Eight

Ⅰ Polar coordinates in $\mathbb{R}^2$

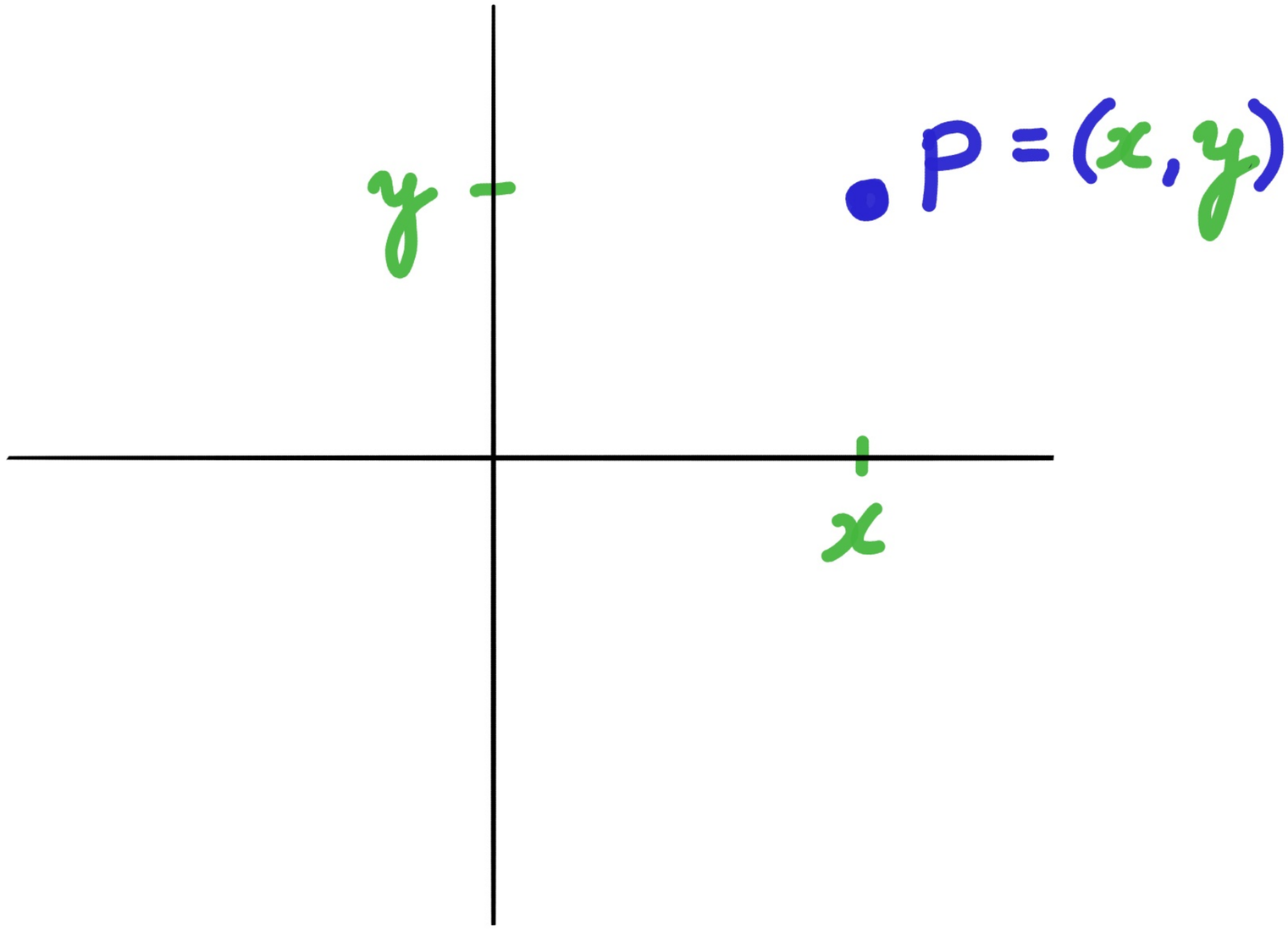
Ⅱ Cylindrical coordinates in $\mathbb{R}^3$

Ⅲ Spherical coordinates in $\mathbb{R}^3$

Ⅳ A word on functions

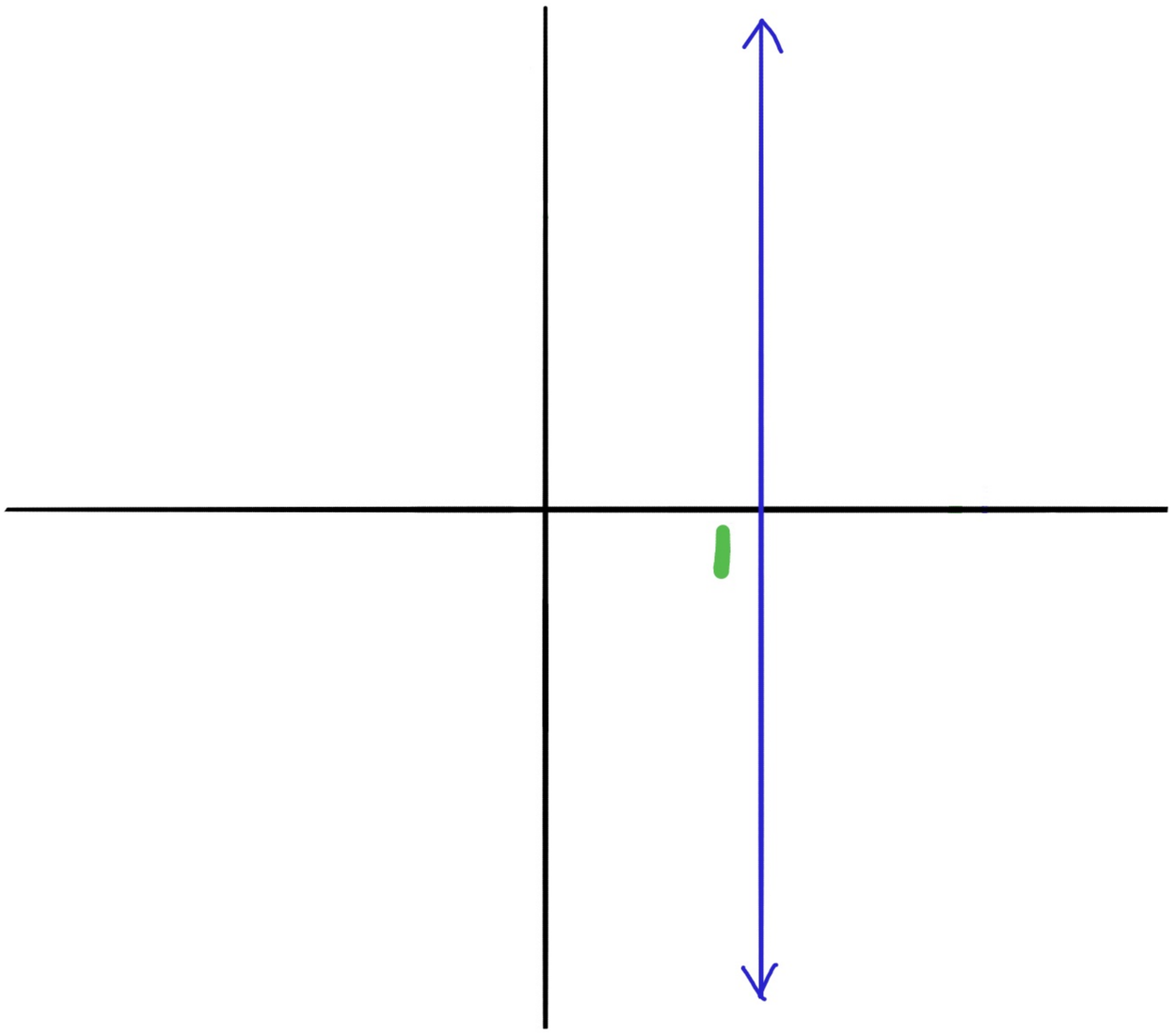
Ⅰ Polar coordinates in $\mathbb{R}^2$

$\mathbb{R}^2$



Cartesian

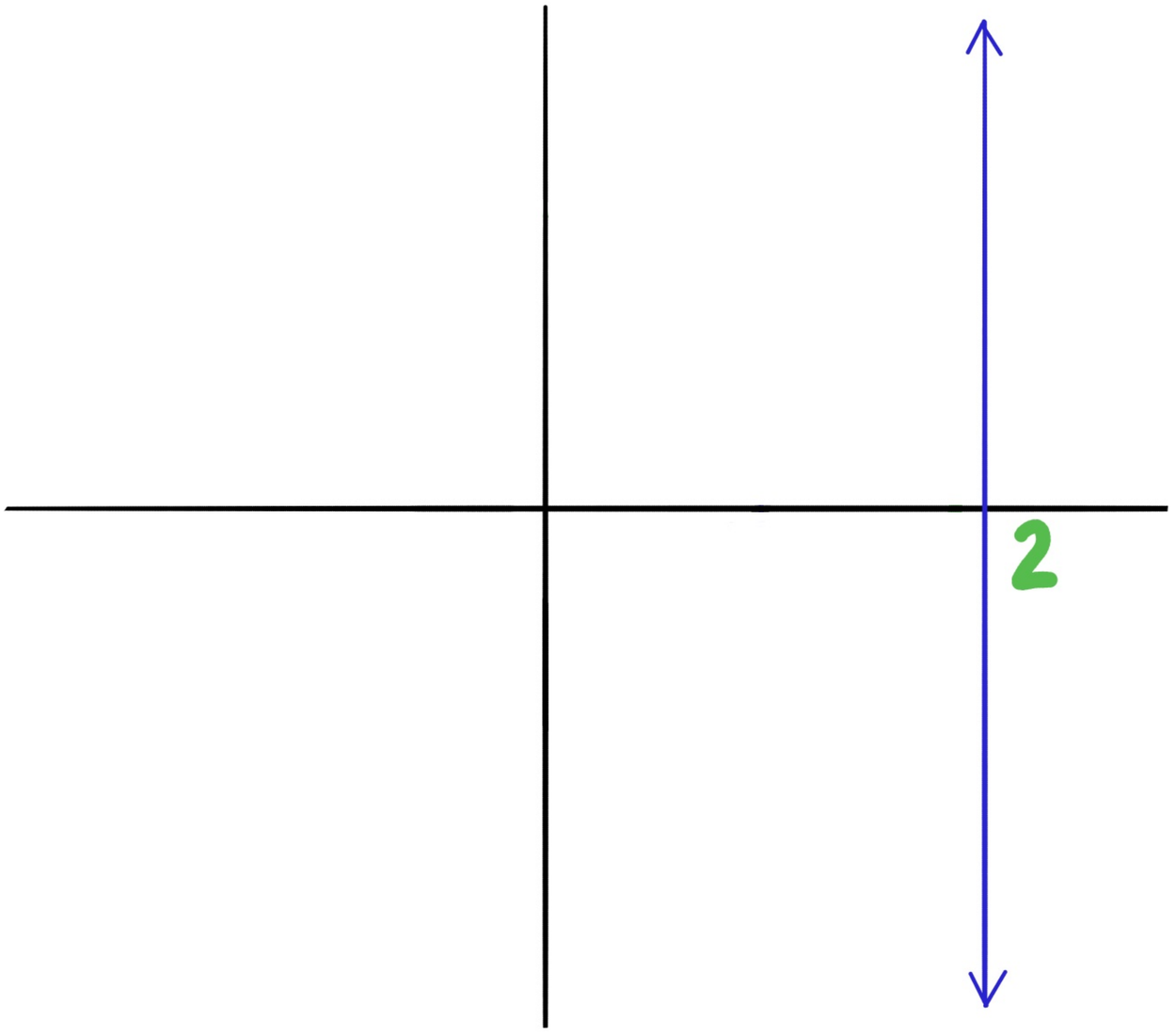
$\mathbb{R}^2$



$x = 1$

Cartesian

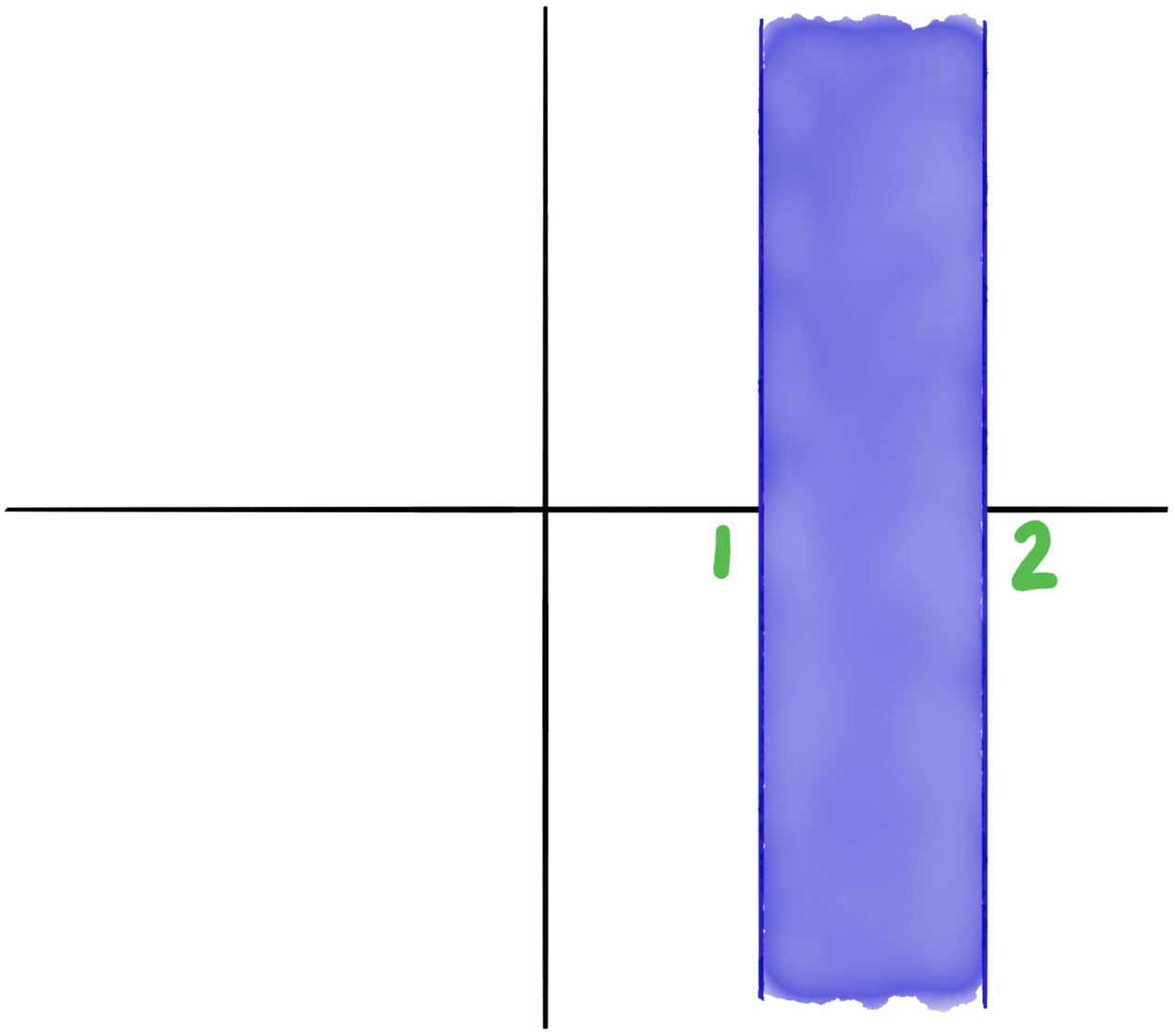
$\mathbb{R}^2$



$x = 2$

Cartesian

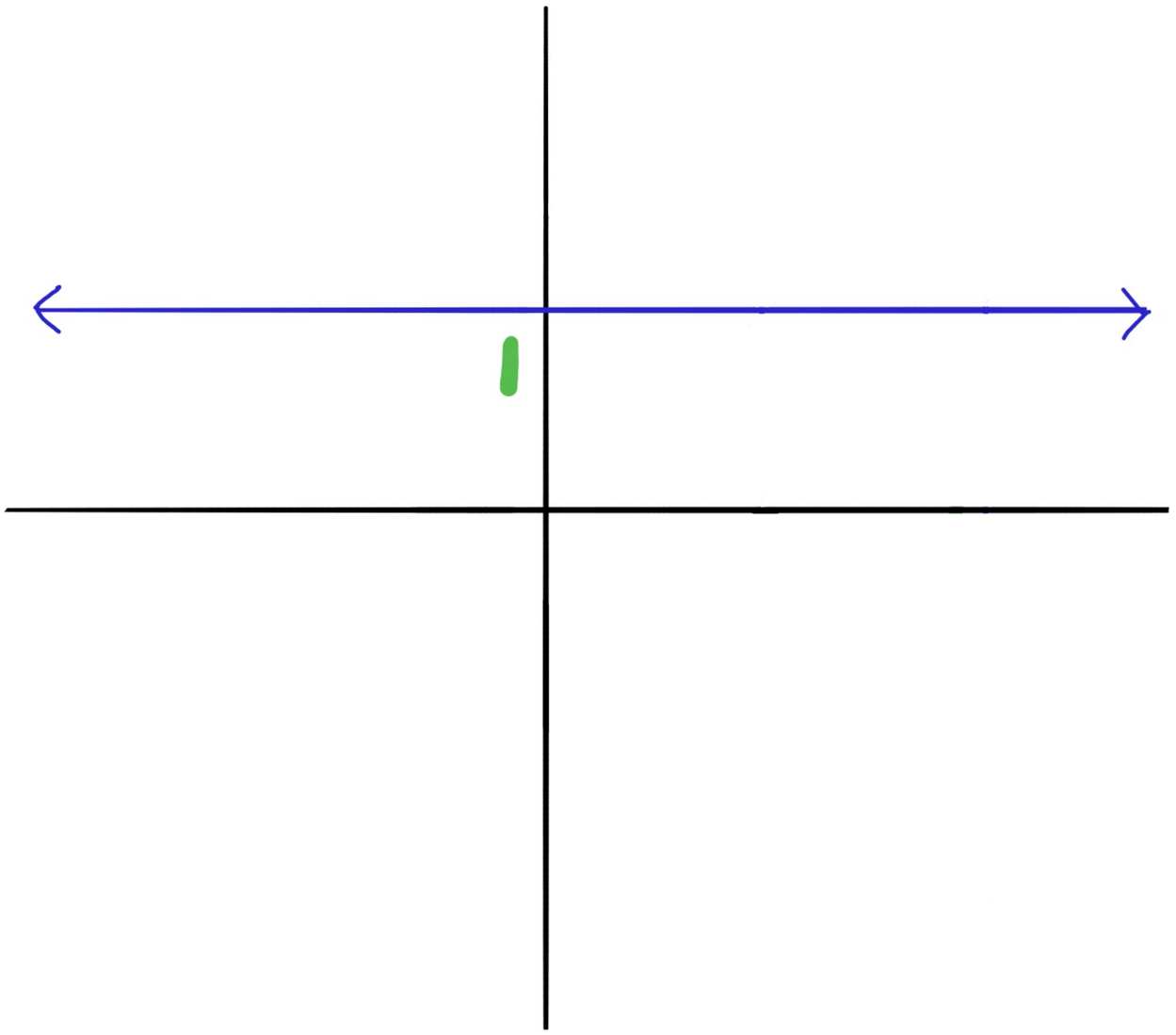
$\mathbb{R}^2$



$1 \leq x \leq 2$

Cartesian

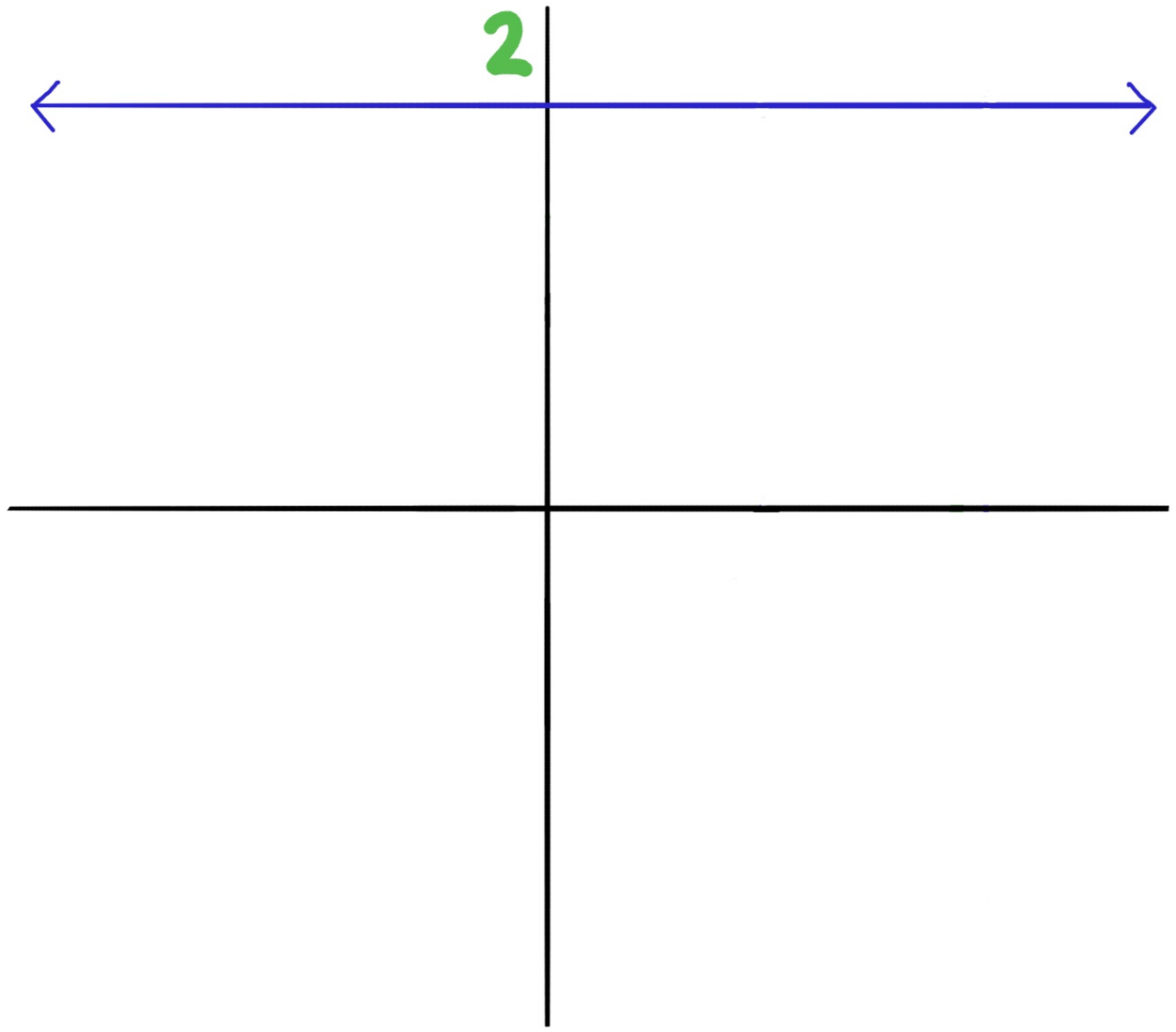
$\mathbb{R}^2$



$y = 1$

Cartesian

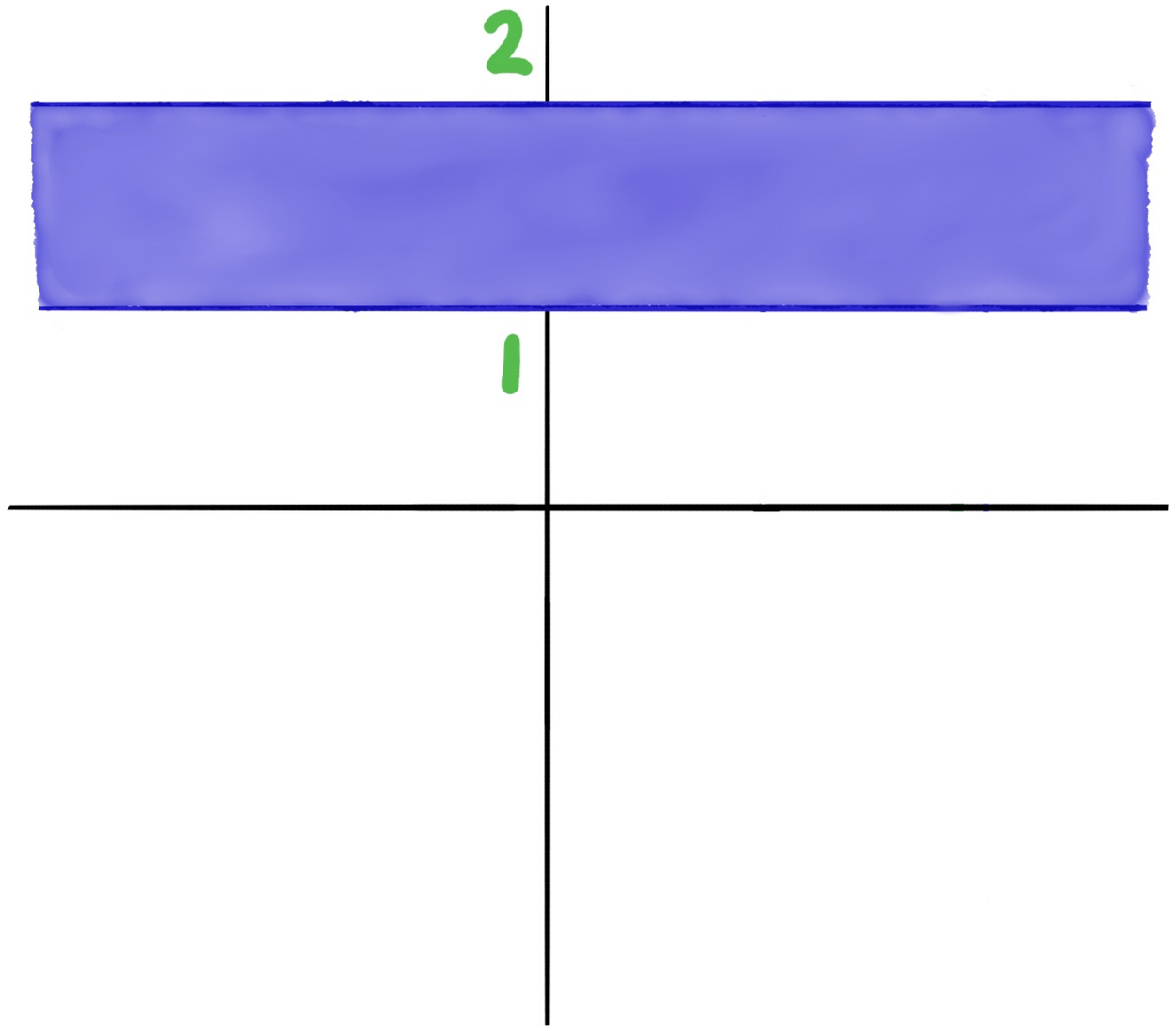
$\mathbb{R}^2$



$y = 2$

Cartesian

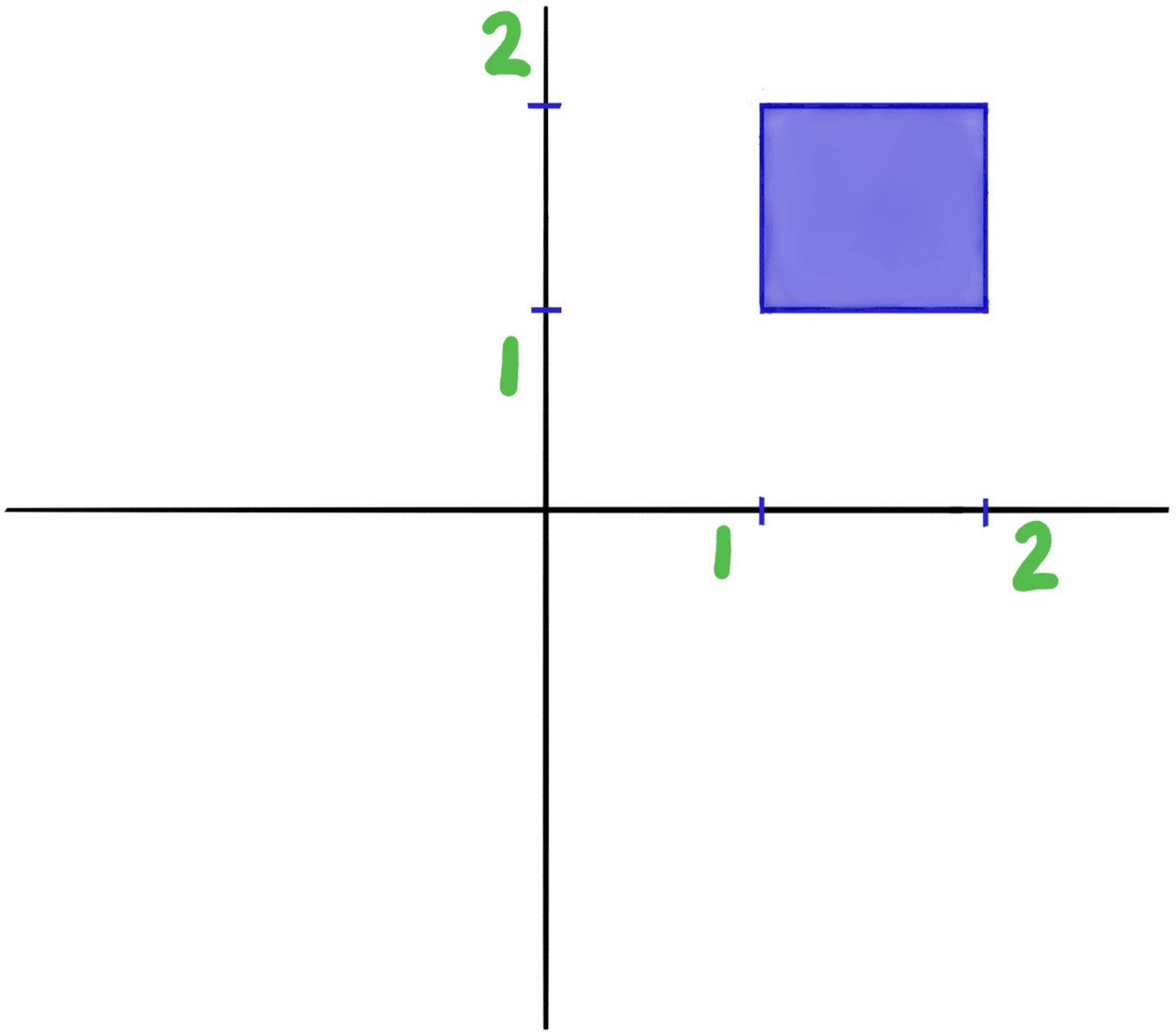
$\mathbb{R}^2$



$1 \leq y \leq 2$

Cartesian

$\mathbb{R}^2$

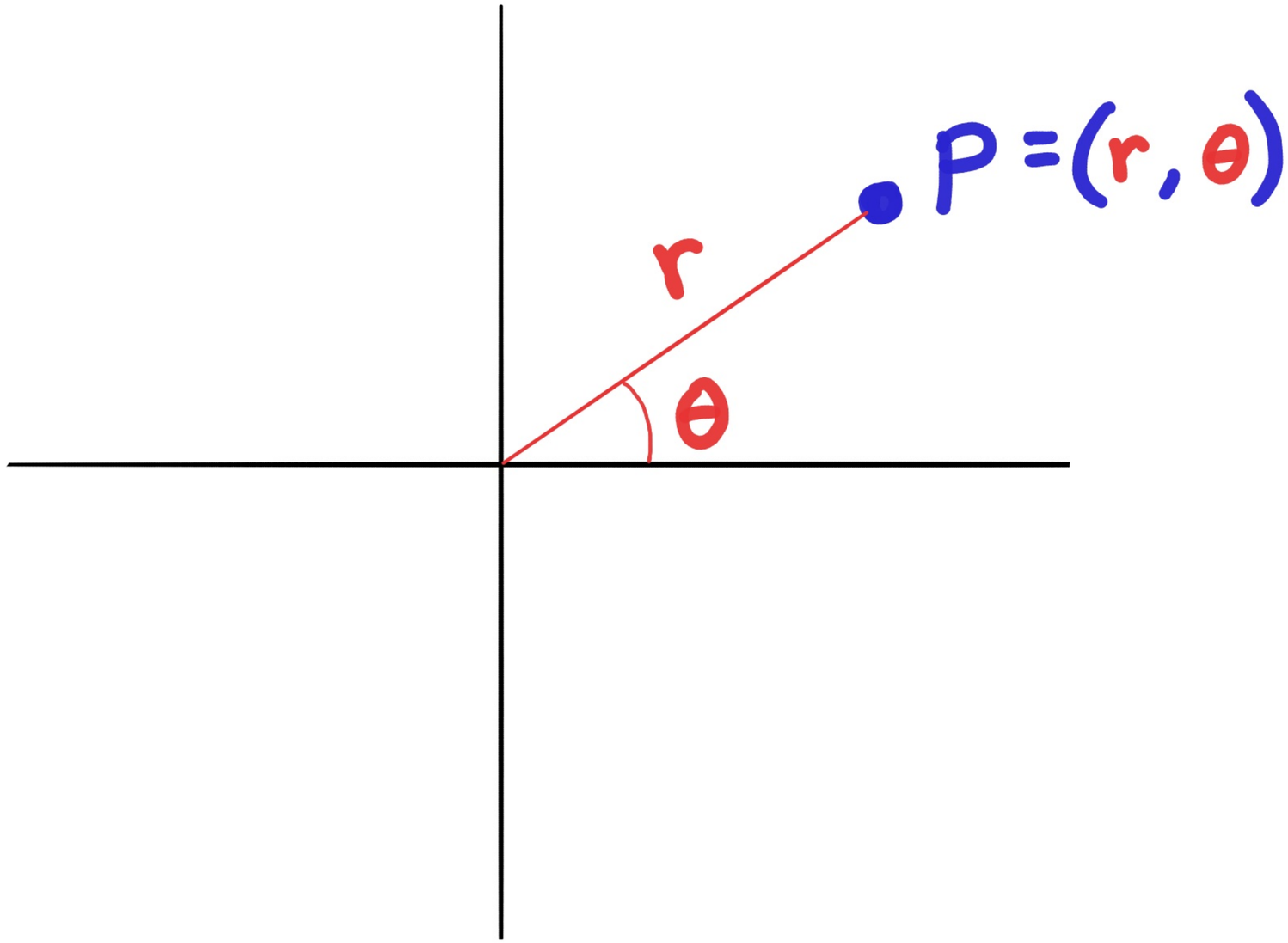


$1 \leq x \leq 2$

$1 \leq y \leq 2$

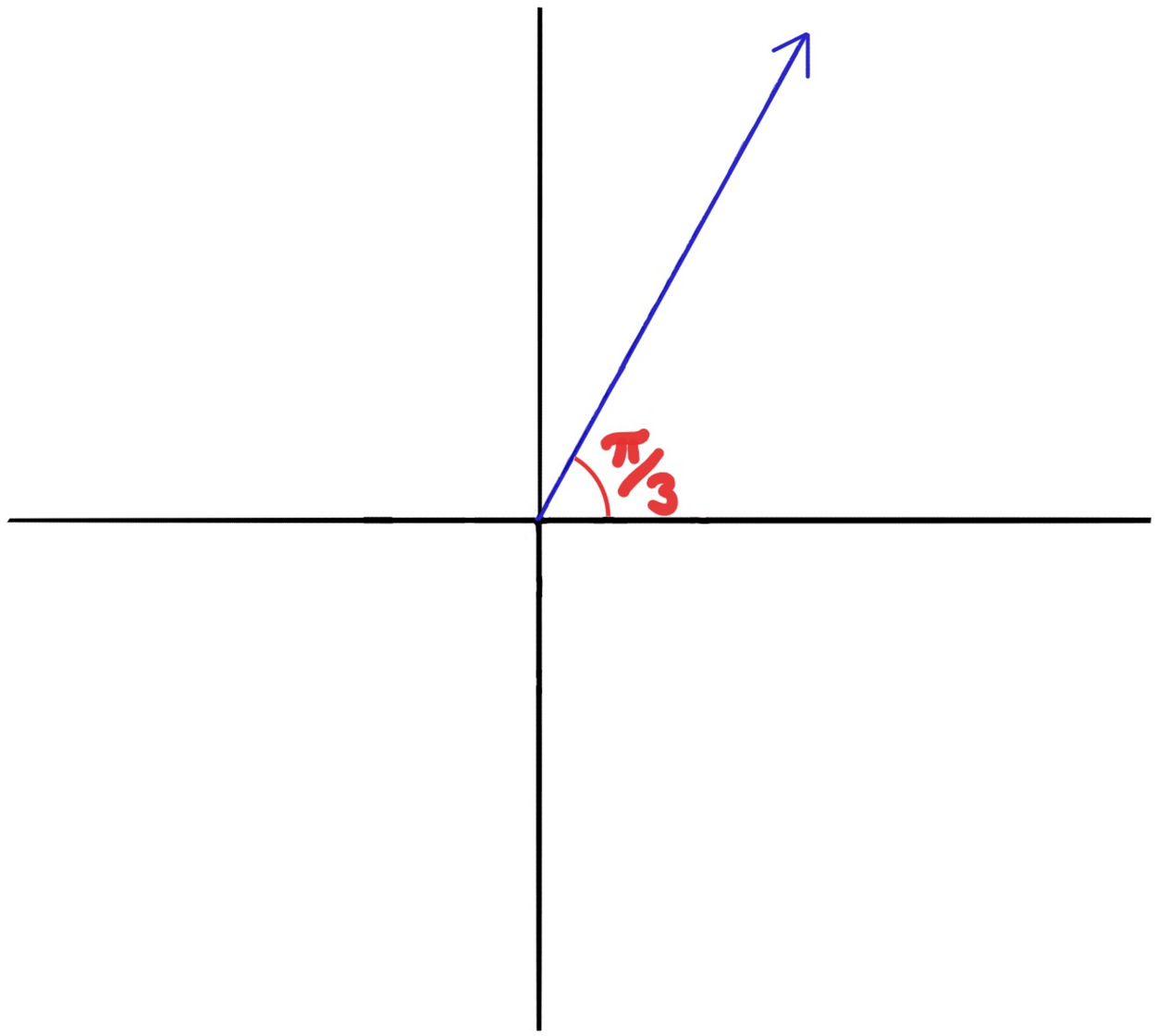
Cartesian

$\mathbb{R}^2$



polar

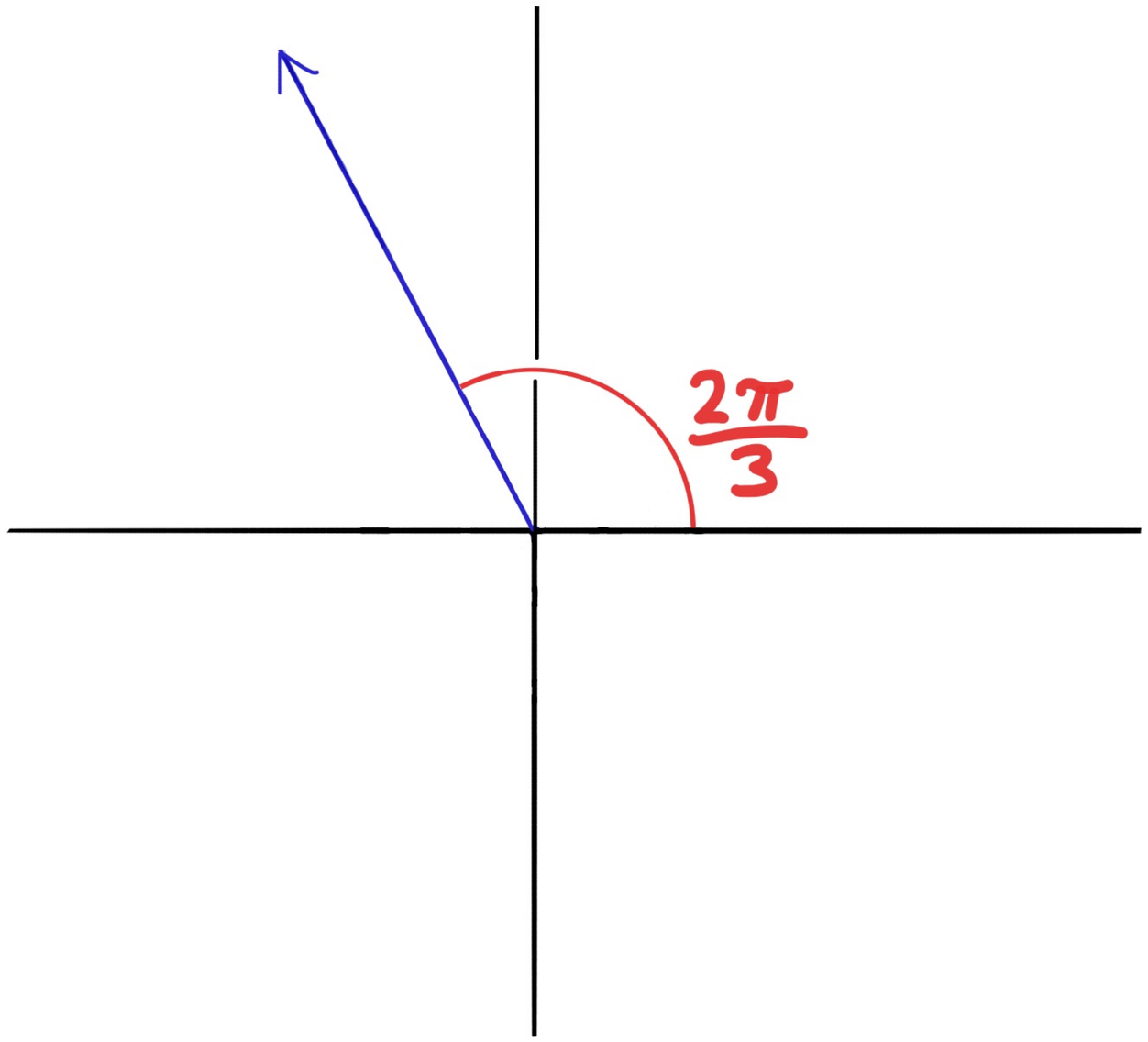
$\mathbb{R}^2$



$\theta = \pi/3$

polar

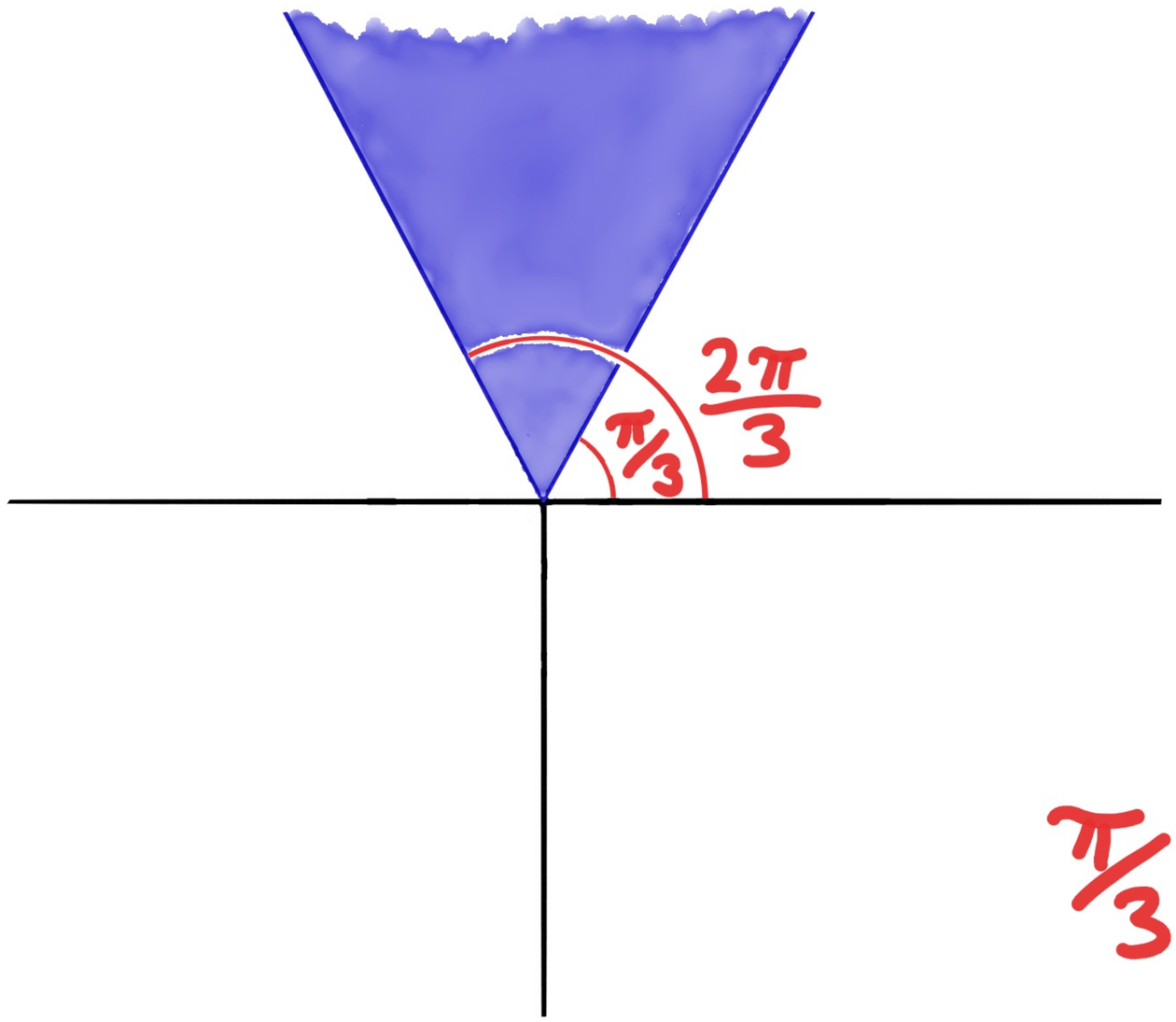
$\mathbb{R}^2$



$\theta = \frac{2\pi}{3}$

polar

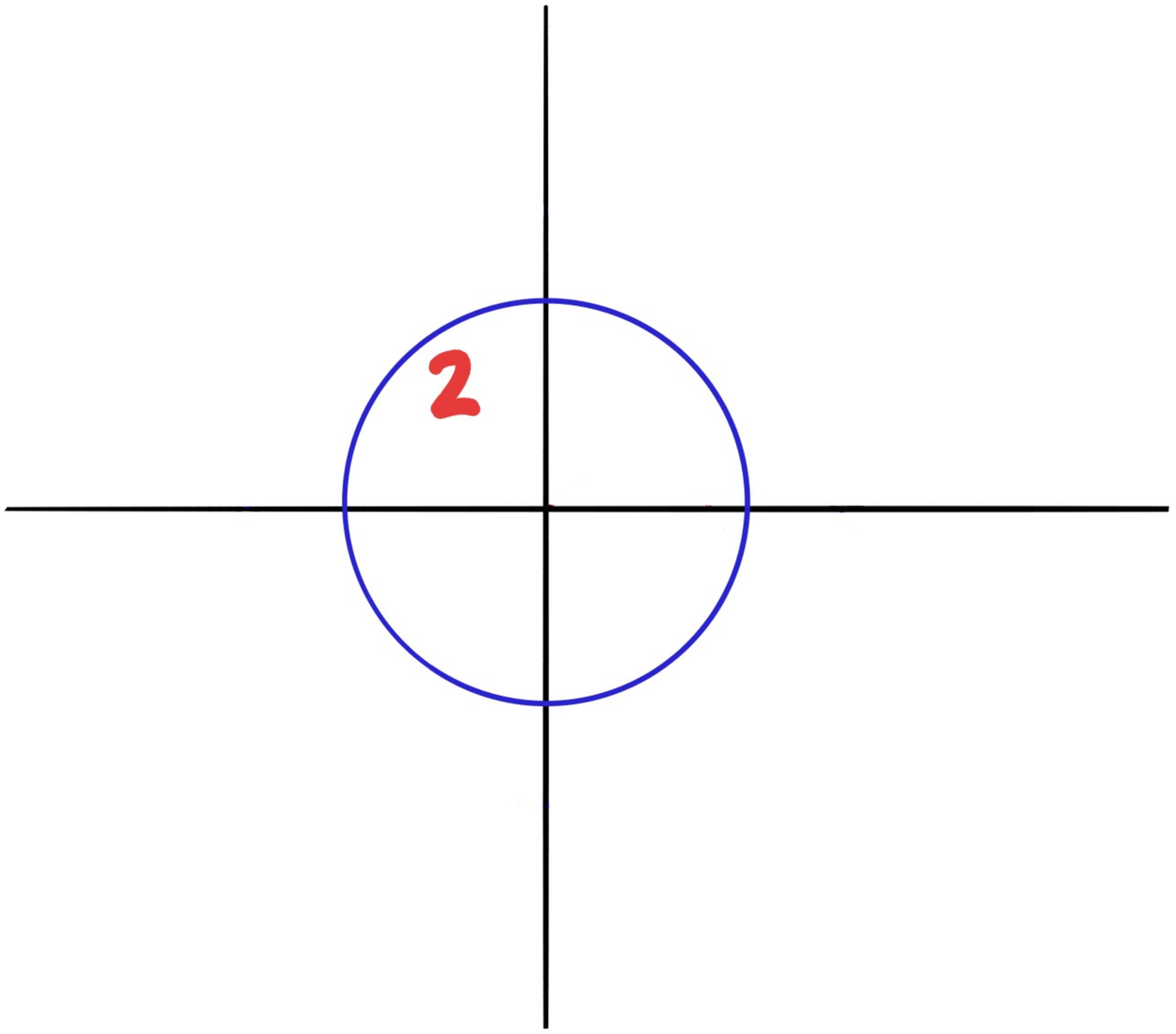
$\mathbb{R}^2$



$\pi/3 \leq \theta \leq \frac{2\pi}{3}$

polar

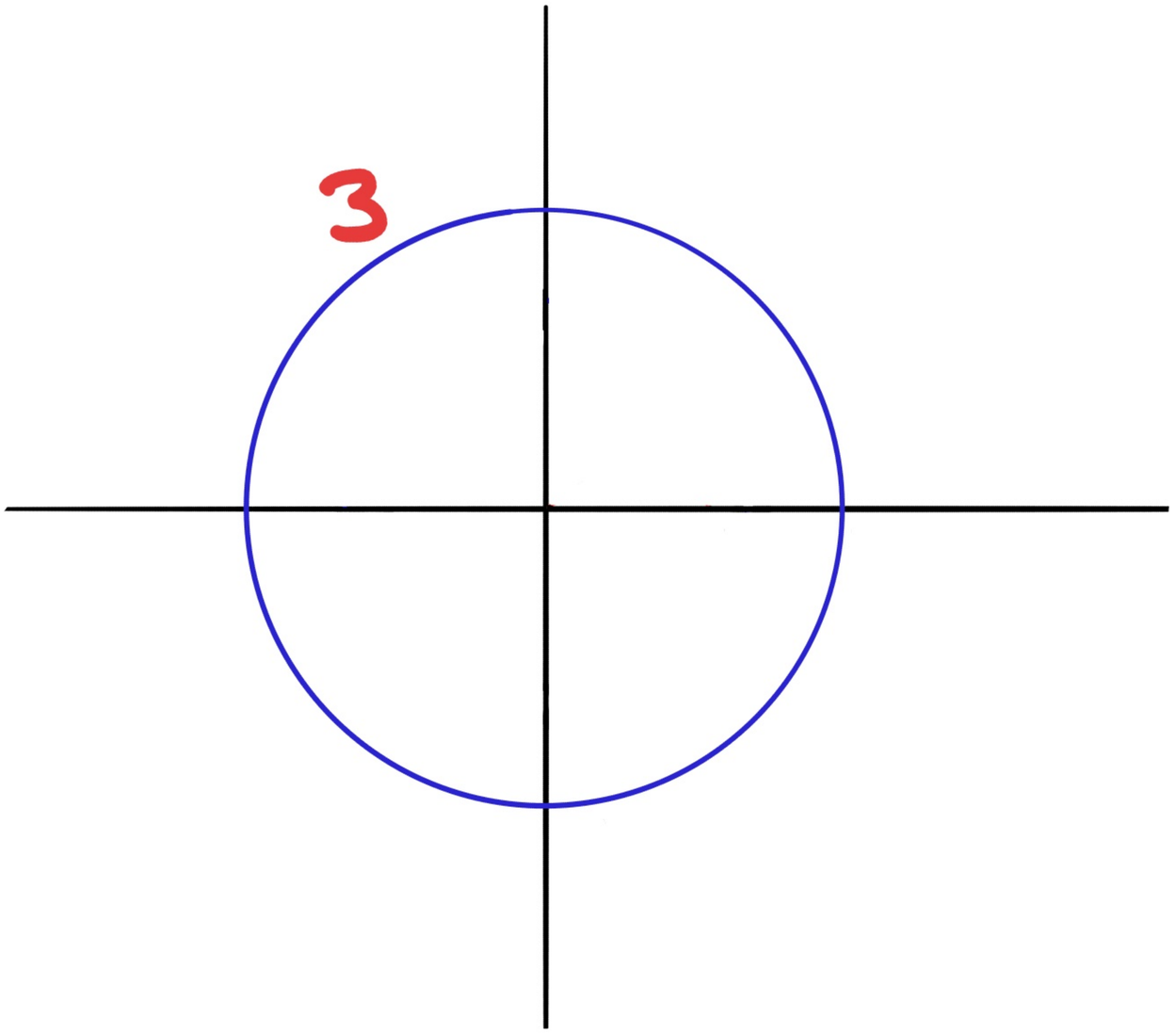
$\mathbb{R}^2$



$r = 2$

polar

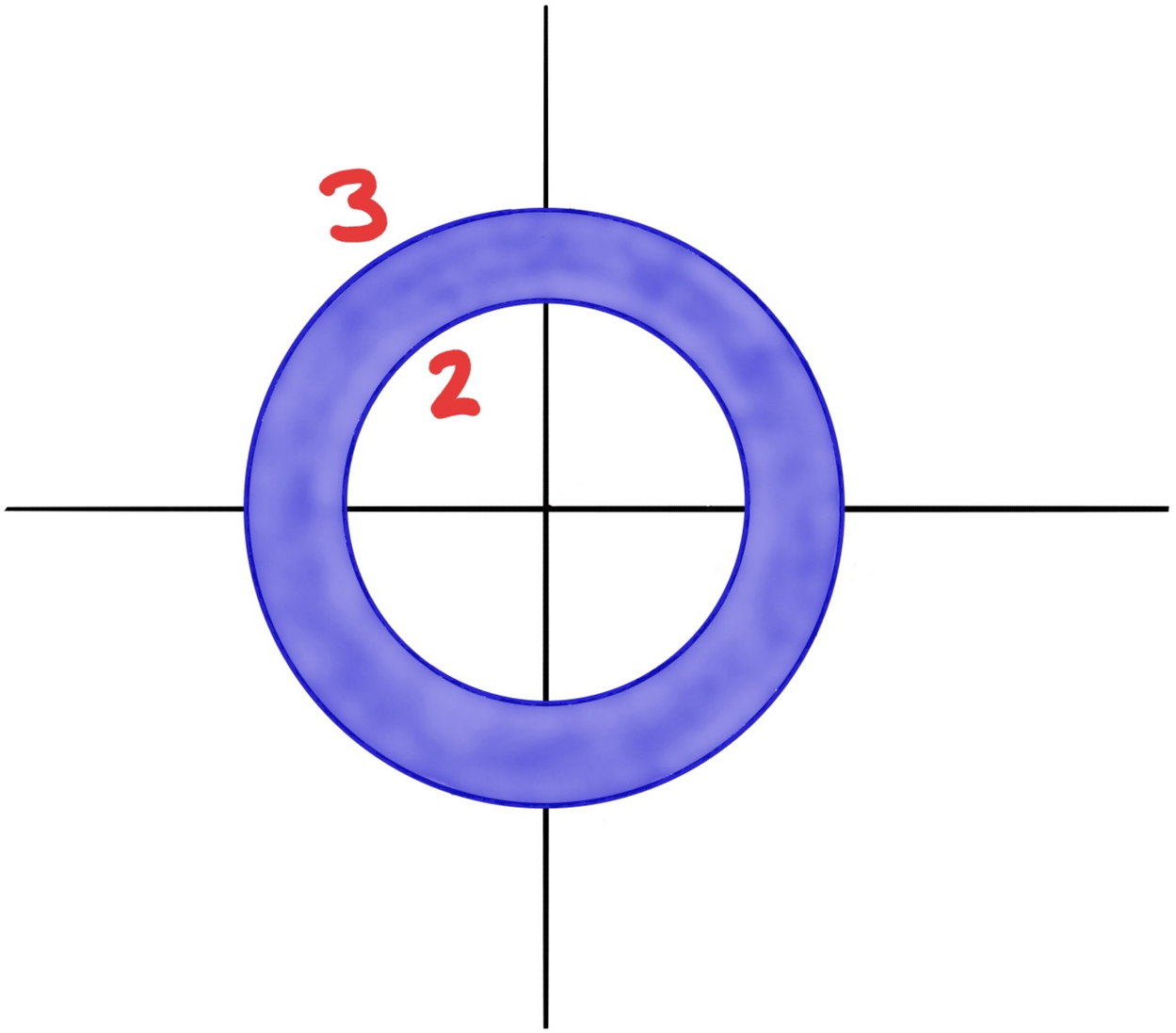
$\mathbb{R}^2$



$r = 3$

polar

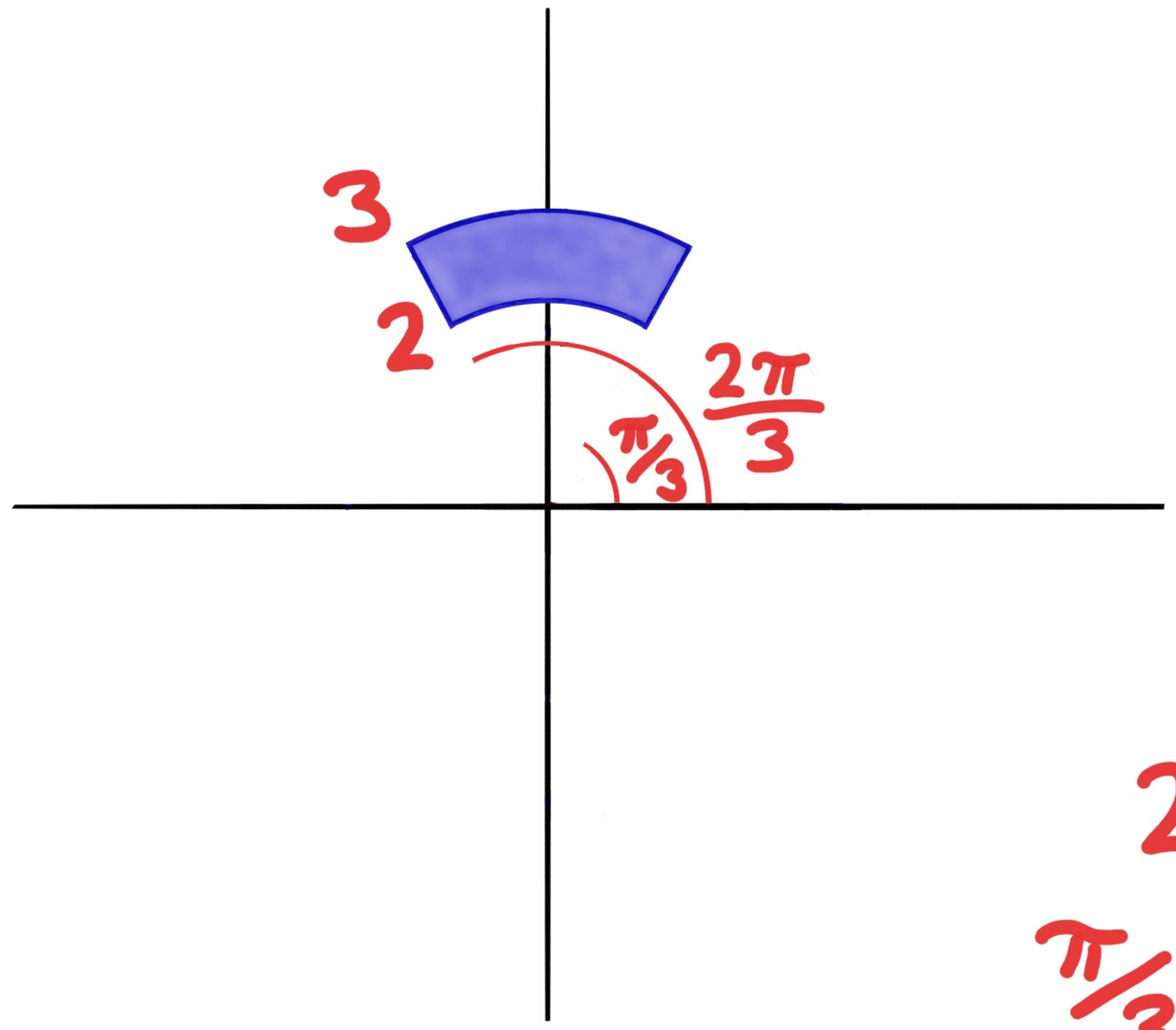
$\mathbb{R}^2$



$2 \leq r \leq 3$

polar

$\mathbb{R}^2$

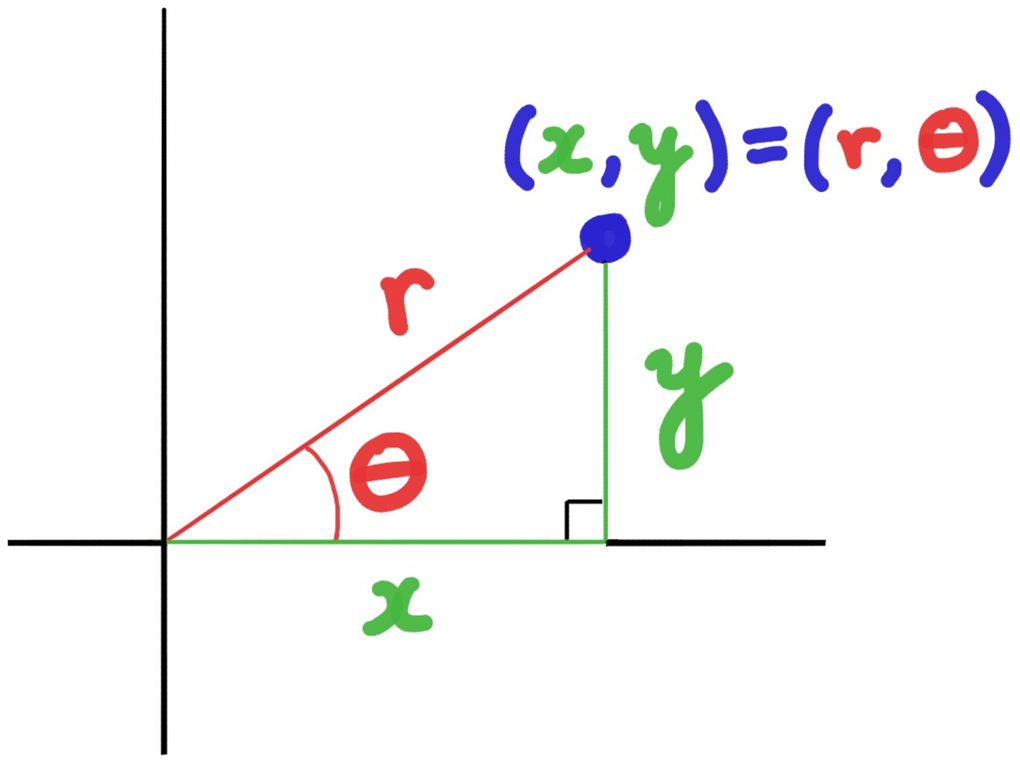


$2 \le r \le 3$

$\pi/3 \le \theta \le 2\pi/3$

polar

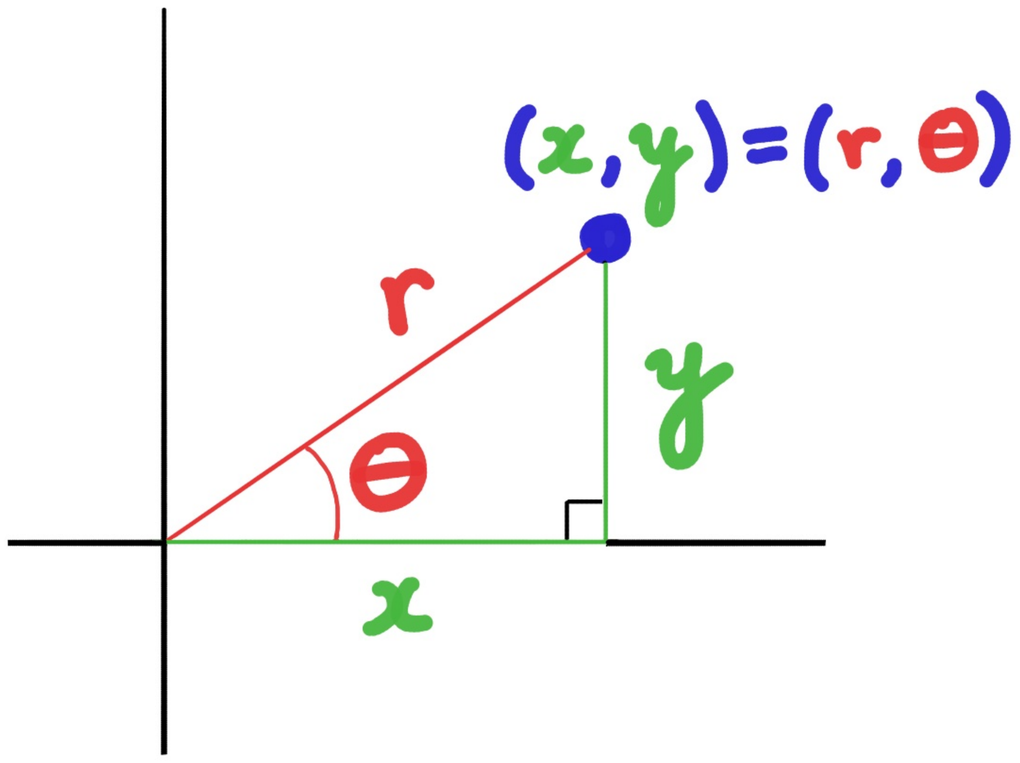
Cartesian / Polar translator



$$\cos\theta = \frac{x}{r}$$

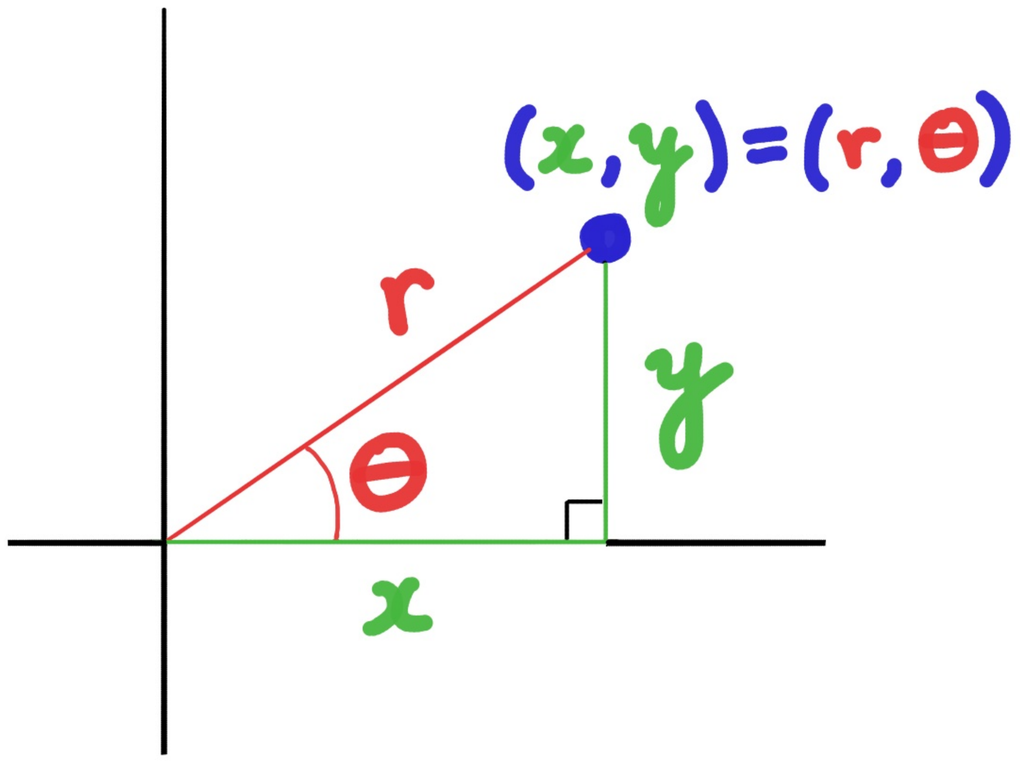
$$\sin\theta = \frac{y}{r}$$

Cartesian / Polor translator



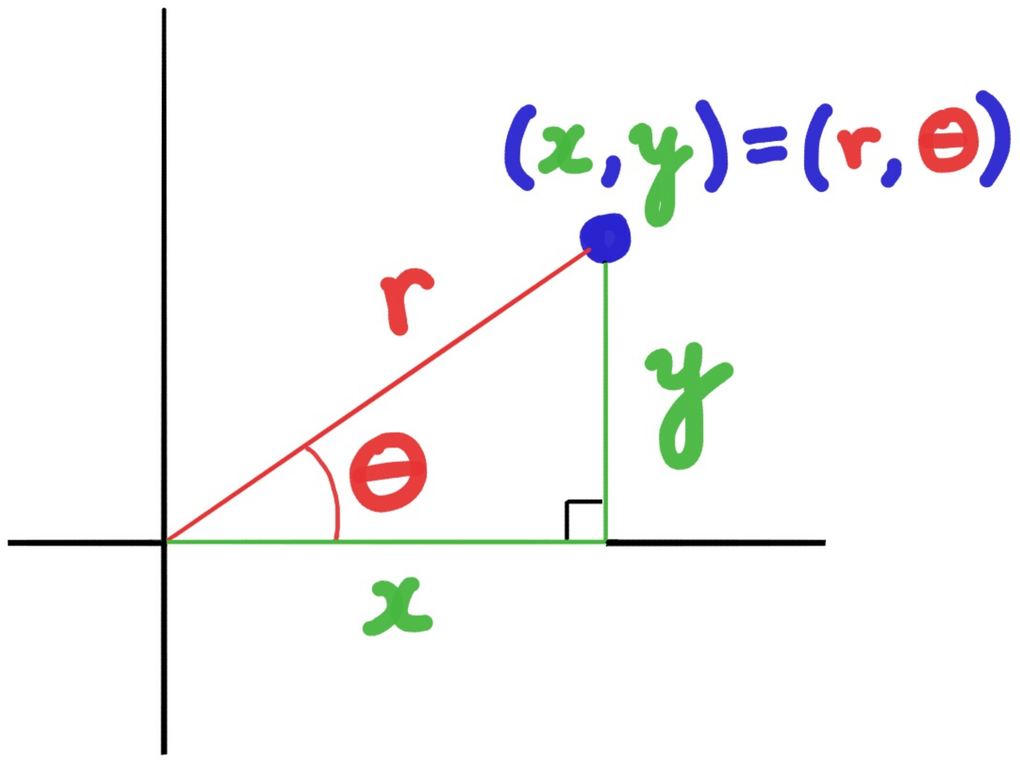
$$\left.\begin{aligned}\cos\theta &= \frac{x}{r}\\ \sin\theta &= \frac{y}{r}\end{aligned}\right\} \Rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Cartesian / Polar translator



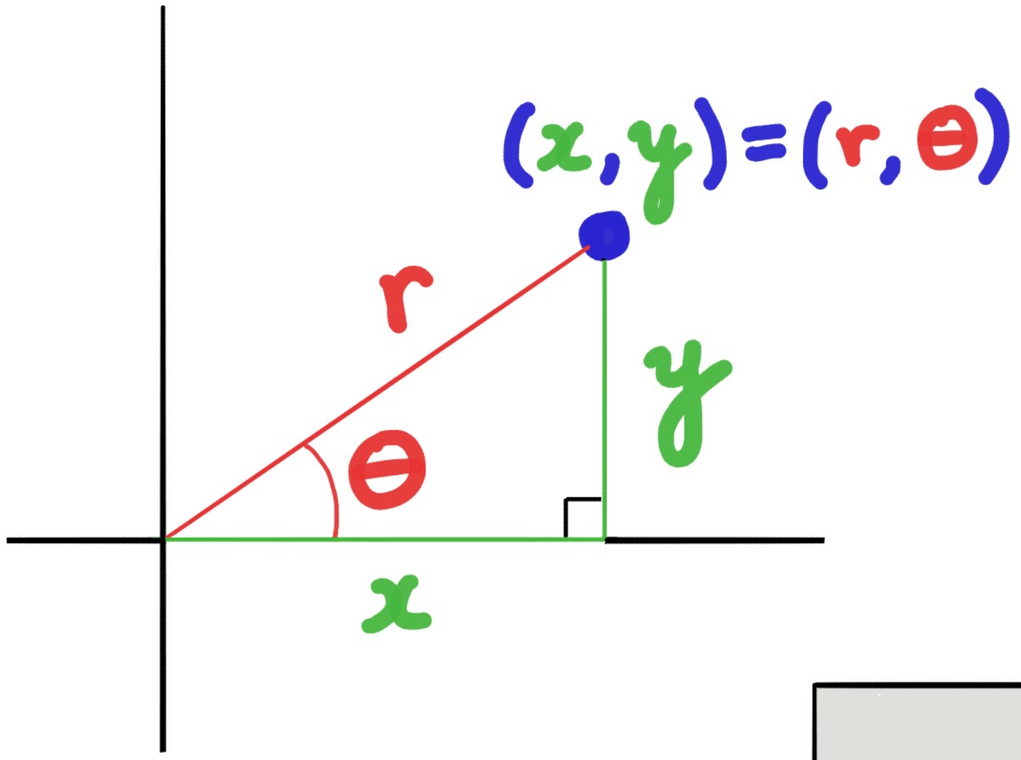
$$r = \|(x, y)\|$$

Cartesian / Polar translator



$$r = \|(x,y)\| \quad \Rightarrow \quad \begin{cases} r = \sqrt{x^2+y^2} \\ \text{or} \\ r^2 = x^2+y^2 \end{cases}$$

Cartesian / Polar translator



$$x = r\cos\theta$$

$$y = r\sin\theta$$

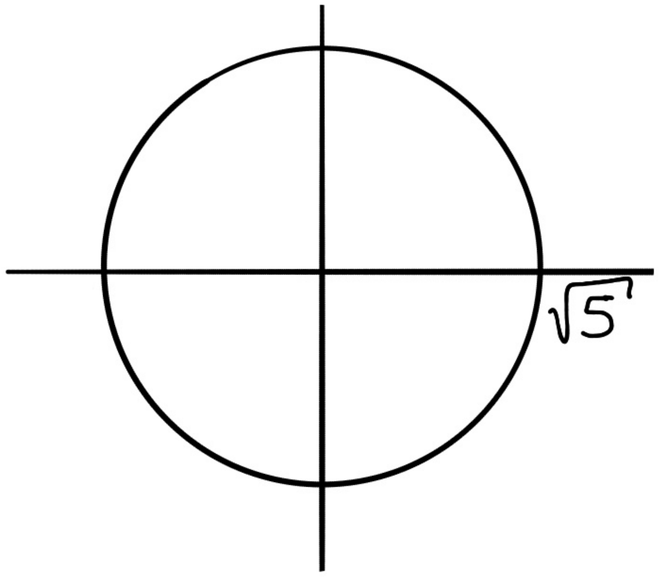
$$r = \sqrt{x^2 + y^2}$$

or

$$r^2 = x^2 + y^2$$

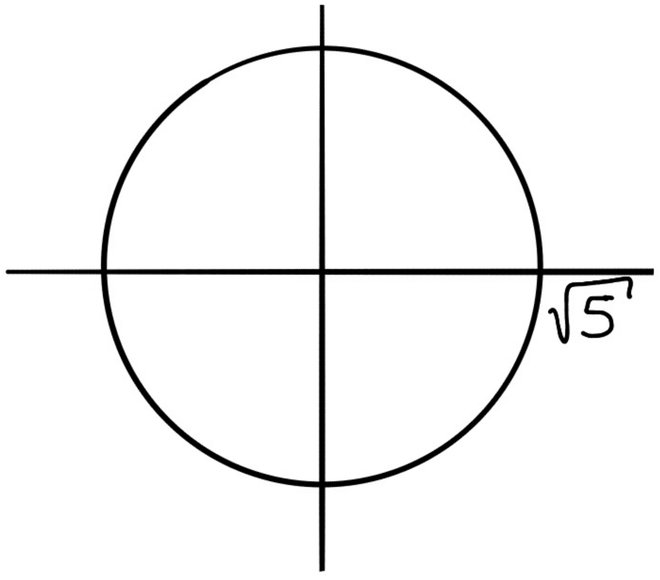
Problem:

Convert the Cartesian equation $x^2+y^2=5$ into a polar equation.



Problem:

Convert the Cartesian equation $x^2+y^2=5$ into a polar equation.



Solution:

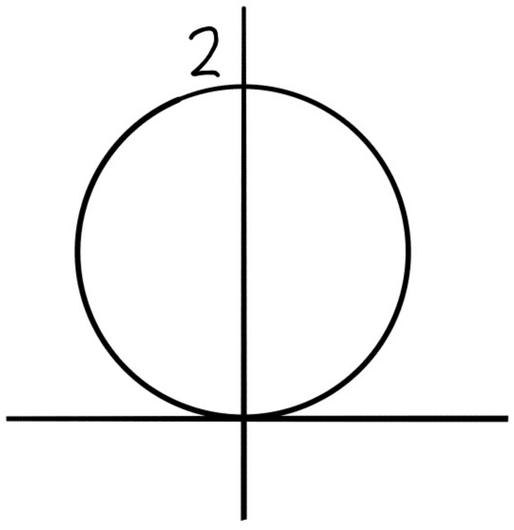
$$x^2+y^2=r^2.$$

Substituting, we have

$$r^2=5 \quad , \text{ or } \quad r=\sqrt{5}.$$

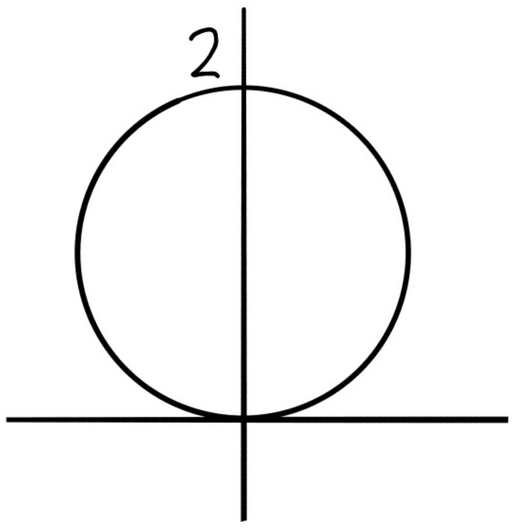
Problem:

Convert the polar equation $r = 2\sin\theta$ into a Cartesian equation.



Problem:

Convert the polar equation $r = 2\sin\theta$ into a Cartesian equation.



Solution:

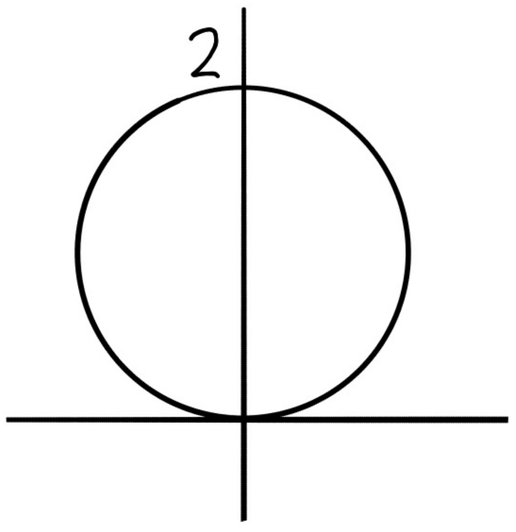
$y = r\sin\theta$, so $\sin\theta = \frac{y}{r}$.

Substituting, $r = 2\frac{y}{r}$, so

$r^2 = 2y$.

Problem:

Convert the polar equation $r = 2\sin\theta$ into a Cartesian equation.



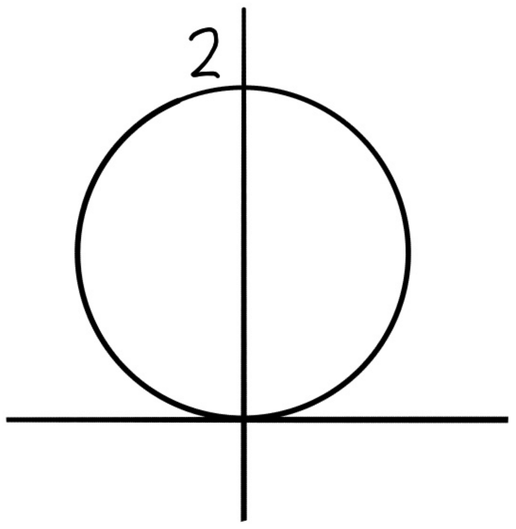
Solution:

$y = r\sin\theta$, so $\sin\theta = \frac{y}{r}$.

Substituting, $r = 2\frac{y}{r}$, so $r^2 = 2y$. Substituting x^2+y^2 for r^2 yields $x^2+y^2 = 2y$

Problem:

Convert the polar equation $r = 2\sin\theta$ into a Cartesian equation.

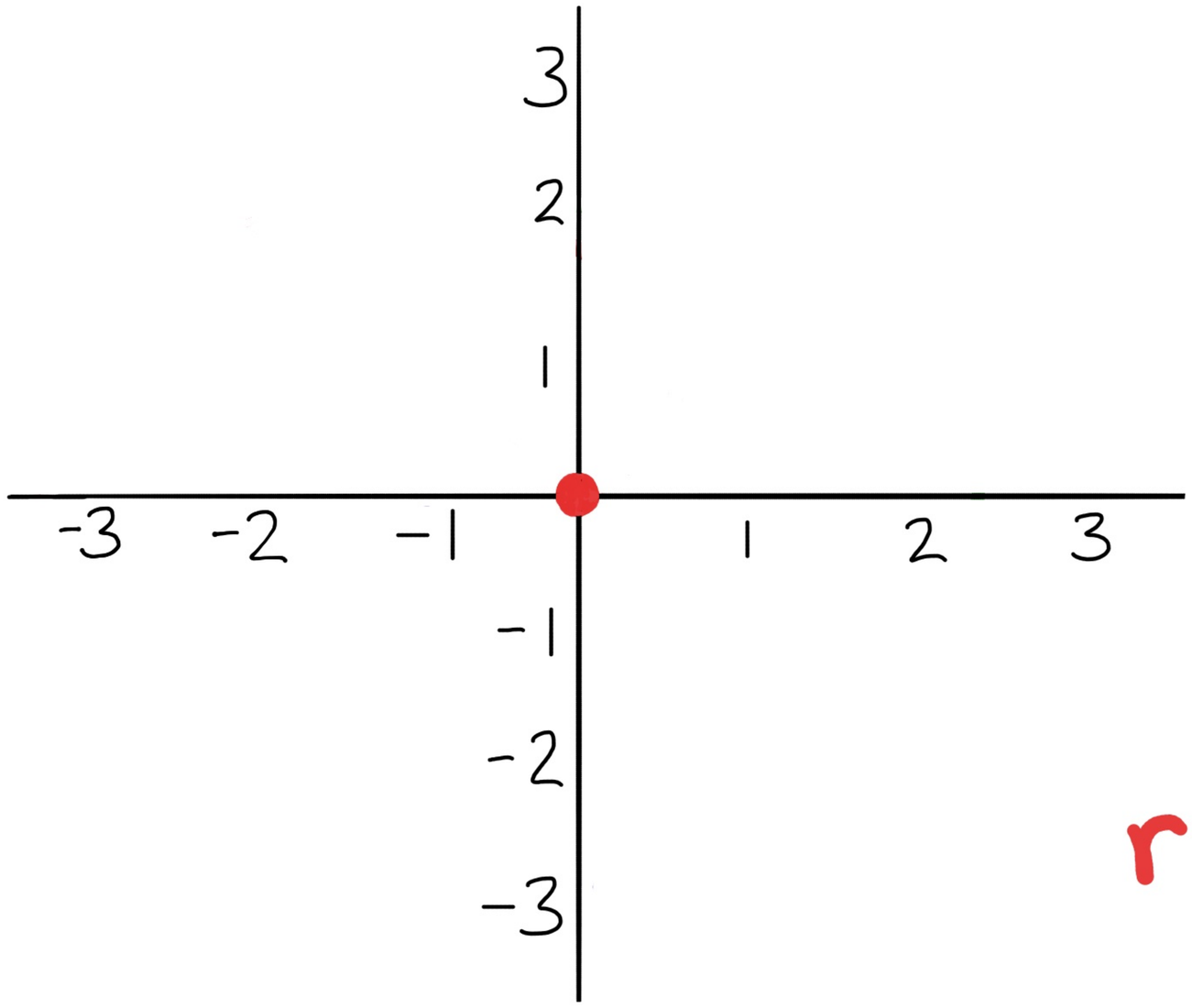


Solution:

$y = r\sin\theta$, so $\sin\theta = \frac{y}{r}$.

Substituting, $r = 2\frac{y}{r}$, so $r^2 = 2y$. Substituting x^2+y^2 for r^2 yields $x^2+y^2 = 2y$, or $(x-0)^2+(y-1)^2 = 1$.

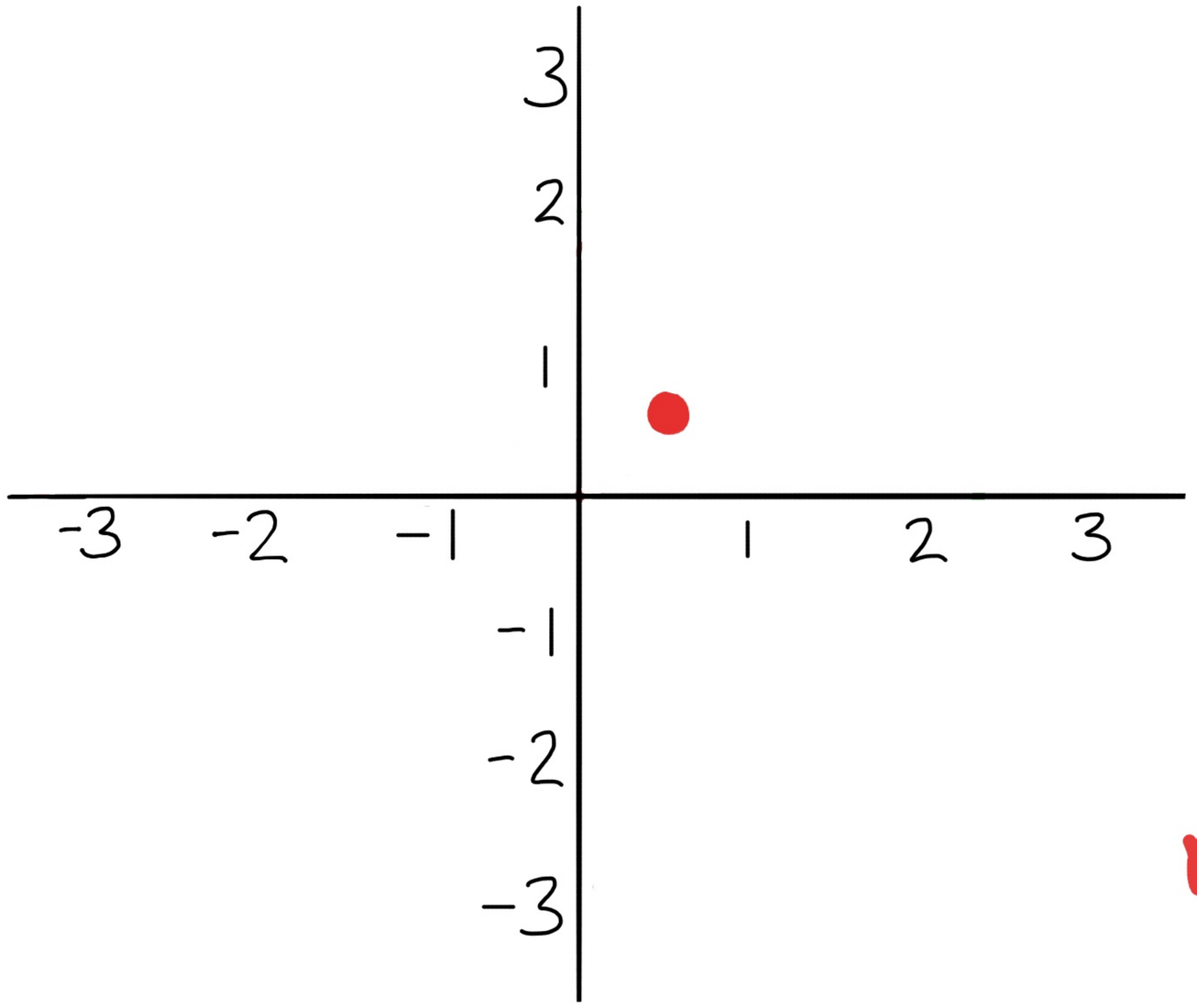
$\mathbb{R}^2$



$r = 0 = \theta$

polar

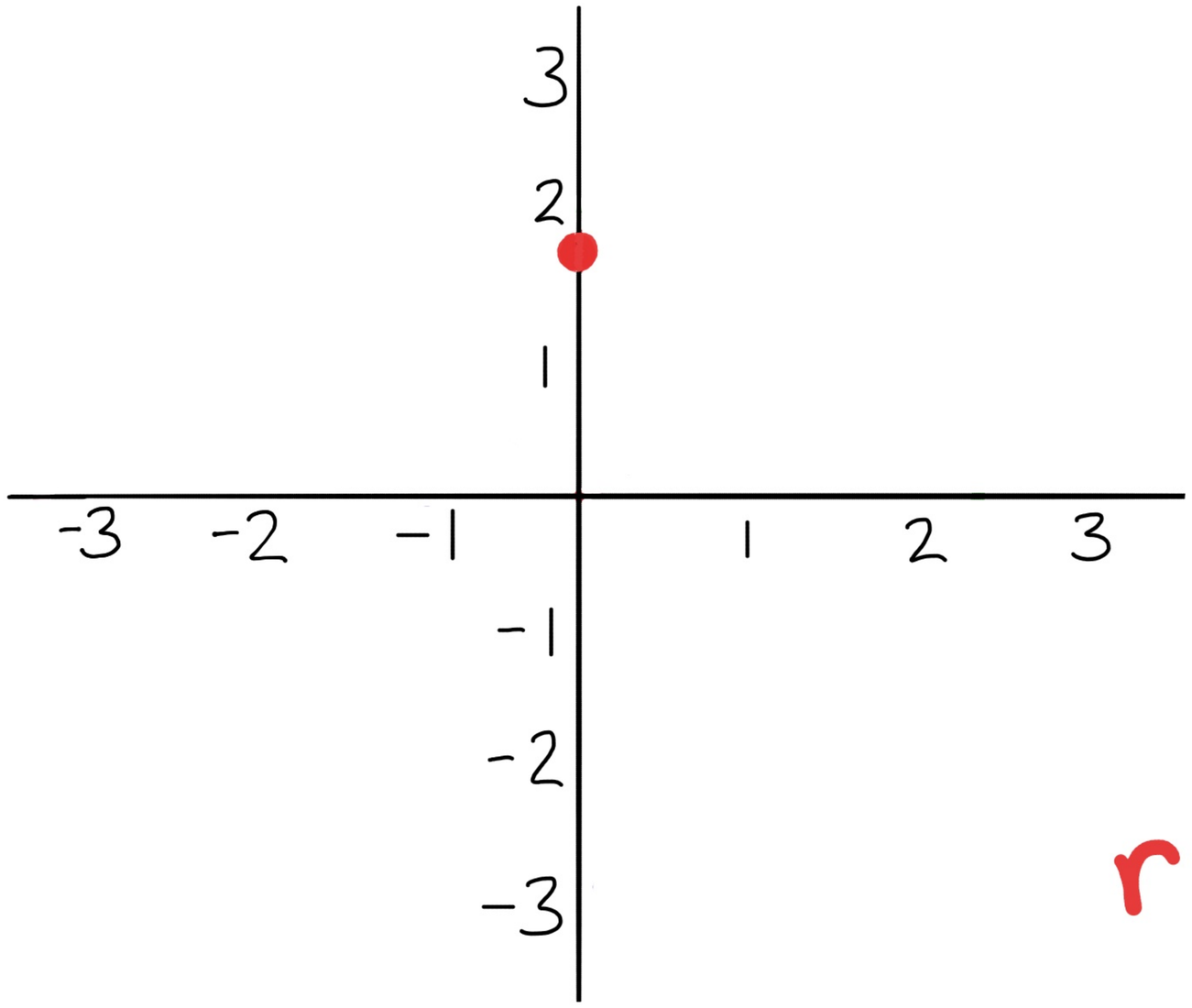
$\mathbb{R}^2$



$r = \frac{\pi}{4} = \theta$

polar

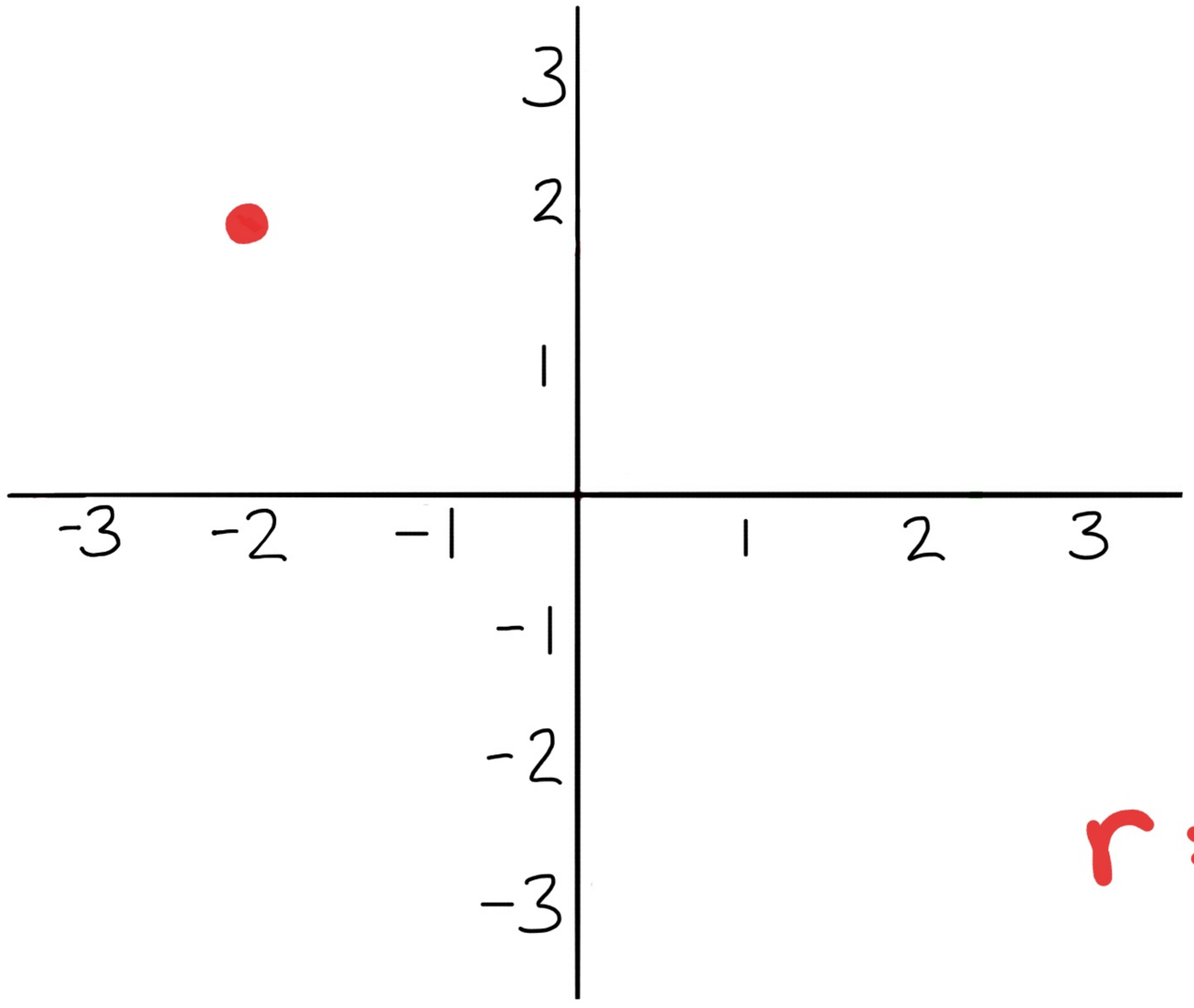
$\mathbb{R}^2$



$r = \frac{\pi}{2} = \theta$

polar

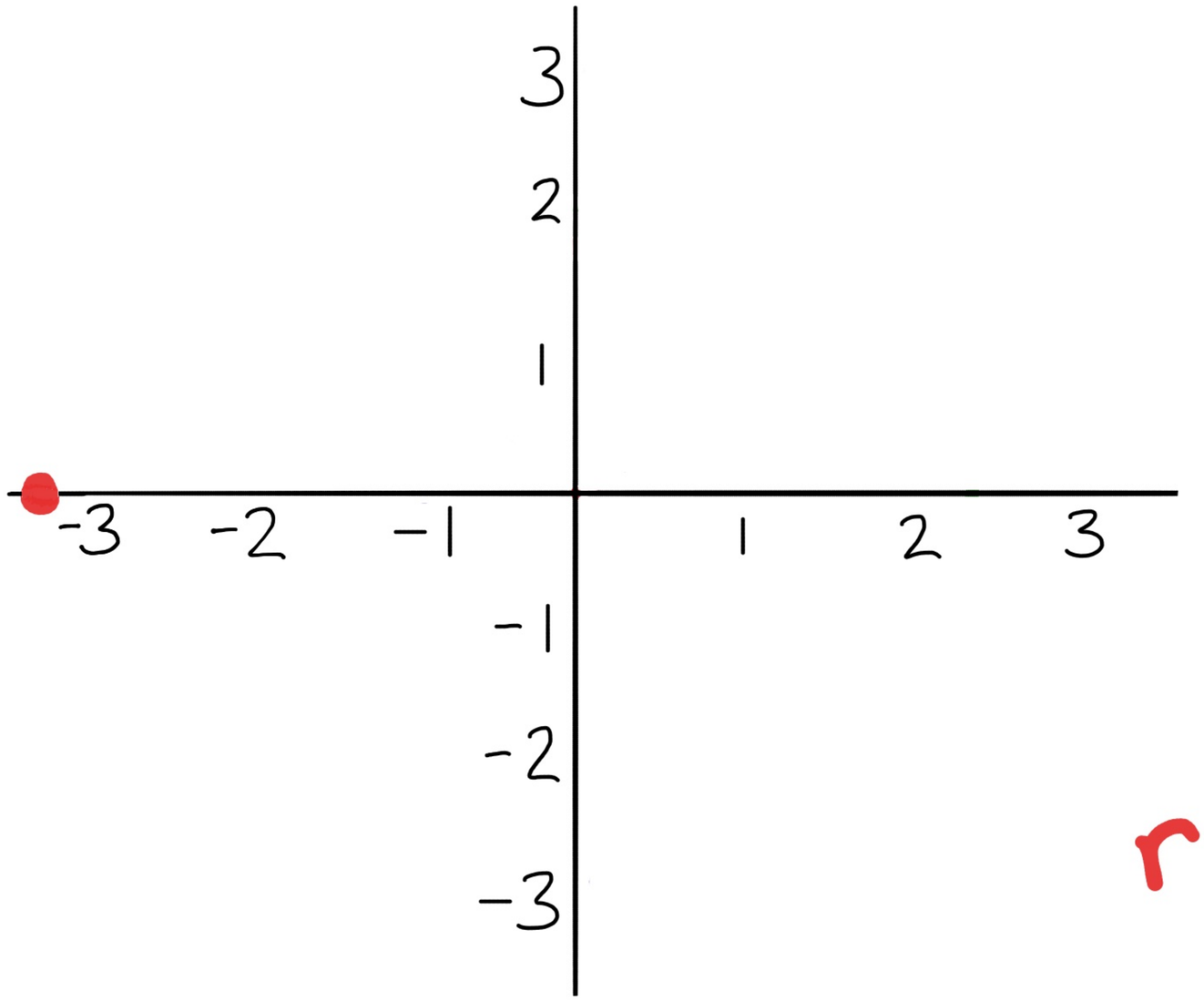
$\mathbb{R}^2$



$r = \frac{3\pi}{4} = \theta$

polar

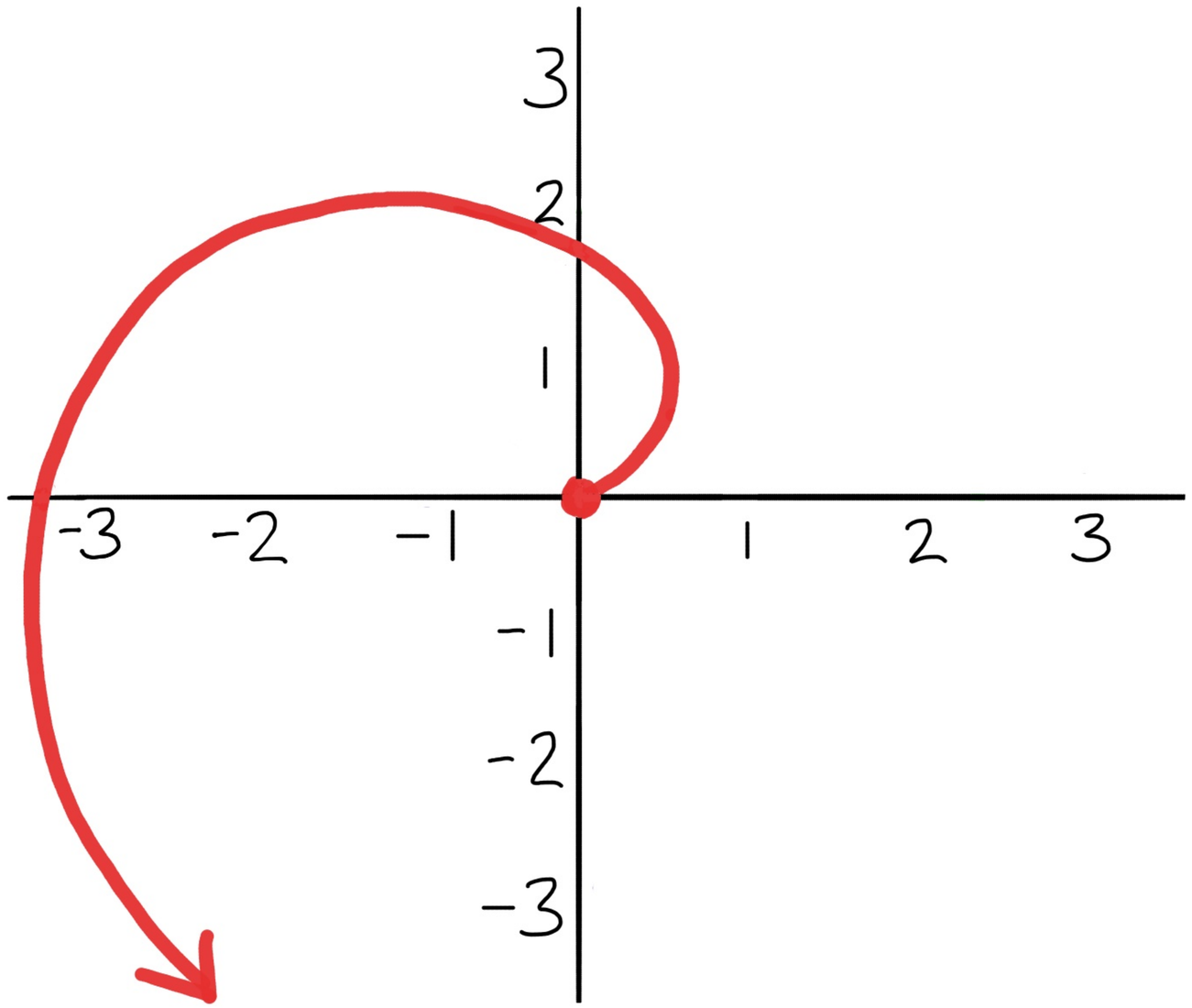
$\mathbb{R}^2$



$r = \pi = \theta$

polar

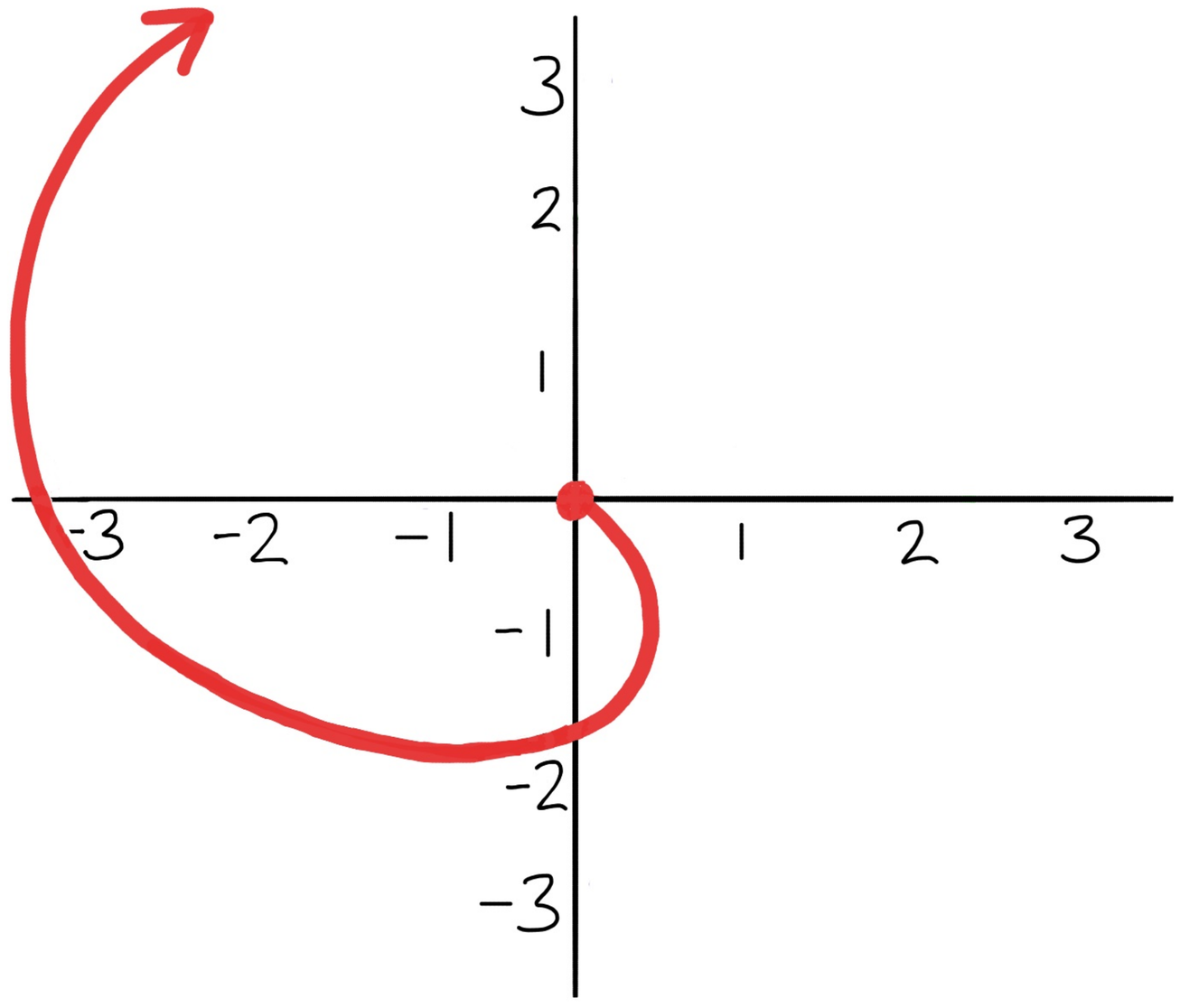
$\mathbb{R}^2$



$r = \theta$

polar

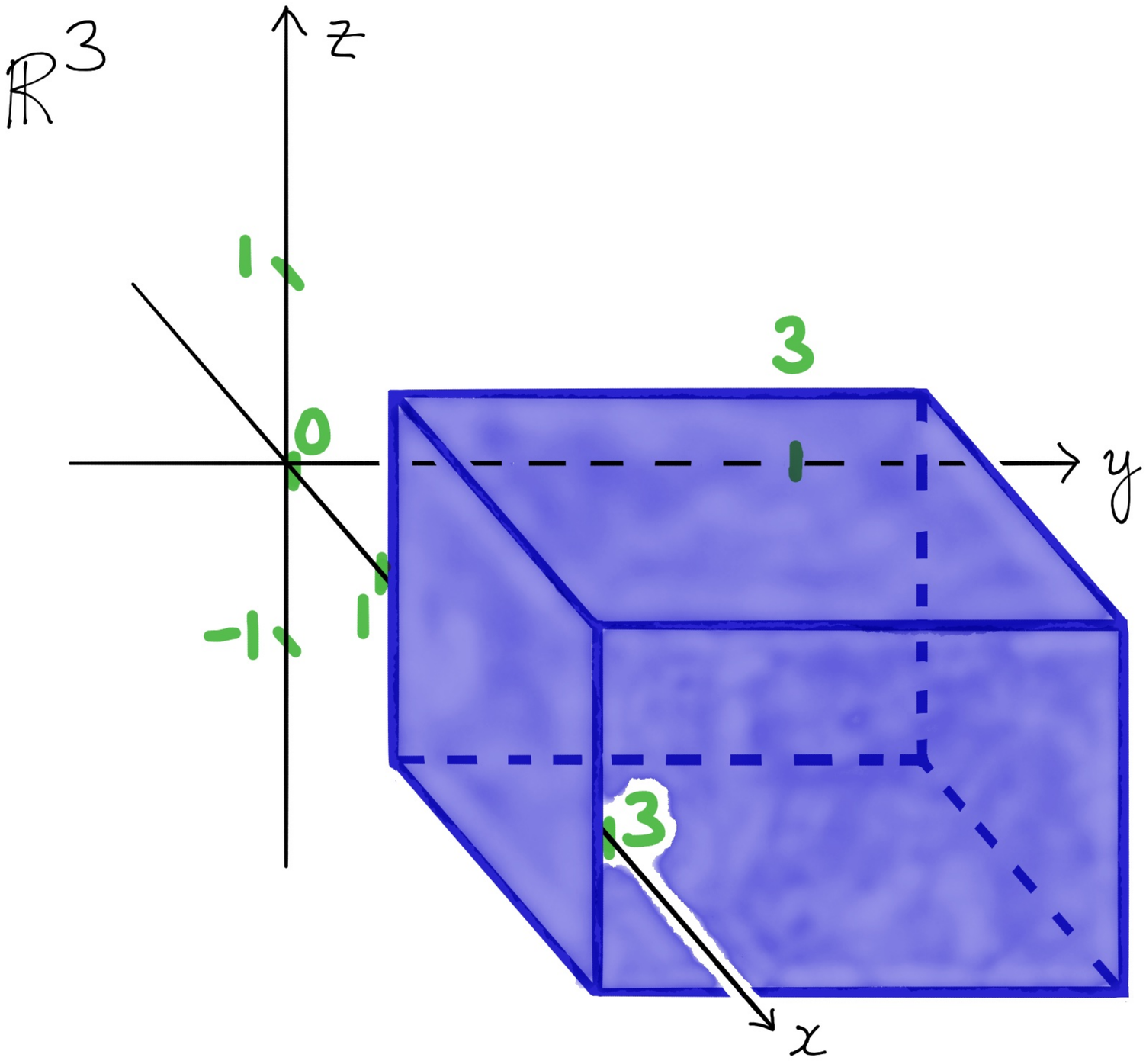
$\mathbb{R}^2$



$r = -\theta$

polar

ⅡCylindrical coordinates in $\mathbb{R}^3$



$1 \leq x \leq 3$

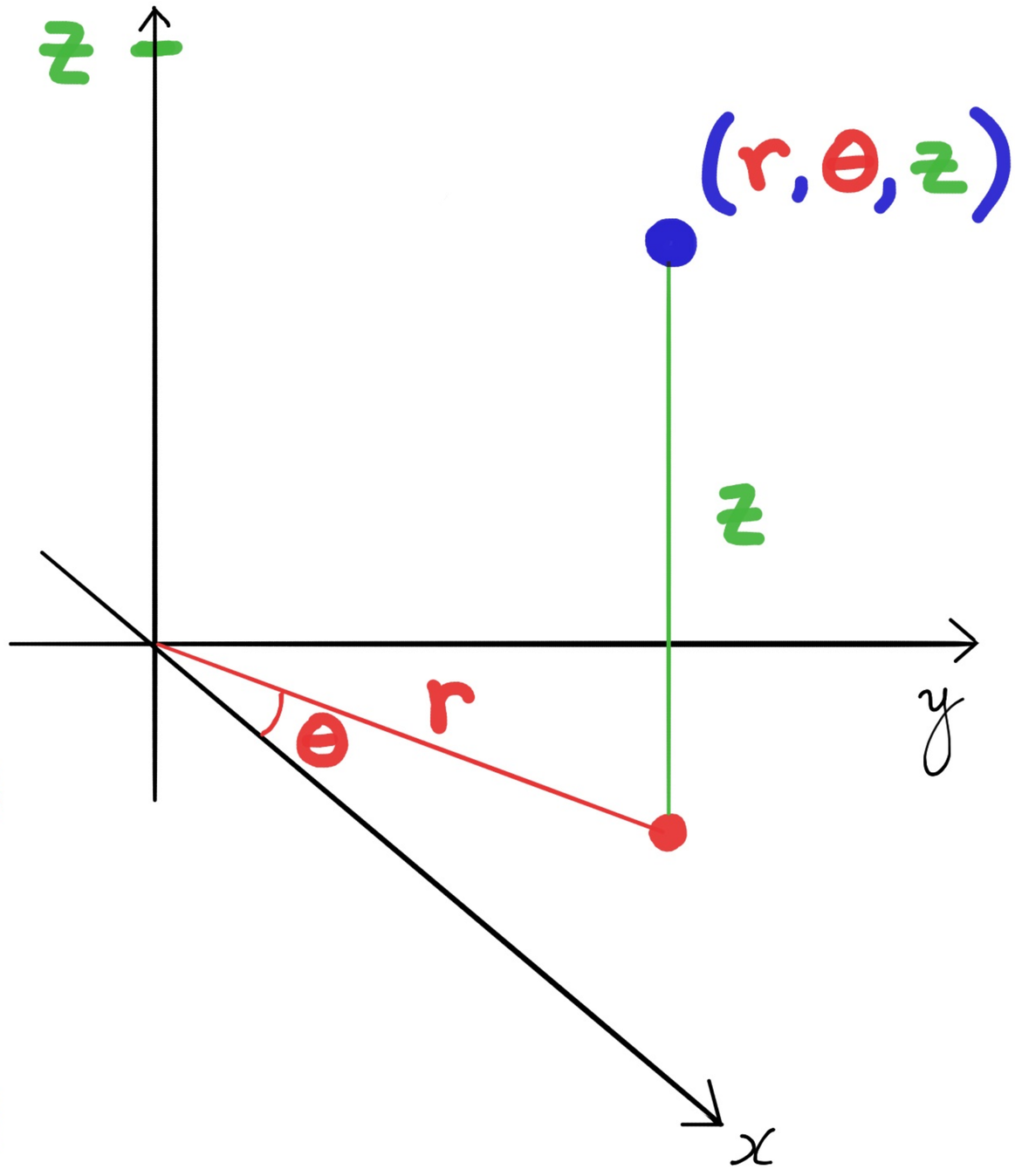
$0 \leq y \leq 3$

$-1 \leq z \leq 1$

Cartesian

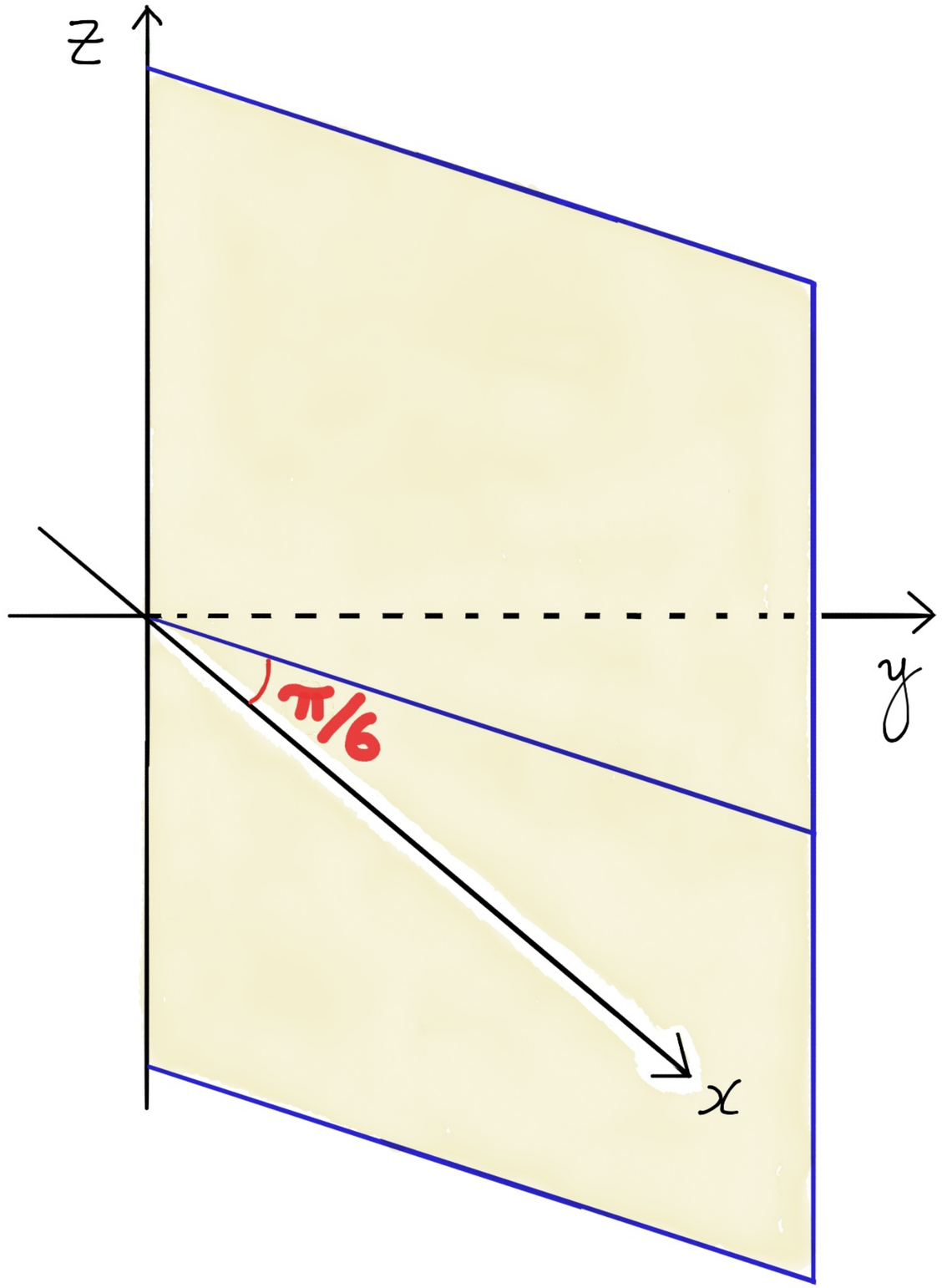
Cylindrical

$\mathbb{R}^3$



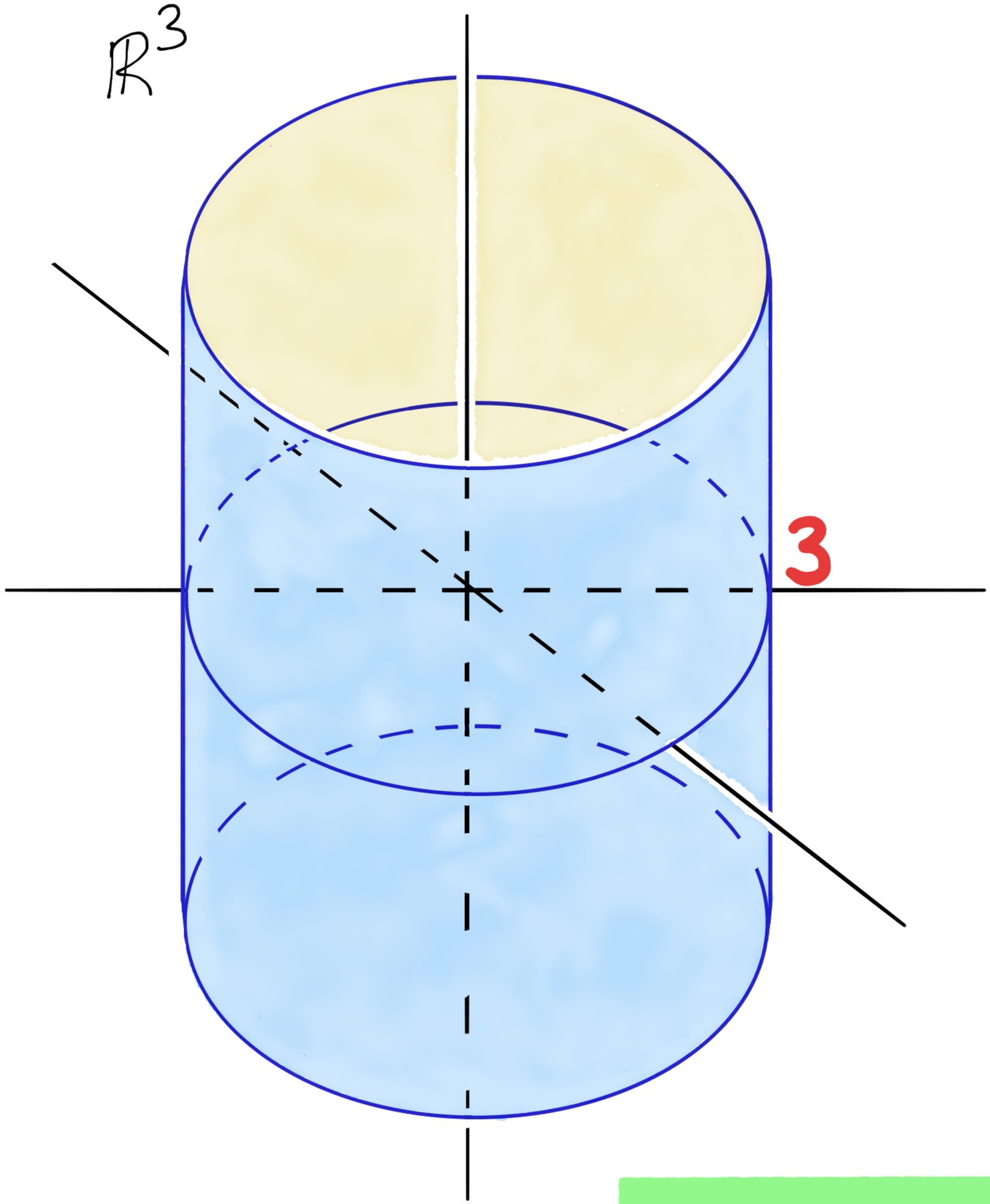
- polar in xy-plane
- Cartesian in z-axis

$\mathbb{R}^3$



$\theta = \pi/6$

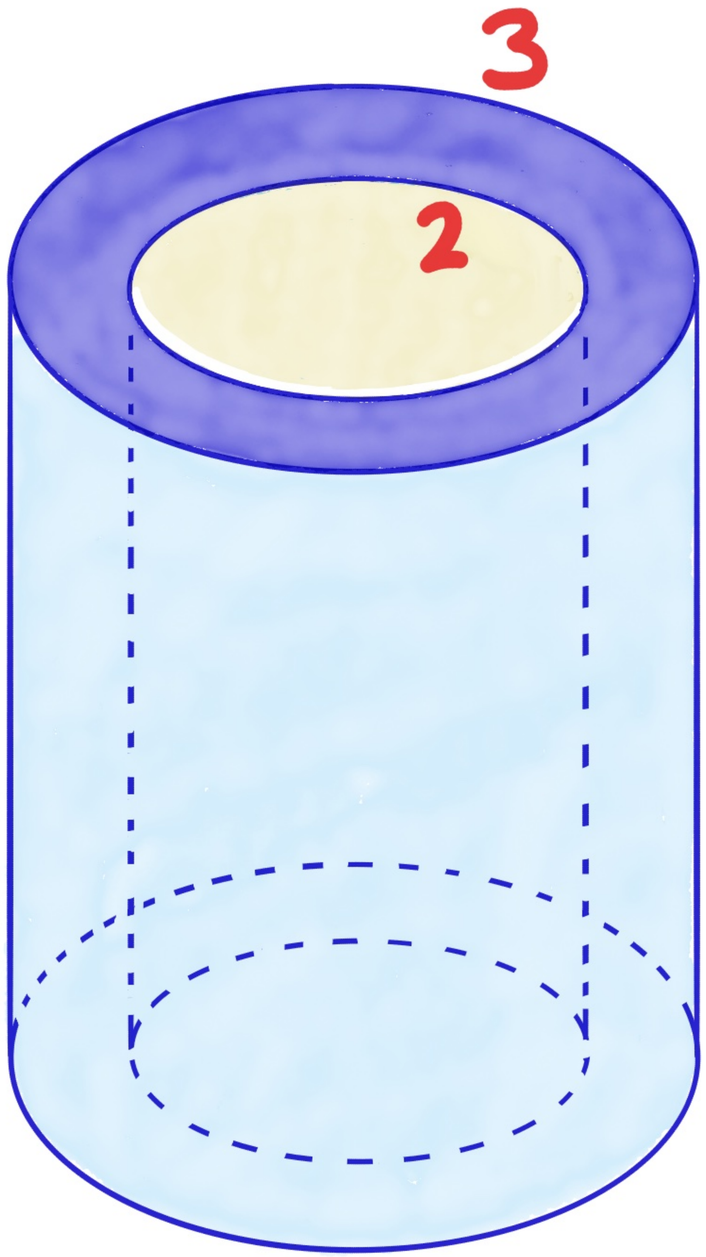
Cylindrical



$r=3$

Cylindrical

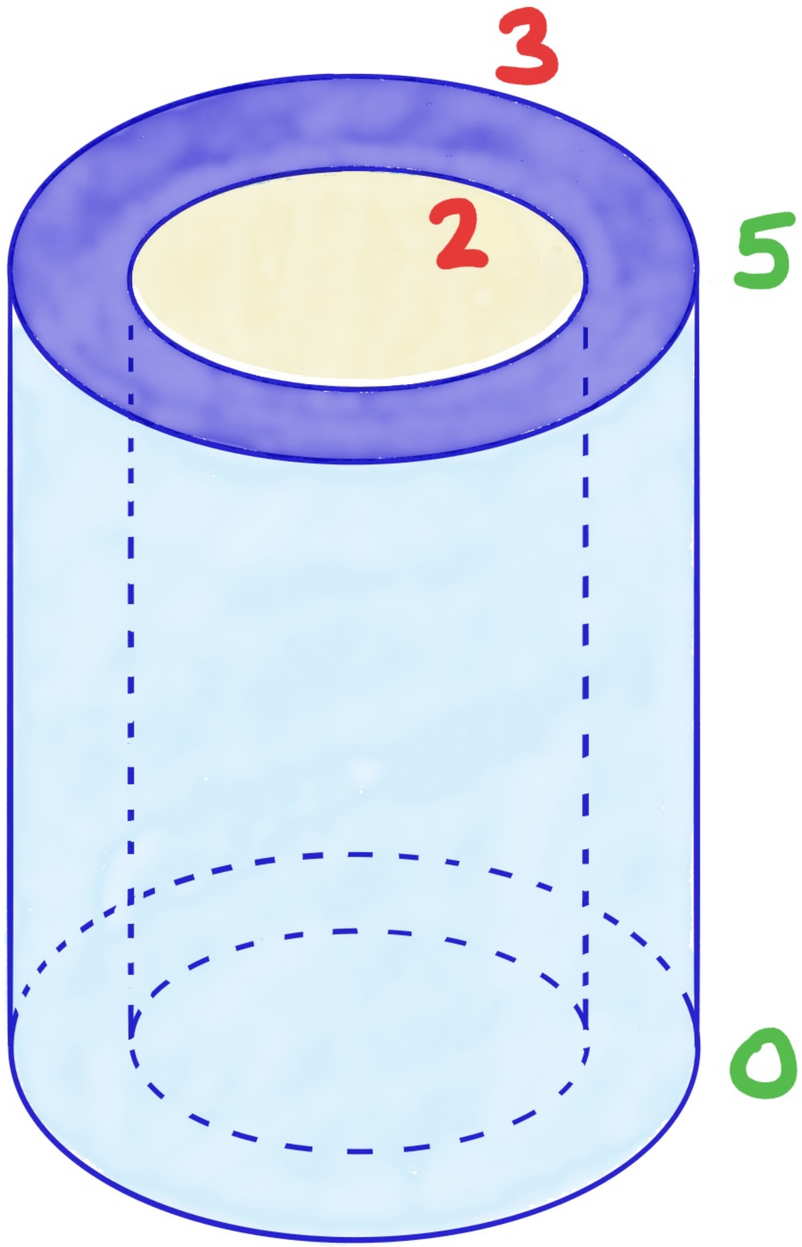
$\mathbb{R}^3$



$2 \leq r \leq 3$

Cylindrical

$\mathbb{R}^3$

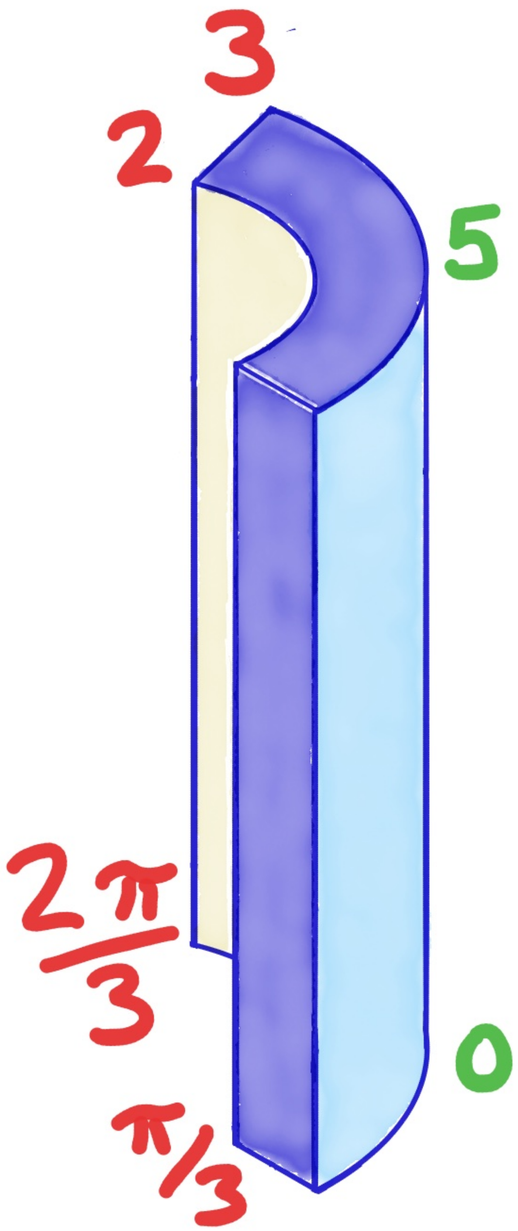


$2 \le r \le 3$

$0 \le z \le 5$

Cylindrical

$\mathbb{R}^3$



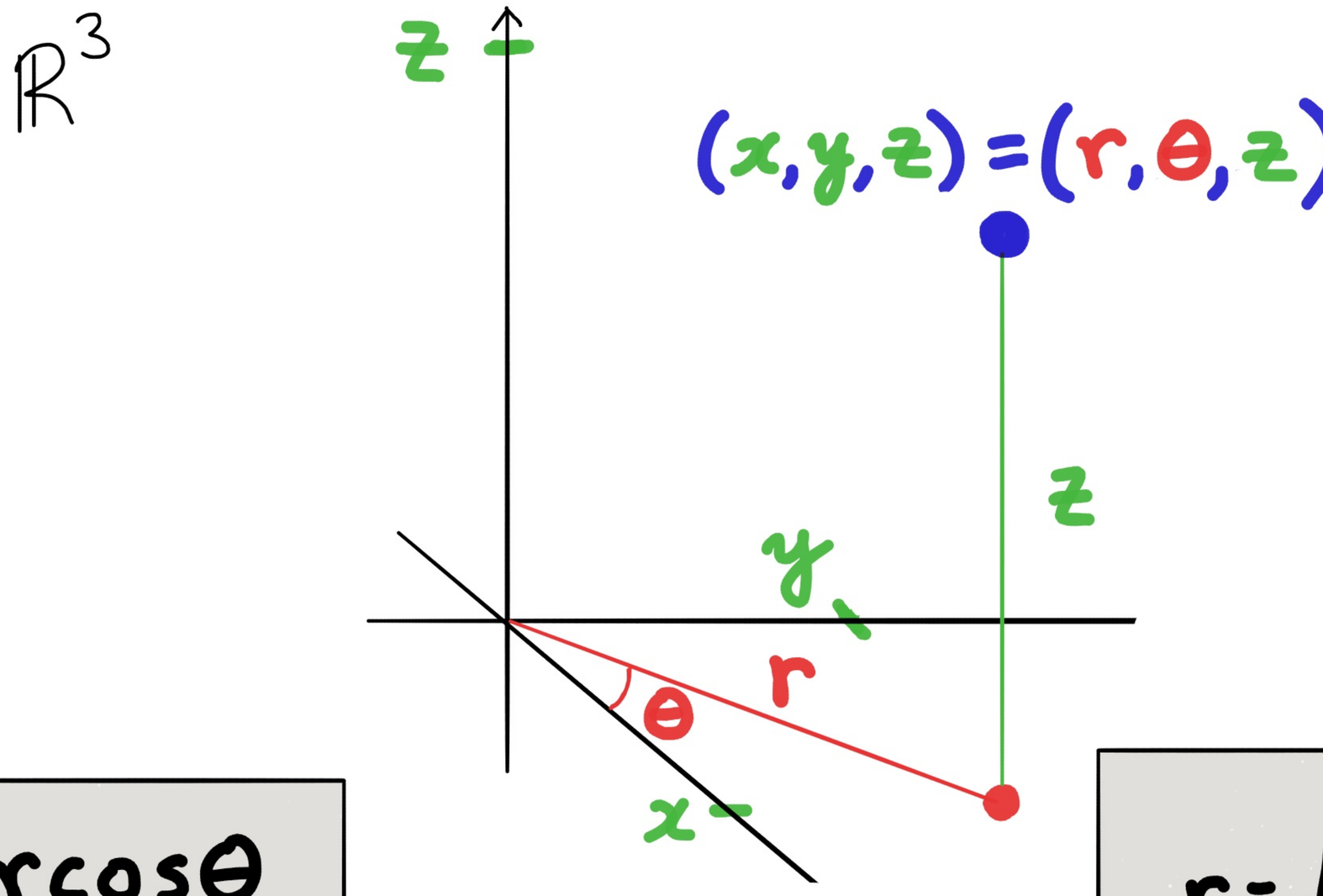
$2 \le r \le 3$

$\pi/3 \le \theta \le 2\pi/3$

$0 \le z \le 5$

Cylindrical

Cartesian/cylindrical translator

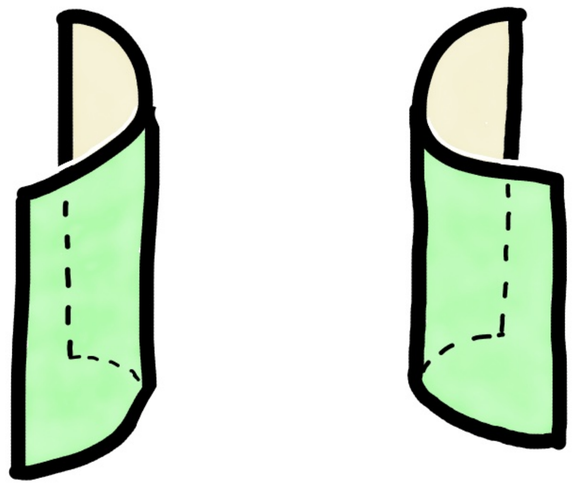


$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$r = \sqrt{x^2+y^2}$$
$$r^2 = x^2+y^2$$
$$z = z$$

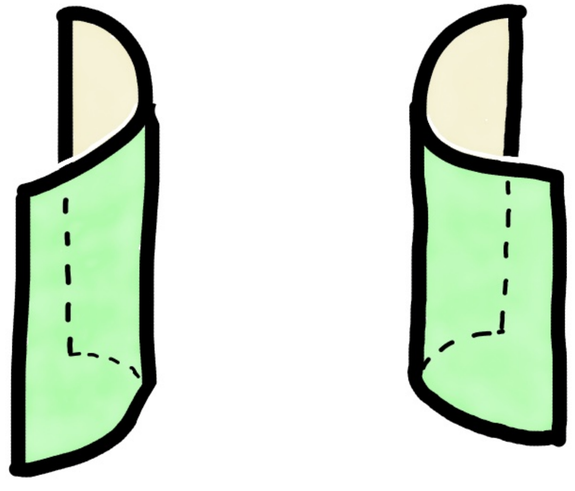
Problem:

Write the Cartesian equation $y^2-x^2=9$ as a cylindrical equation.



Problem:

Write the Cartesian equation $y^2-x^2=9$ as a cylindrical equation.



Solution:

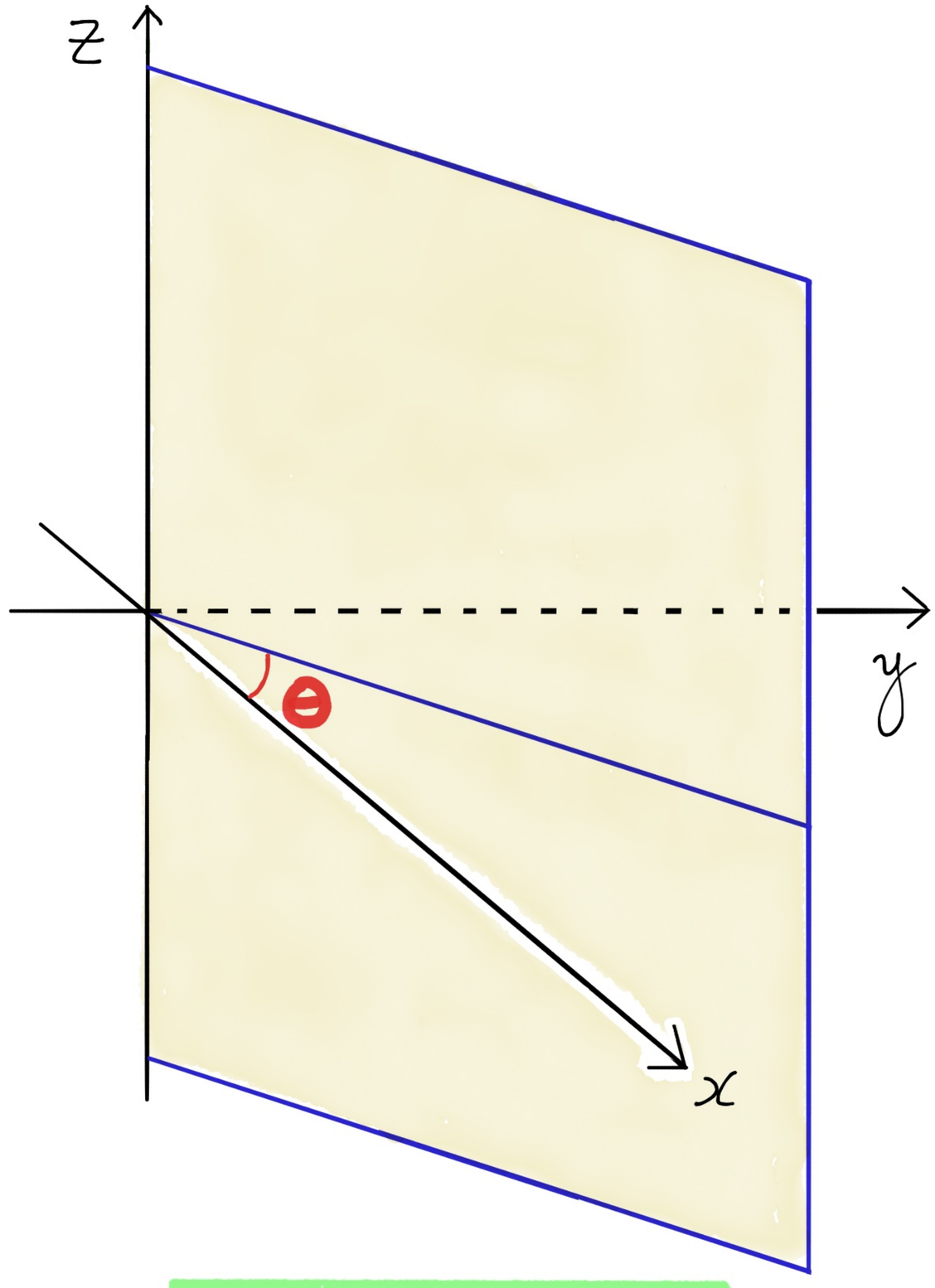
$$\begin{aligned} 9 &= y^2-x^2 \\ &= (r\sin\theta)^2-(r\cos\theta)^2 \\ &= r^2(\sin^2\theta-\cos^2\theta) \\ &= r^2(-\cos(2\theta)) \end{aligned}$$

$$r^2\cos(2\theta)=-9$$

Ⅲ Spherical coordinates in $\mathbb{R}^3$

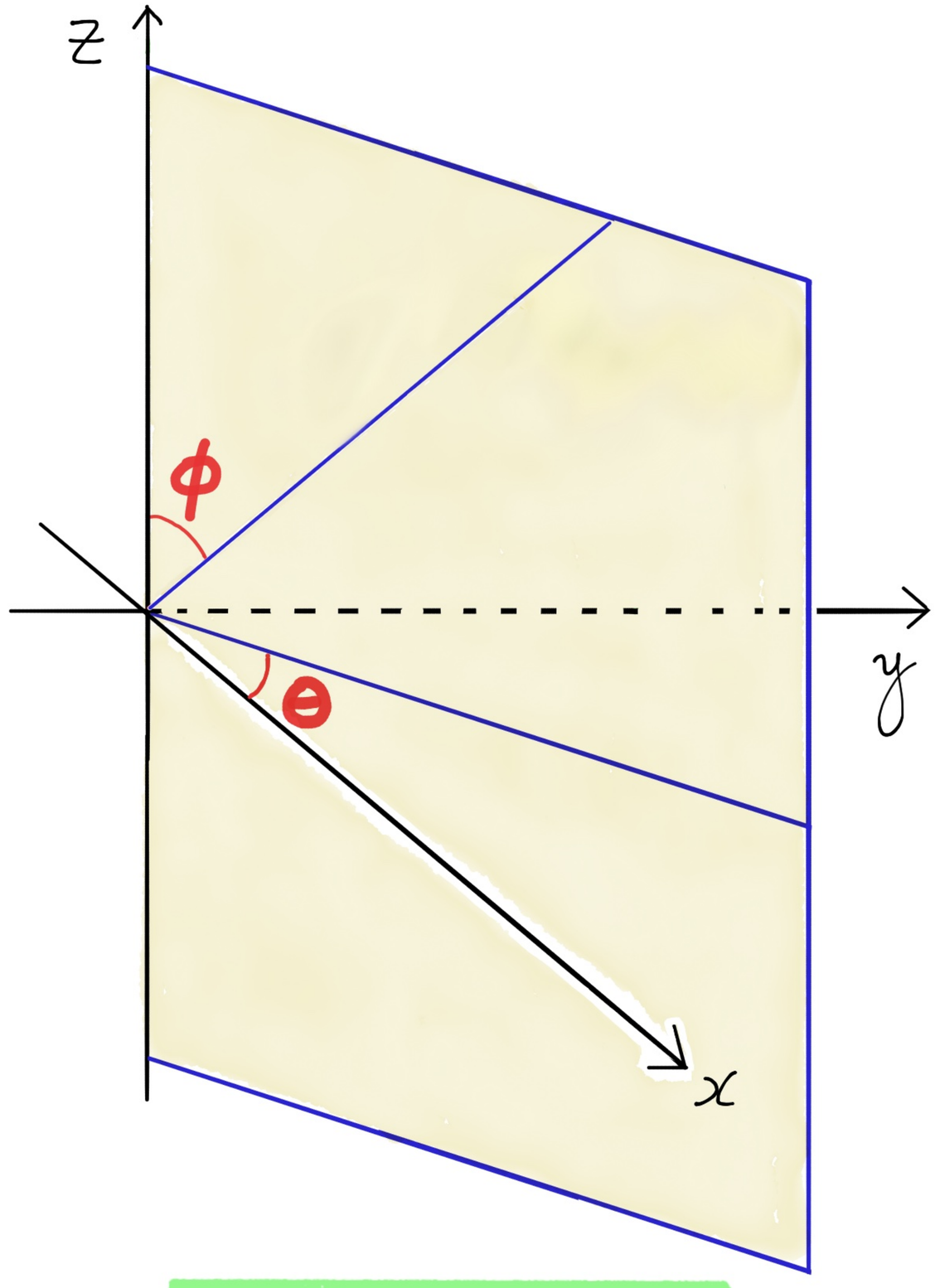
$\mathbb{R}^3$

- $0 \le \theta \le 2\pi$



Spherical

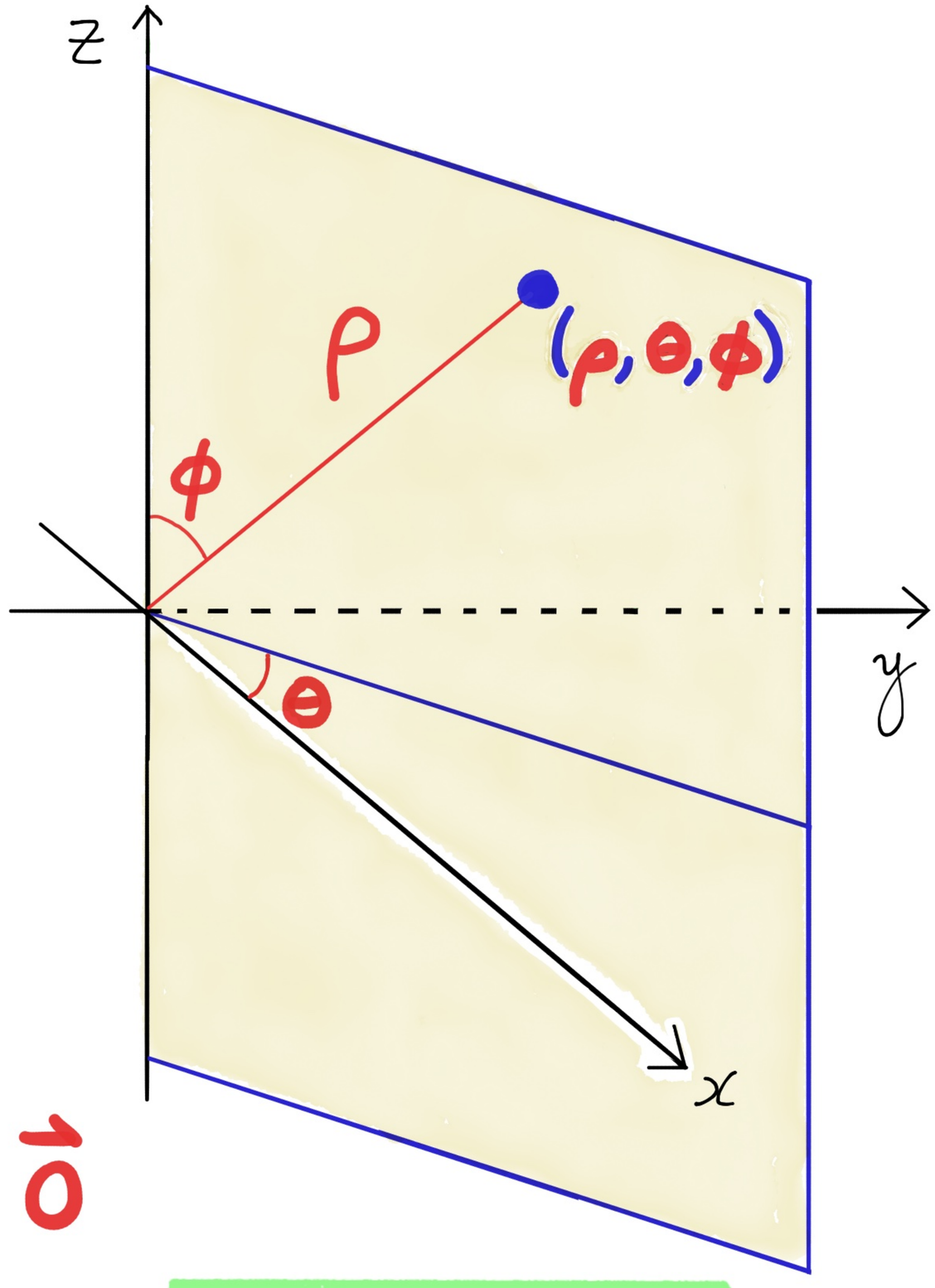
$\mathbb{R}^3$



- $0 \le \theta \le 2\pi$
- $0 \le \phi \le \pi$

Spherical

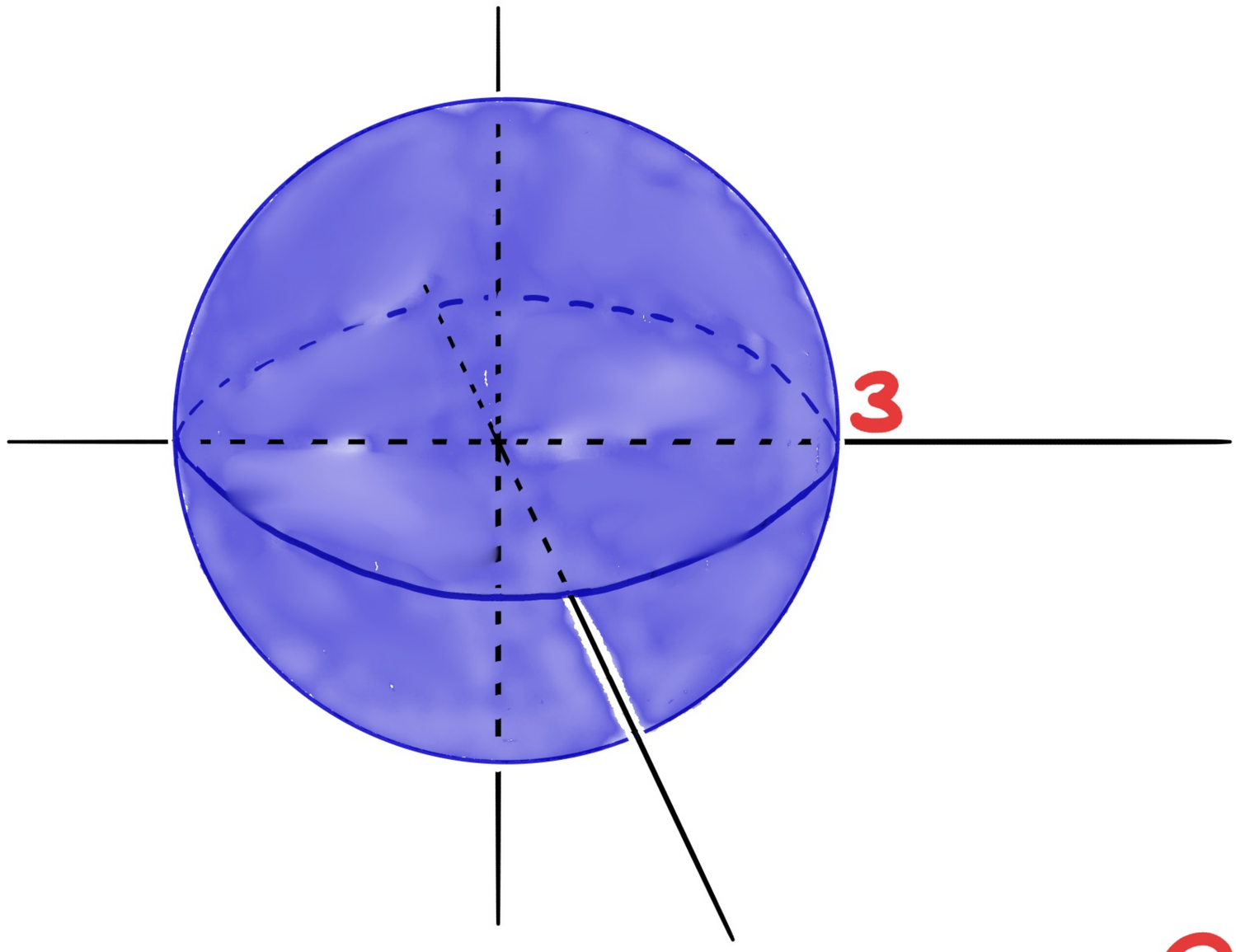
$\mathbb{R}^3$



- $0 \le \theta \le 2\pi$
- $0 \le \phi \le \pi$
- $\rho \ge 0$ is distance to $\vec{0}$

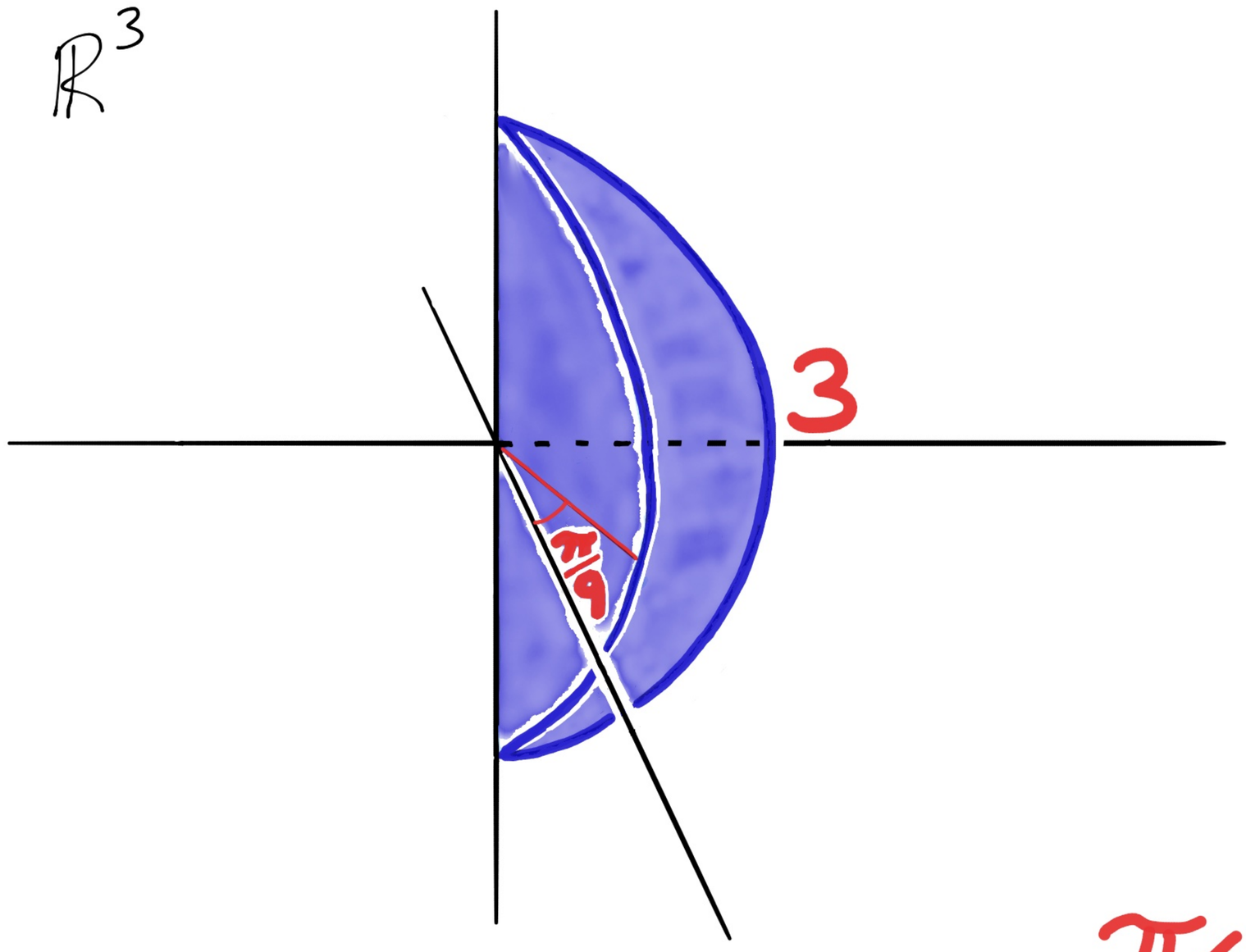
Spherical

$\mathbb{R}^3$



$\rho = 3$

Spherical

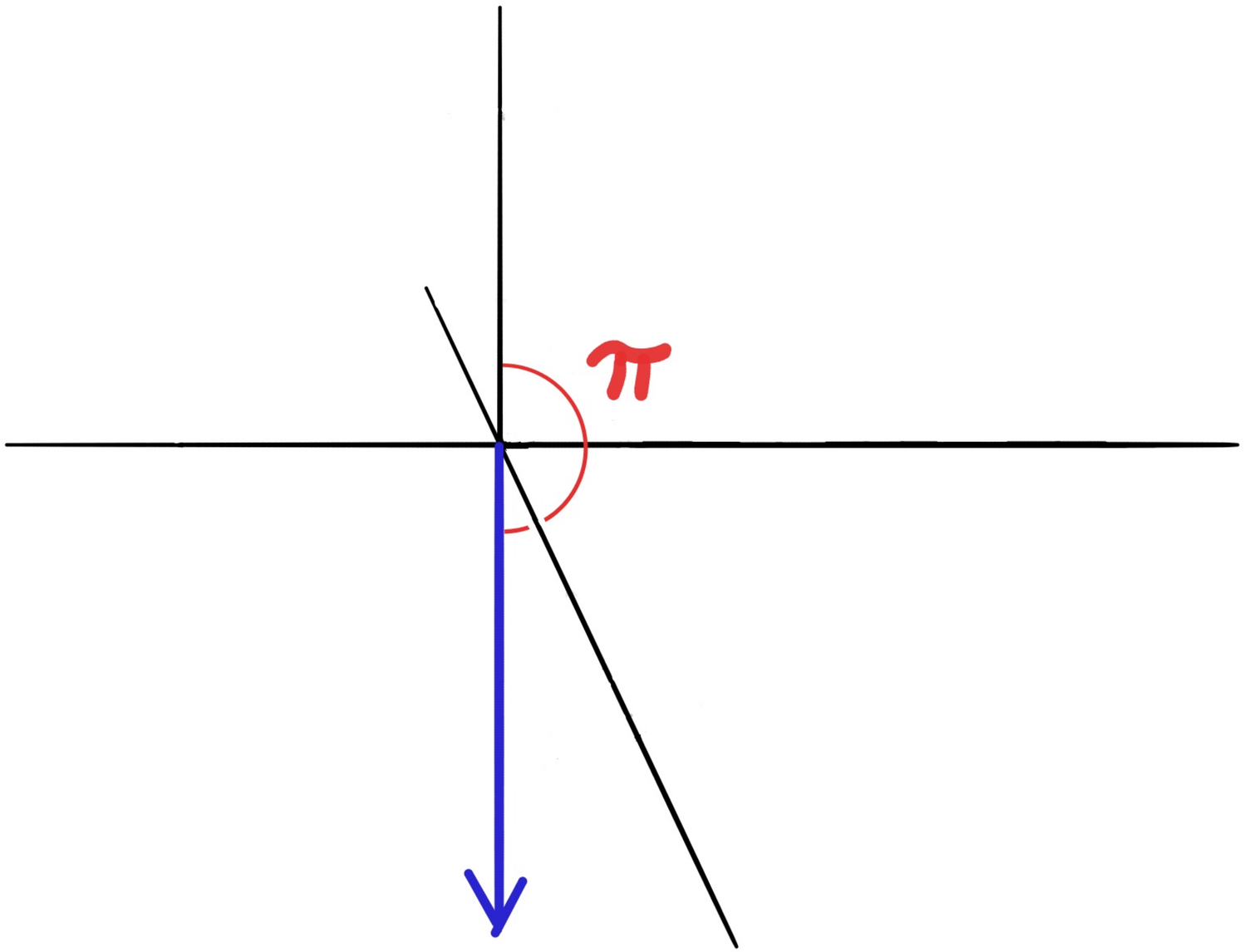


$\rho \leq 3$

$\pi/9 \leq \theta \leq \pi/4$

Spherical

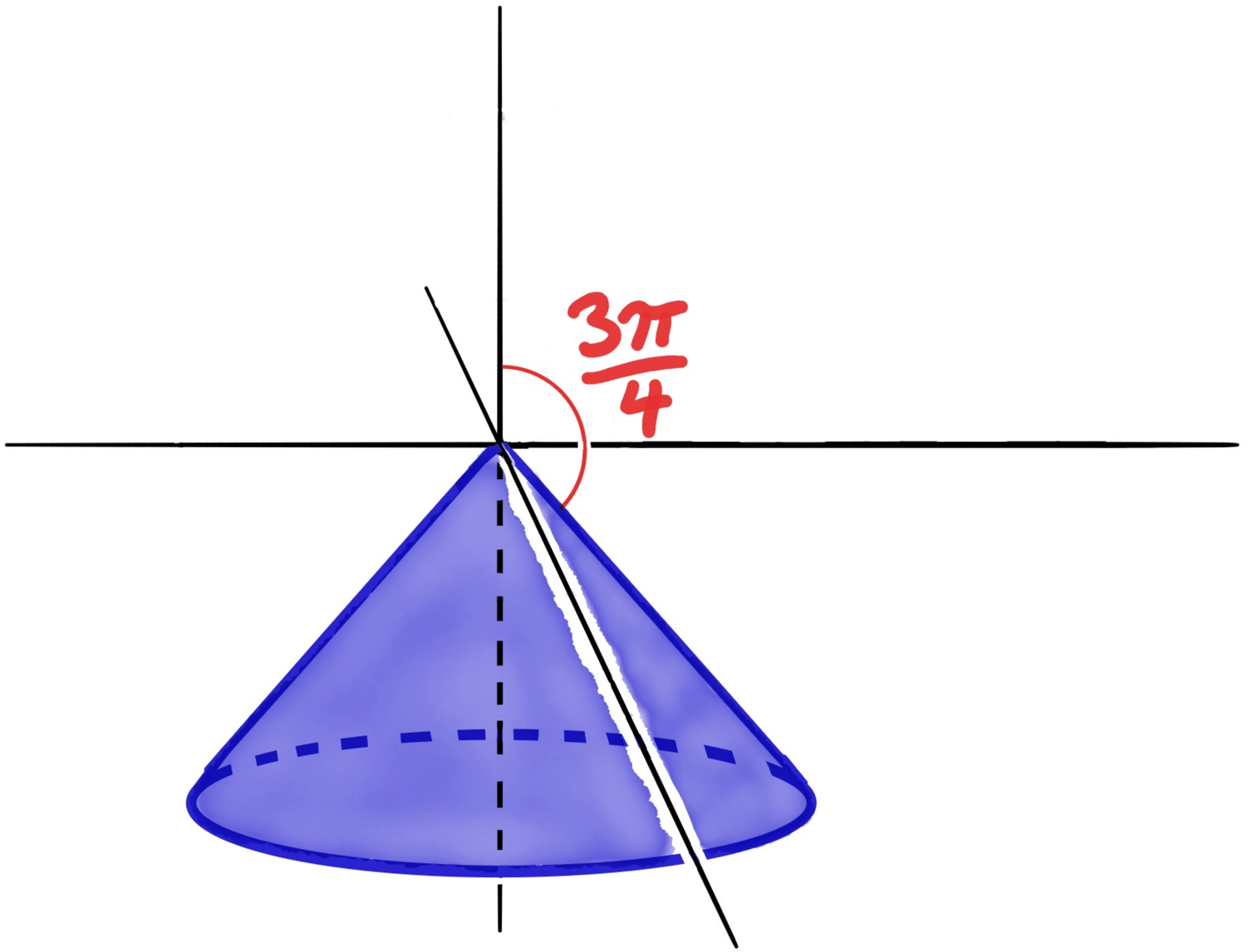
$\mathbb{R}^3$



$\phi = \pi$

Spherical

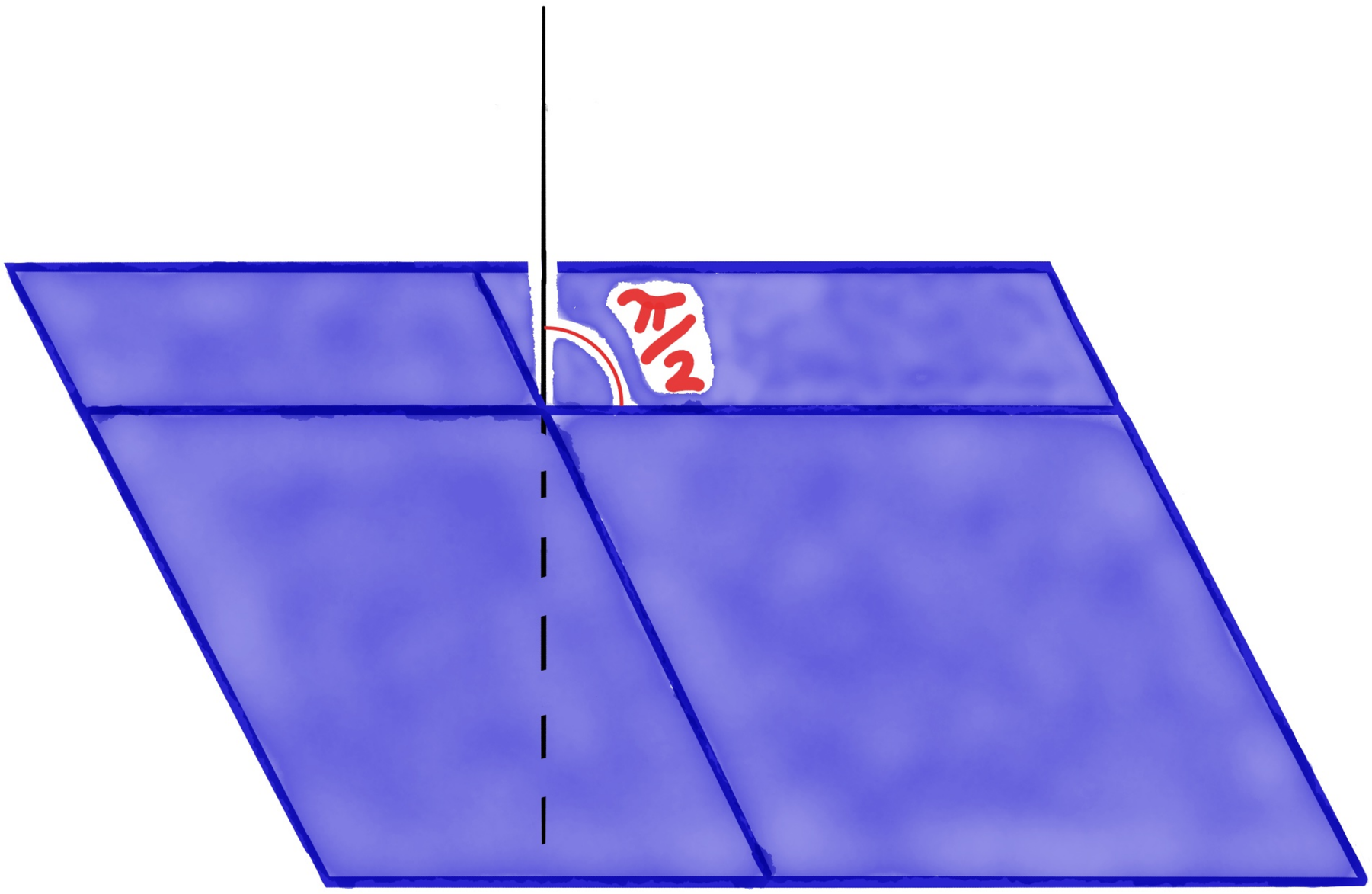
$\mathbb{R}^3$



$\phi = \frac{3\pi}{4}$

Spherical

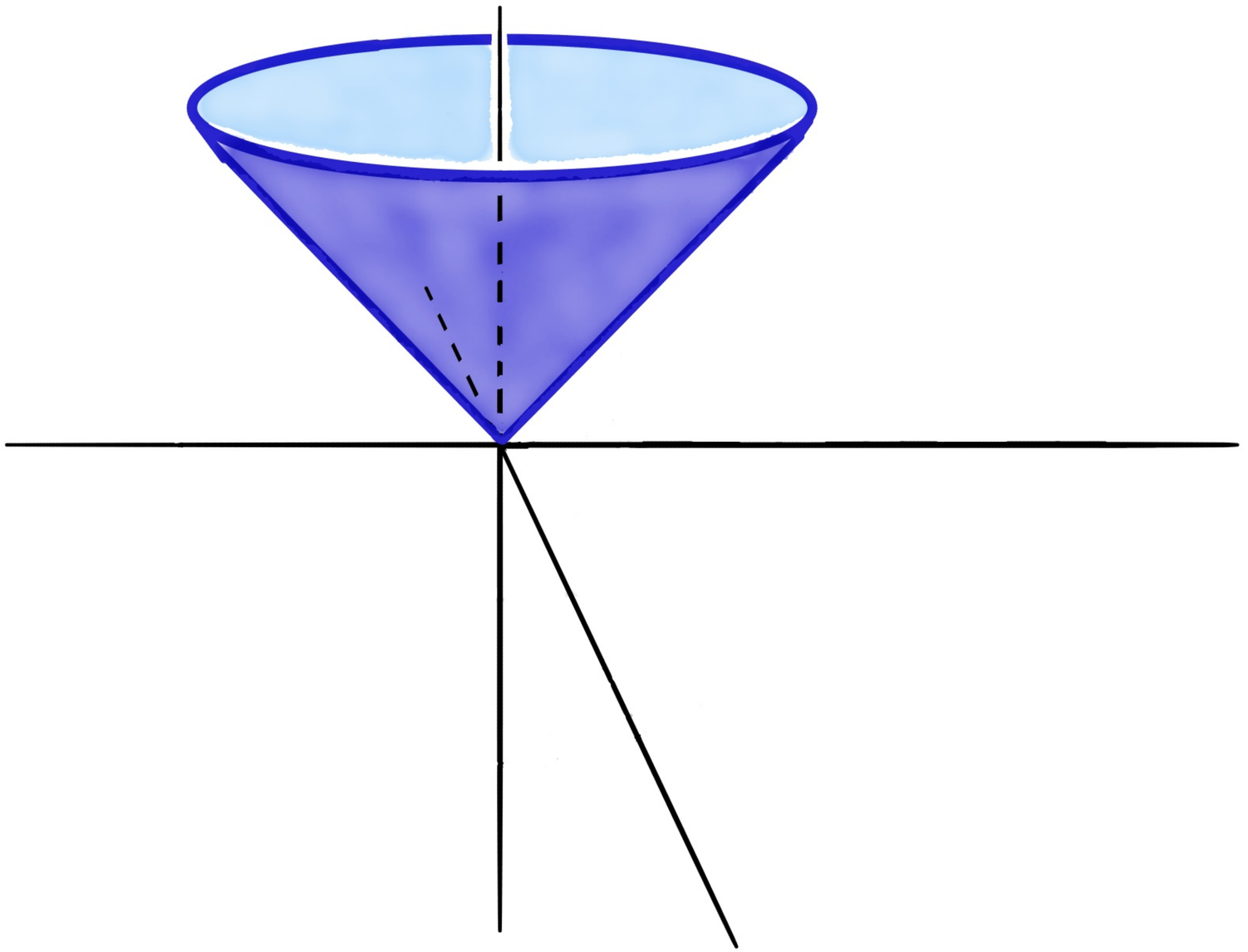
$\mathbb{R}^3$



$$\phi = \frac{\pi}{2}$$

Spherical

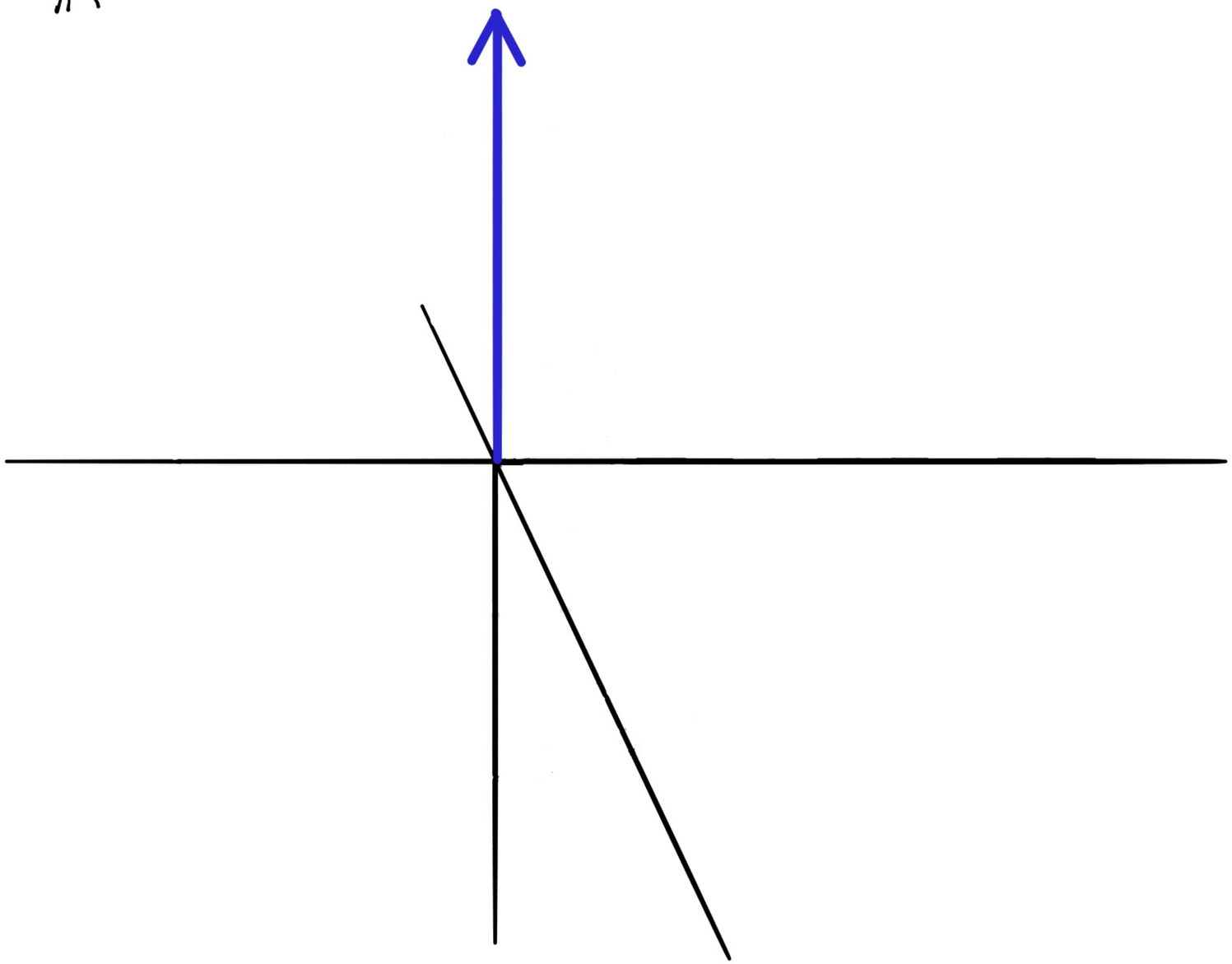
$\mathbb{R}^3$



$\phi = \frac{\pi}{4}$

Spherical

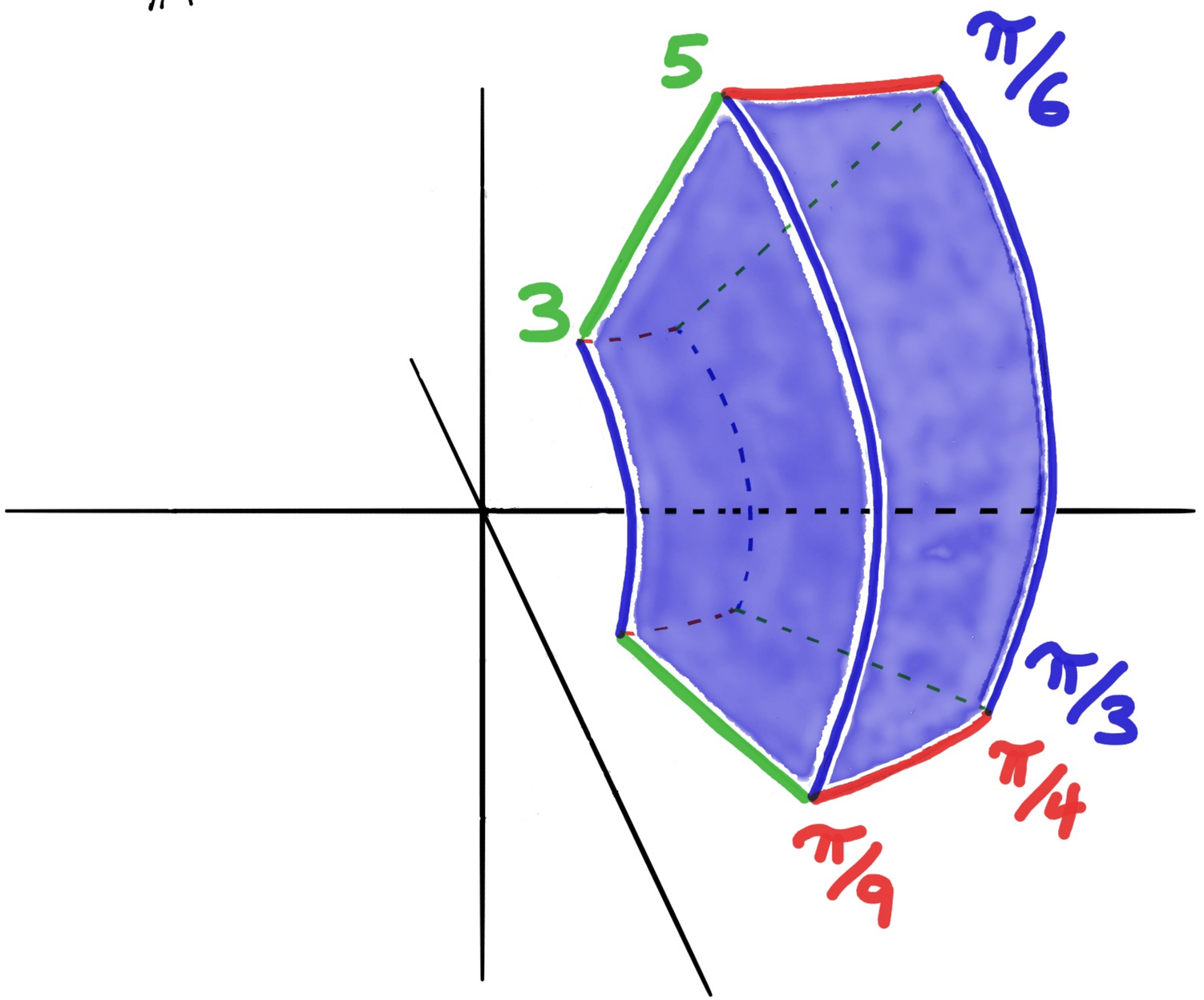
$\mathbb{R}^3$



$\phi = 0$

Spherical

$\mathbb{R}^3$



$3 \le \rho \le 5$

$\pi/9 \le \theta \le \pi/4$

$\pi/6 \le \phi \le \pi/3$

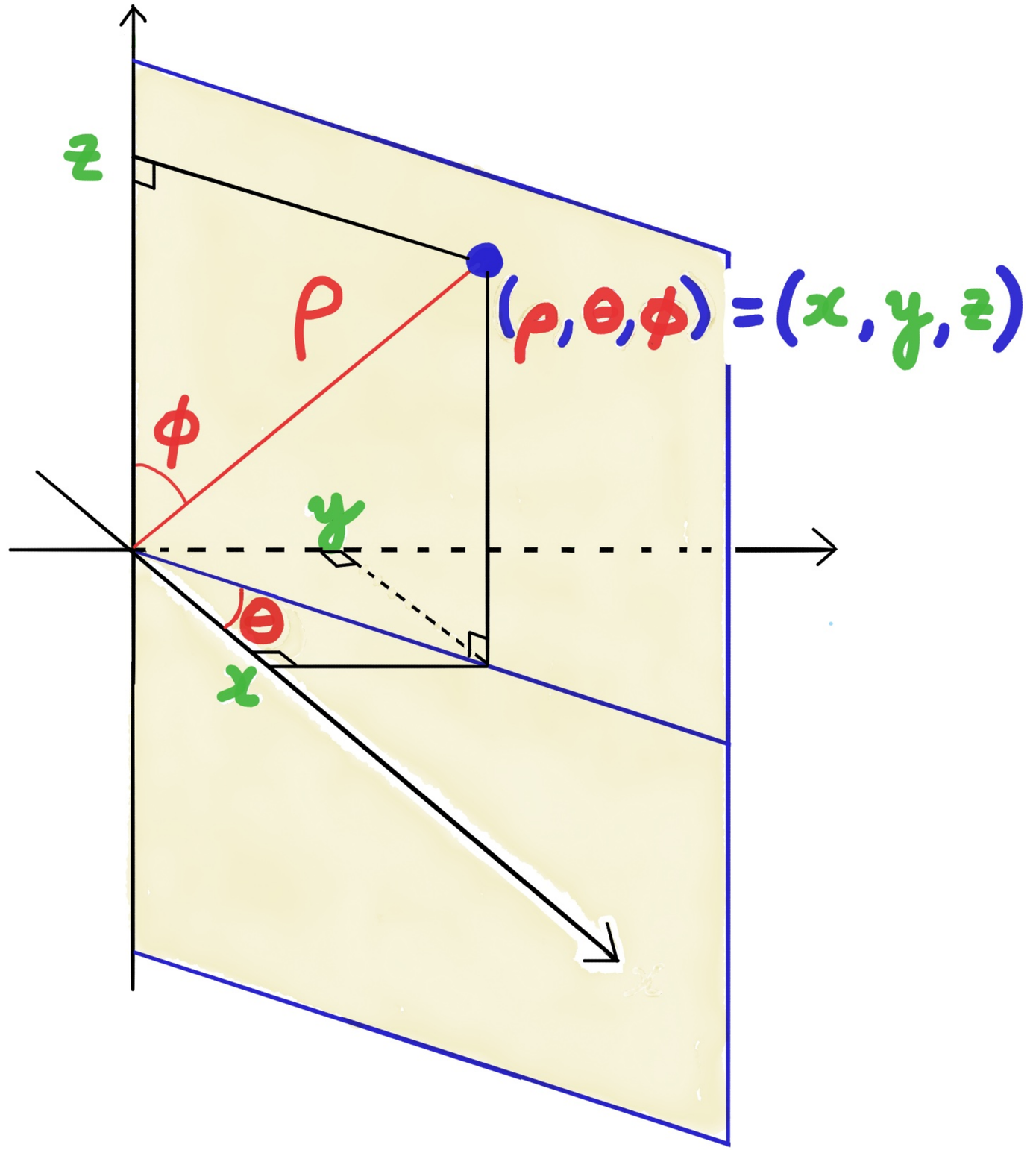
Spherical

Cartesian/spherical translator

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\rho^2 = x^2 + y^2 + z^2$$



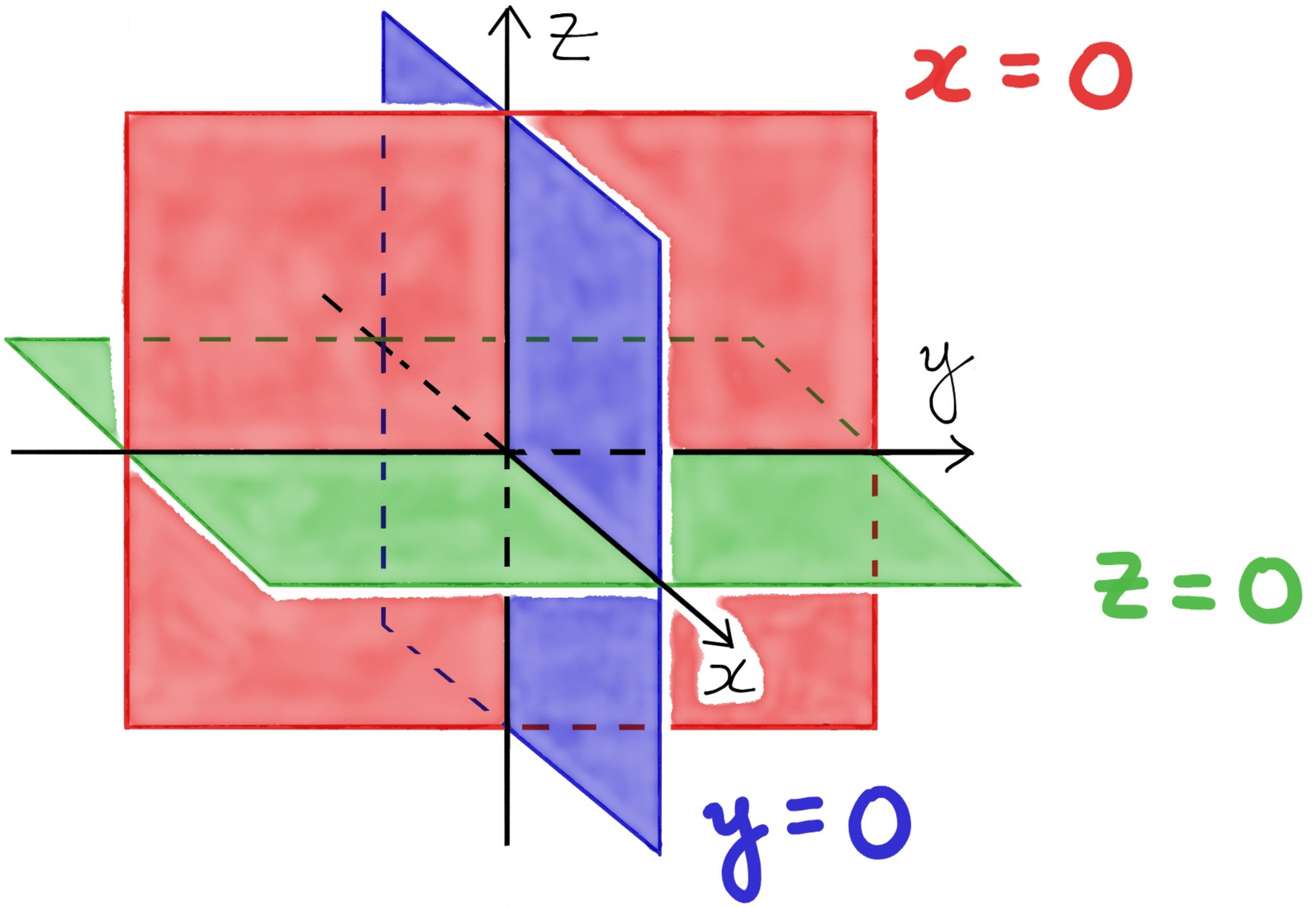
$$xyz = 0$$

$$xyz = 0$$

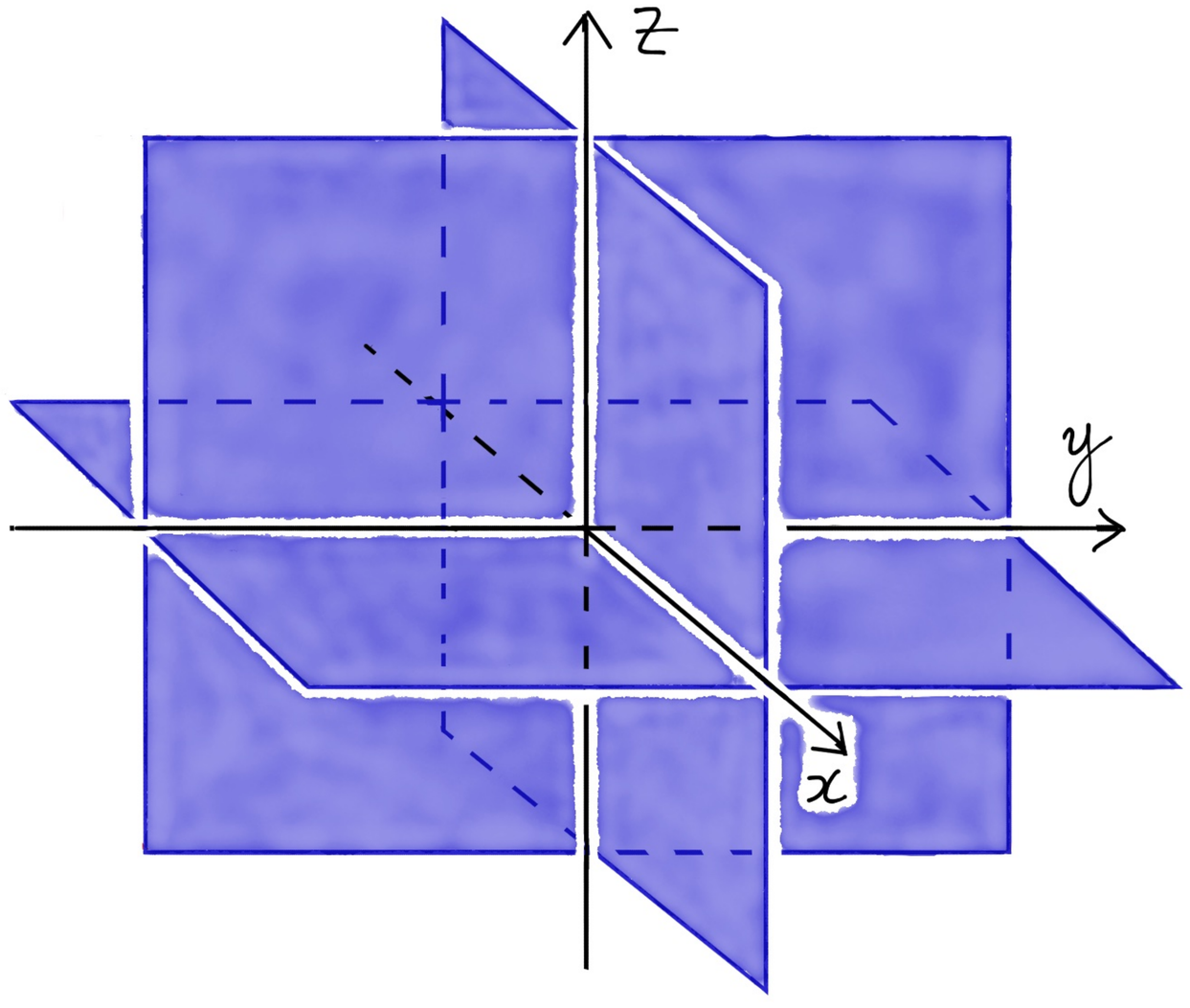
$x = 0$ or $y = 0$ or $z = 0$

$$xyz = 0$$

$x = 0$ or $y = 0$ or $z = 0$



$$xyz = 0$$



Problem:

Write the Cartesian equation $xyz = 0$ as a spherical equation.

Problem:

Write the Cartesian equation $xyz = 0$ as a spherical equation.

Solution:

$$0 = xyz$$

$$= (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\phi)$$

$$= \rho^3 \sin^2\phi \cos\phi \cos\theta \sin\theta$$

$$0 = \rho^3 \sin^2\phi \cos\phi \cos\theta \sin\theta$$

$$0 = \rho^3 \sin^2\phi \cos\phi \cos\theta \sin\theta$$

$\rho = 0$ or $\sin\phi = 0$ or $\cos\phi = 0$

or $\cos\theta = 0$ or $\sin\theta = 0$

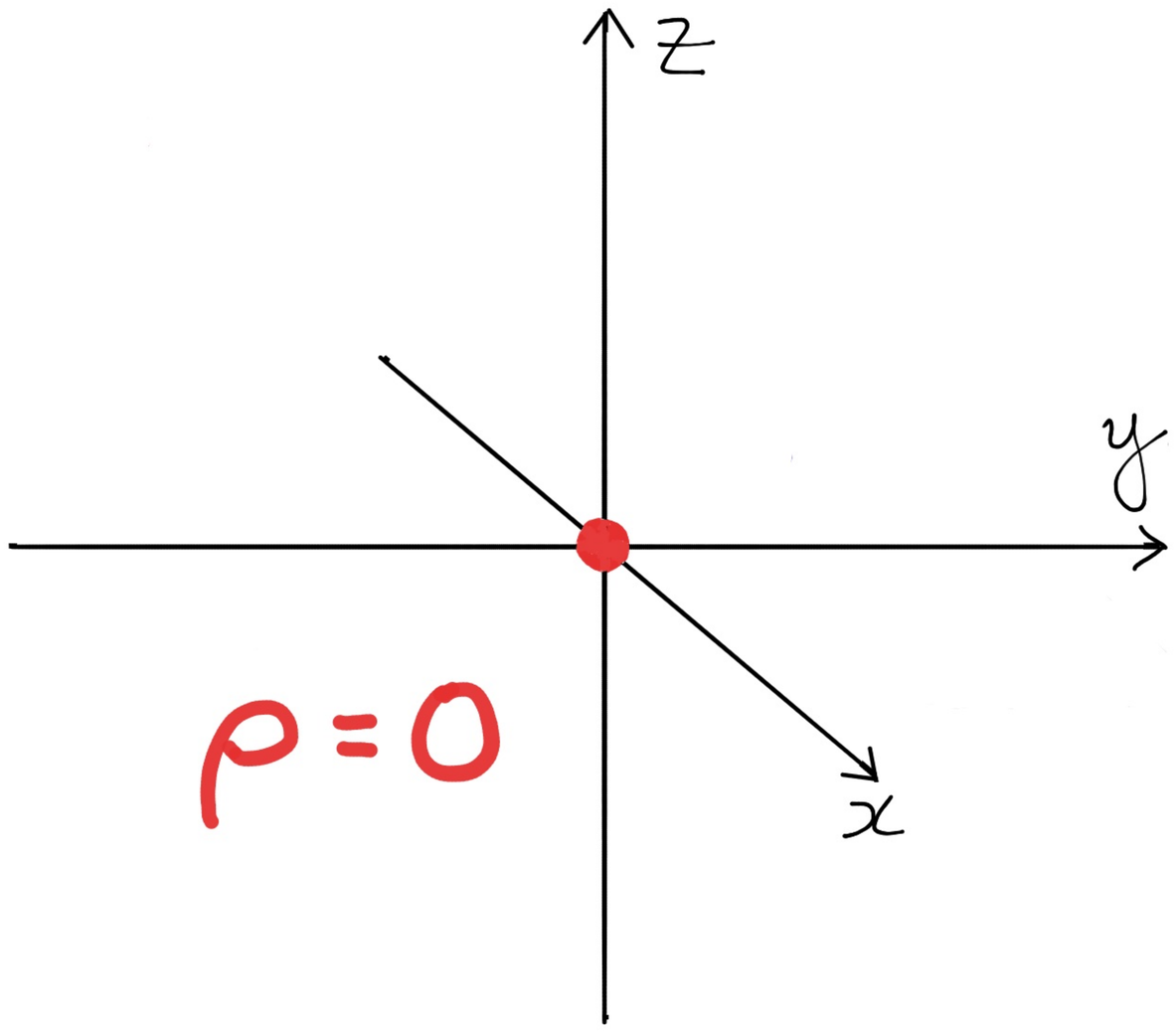
$$0 = \rho^3 \sin^2\phi \cos\phi \cos\theta \sin\theta$$

$\rho = 0$ or $\sin\phi = 0$ or $\cos\phi = 0$

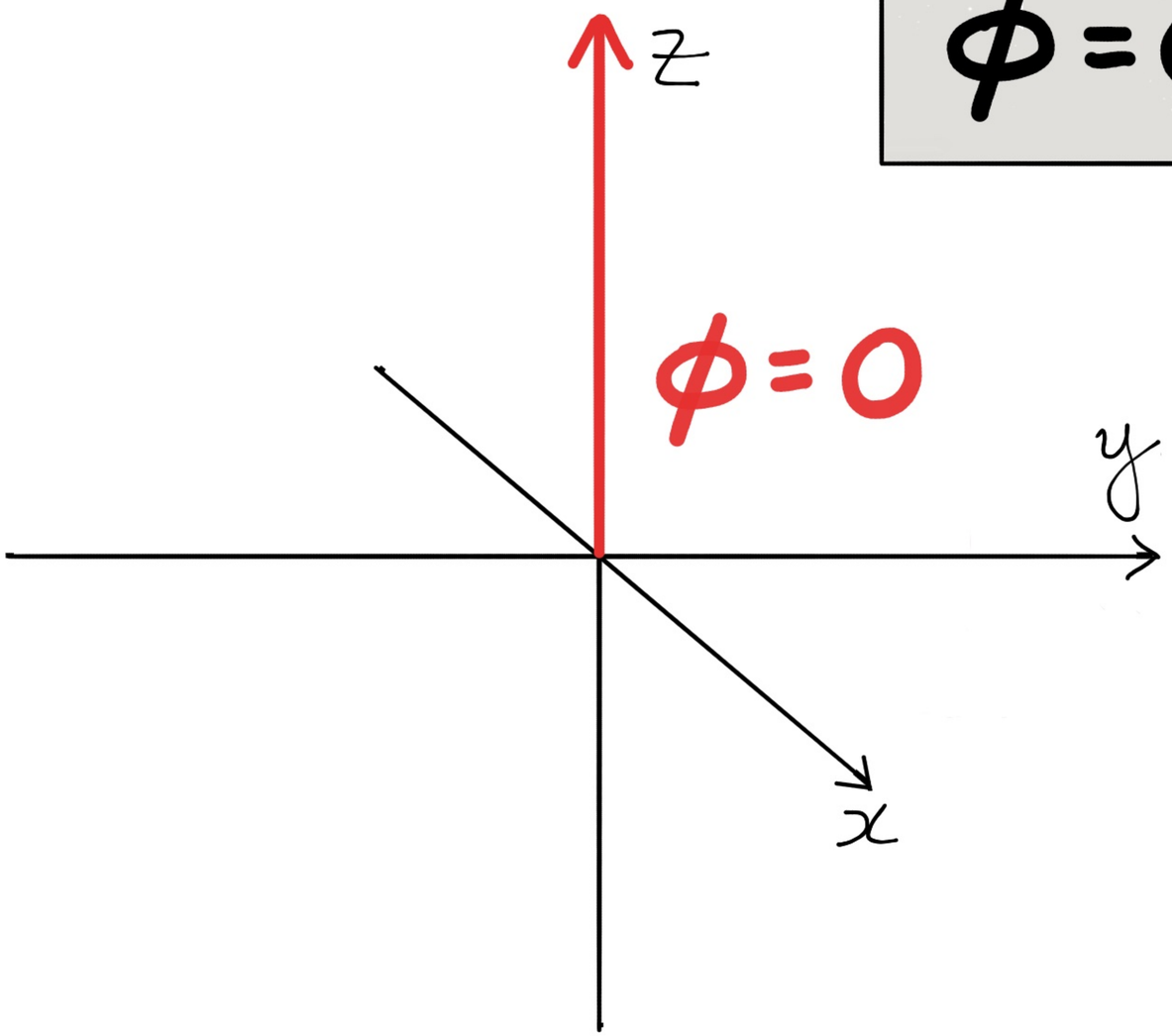
or $\cos\theta = 0$ or $\sin\theta = 0$

$\rho = 0$ or $\phi = 0, \frac{\pi}{2}, \pi$ or $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

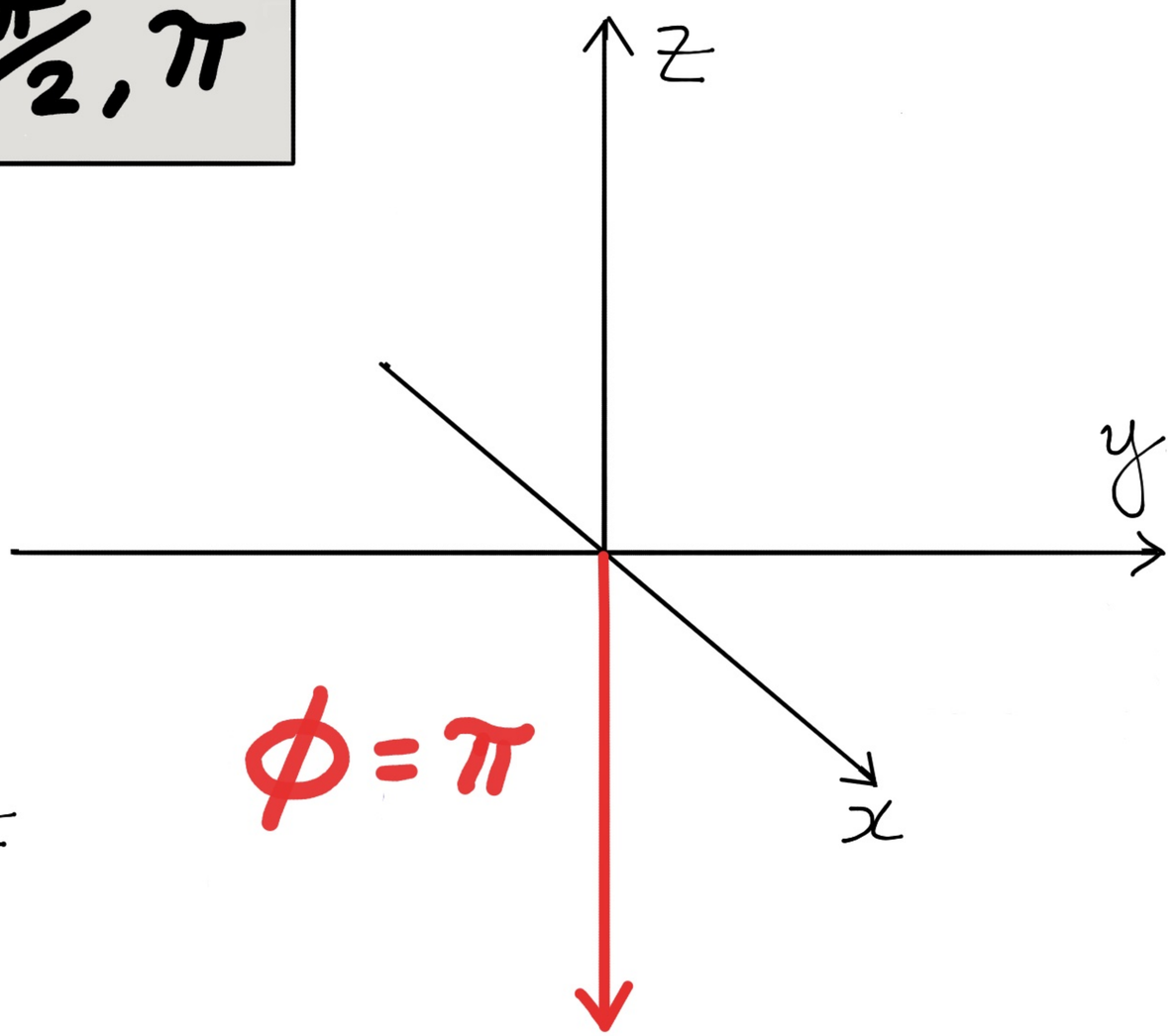
$$\rho = 0$$



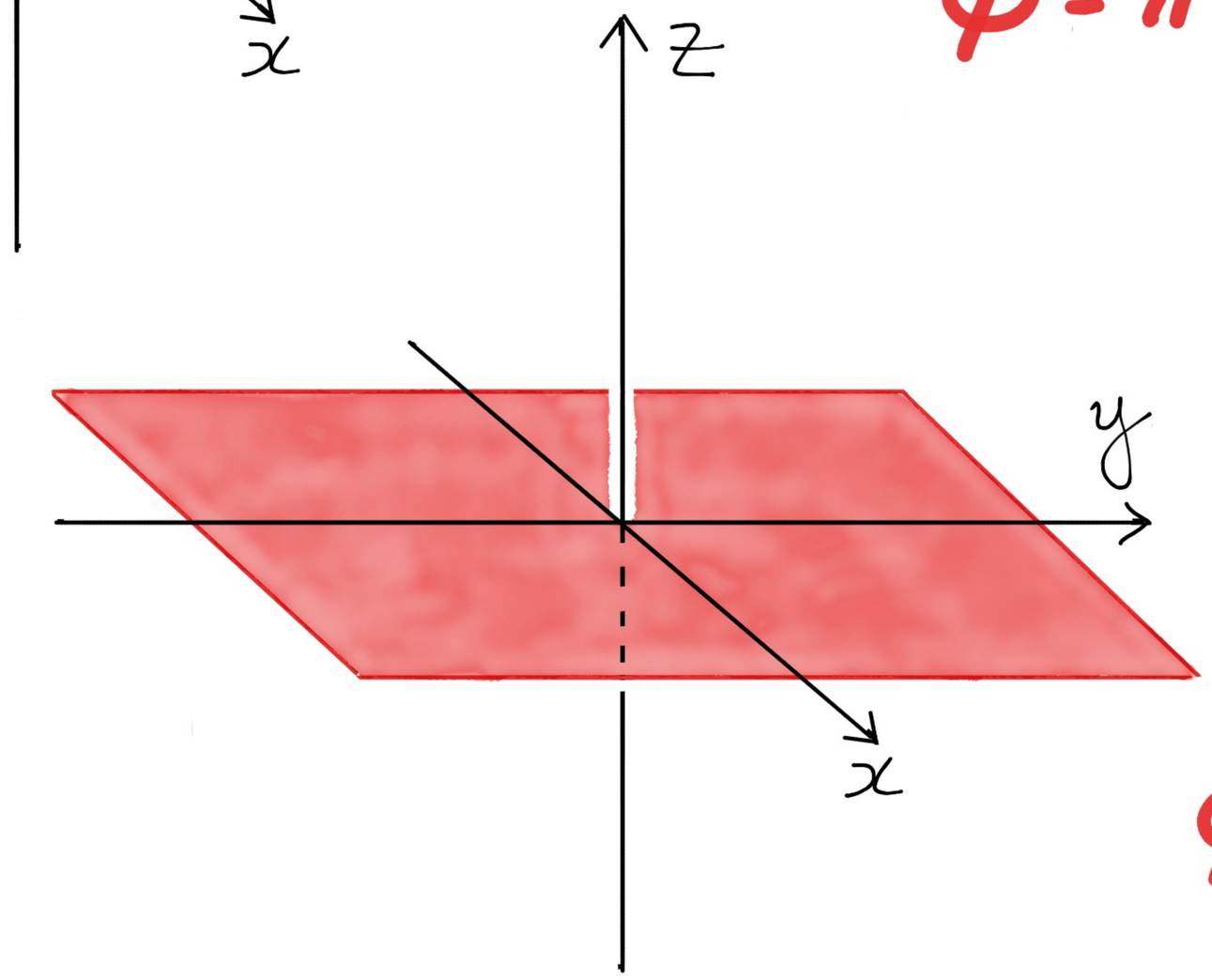
$$\phi = 0, \pi/2, \pi$$



$\phi = 0$

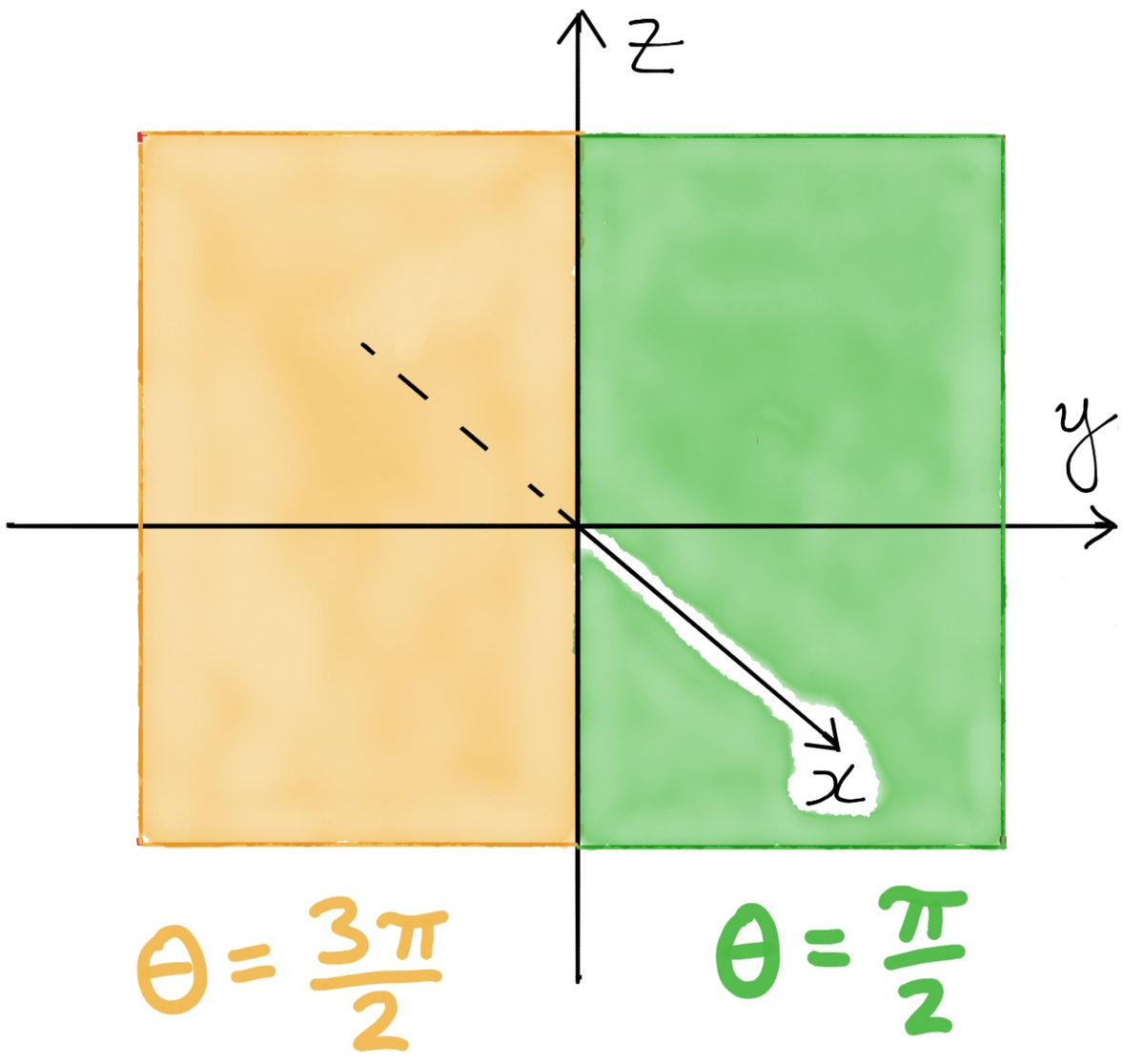


$\phi = \pi$



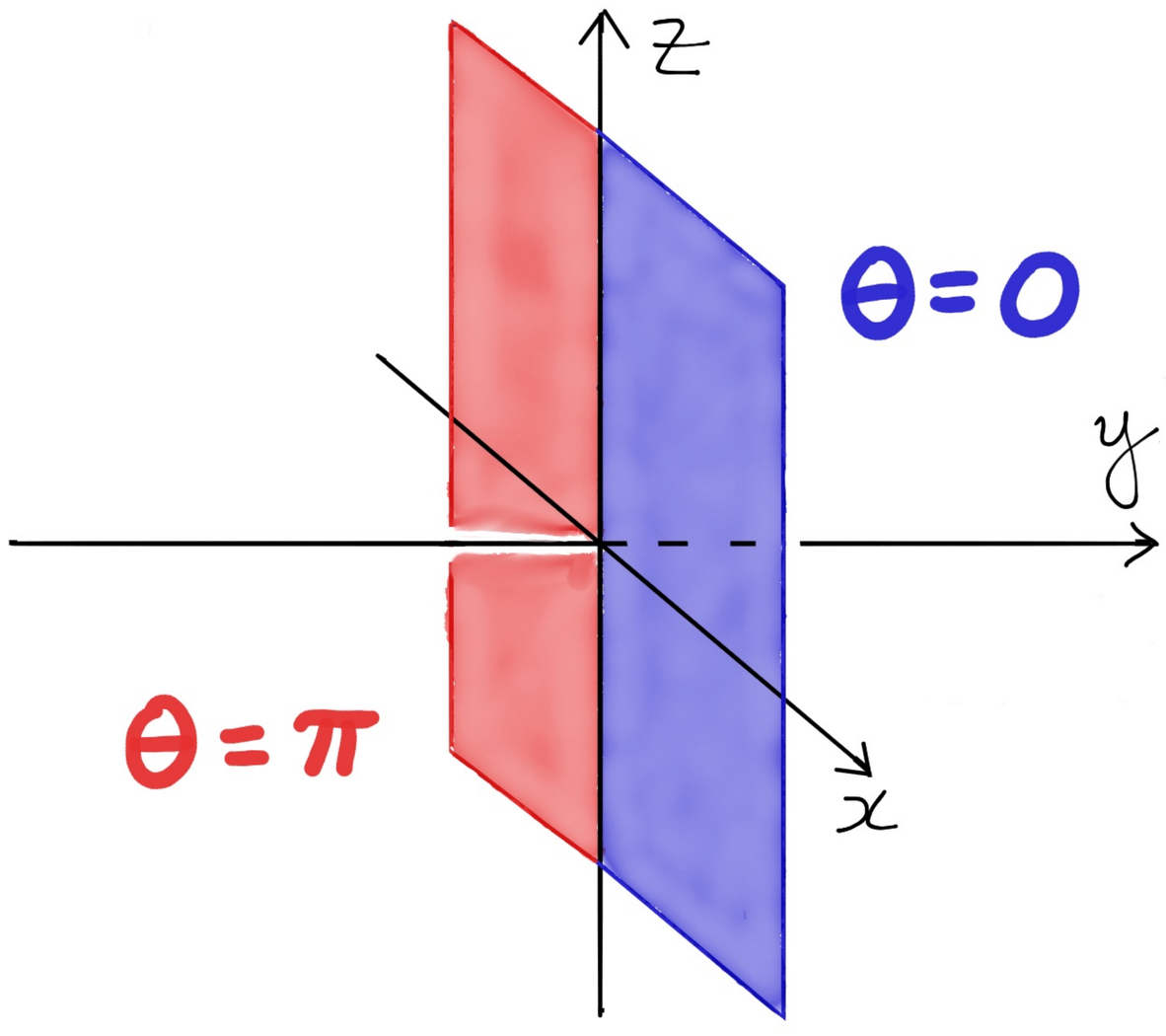
$\phi = \pi/2$

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$



$\theta = \frac{3\pi}{2}$

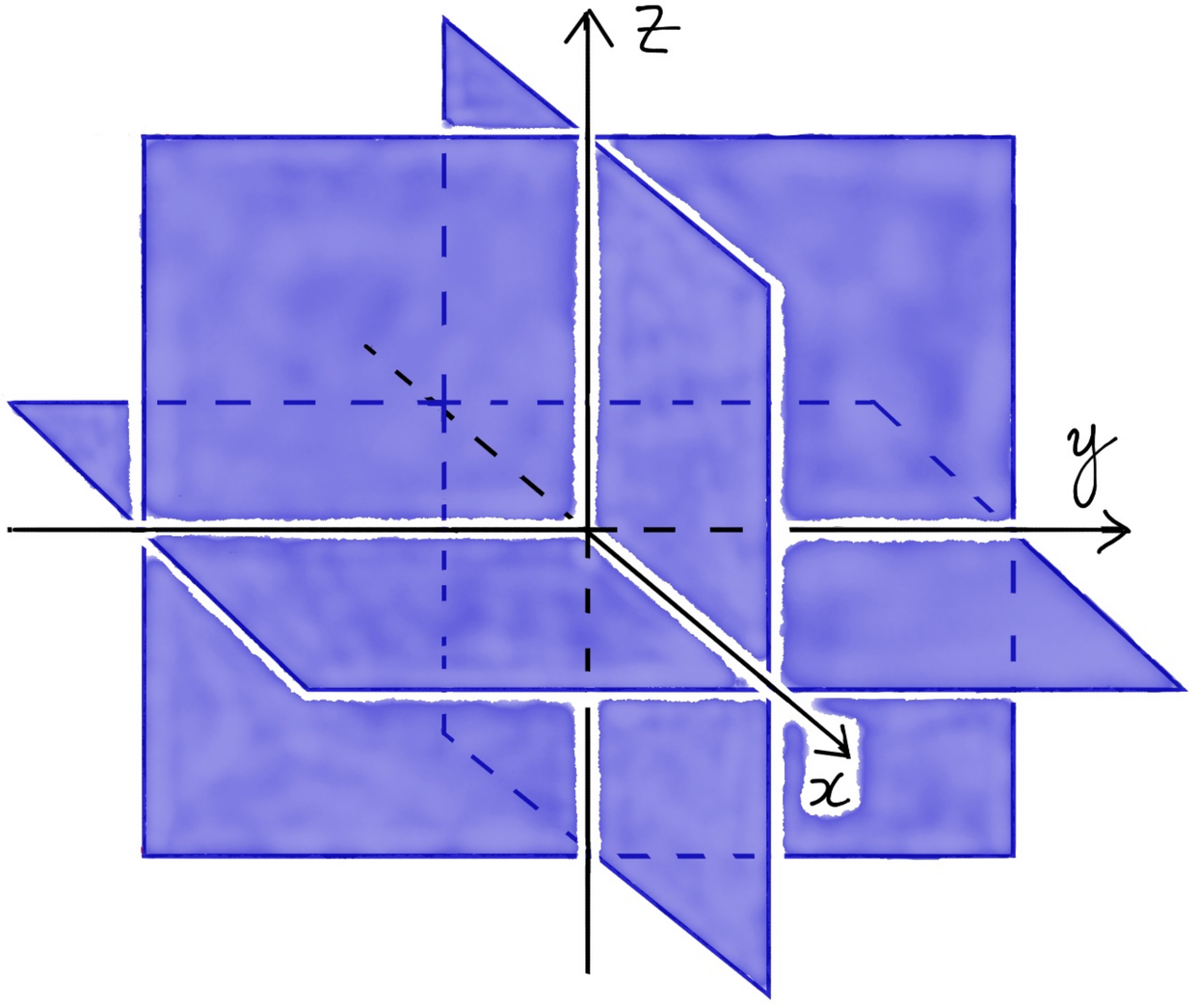
$\theta = \frac{\pi}{2}$



$\theta = 0$

$\theta = \pi$

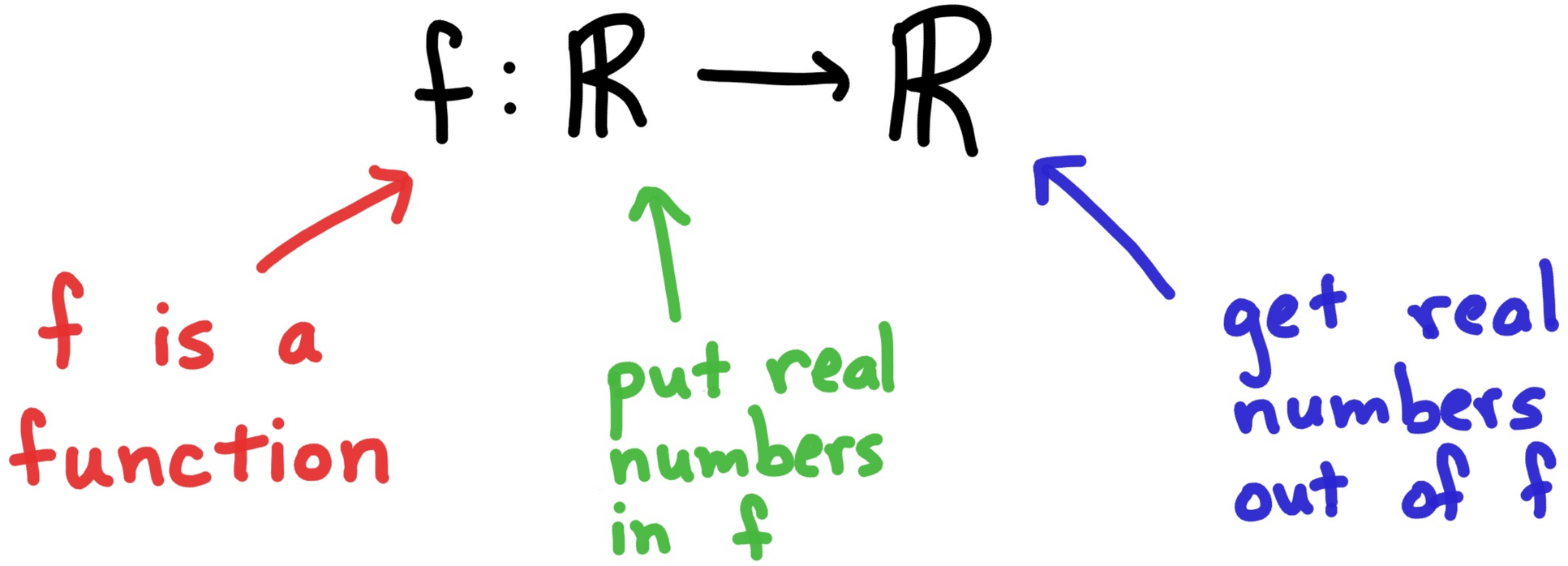
$$xyz = 0$$

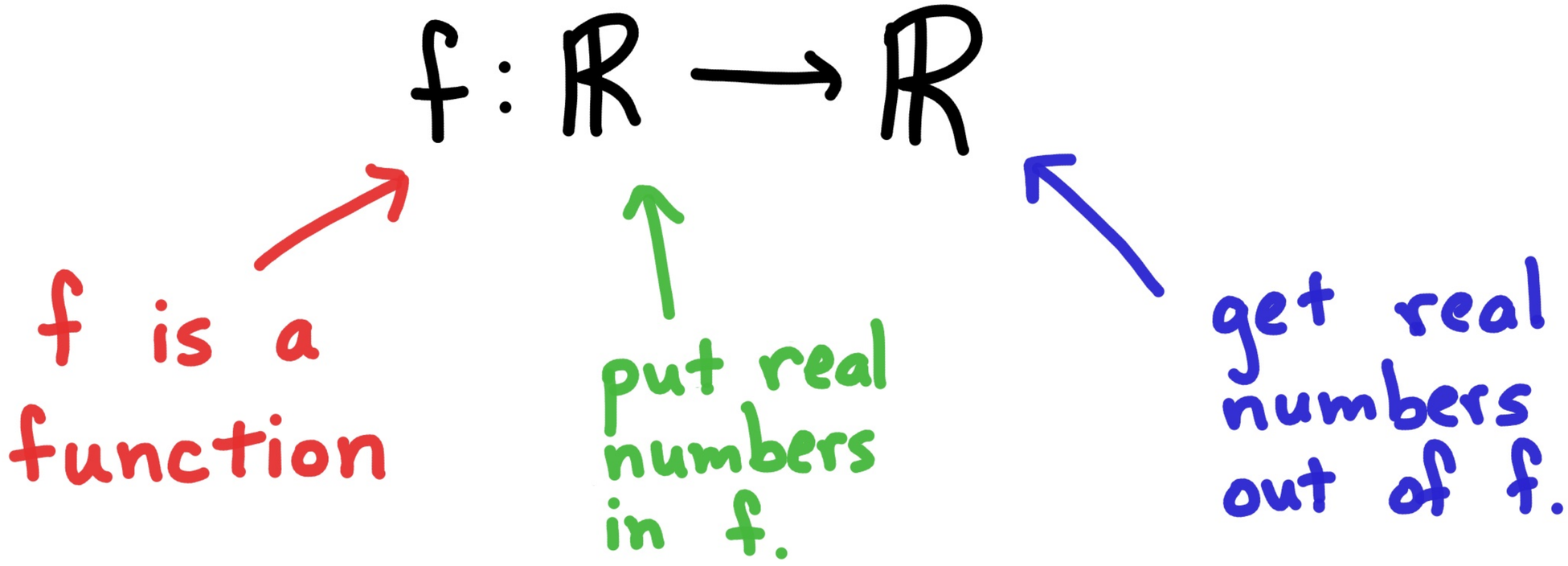


$$\rho^3 \sin^2\phi \cos\phi \cos\theta \sin\theta = 0$$

Ⅳ A word on functions

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$


$f : \mathbb{R} \to \mathbb{R}$
f is a function
put real numbers in f
get real numbers out of f



Example:

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

f is a function

put real numbers in f.

get real numbers out of f.

Example:

$f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$.

$f(2) = 4$, $f(3) = 9$, etc.

Example:

$$g: \mathbb{R}^2 \to \mathbb{R}^3, \quad g(x,y) = (2x, 3y, x+y)$$

Example:

$$g: \mathbb{R}^2 \to \mathbb{R}^3, \quad g(x,y) = (2x, 3y, x+y)$$

$(1,2) \in \mathbb{R}^2$ and

$g(1,2) =$

Example:

$$g: \mathbb{R}^2 \to \mathbb{R}^3, \quad g(x,y) = (2x, 3y, x+y).$$

$(1,2) \in \mathbb{R}^2$ and

$g(1,2) = (2\cdot 1, 3\cdot 2, 1+2) = (2,6,3) \in \mathbb{R}^3$.