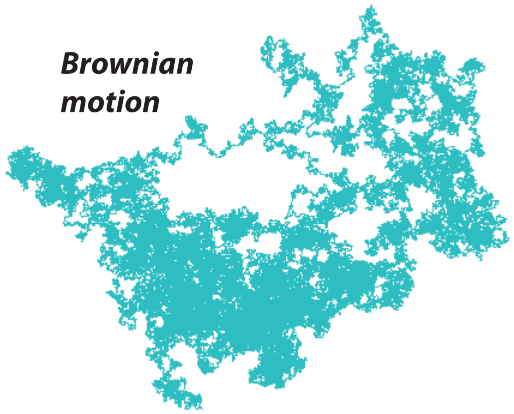
$D = 1$



Koch snowflake

$D = 1.26$

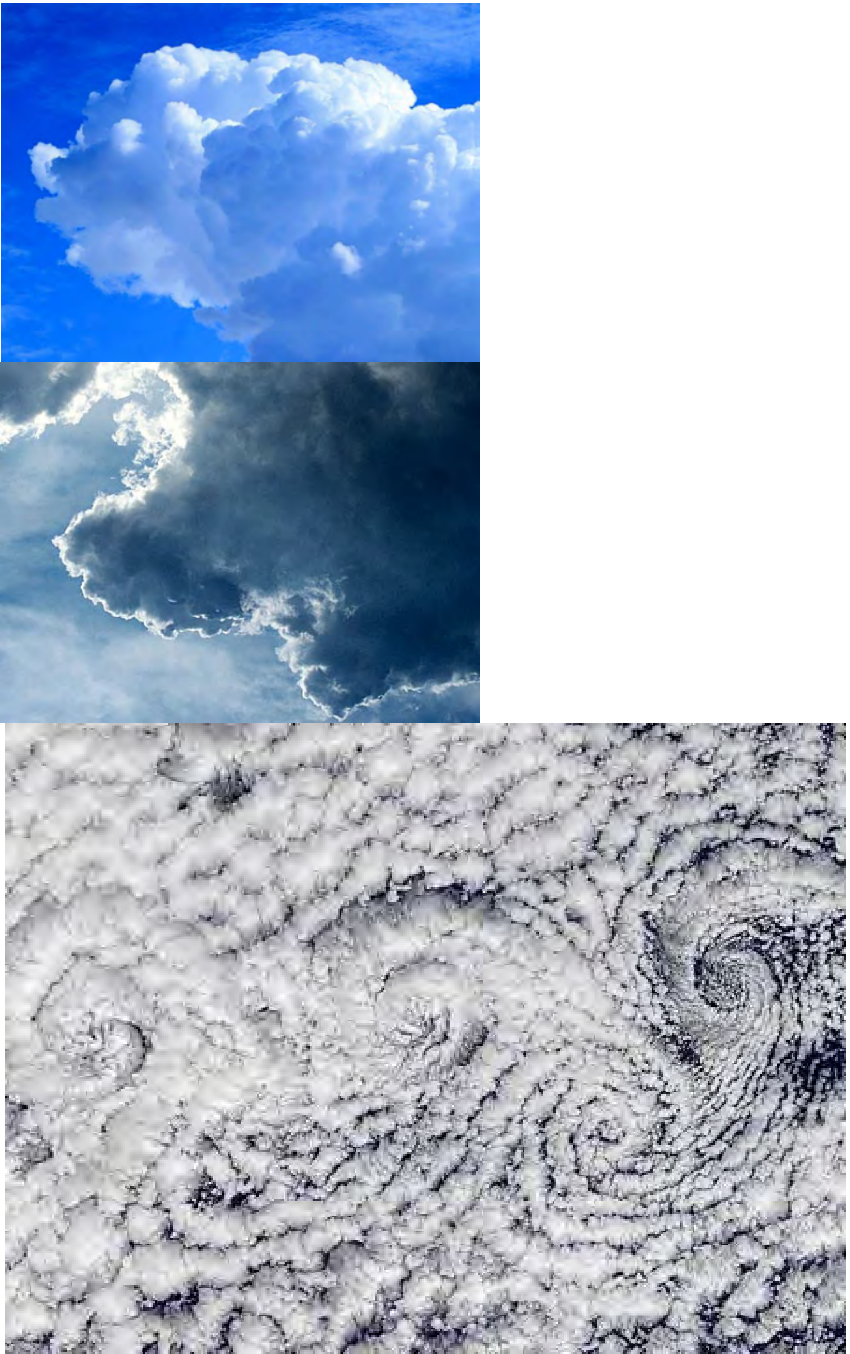




space filling curves

$D = 2$

clouds exhibit fractal behavior from 1 to 1000 km



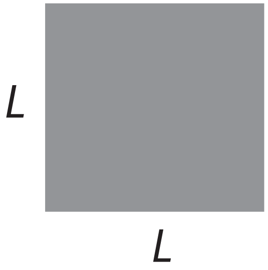
use ***perimeter-area*** data to find that cloud and rain boundaries are fractals

$$D \approx 1.35$$

S. Lovejoy, Science, 1982

$$P \sim \sqrt{A}$$

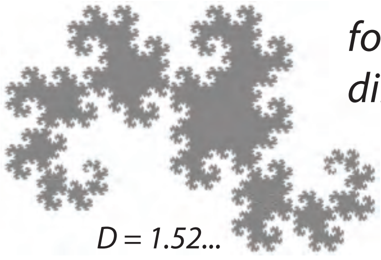
simple shapes



$A = L^2$

$P = 4L = 4\sqrt{A}$

$$P \sim \sqrt{A}^{D}$$



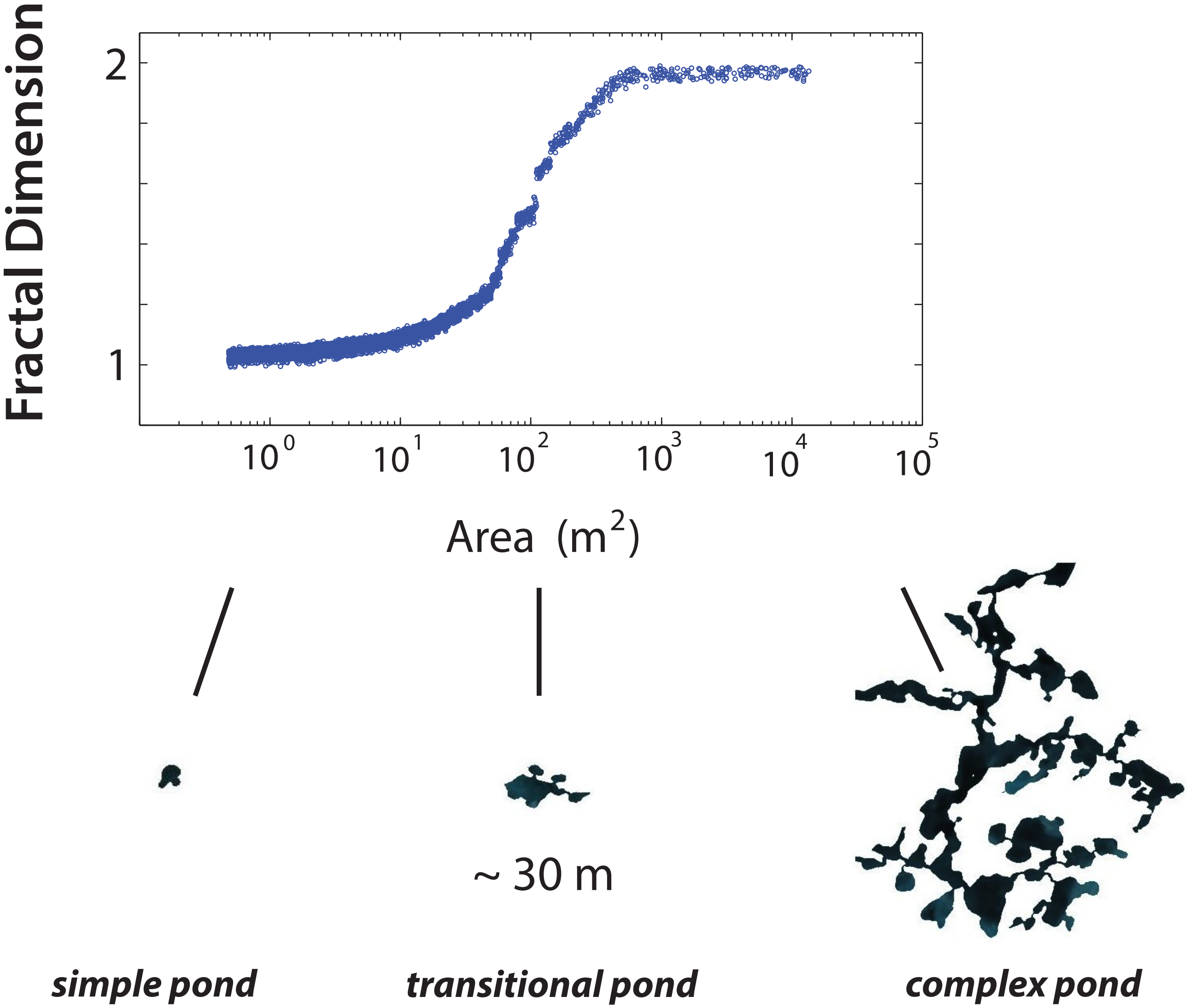
for fractals with dimension D

Transition in the fractal geometry of Arctic melt ponds

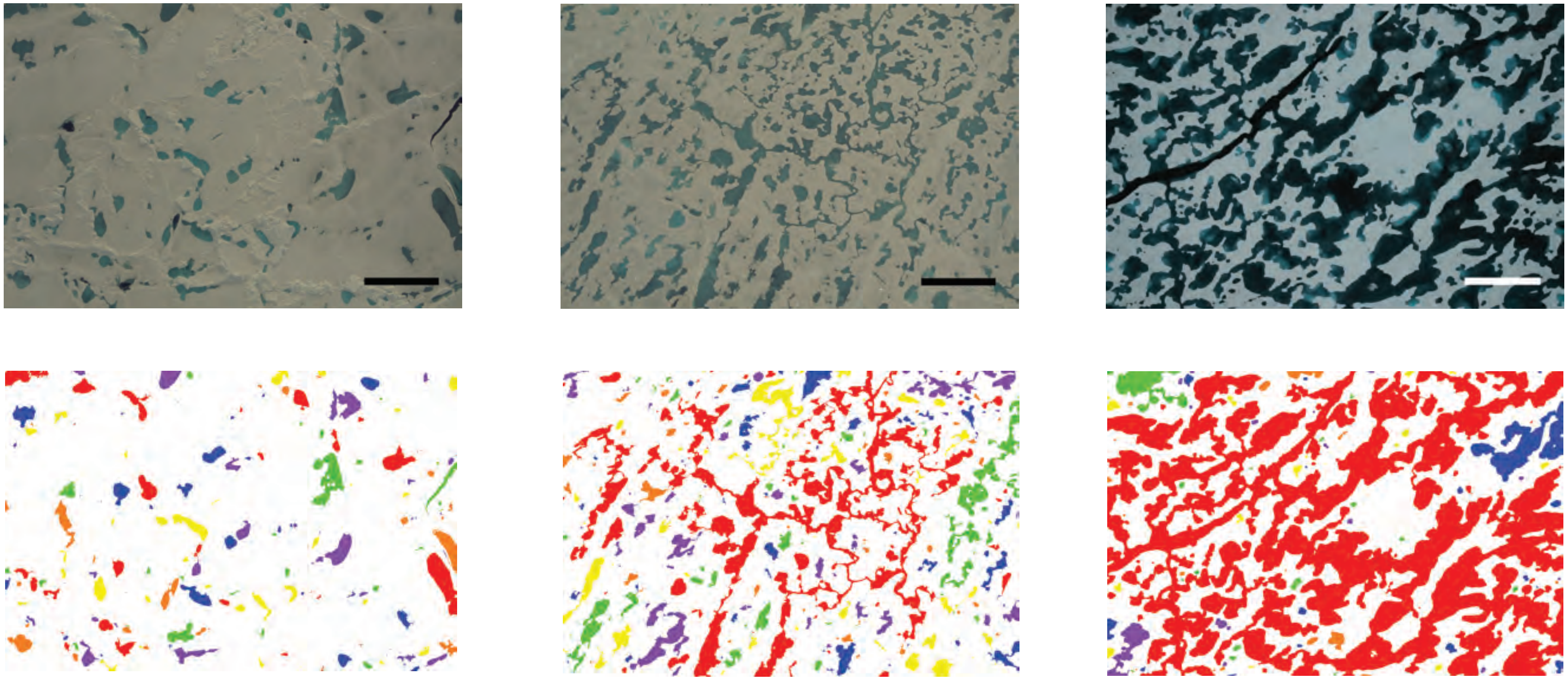
Christel Hohenegger, Bacim Alali, Kyle Steffen, Don Perovich, Ken Golden

The Cryosphere, 2012

complexity grows with length scale



small simple ponds coalesce to form large connected structures with complex boundaries



melt pond percolation

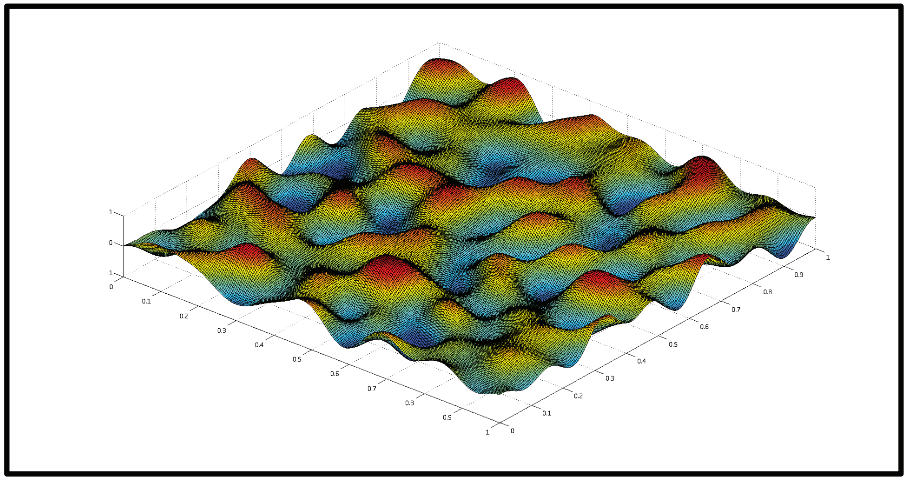
results on percolation threshold, correlation length, cluster behavior

Anthony Cheng (Hillcrest HS), Dylan Webb (Skyline HS), Court Strong, Ken Golden

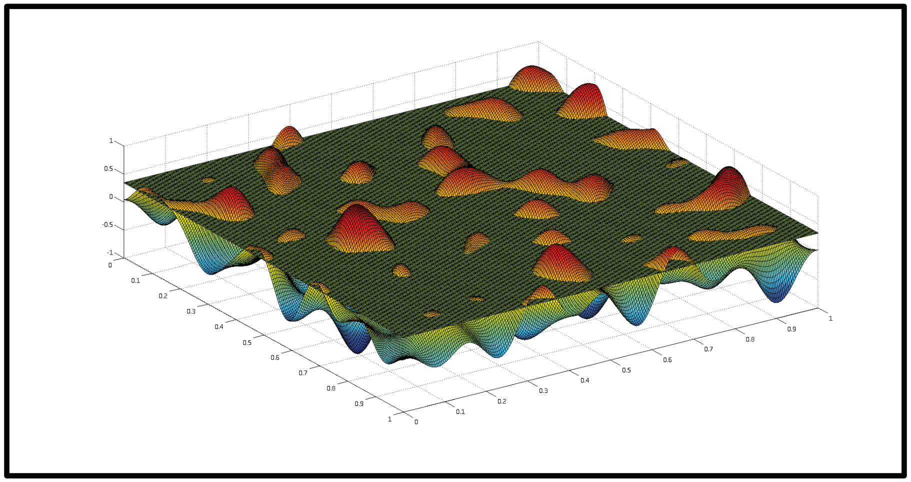
Continuum percolation model for melt pond evolution

level sets of random surfaces

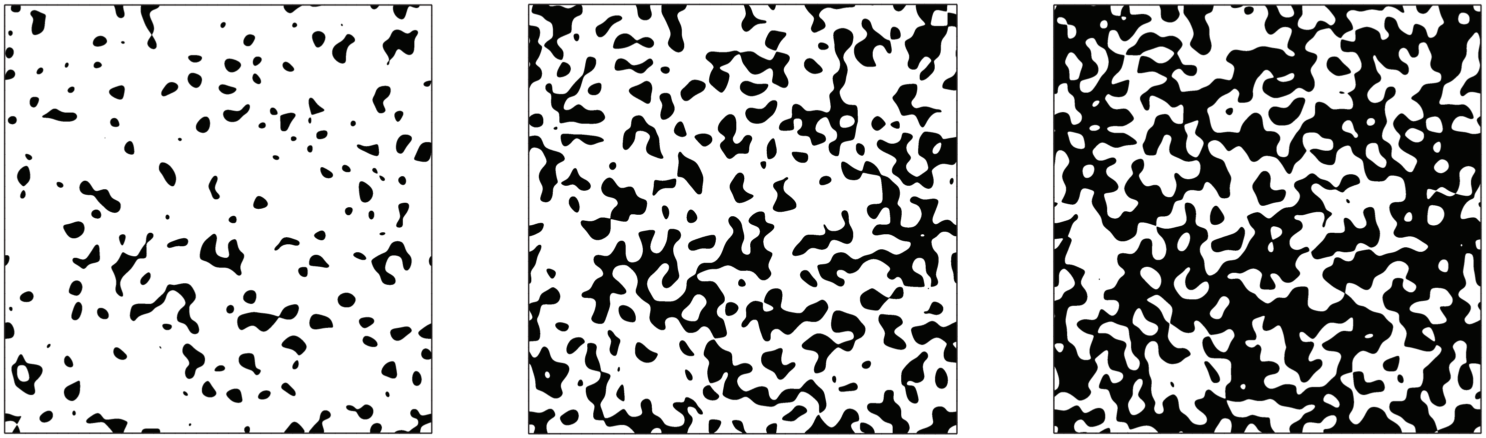
Brady Bowen, Court Strong, Ken Golden, J. Fractal Geometry 2018



random Fourier series representation of surface topography

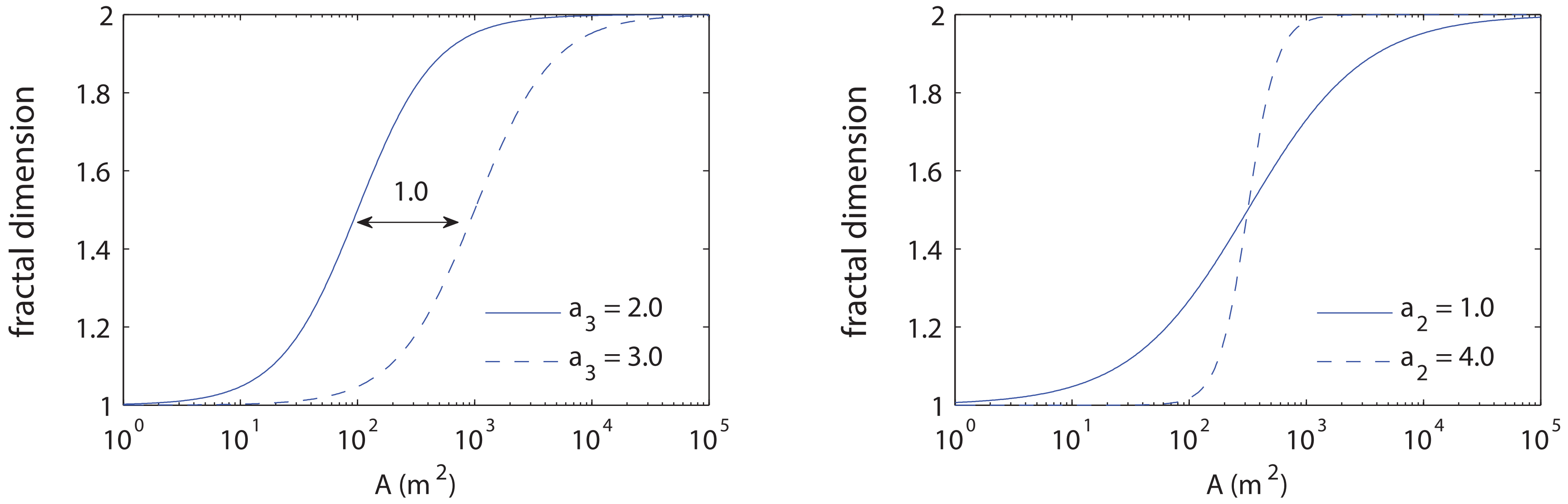


intersections of a plane with the surface define melt ponds

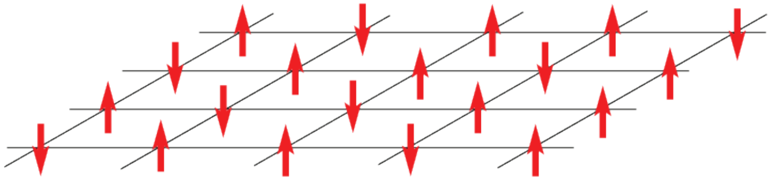


electronic transport in disordered media *diffusion in turbulent plasmas* *Isichenko, Rev. Mod. Phys., 1992*

fractal dimension curves depend on statistical parameters defining random surface



Ising Model for a Ferromagnet



$$s_i = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$$

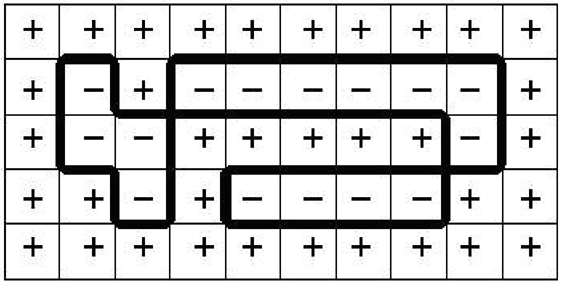
applied magnetic field

$$\mathcal{H}_\omega = -H \sum_i s_i - J \sum_{<i,j>} s_i s_j$$

nearest neighbor Ising Hamiltonian

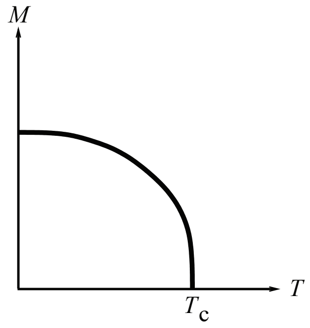
for any configuration $\omega \in \Omega = \{-1, 1\}^N$ of the spins

ferromagnetic interaction $J \geq 0$



+	+	+	+	+	+	+	+	+	+
+	−	+	−	−	−	−	−	−	+
+	−	−	+	+	+	+	+	−	+
+	+	−	+	−	−	−	−	+	+
+	+	+	+	+	+	+	+	+	+

magnetization



Curie point
critical temperature

$$M(T, H) = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle = -\frac{\partial f}{\partial H}$$

homogenized parameter
like effective conductivity

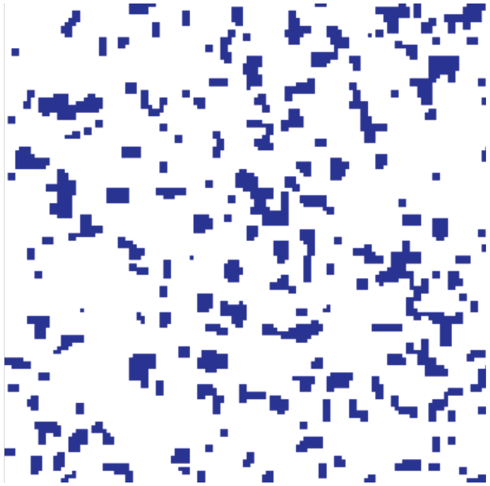
Ising model for ferromagnets ⟶ Ising model for melt ponds

$$\mathcal{H}_\omega = -J \sum_{<i,j>}^{N} s_i s_j - \sum_{i}^{N} H_i\, s_i \qquad s_i = \begin{cases} \uparrow & +1 \quad \text{water} \quad \text{(spin up)} \\ \downarrow & -1 \quad \text{ice} \quad \text{(spin down)} \end{cases}$$

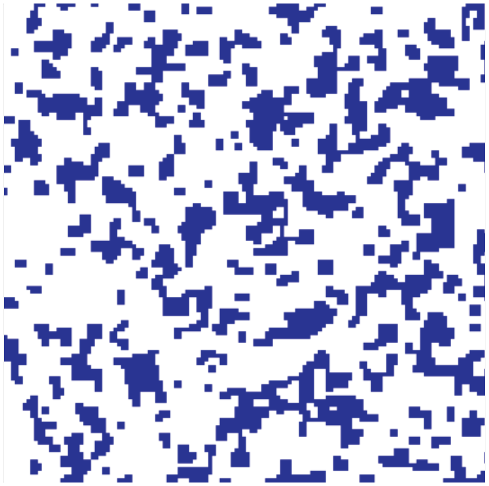
magnetization $M = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_j s_j \right\rangle$

pond coverage $\frac{(M+1)}{2}$

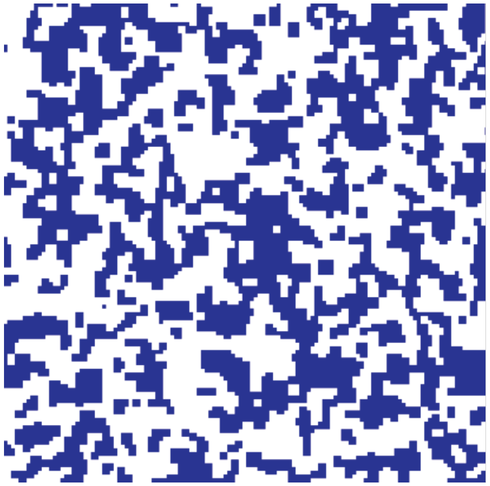
~ albedo



$F_{\text{out}} = 0.15$



$F_{\text{out}} = 0.30$



$F_{\text{out}} = 0.45$

Melt ponds are metastable islands of like spins in our random field Ising model.

input spin configuration independent binary random variables = +1 with probability F_{in}

Ma, Sudakov, Strong, Golden 2018

Glauber Dynamics (Metropolis at $T=0$):

if spin flip lowers energy, accept
if spin flip raises energy, reject

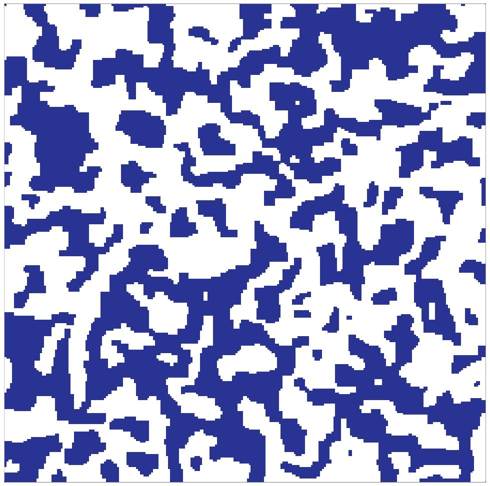
majority wins, water fills troughs

Metropolis algorithm: if lower accept
if raises accept with prob = Gibbs factor

Random initial configuration; as energy is minimized system "flows" toward metastable equilbrium

Order from Disorder

Ising model



melt pond photo

(Perovich)

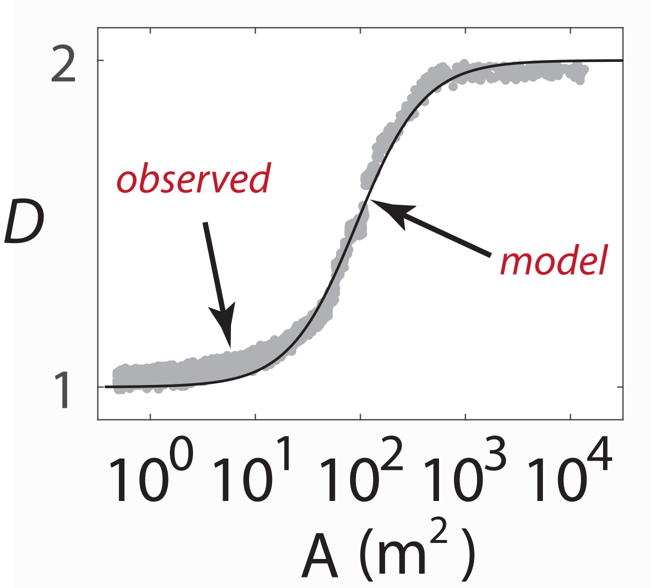
ONLY MEASURED INPUT = LENGTH SCALE (GRID SIZE)

Ising model results

Ma, Sudakov, Strong, Golden 2018

Minimize Ising Hamiltonian energy

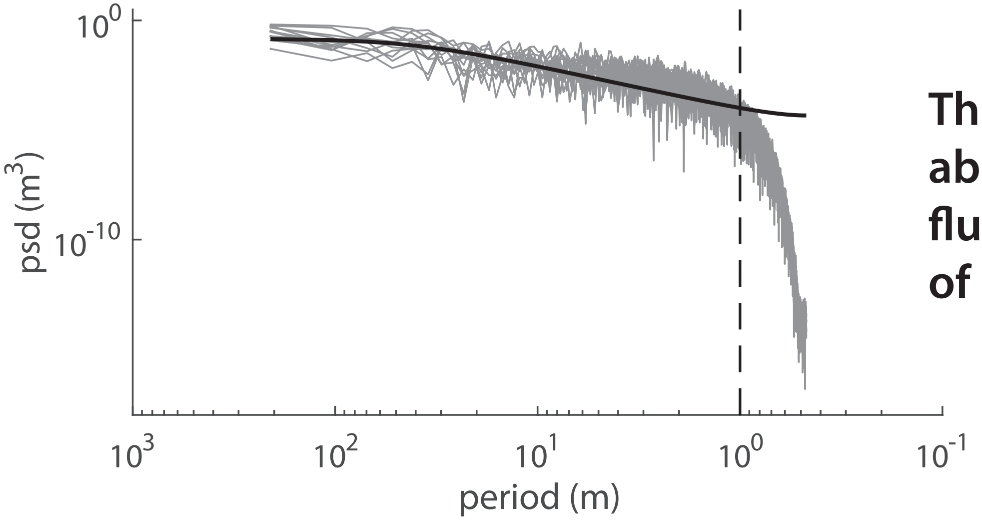
Random magnetic field represents snow topography; interaction term represents horizontal heat transfer.



pond size distribution exponent

observed -1.5 (Perovich, et al 2002)

model -1.58



The lattice constant (1 m) is the length scale above which important spatially correlated fluctuations occur in the power spectrum of snow topography.

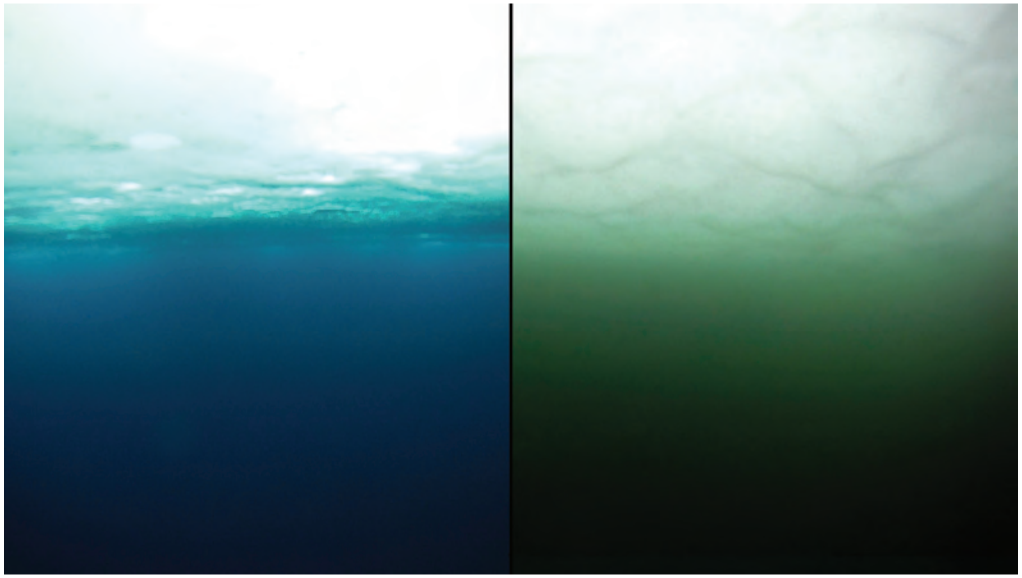
2011 massive under-ice algal bloom

Arrigo et al., *Science* 2012

melt ponds act as

WINDOWS

allowing light through sea ice



no bloom

bloom

Have we crossed into a new ecological regime?

The frequency and extent of sub-ice phytoplankton blooms in the Arctic Ocean

Horvat, Rees Jones, Iams, Schroeder, Flocco, Feltham, ***Science Advances***, 2017

The distribution of solar energy under ponded sea ice

Horvat, Flocco, Rees Jones, Roach, Golden, 2018

(2015 AMS MRC)

The Melt Pond Conundrum:

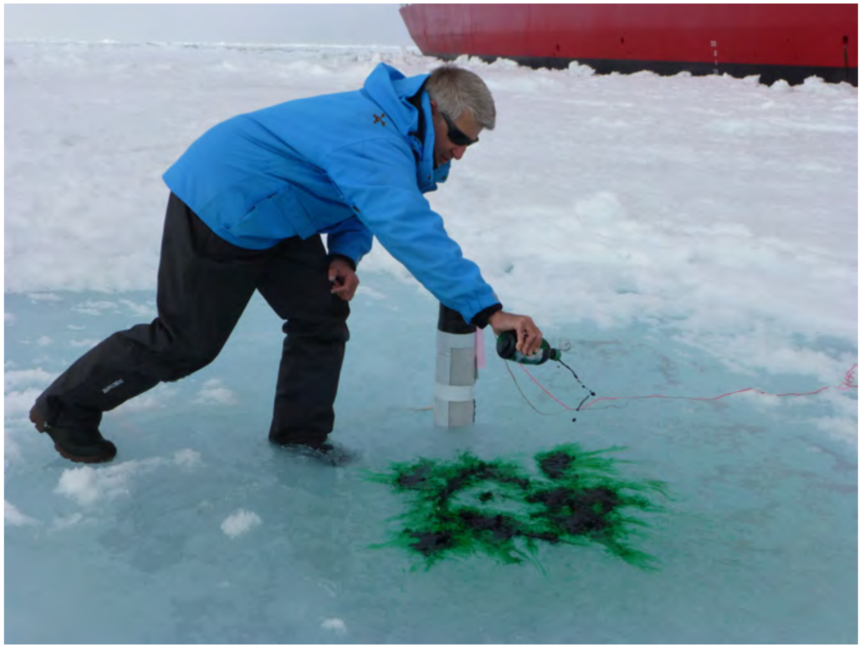
How can ponds form on top of sea ice that is highly permeable?

C. Polashenski, K. M. Golden, D. K. Perovich, E. Skyllingstad, A. Arnsten, C. Stwertka, N. Wright

Percolation Blockage: A Process that Enables Melt Pond Formation on First Year Arctic Sea Ice

J. Geophys. Res. Oceans 2017

2014 Study of Under Ice Blooms in the Chuckchi Ecosystem (SUBICE)
aboard USCGC Healy



Conclusions

1. Summer Arctic sea ice is melting rapidly, and melt ponds and other processes must be accounted for in order to predict melting rates.

2. Fluid flow through sea ice mediates melt pond evolution and many processes important to climate change and polar ecosystems.

3. Statistical physics and homogenization help ***link scales***, provide rigorous methods for finding effective behavior, and advance how sea ice is represented in climate models.

4. Critical behavior (in many forms) is inherent in the climate system.

5. Field experiments are essential to developing relevant mathematics.

6. Our research will help to improve projections of climate change, the fate of Earth's sea ice packs, and the ecosystems they support.

THANK YOU

Office of Naval Research

Arctic and Global Prediction Program

Applied and Computational Analysis Program

National Science Foundation

Division of Mathematical Sciences

Division of Polar Programs

United States Coast Guard

Department of the Navy ONR Science & Technology

MCRN Mathematics and Climate Research Network

Australian Government
Department of the Environment and Water Resources
Australian Antarctic Division

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THE UNIVERSITY OF UTAH

Buchanan Bay, Antarctica   Mertz Glacier Polynya Experiment   July 1999

2 — THE MERCURY Thursday, July 23, 1998

Fire endangers Hobart's ice ship

By DAVID CARRIGG

AN engine-room fire has left the Hobart-based Antarctic research ship Aurora Australis without power in dangerous sea ice off the Antarctic coast.

None of the 79 people on board was injured in the blaze, which broke out early yesterday morning while the ship was in deep water 185km off the coast.

The extent of the damage is not known.

Australian Antarctic Division director Rex Moncur said the fire was extinguished by flooding the engine room with an inert gas.

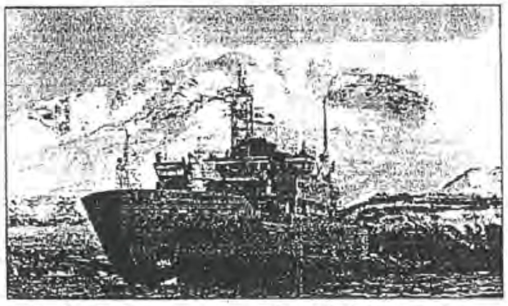
The gas had to be cleared before crew wearing breathing apparatus could enter and assess the situation.

He said it could be some time before the extent of damage was known.

The 25 crew and 54 expeditioners, mostly from Hobart, would wear thermal clothing and stay below decks to keep warm.

"There is always a risk of becoming ice-bound in these waters at this time of the year but at this stage we don't expect to launch a rescue mission from Hobart," Mr Moncur said.

The ship was in regular radio contact with the Antarctic Division's Hobart office.



A file photo of the Aurora Australis in Antarctica.

He expected the expeditioners and crew to abandon the pioneering winter voyage and return the ship to Hobart for repairs in about a week.

The Antarctic Division, which hires the ship from P&O Australia, would not be hiring another vessel for the expedition.

"It's a pretty specialist vessel so you couldn't get the sort of research capability that this ship has got readily available," Mr Moncur said.

"We hope the next voyage can still proceed on schedule, which is early September."

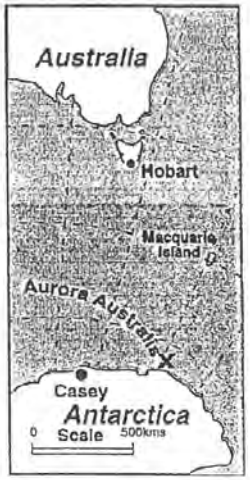
The Aurora Australis is owned by P&O Australia and charted by the Antarctic Division for about $11 million a year.

P&O Australia managing director Richard Hein said yesterday the company was assessing the situation and a number of rescue options were being considered.

It was too early to say whether P&O would be liable for the cost of the aborted mission.

The vessel left Hobart last Wednesday for a seven-week voyage mainly to study a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

The ship was nearing the polynya when the fire broke out.



Oceanographers believe a closer study of the phenomenon will lead to a better understanding of climate change.

CSIRO Marine Research oceanographer Steve Rintoul said the dense bottom water, created only in a few places in Antarctica and to a lesser extent in the North Atlantic, was critical to the chemistry and biology of the world's oceans.

THE ADVERTISER (Adelaide)
Thurs 23 July 1998 Page 14

Fire strands Antarctic ship in sea ice

AN engine room fire has disabled the icebreaker Aurora Australis in sea ice, deep in Antarctic waters.

There were no injuries and the ship was not in danger after Tuesday night's fire, Australian Antarctic Division director Mr Rex Moncur said. But Mr Moncur said he expected it would have to abandon its pioneering mid-winter voyage to the edge of the Antarctic continent and return to Hobart for repairs.

The cause of the fire was not known but the engines have been turned off, with the ship 100 nautical miles from the Antarctic coast.

THE CANBERRA TIMES
Thursday 23 July 1998 Page 4

Antarctic voyage stopped by fire

HOBART: An engine room fire has disabled the Australian icebreaker Aurora Australis in sea ice, deep in Antarctic waters.

Australian Antarctic Division director Rex Moncur said there were no injuries and the ship was not in danger after Tuesday night's fire.

But Mr Moncur said he expected Aurora Australis would have to abandon its pioneering mid-winter voyage to the edge of the Antarctic continent to return to Hobart for repairs.

The fire had been extinguished and the engines were turned off, leaving the ship in sea ice about 100 nautical miles from the Antarctic coast, he said. The weather was good.

Crew had to wear breathing apparatus to enter the engine room and it was likely to be 24 hours before the damage could be fully assessed.

The Aurora, with 54 expeditioners and 25 crew, left Hobart last Wednesday for a seven-week voyage which was to have focused on a polynya, an area where savage winds break up the sea ice and cause heavy, salt-laden water to sink to the bottom.

Mr Moncur said. the cause of the fire was not yet known.

Sydney Morning Herald
23 July, 1998

ICEBREAKER BURNS

A pioneering $2 million Australian scientific voyage to the mid-winter Antarctic pack-ice is expected to be scrapped following an engine-room fire on the Aurora Australis yesterday.

The 54 people on board were forced on deck in the

2:45 am July 22, 1998

``Please don't be alarmed but we have an uncontrolled fire in the engine room ...."

about 10 minutes later ...

``Please don't be alarmed but we're lowering the lifeboats ...."