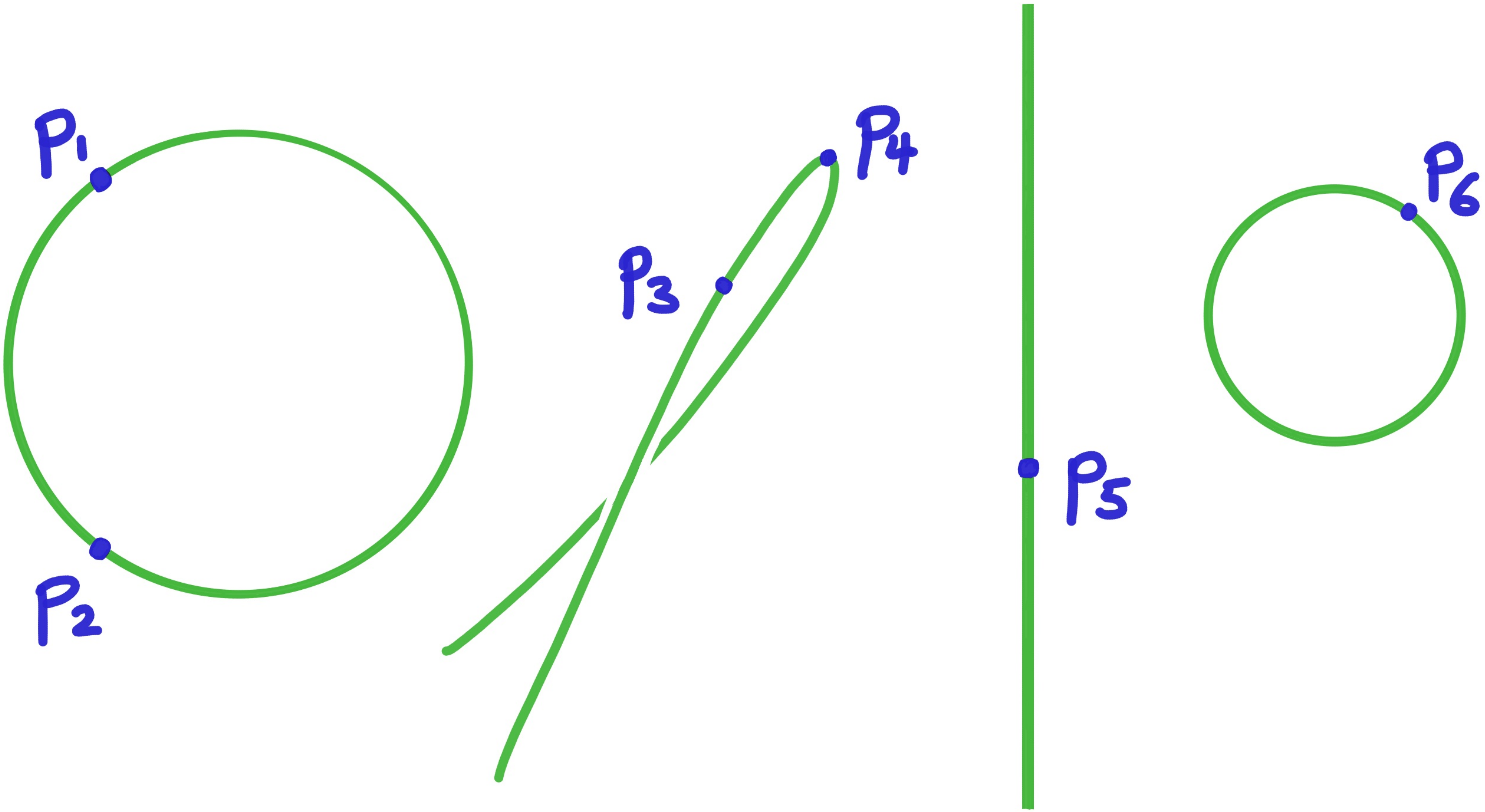
Six

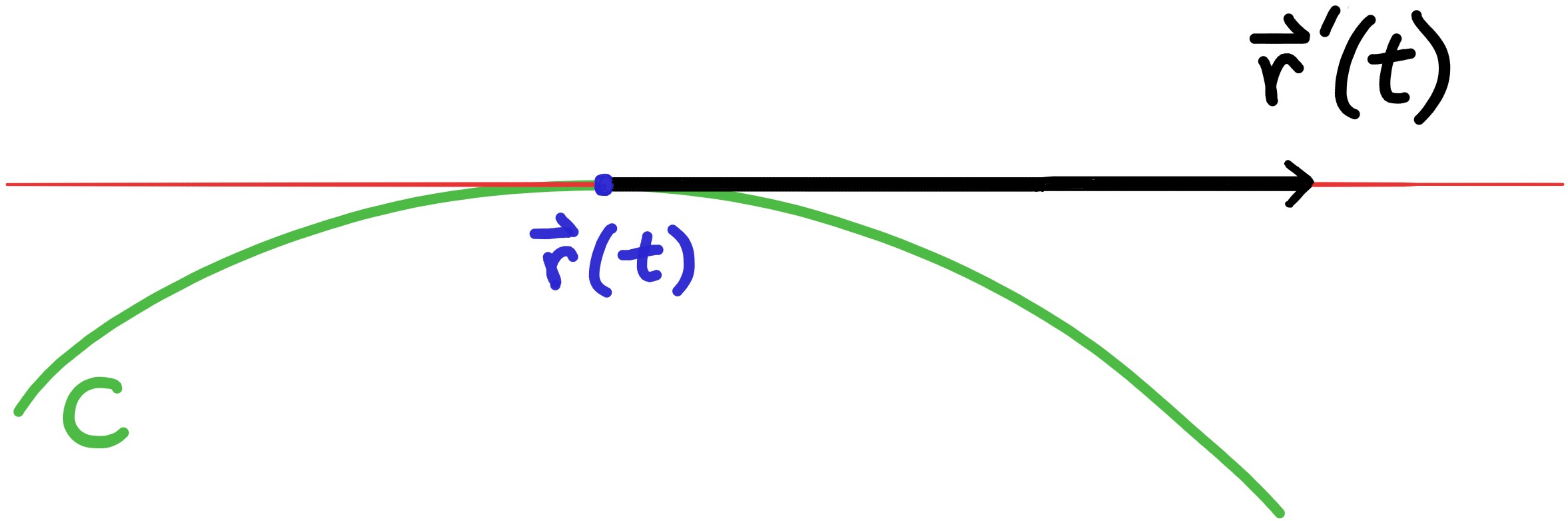
I Curvature

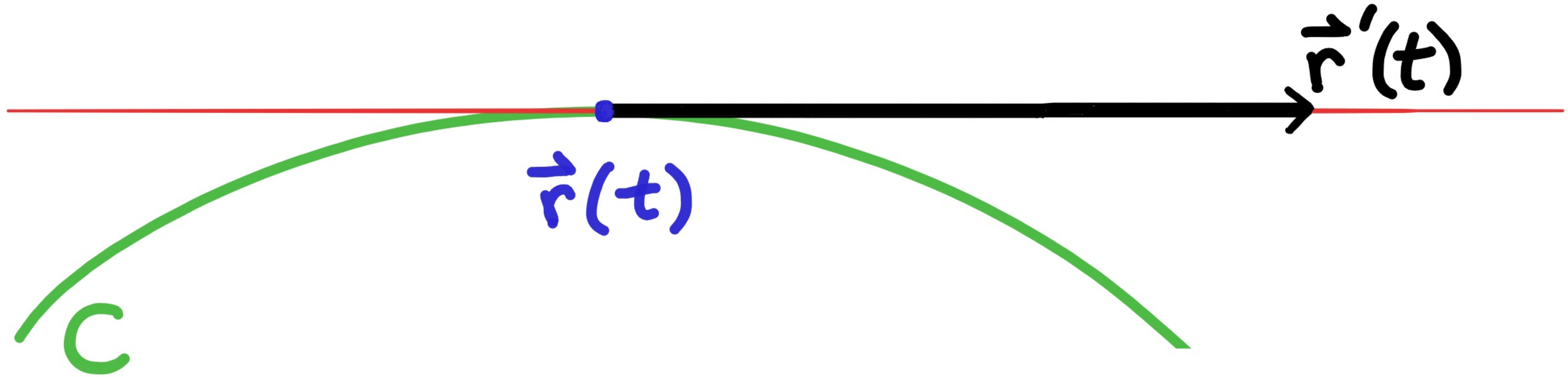
II Curvature of graphs

I Curvature

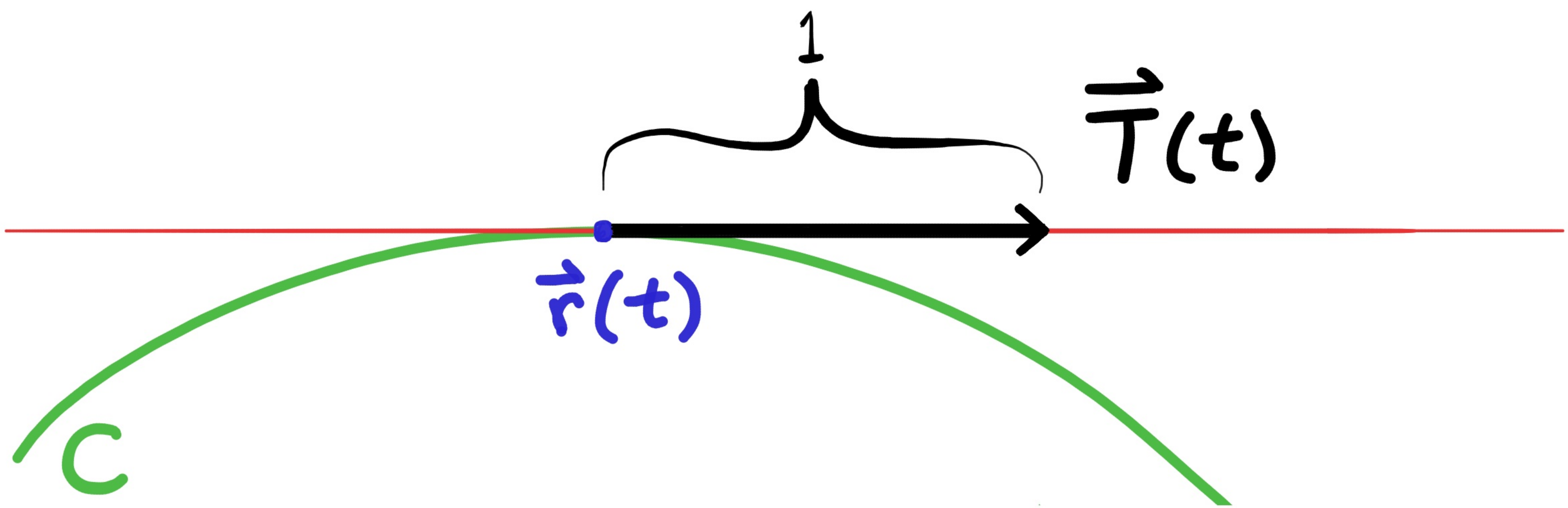

P1
P2
P3
P4
P5
P6

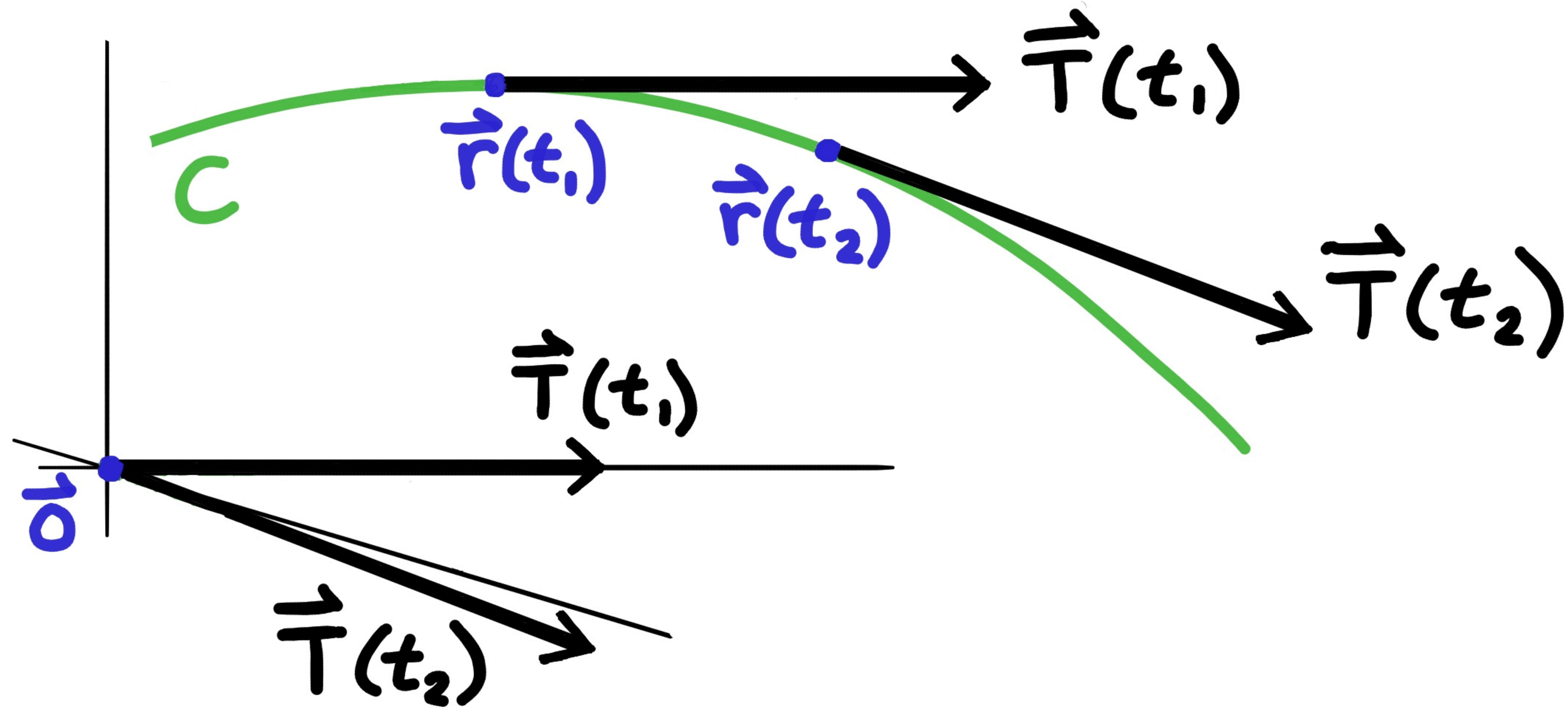
Recall, if a curve C in $\mathbb{R}^n$ is parametrized by $\vec{r}:\mathbb{R}\to\mathbb{R}^n$, then $\vec{r}\,'(t)$ is a vector tangent to C at $\vec{r}(t)$.





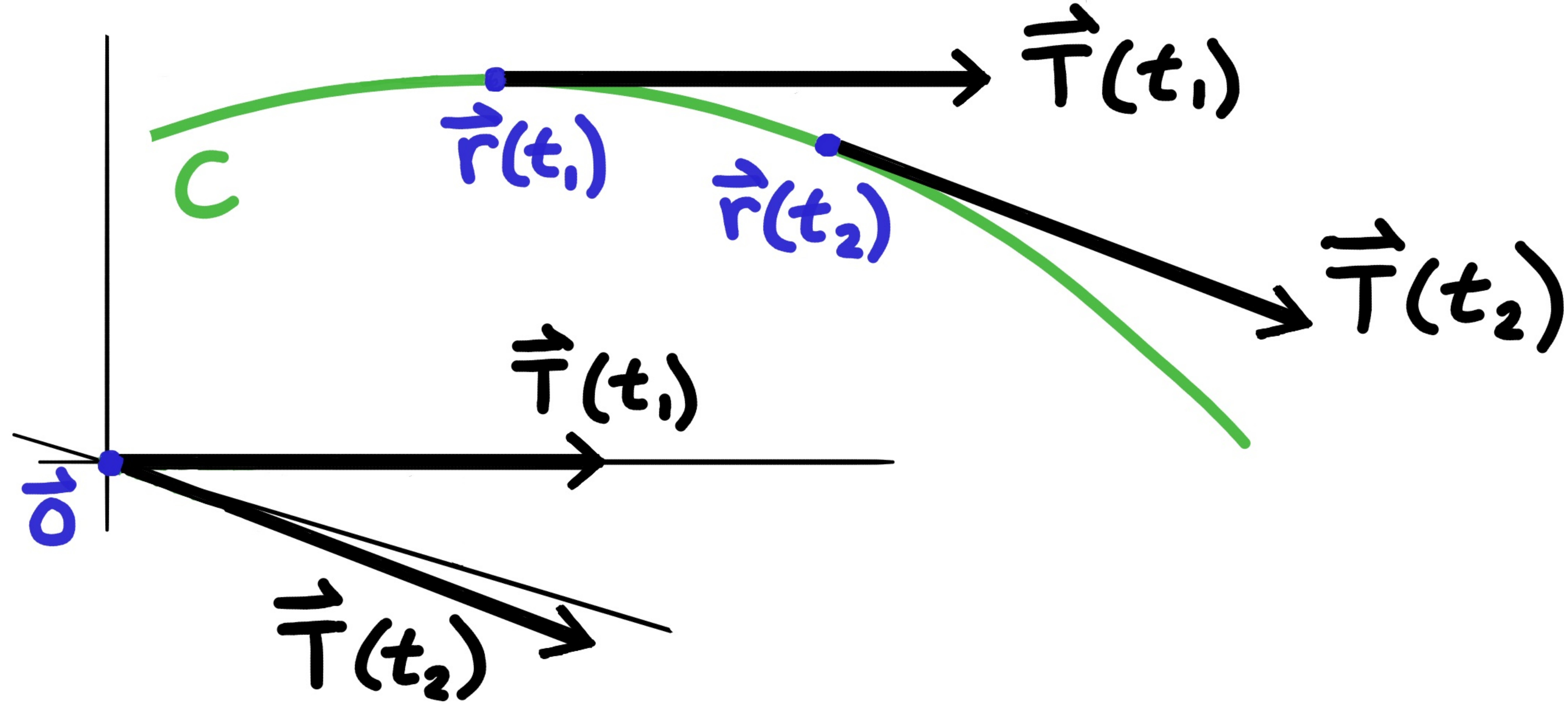
Let $\vec{T}(t)$ be the unit vector in the direction of $\vec{r}\,'(t)$. That is, $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$.





Curvature measures how much $\vec{T}(t)$ twists, changes, as t varies.

$$\kappa(t) = \boxed{\text{curvature at } \vec{r}(t)} =$$



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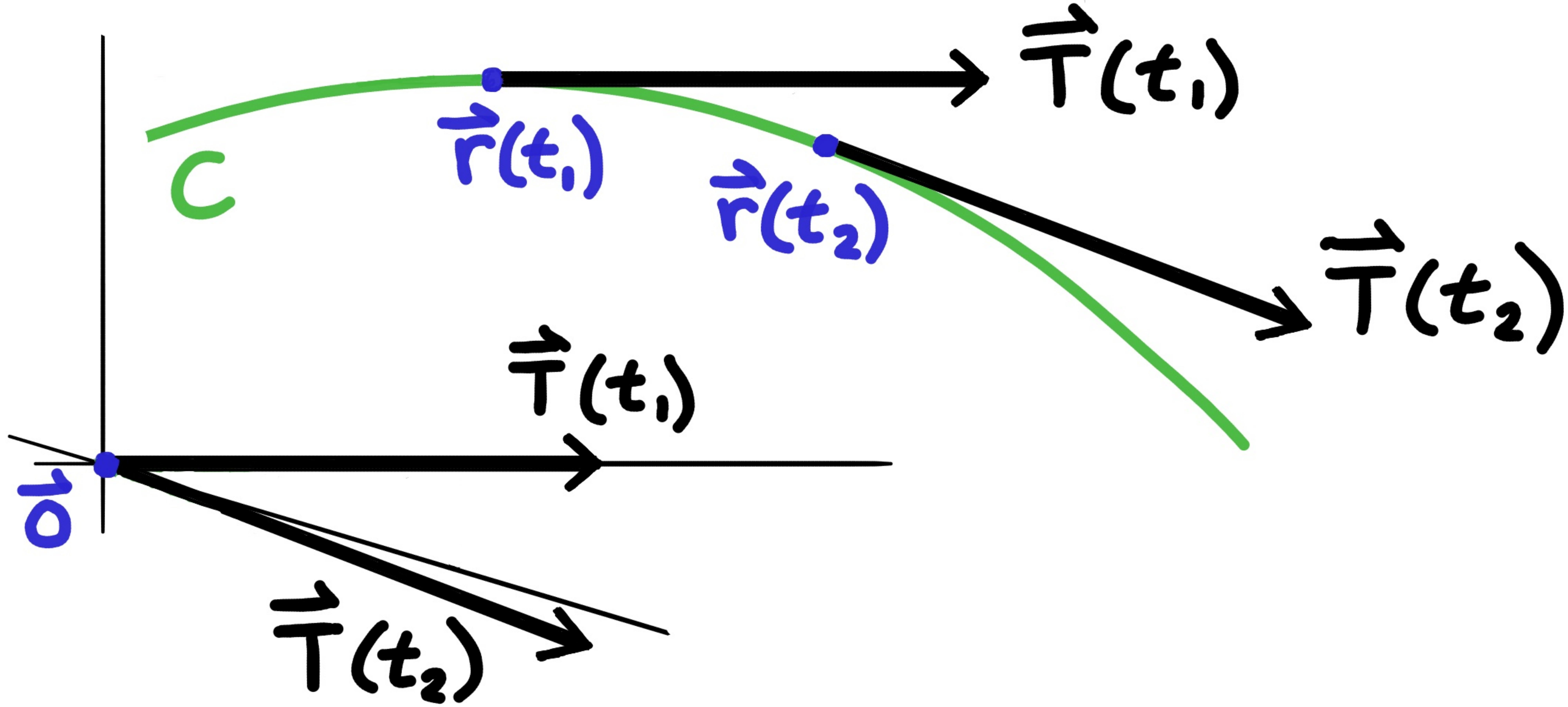

Curvature measures how much $\vec{T}(t)$ twists, changes, as t varies.

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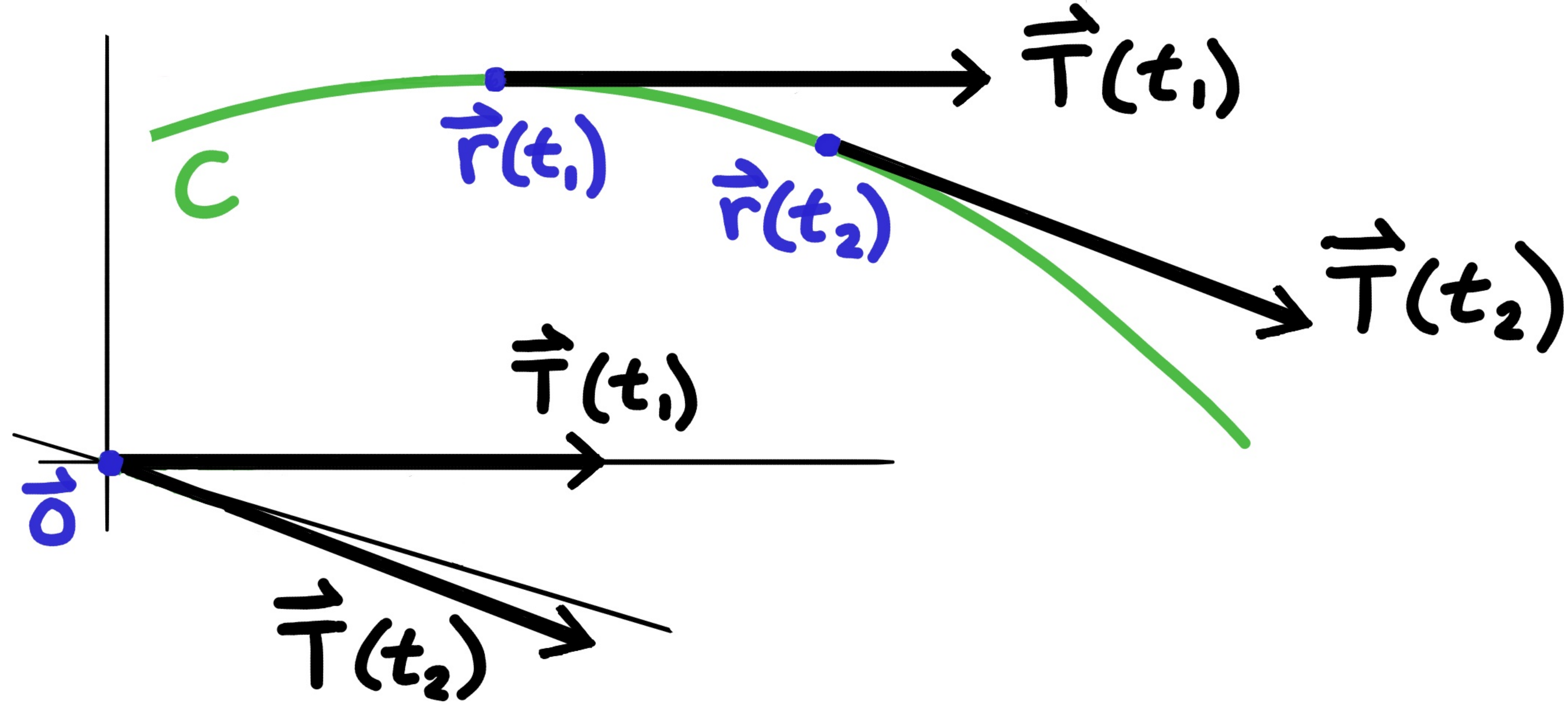

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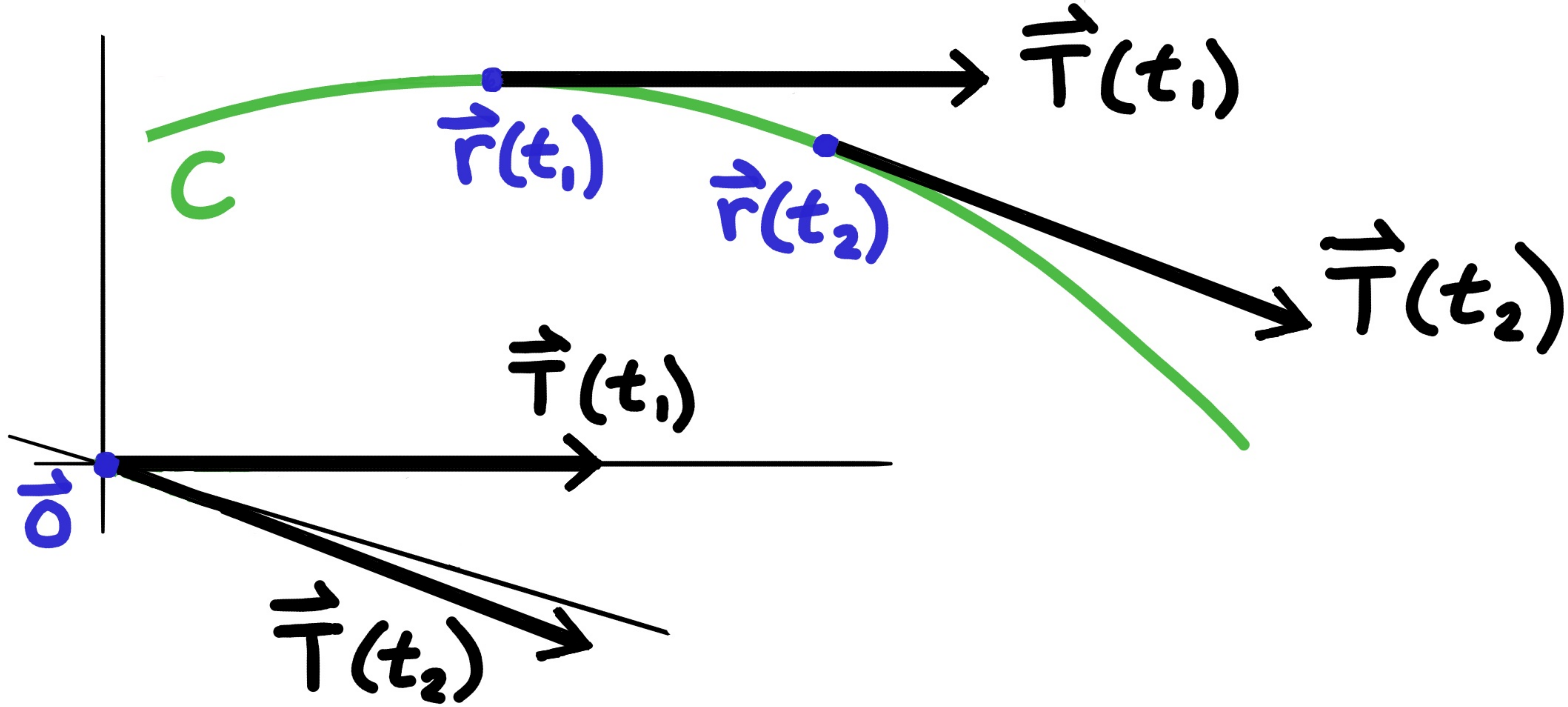

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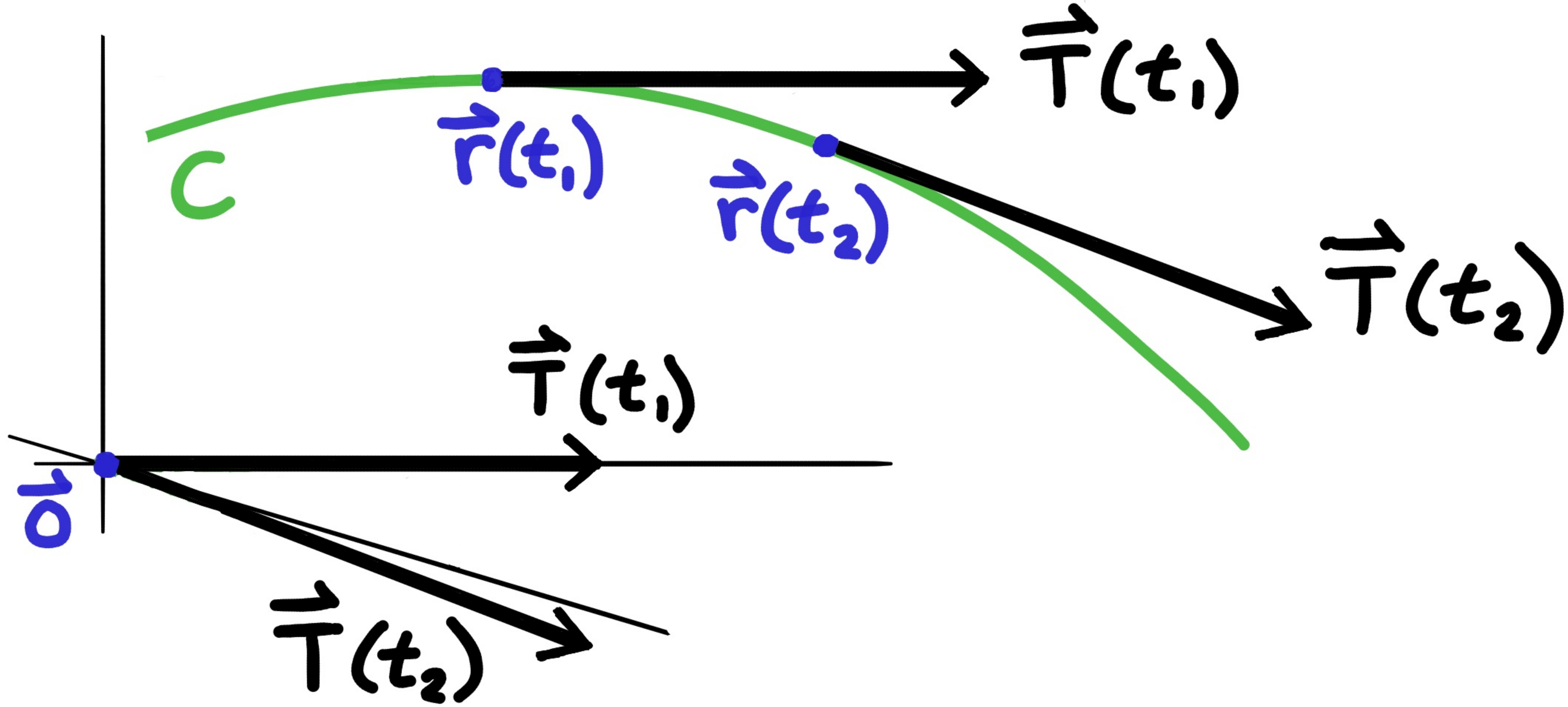
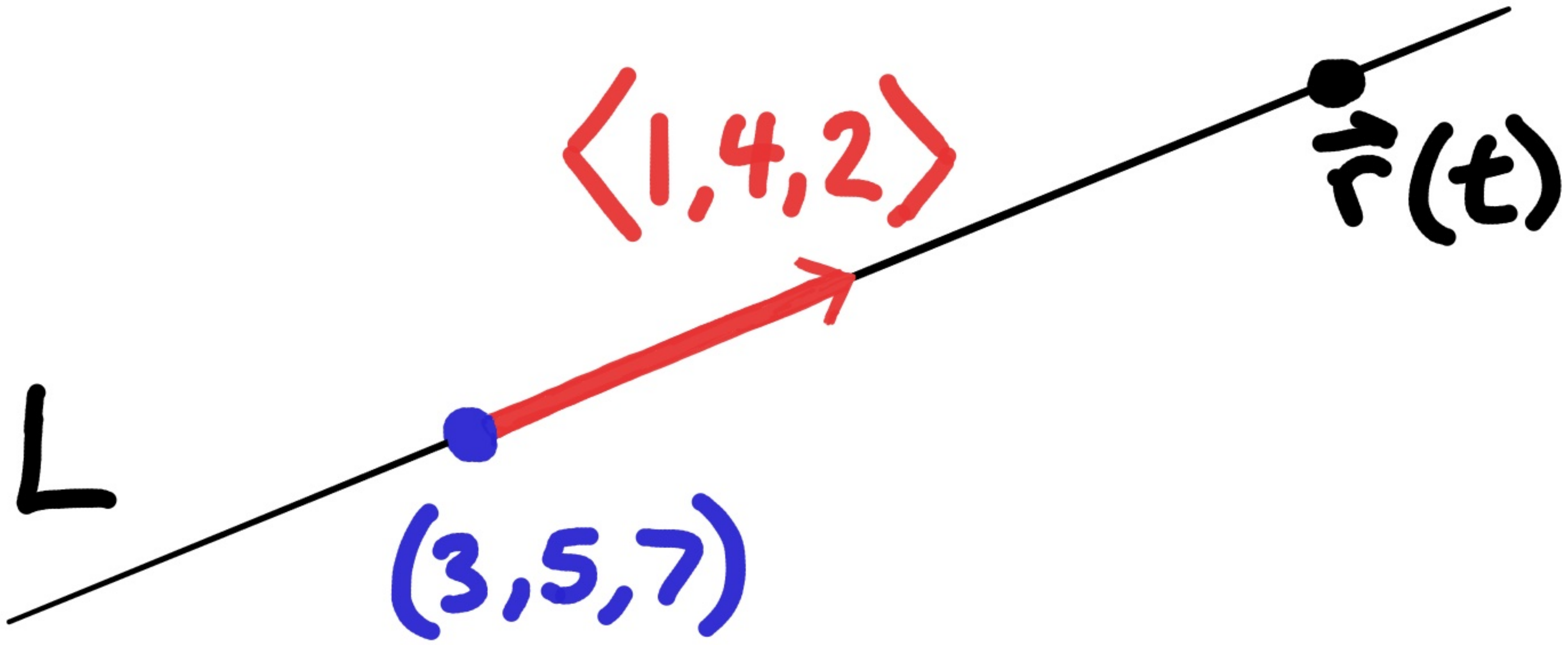


Curvature measures how much $\vec{T}(t)$ twists, changes, as t varies.

$$\kappa(t) = \boxed{\begin{array}{c}\text{curvature}\\ \text{at } \vec{r}(t)\end{array}} = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

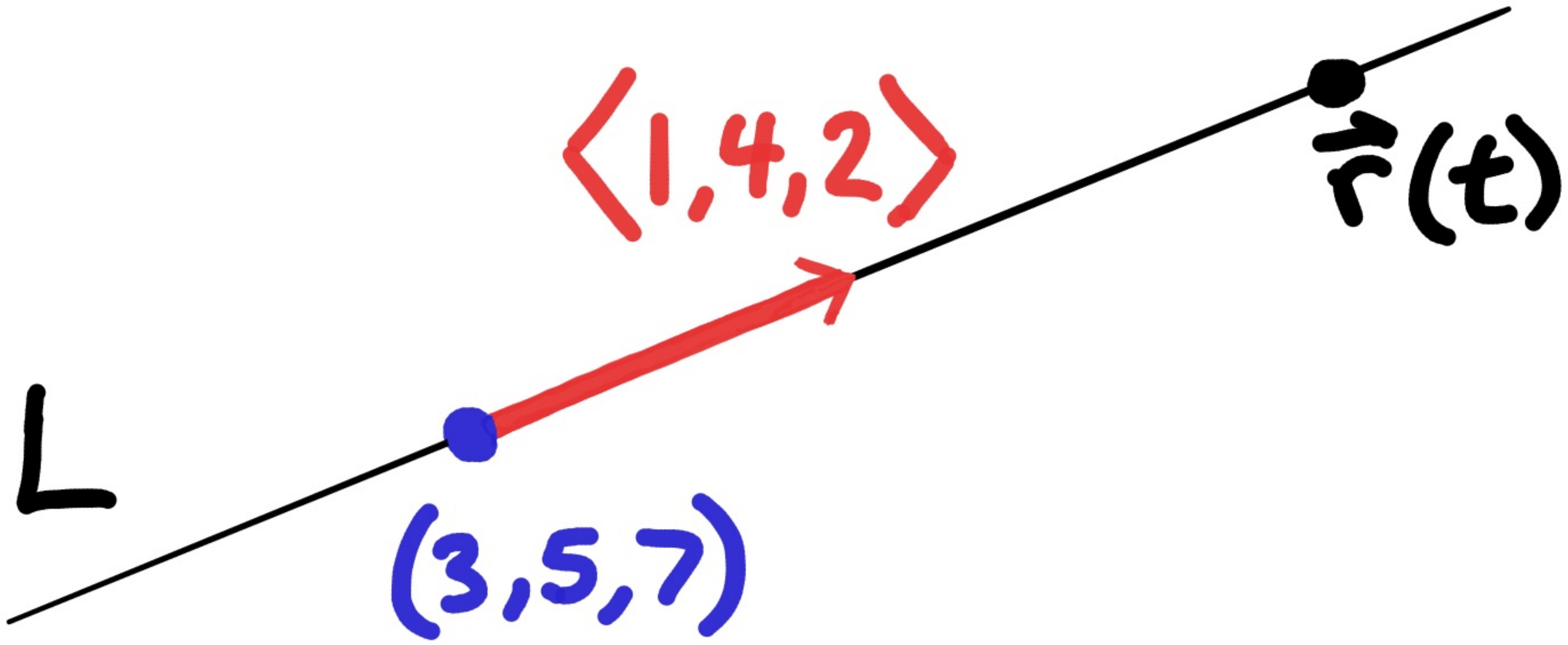
Curvature at a point on a line



L is parametrized by

$$\vec{r}(t) = t\langle 1,4,2\rangle + (3,5,7)$$
$$= (t+3,\, 4t+5,\, 2t+7)$$

Curvature at a point on a line



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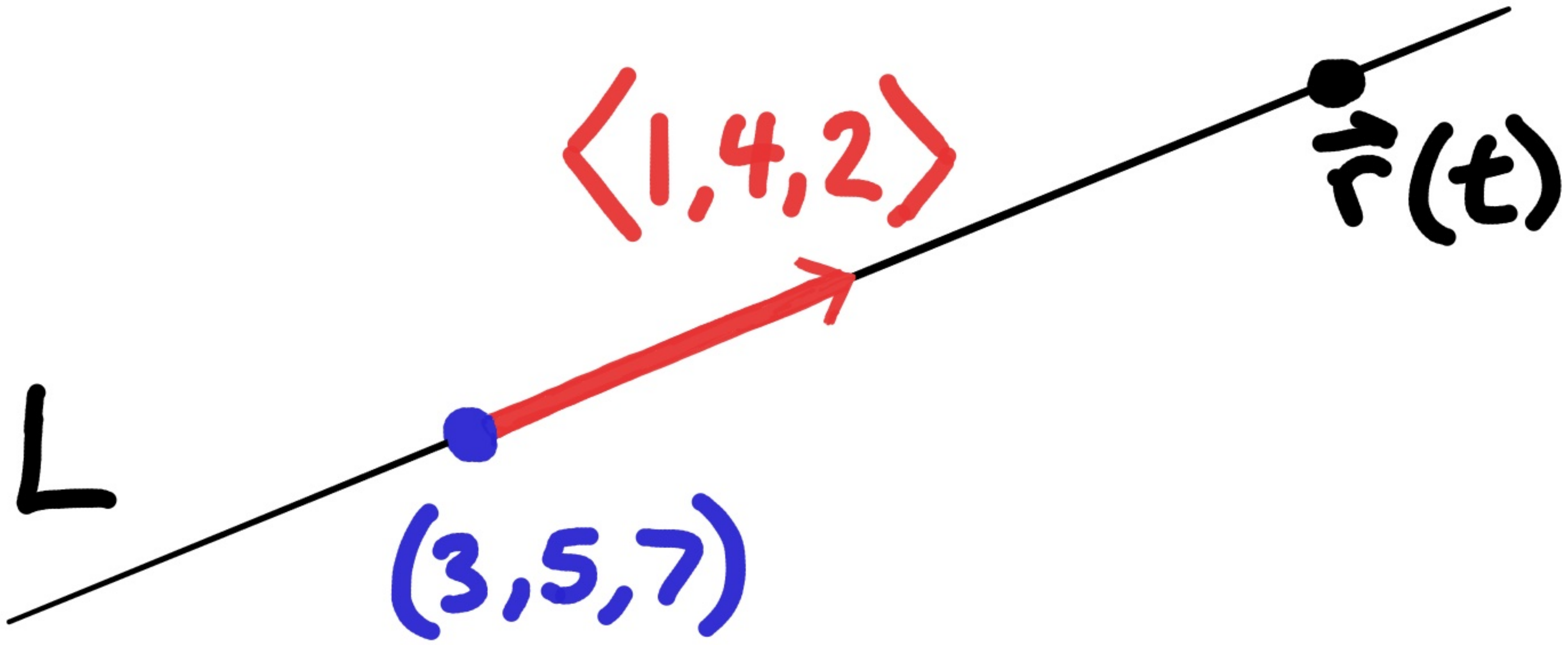

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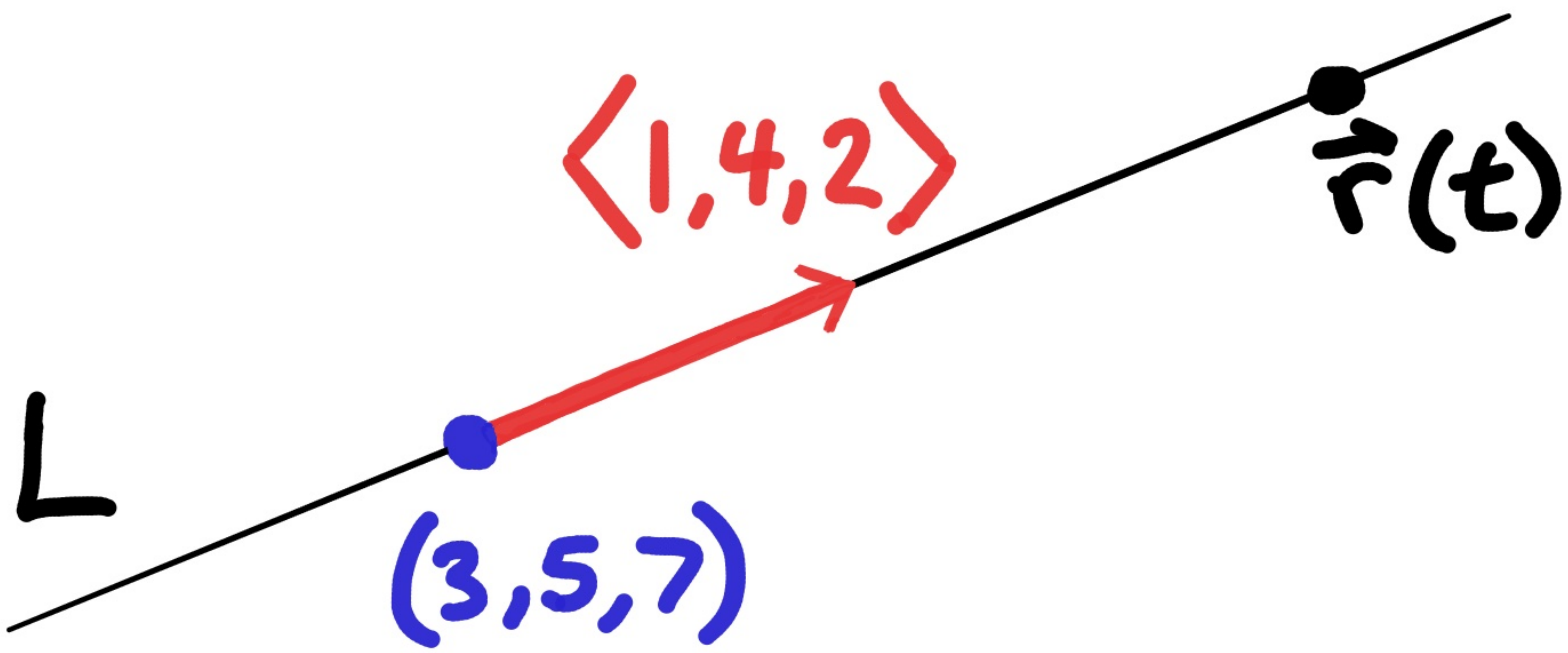

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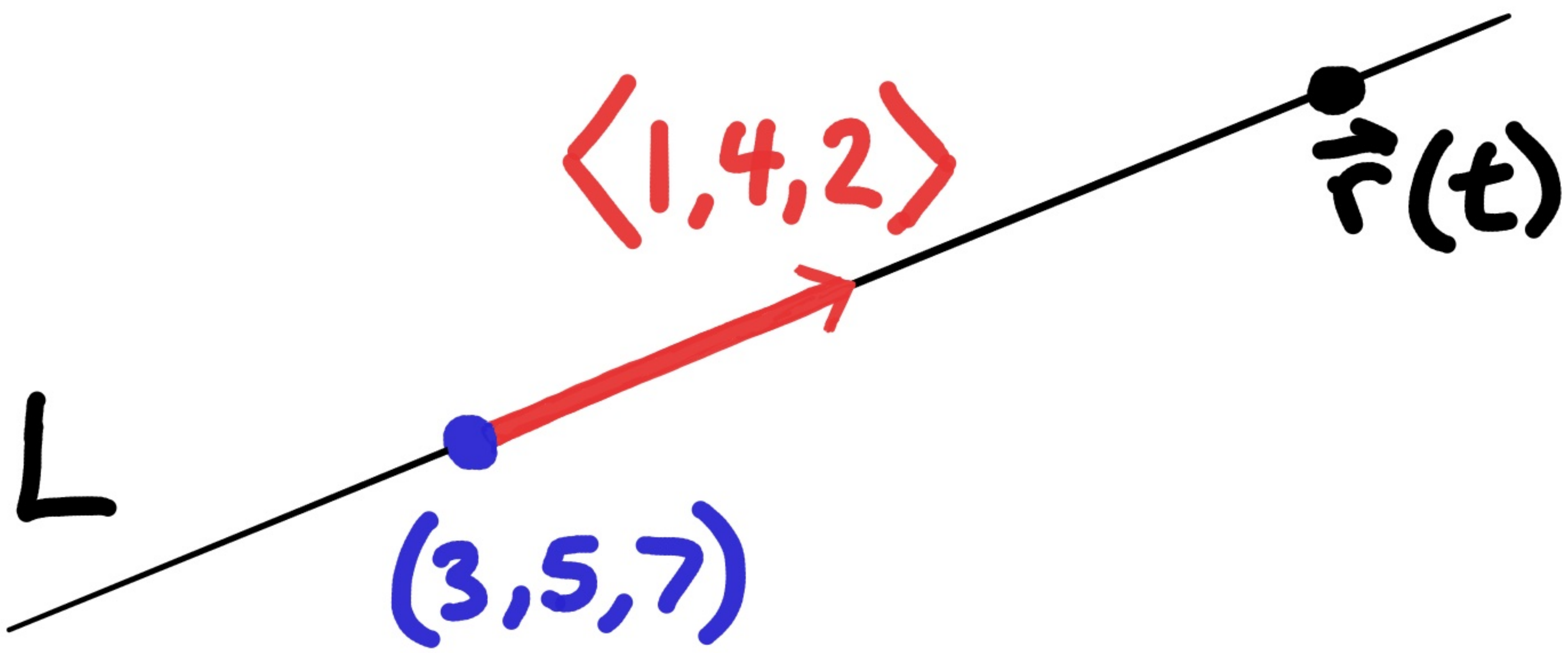

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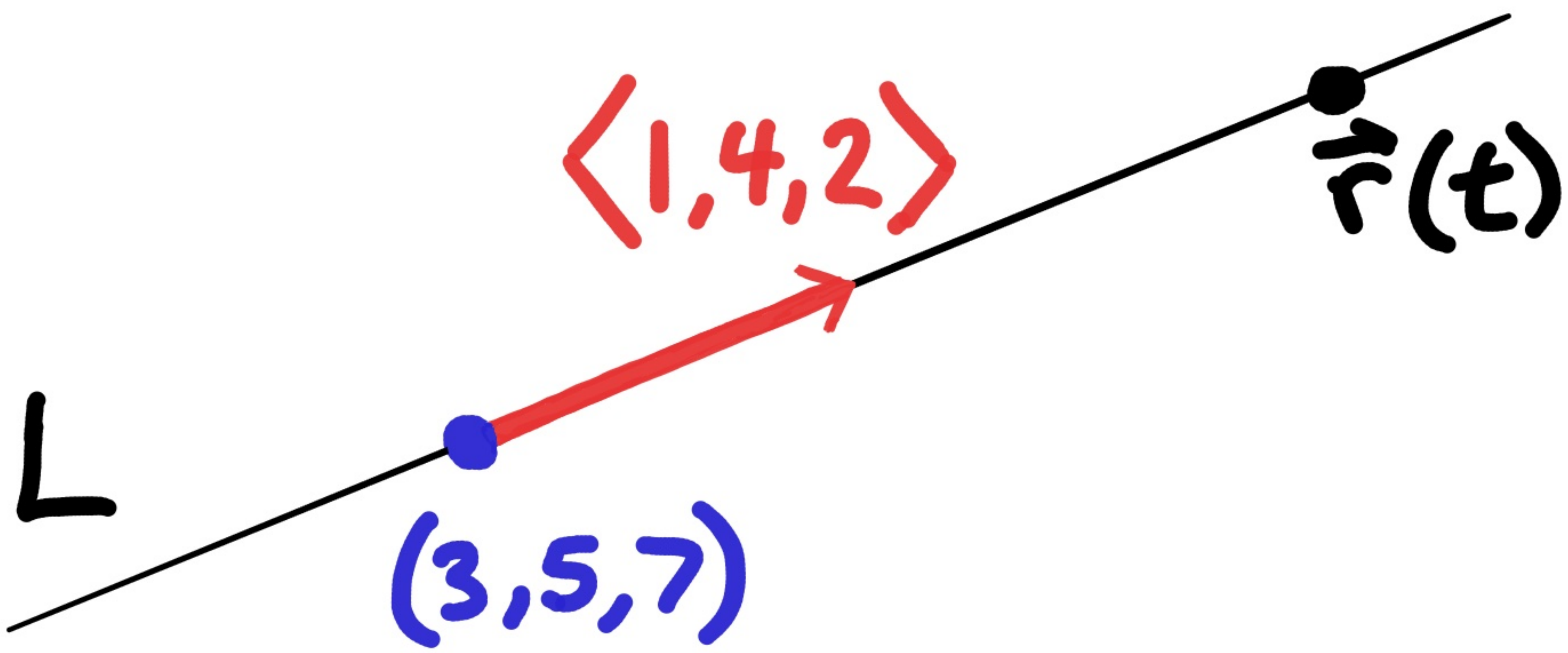

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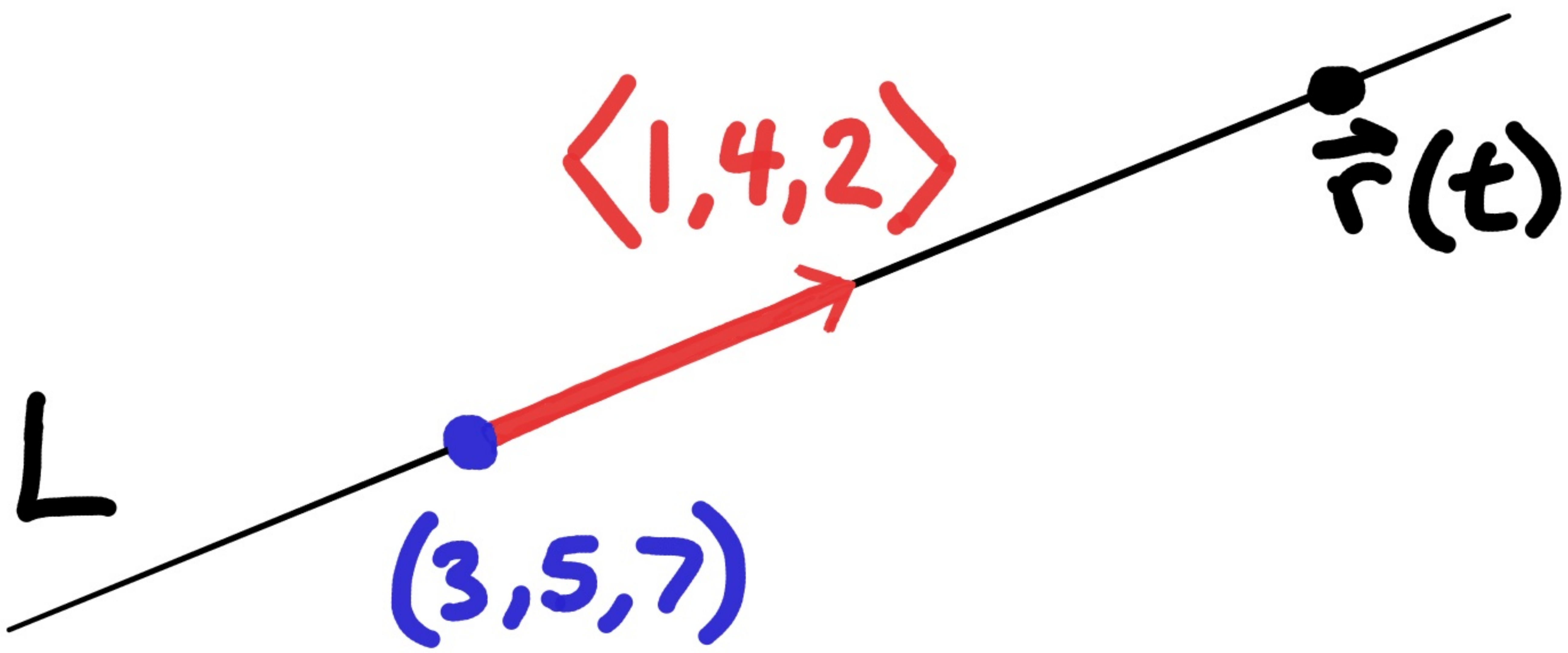

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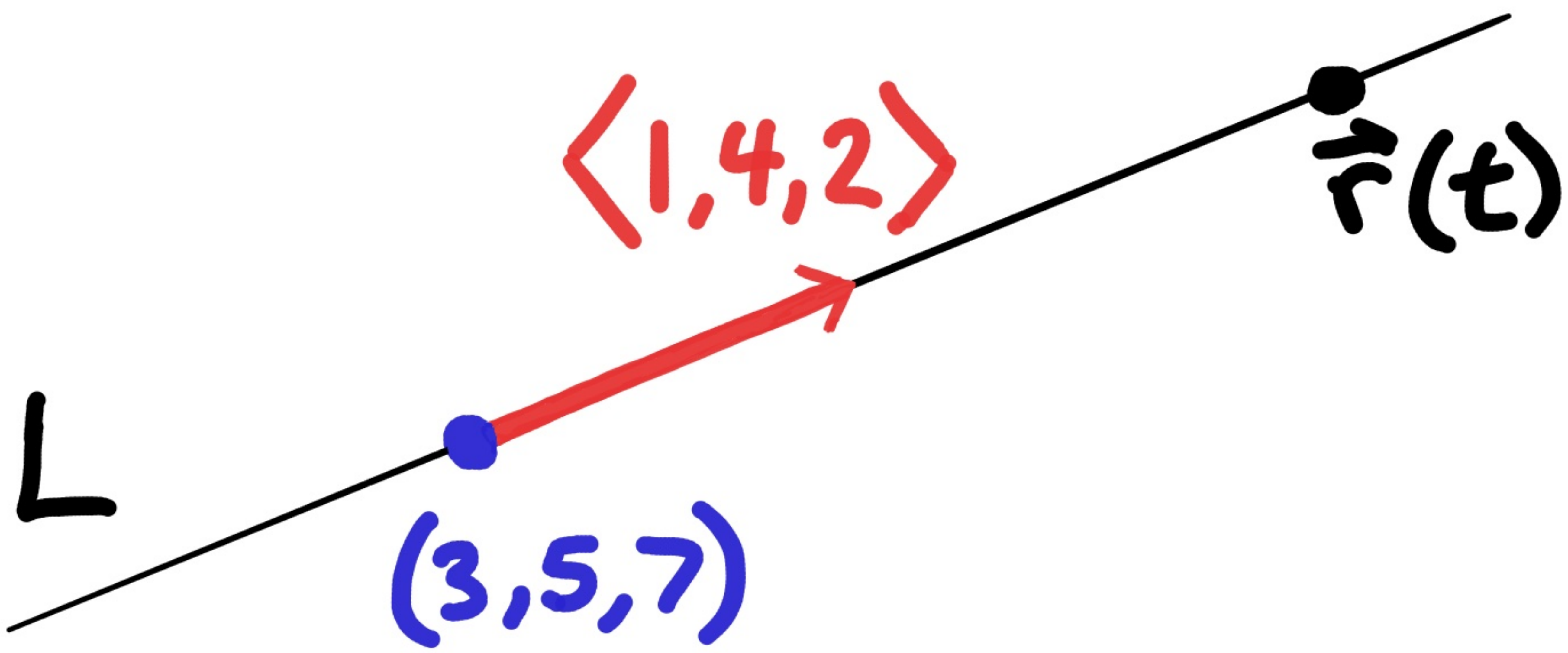

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$$K(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}$$

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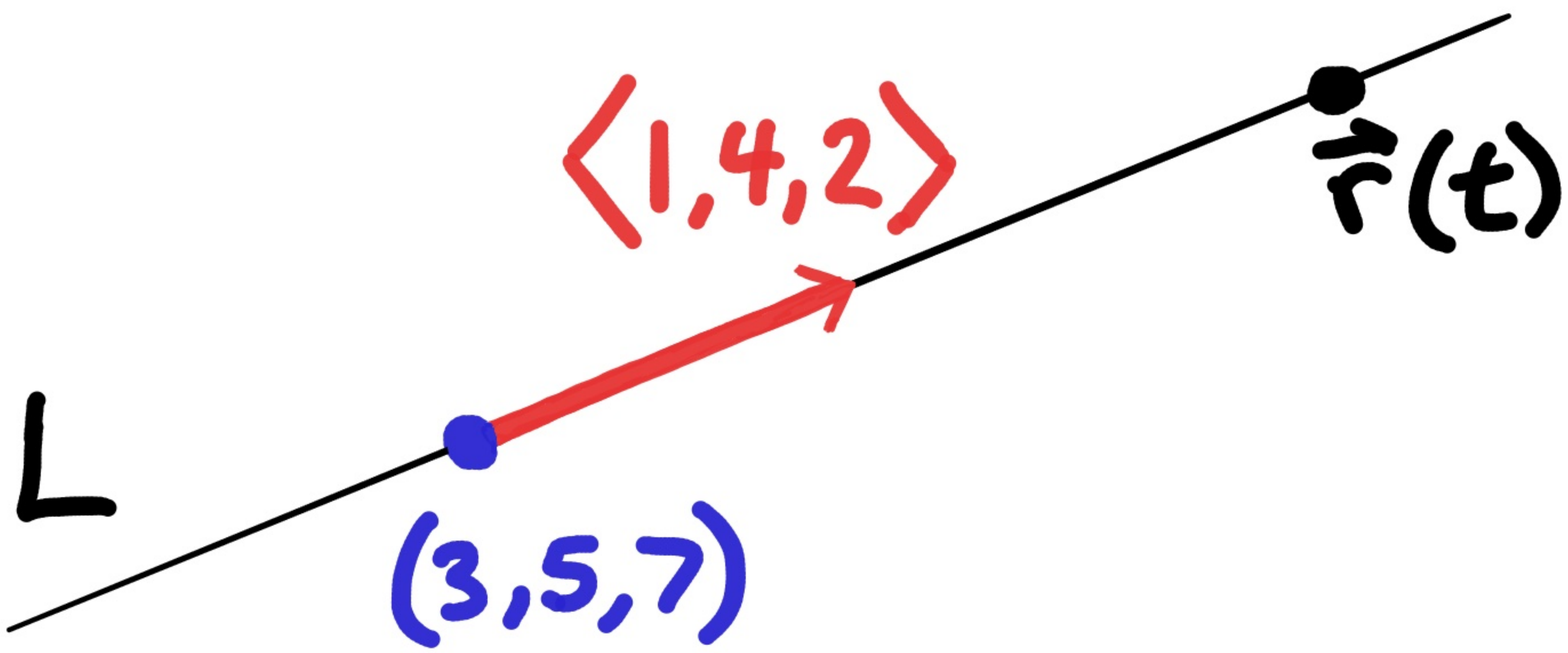

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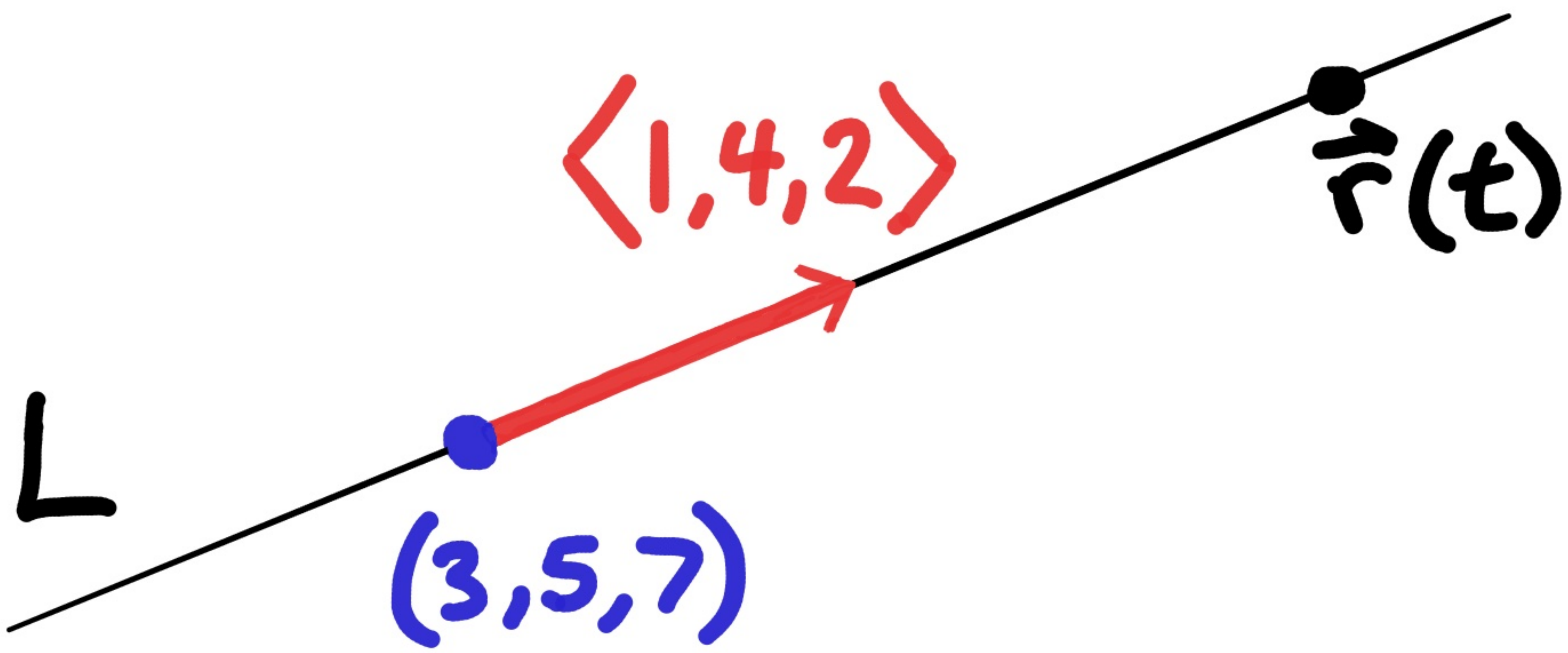

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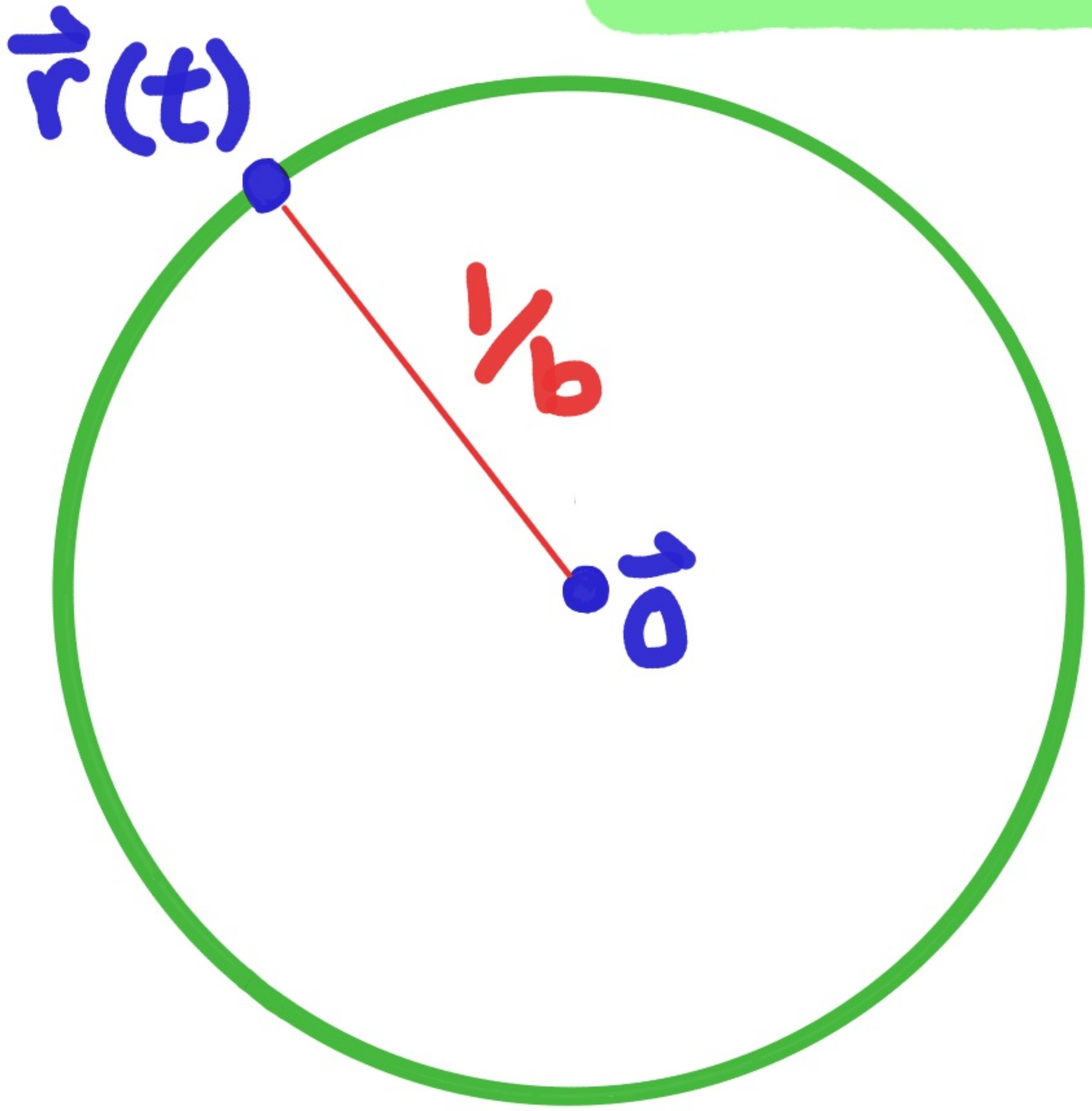
$$\vec{T}\,'(t) = \langle 0,0,0\rangle$$

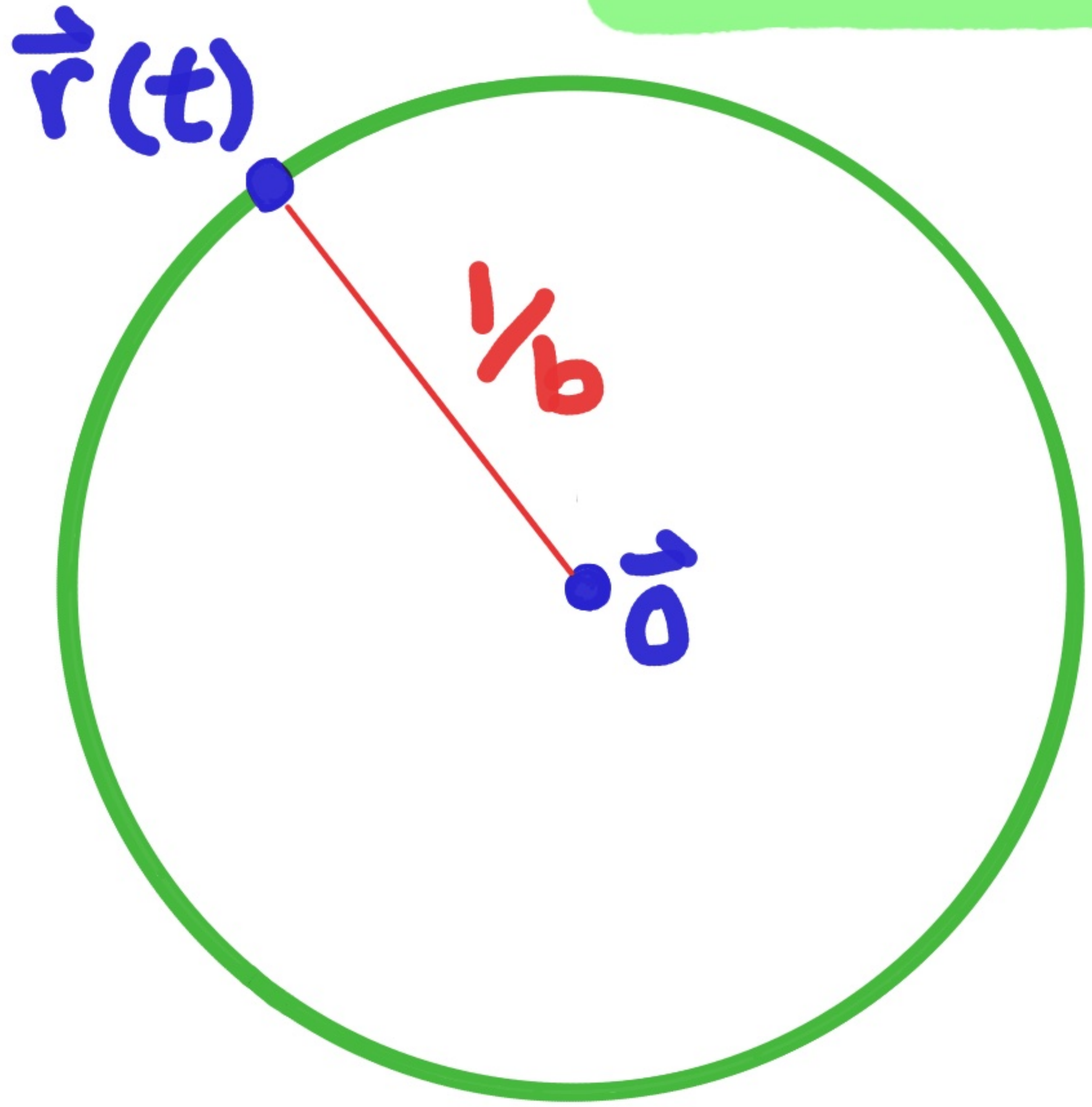
$$K(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = 0$$

Curvature at a point on a circle of radius $1/b$



C is parametrized by $\vec{r}(t) = \langle \frac{1}{b}\cos t, \frac{1}{b}\sin t \rangle$

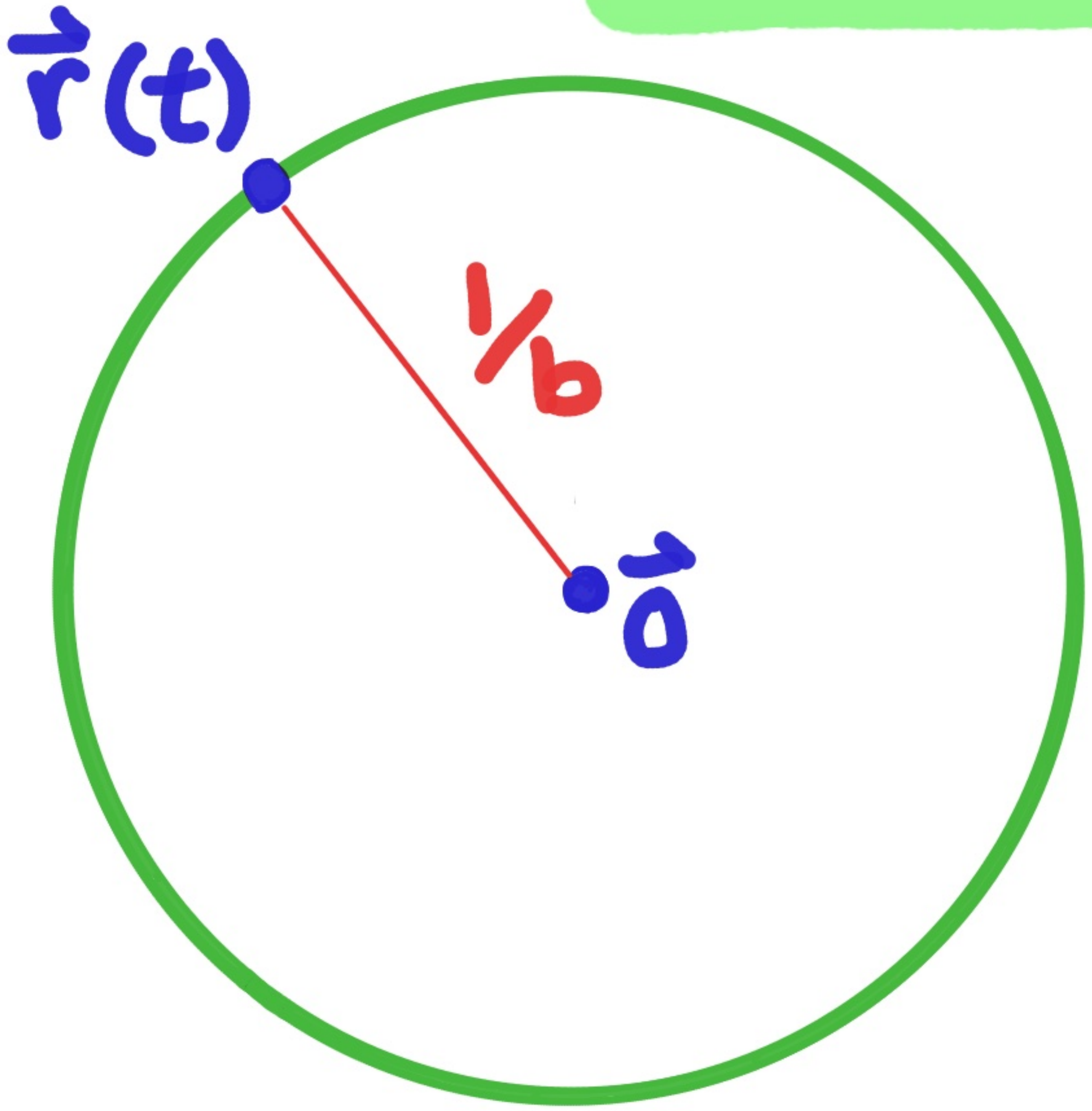
Curvature at a point on a circle of radius $1/b$



C is parametrized by $\vec{r}(t) = \langle \frac{1}{b}\cos t, \frac{1}{b}\sin t \rangle$

$\vec{r}\,'(t) = \langle -\frac{1}{b}\sin t, \frac{1}{b}\cos t \rangle$

Curvature at a point on a circle of radius $1/b$



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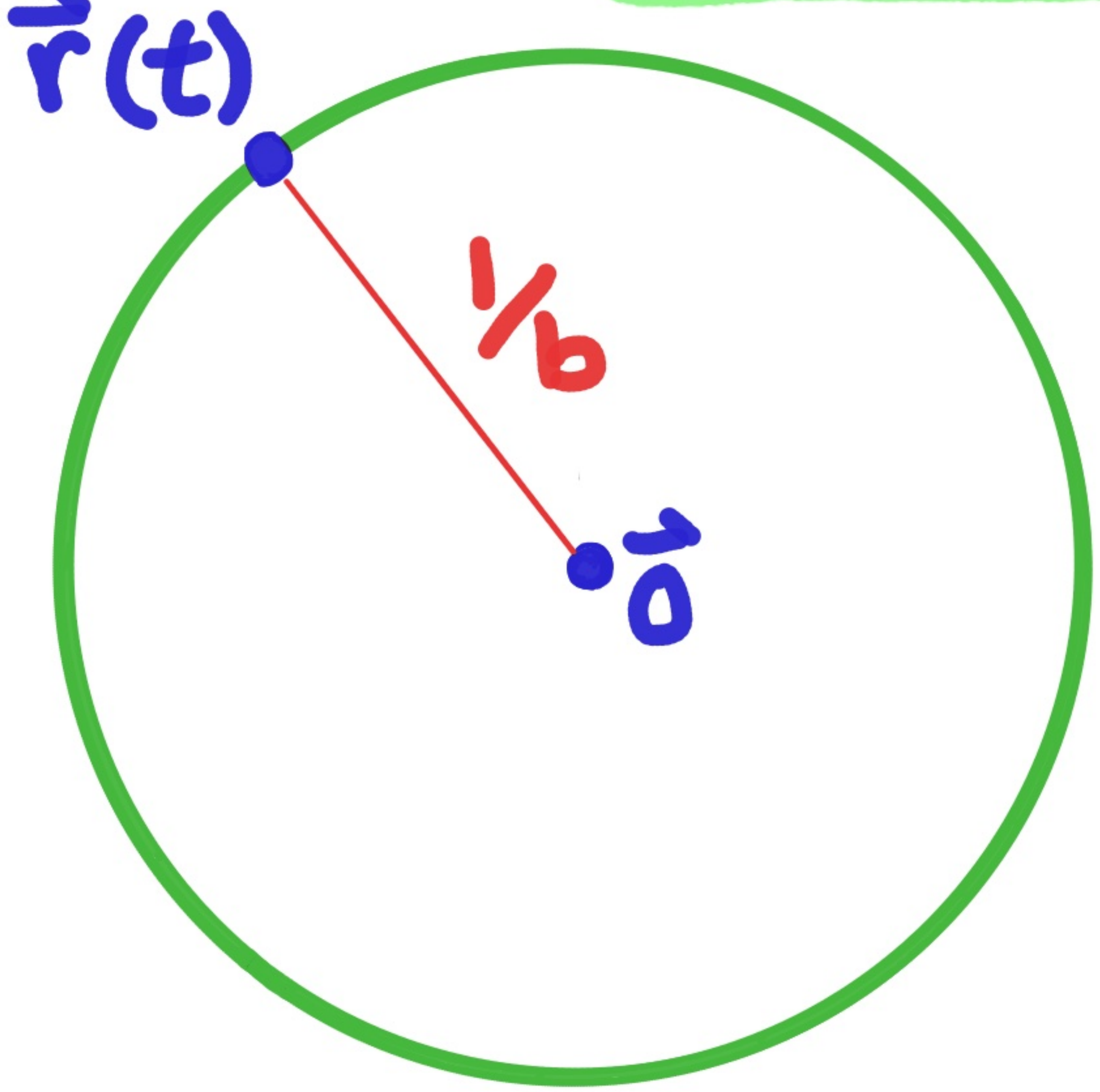
$\|r'(t)\| =$

Curvature at a point on a circle of radius $1/b$



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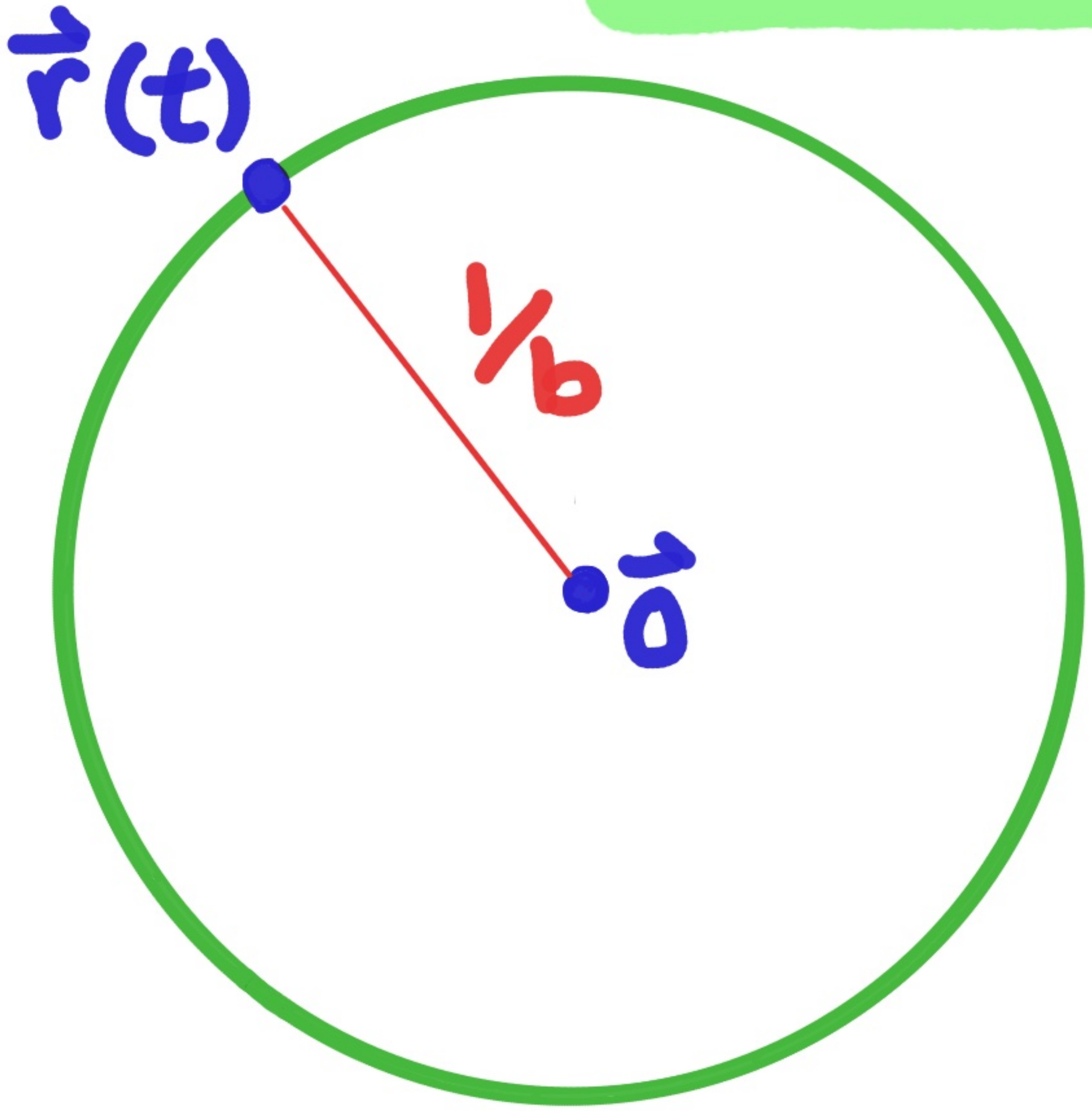
$$\vec{r}\,'(t) = \langle -\tfrac{1}{b}\sin t, \tfrac{1}{b}\cos t \rangle$$

$$\|r'(t)\| = \tfrac{1}{b}\sqrt{\sin^2 t + \cos^2 t}$$

Curvature at a point on a circle of radius $1/b$



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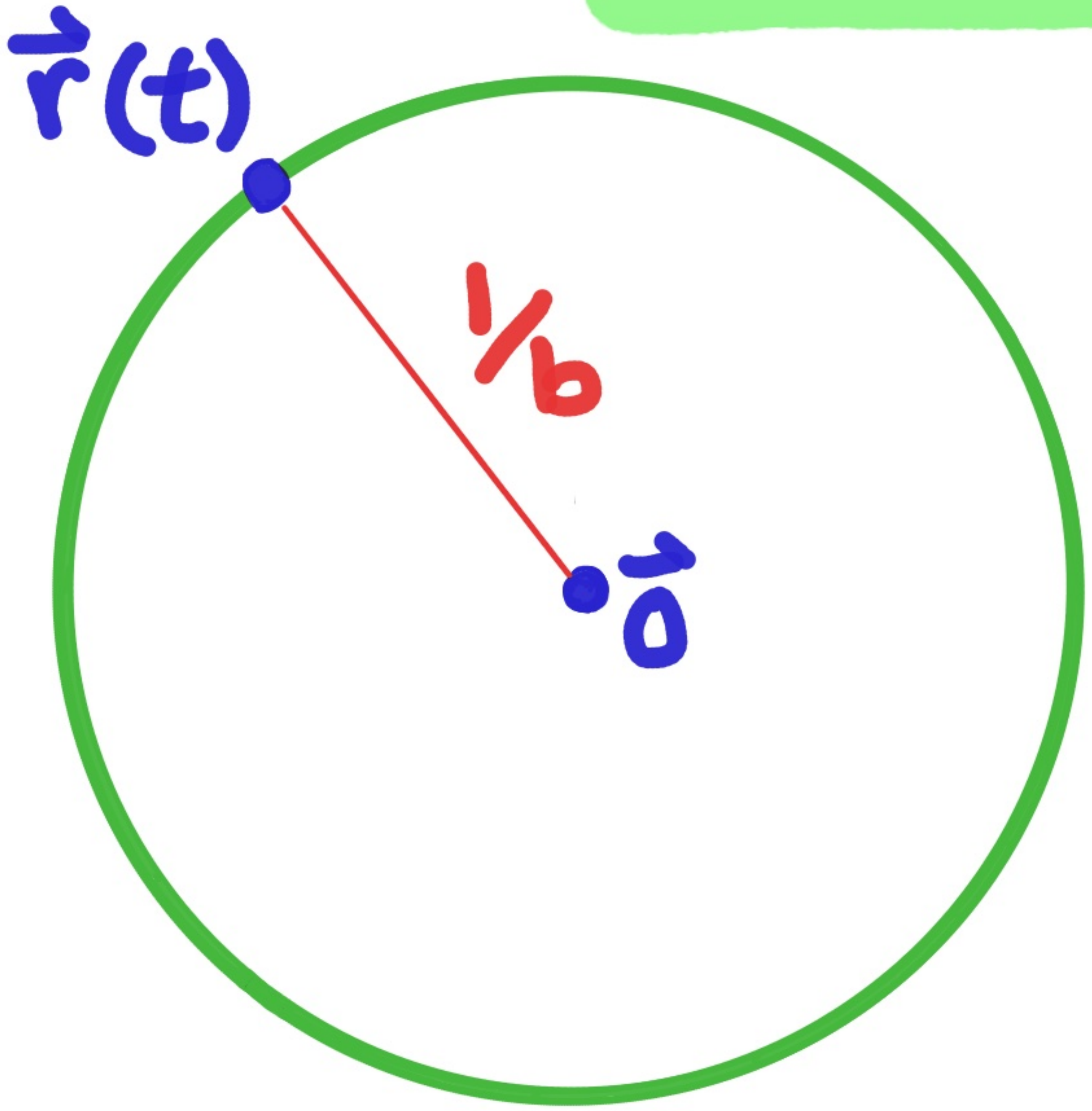
$\|r'(t)\| = \frac{1}{b}\sqrt{\sin^2 t + \cos^2 t} = \frac{1}{b}$

Curvature at a point on a circle of radius $1/b$



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$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

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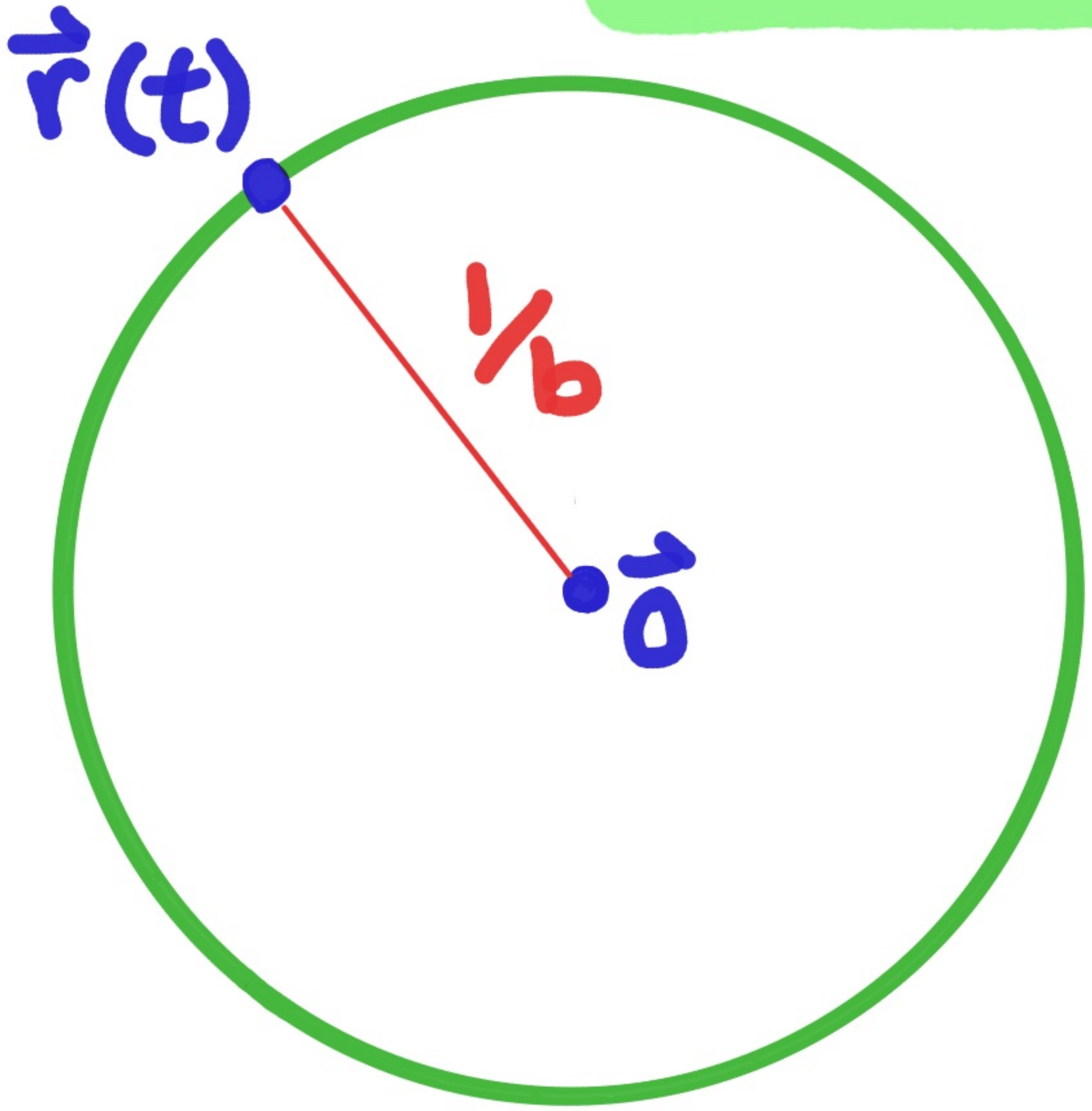

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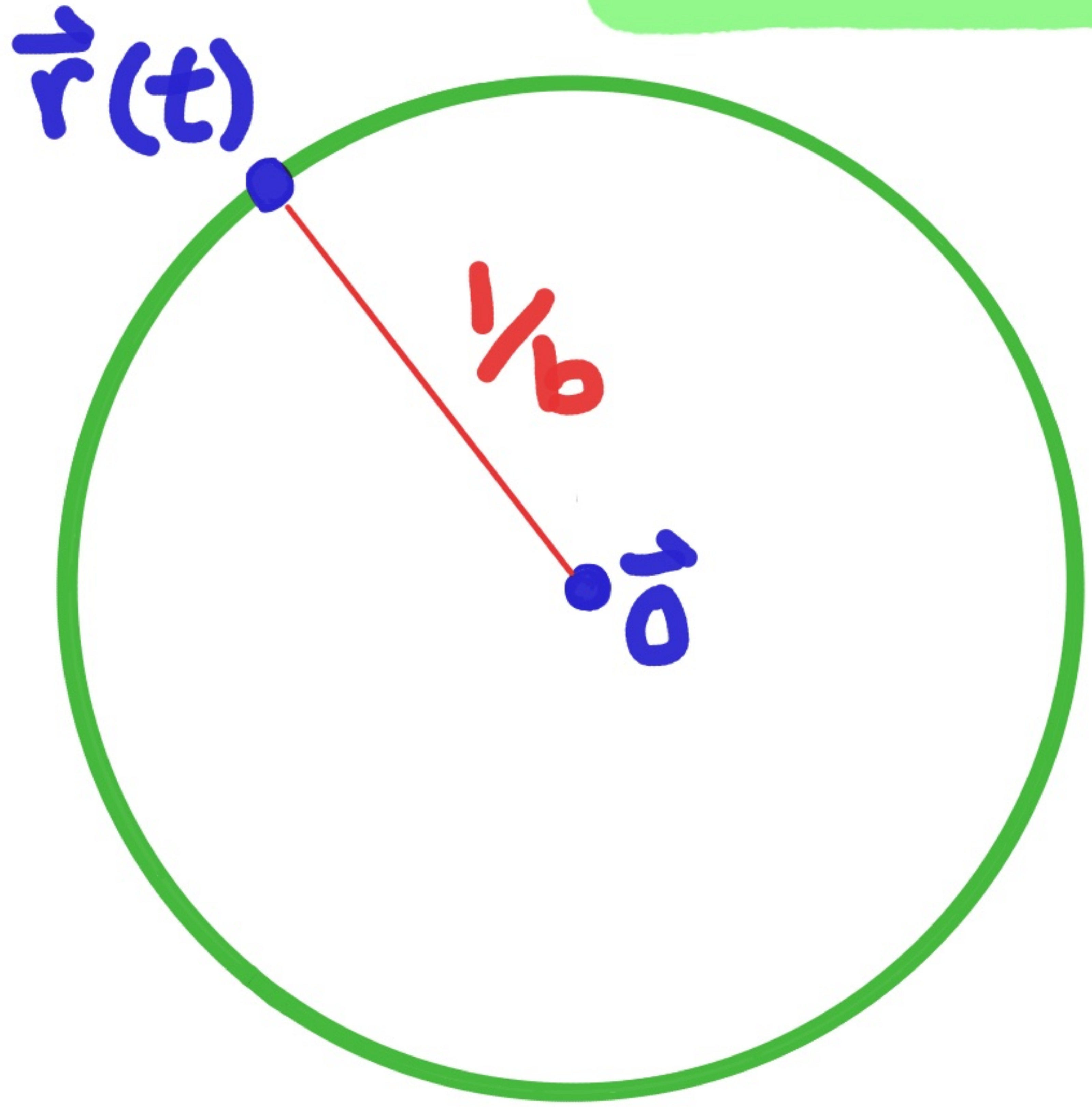
$$\|r'(t)\| = \tfrac{1}{b}\sqrt{\sin^2 t + \cos^2 t} = \tfrac{1}{b}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$= \frac{\langle -\frac{1}{b}\sin t, \frac{1}{b}\cos t \rangle}{1/b}$$

Curvature at a point on a circle of radius $1/b$



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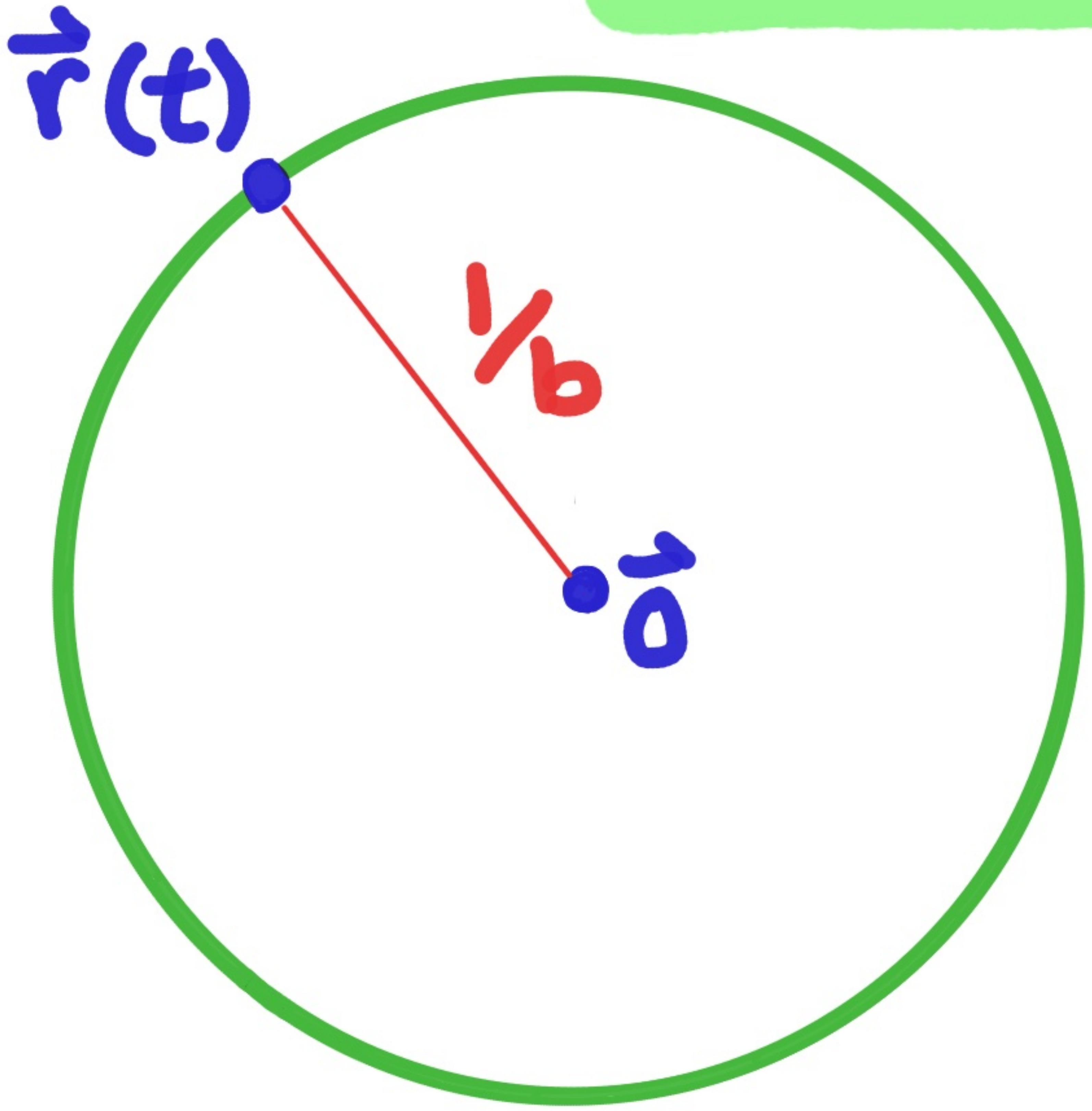
$$\|r'(t)\| = \tfrac{1}{b}\sqrt{\sin^2 t + \cos^2 t} = \tfrac{1}{b}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

$$= \frac{\langle -\frac{1}{b}\sin t, \frac{1}{b}\cos t \rangle}{\frac{1}{b}}$$

$$= \langle -\sin t, \cos t \rangle$$

Curvature at a point on a circle of radius $1/b$



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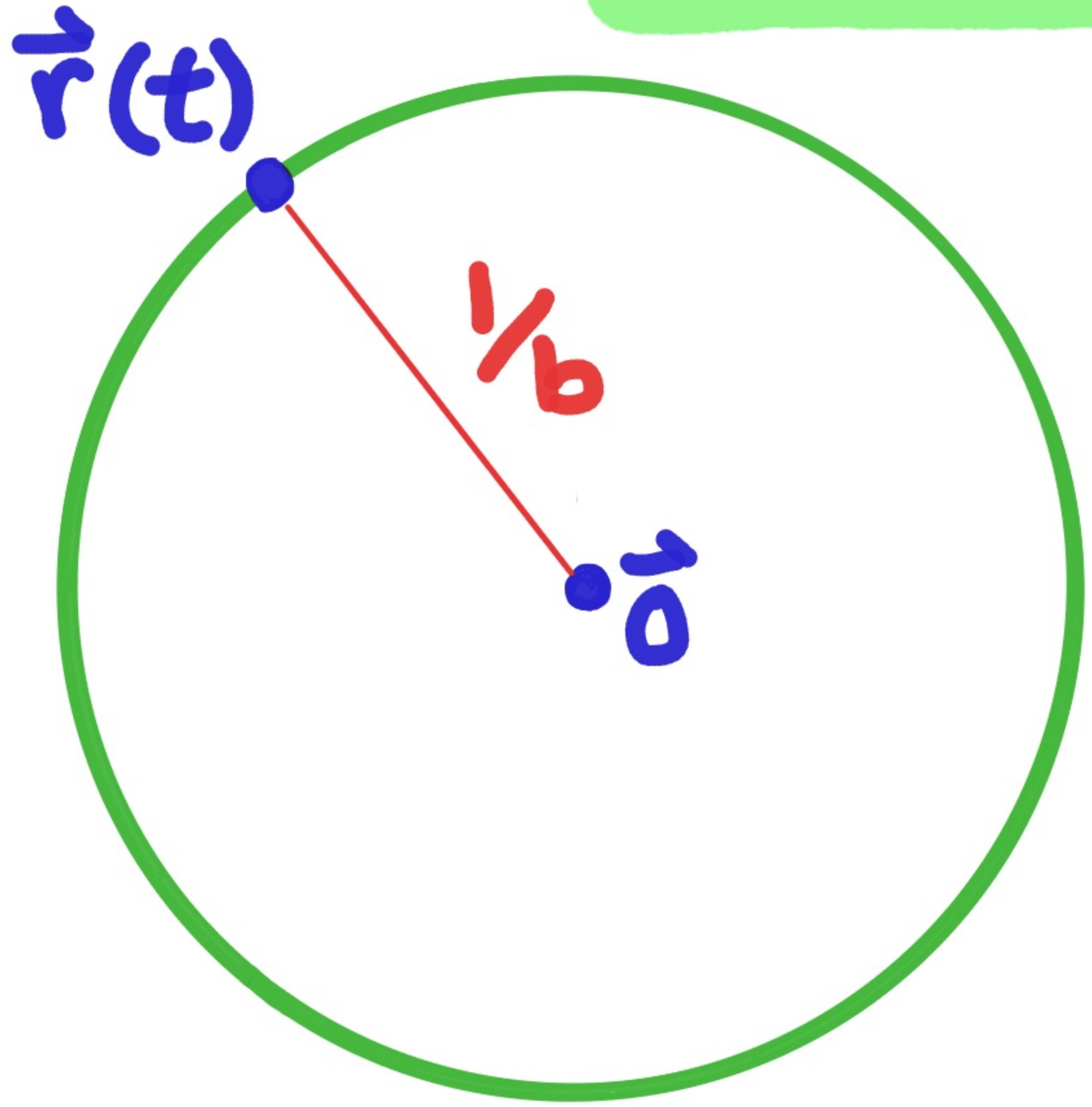
$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$= \frac{\langle -\frac{1}{b}\sin t, \frac{1}{b}\cos t \rangle}{\frac{1}{b}}$$

$$= \langle -\sin t, \cos t \rangle$$

$$\vec{T}\,'(t) =$$

Curvature at a point on a circle of radius $1/b$



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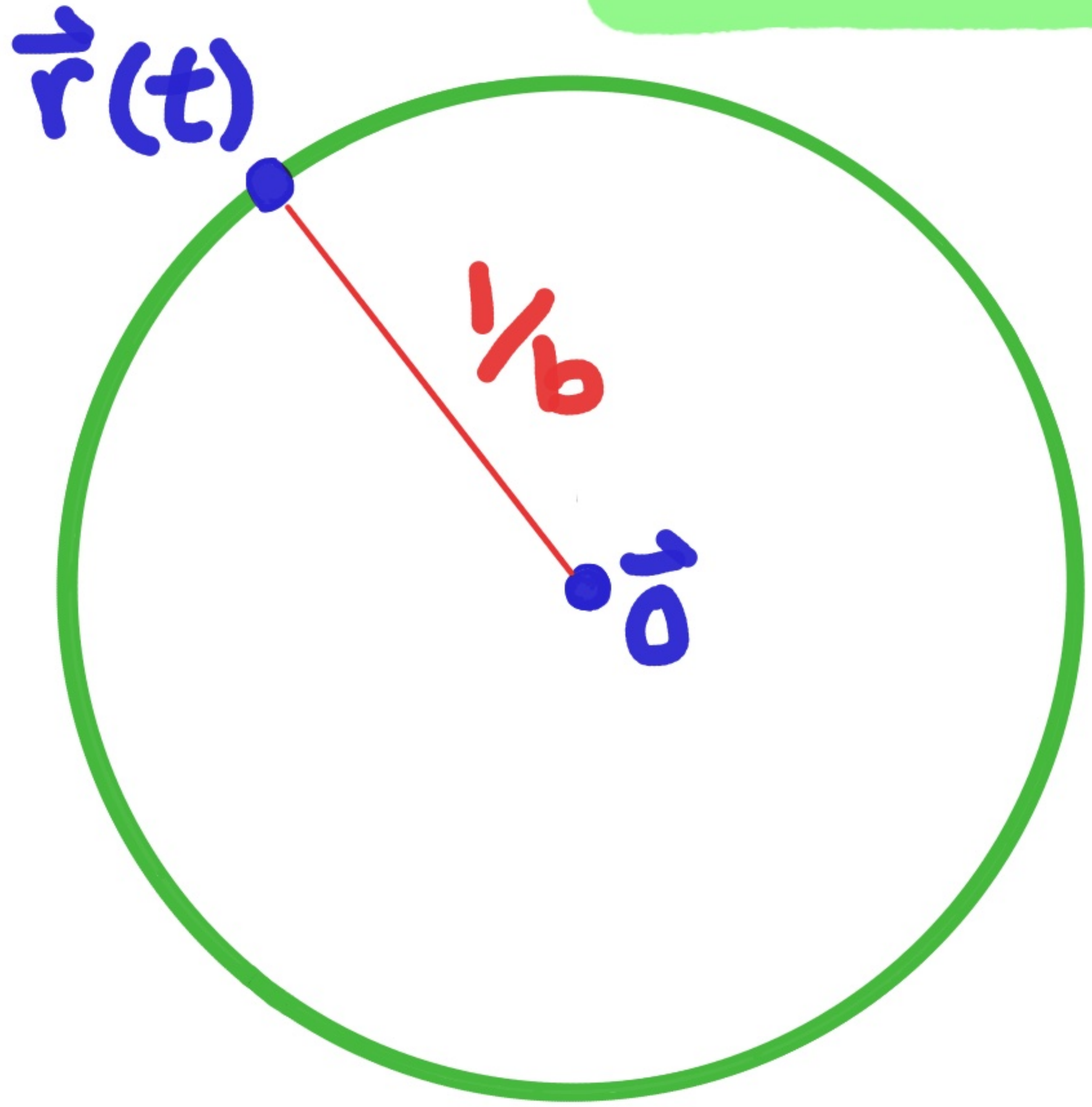
$$= \frac{\langle -\frac{1}{b}\sin t, \frac{1}{b}\cos t \rangle}{\frac{1}{b}}$$

$$= \langle -\sin t, \cos t \rangle$$

$$\vec{T}\,'(t) = \langle -\cos t, -\sin t \rangle$$

Curvature at a point on a circle of radius $1/b$



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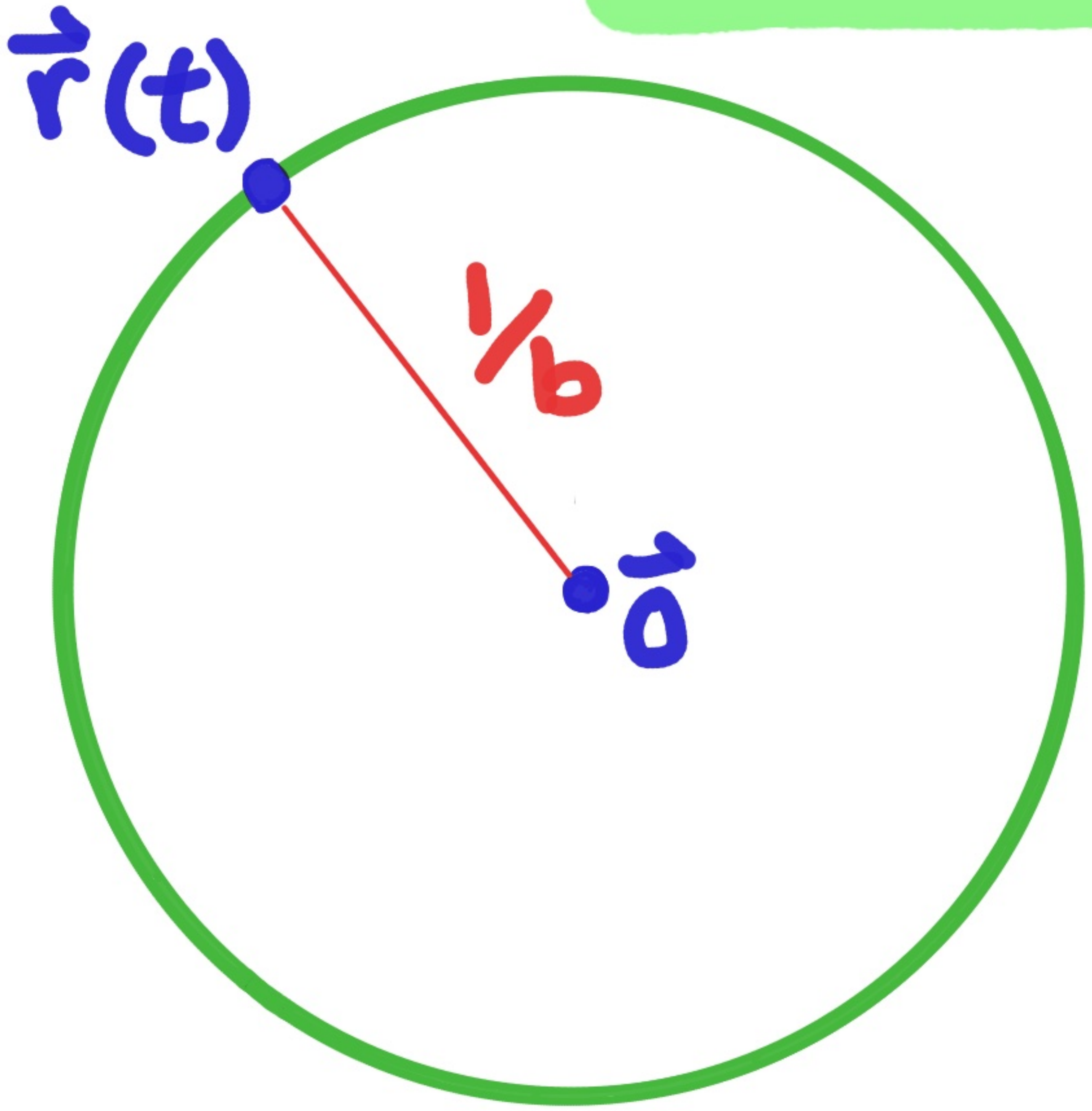
$$\|r'(t)\| = \tfrac{1}{b}\sqrt{\sin^2 t + \cos^2 t} = \tfrac{1}{b}$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle -\frac{1}{b}\sin t, \frac{1}{b}\cos t \rangle}{\frac{1}{b}} = \langle -\sin t, \cos t \rangle$$

$$\vec{T}'(t) = \langle -\cos t, -\sin t \rangle$$

$$K(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

Curvature at a point on a circle of radius $1/b$



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$$\|r'(t)\| = \tfrac{1}{b}\sqrt{\sin^2 t + \cos^2 t} = \tfrac{1}{b}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$= \frac{\langle -\frac{1}{b}\sin t, \frac{1}{b}\cos t \rangle}{1/b}$$

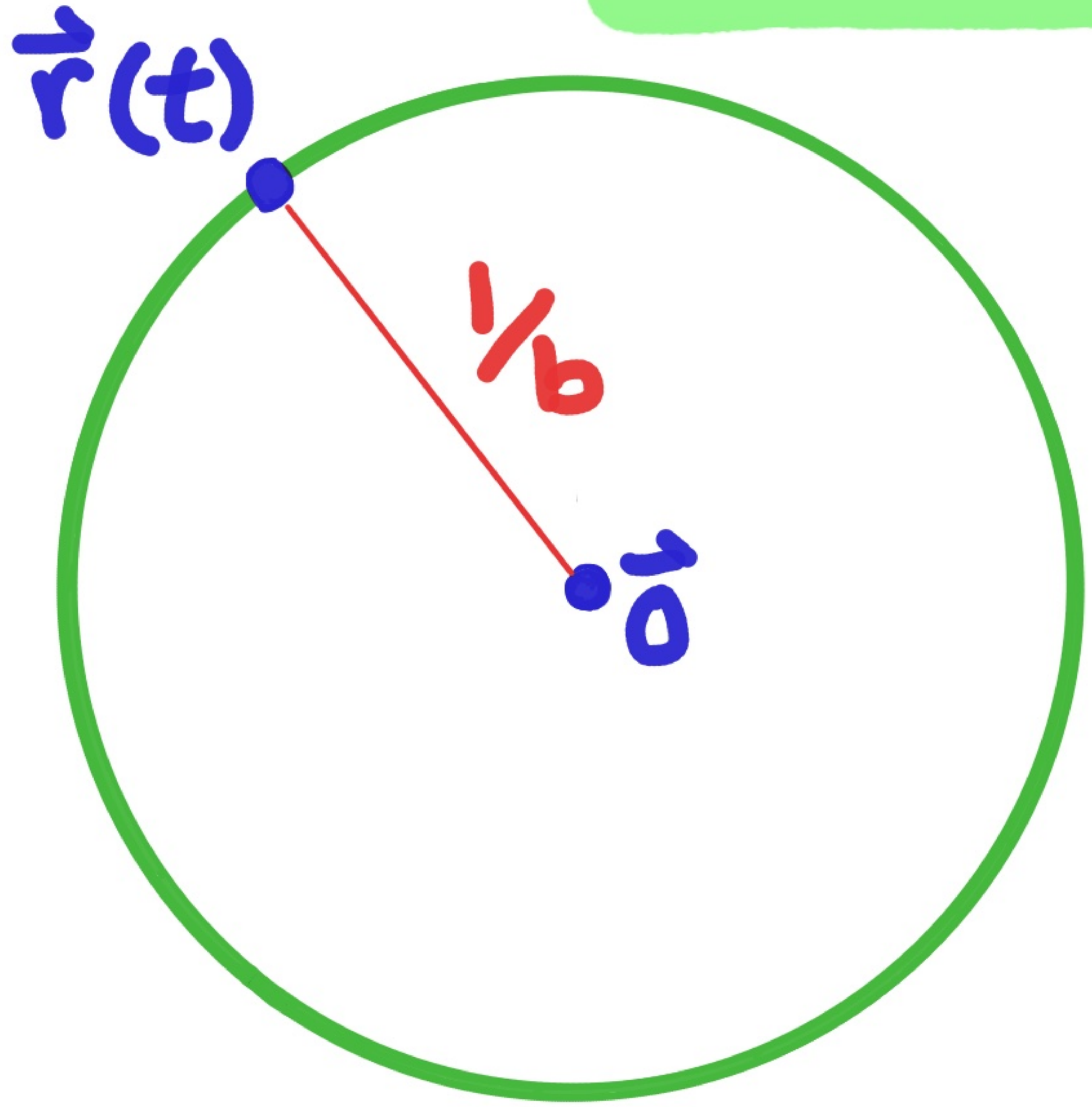
$$= \langle -\sin t, \cos t \rangle$$

$$\vec{T}\,'(t) = \langle -\cos t, -\sin t \rangle$$

$$K(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}$$

$$= \frac{1}{1/b}$$

Curvature at a point on a circle of radius $1/b$



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$\|r'(t)\| = 1/b \sqrt{\sin^2 t + \cos^2 t} = 1/b$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$= \frac{\langle -1/b \sin t, 1/b \cos t \rangle}{1/b}$$

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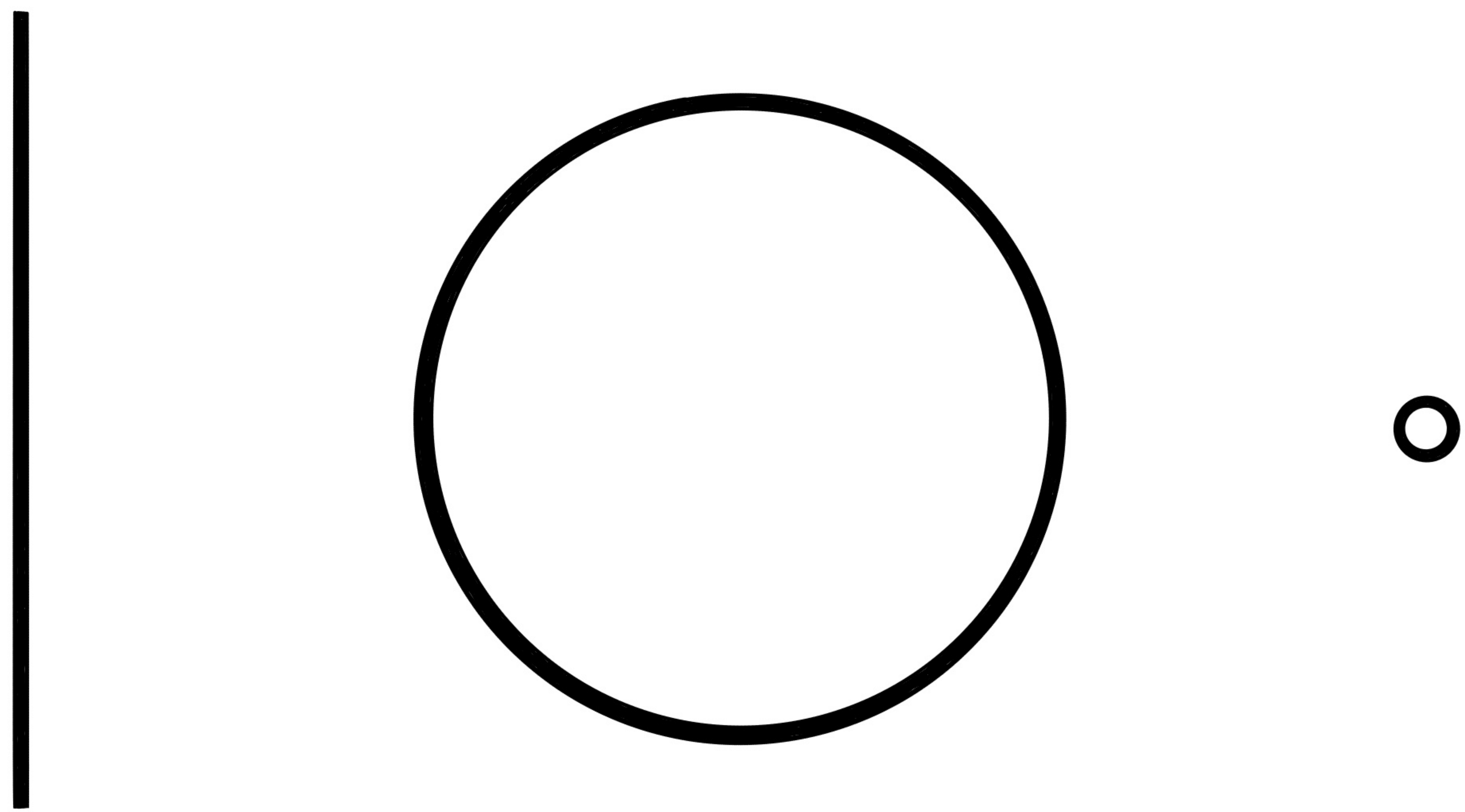
$$\vec{T}\,'(t) = \langle -\cos t, -\sin t \rangle$$

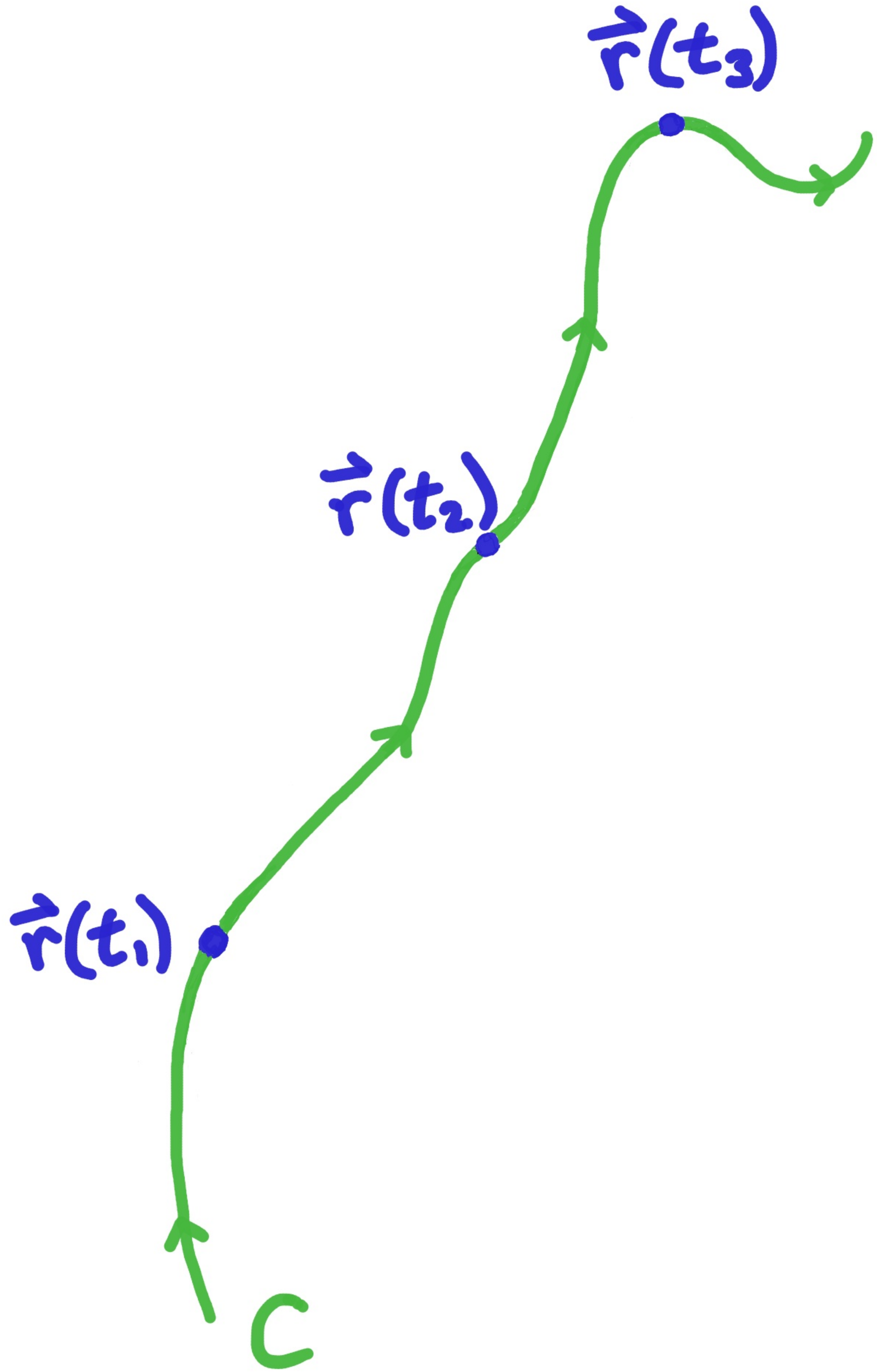
$$K(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}$$

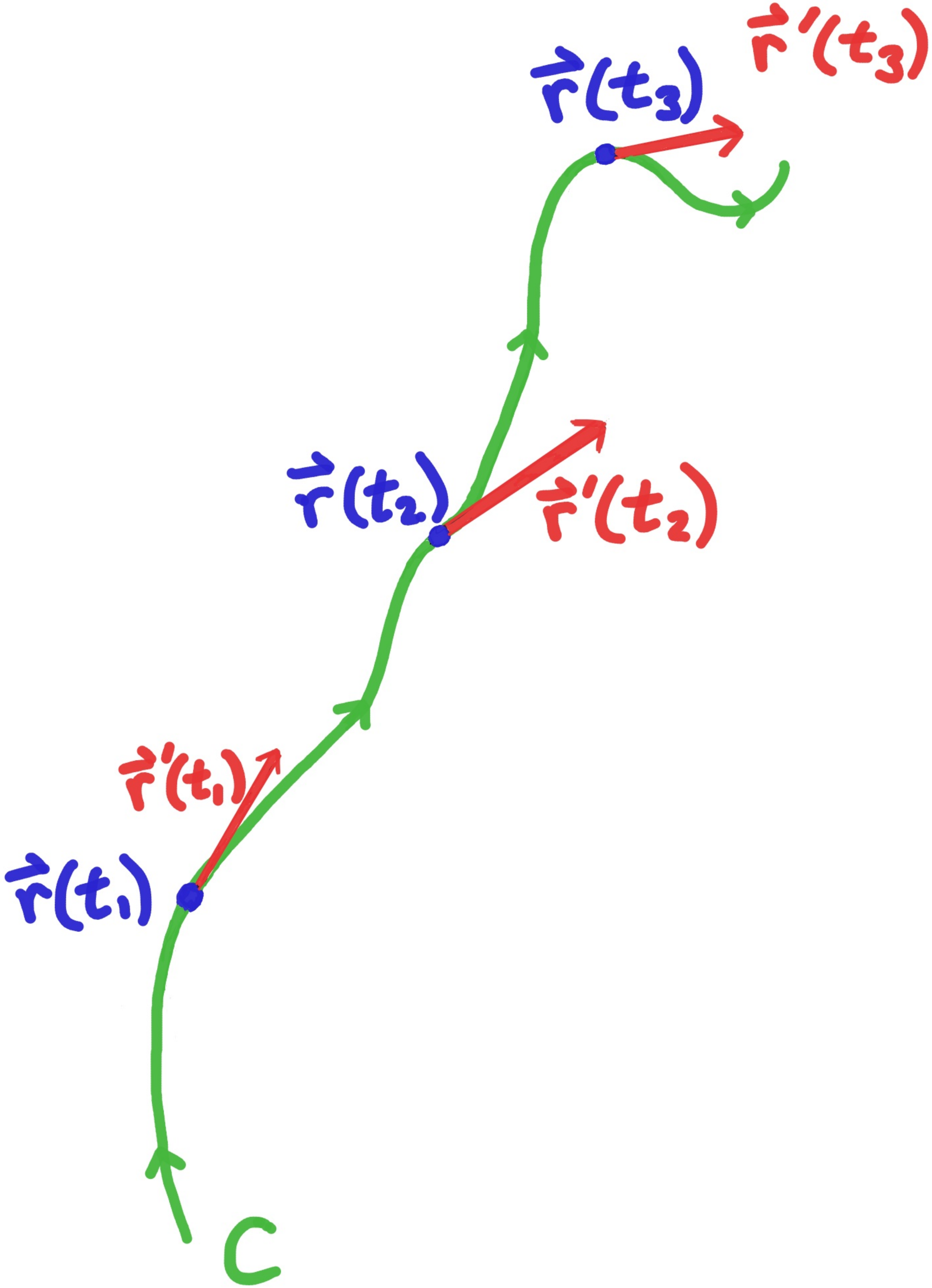
$$= \frac{1}{1/b} = b.$$

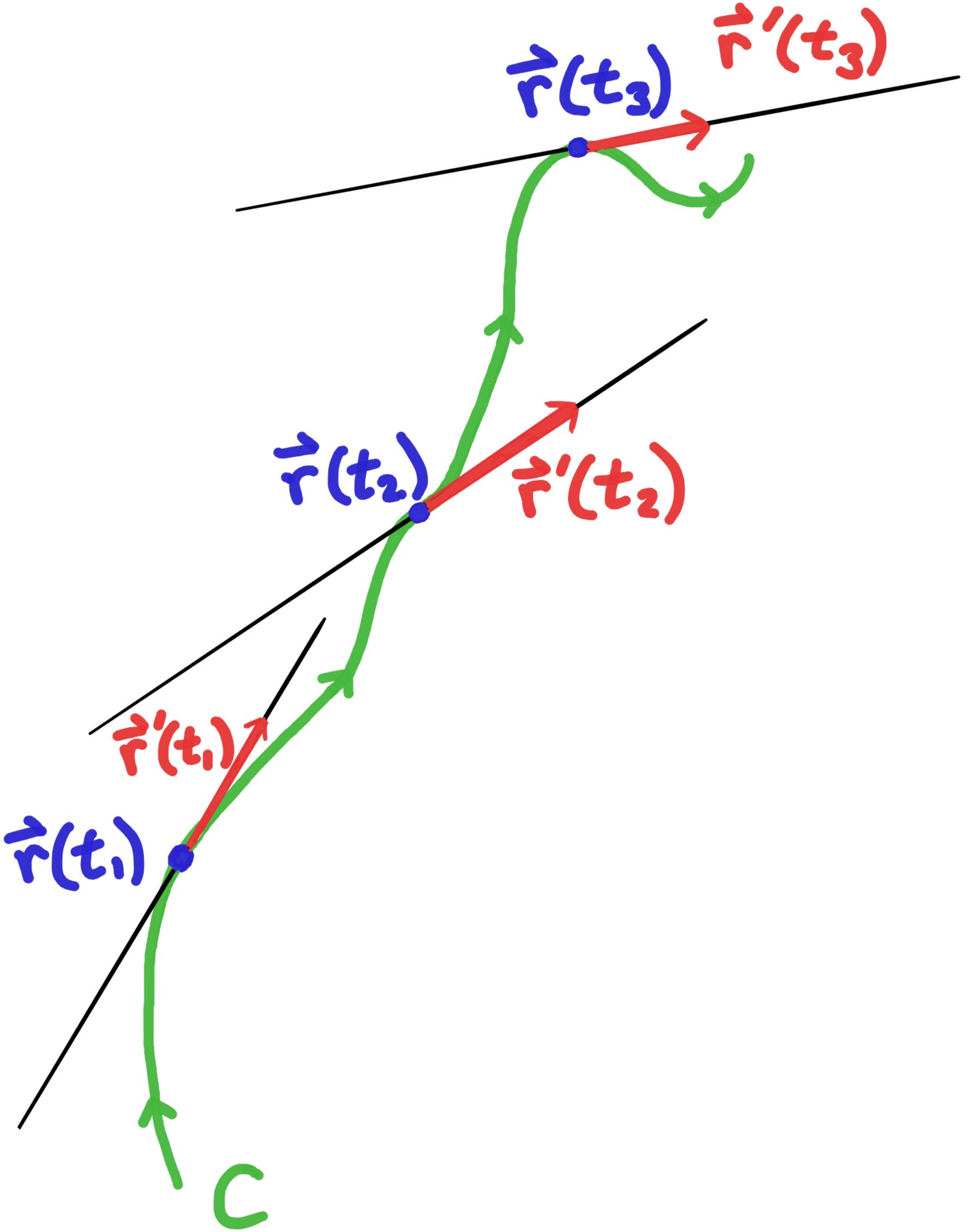
Curves with constant curvature



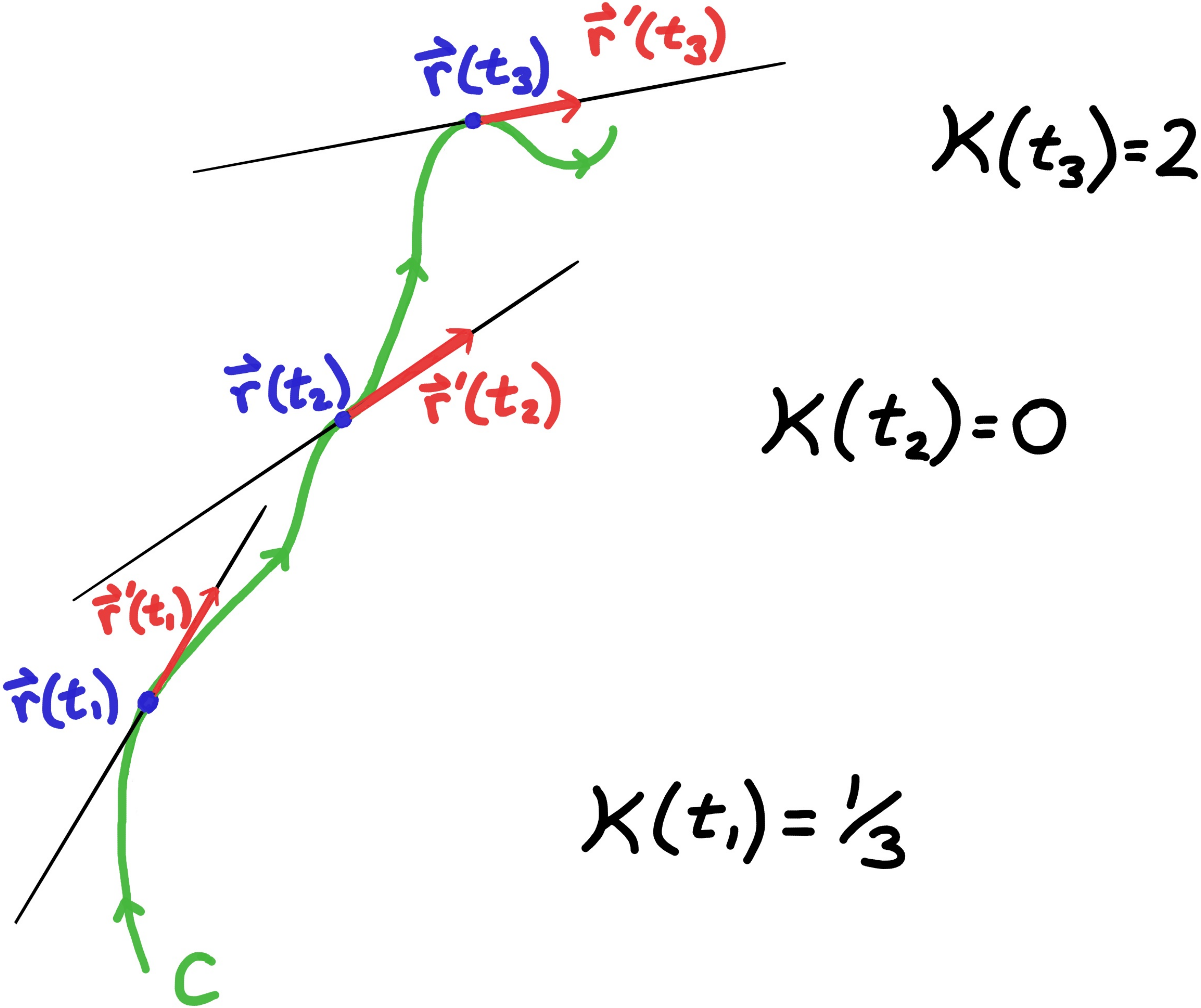
$\kappa = 0$ $\kappa = 1/3$ $\kappa = 100$

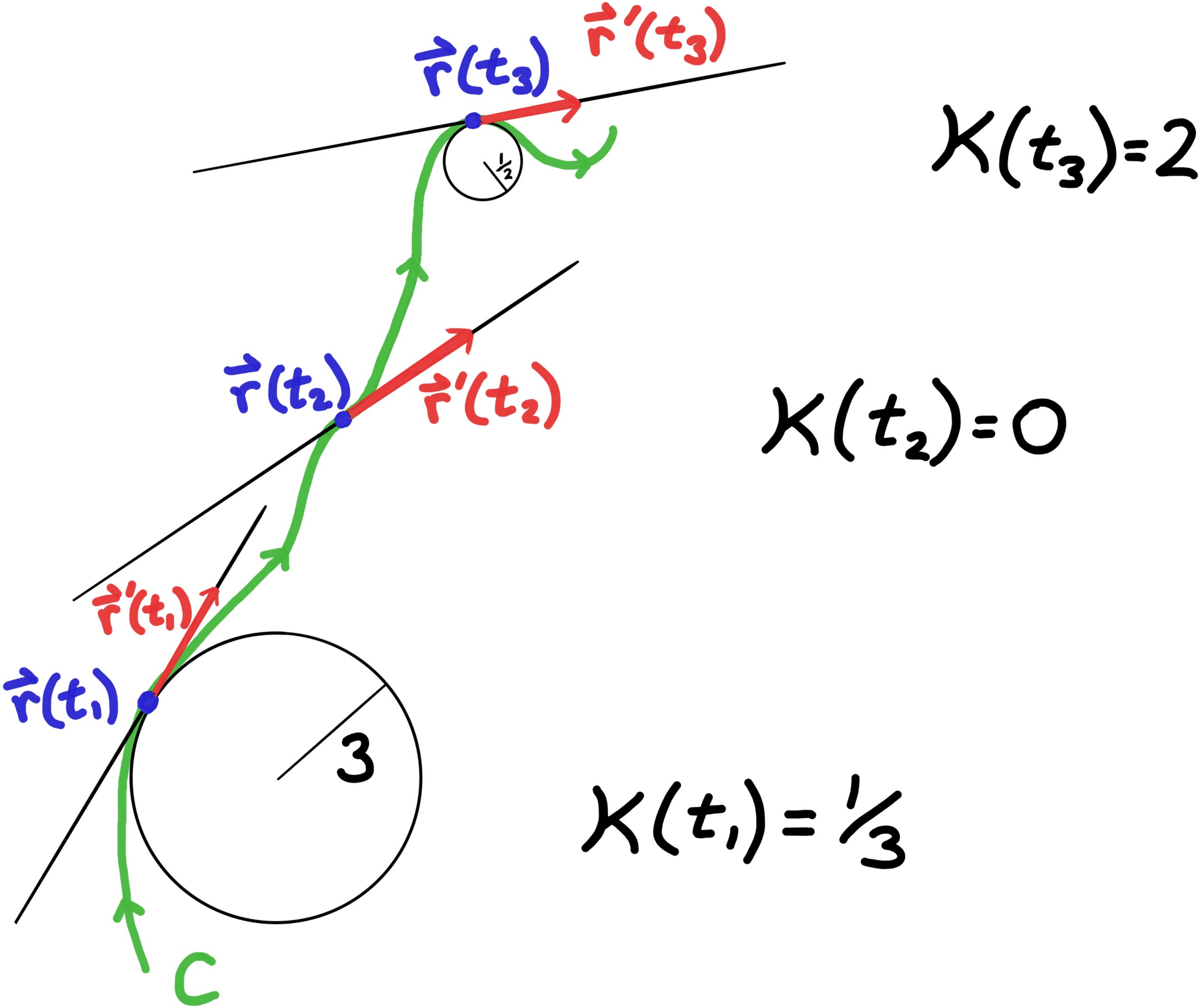

$\vec{r}(t_3)$
$\vec{r}(t_2)$
$\vec{r}(t_1)$
C


$\vec{r}(t_3)$
$\vec{r}\,'(t_3)$
$\vec{r}(t_2)$
$\vec{r}\,'(t_2)$
$\vec{r}\,'(t_1)$
$\vec{r}(t_1)$
C



$\vec{r}(t_3)$
$\vec{r}'(t_3)$
$\vec{r}(t_2)$
$\vec{r}'(t_2)$
$\vec{r}'(t_1)$
$\vec{r}(t_1)$
C


$\vec{r}(t_3)$
$\vec{r}'(t_3)$
$\kappa(t_3) = 2$
$\vec{r}(t_2)$
$\vec{r}'(t_2)$
$\kappa(t_2) = 0$
$\vec{r}'(t_1)$
$\vec{r}(t_1)$
$\kappa(t_1) = 1/3$
C

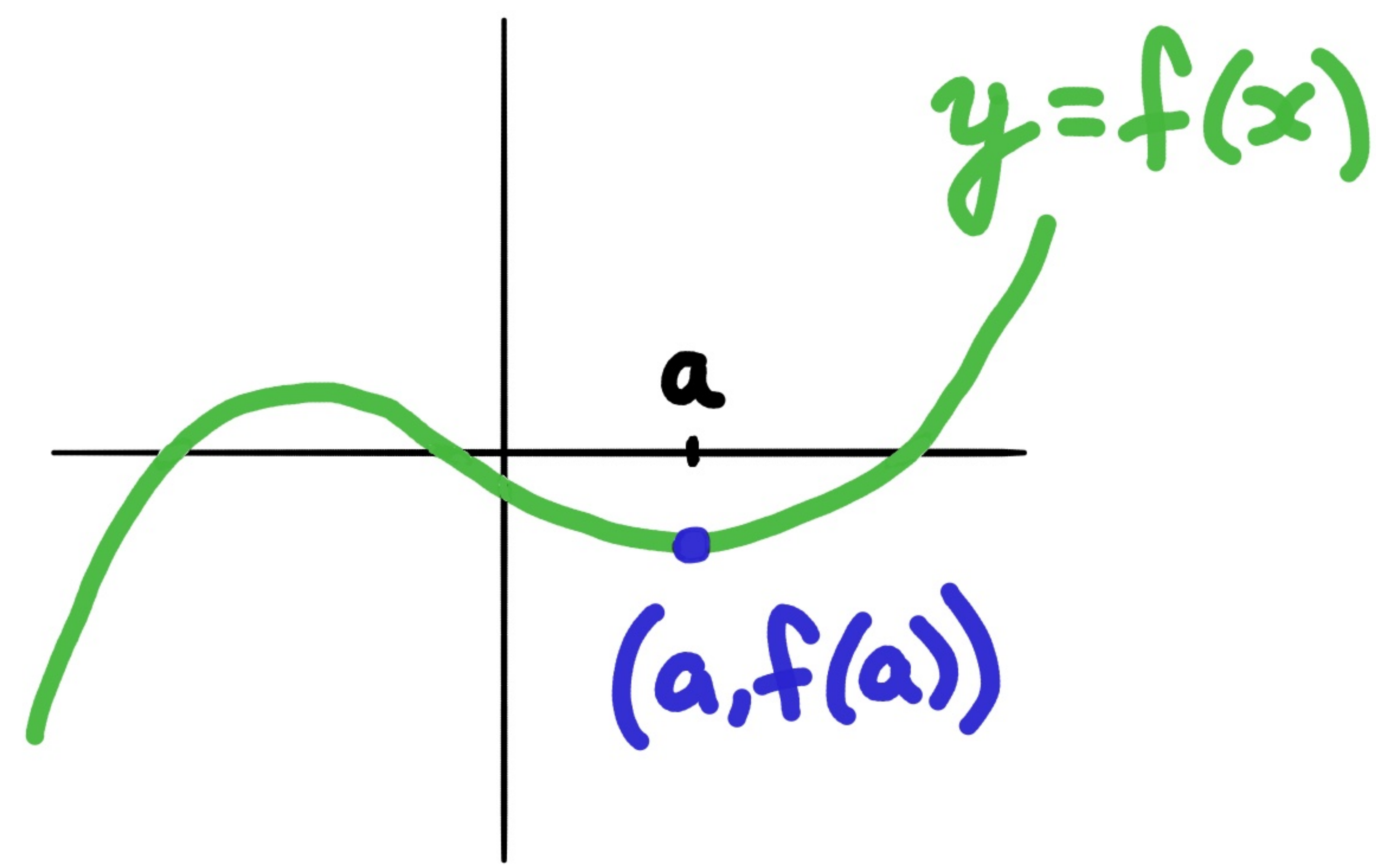


$\vec{r}(t_3)$
$\vec{r}\,'(t_3)$
$\frac{1}{2}$
$K(t_3)=2$
$\vec{r}(t_2)$
$\vec{r}\,'(t_2)$
$K(t_2)=0$
$\vec{r}\,'(t_1)$
$\vec{r}(t_1)$
3
$K(t_1)=1/3$
C

II Curvature of graphs

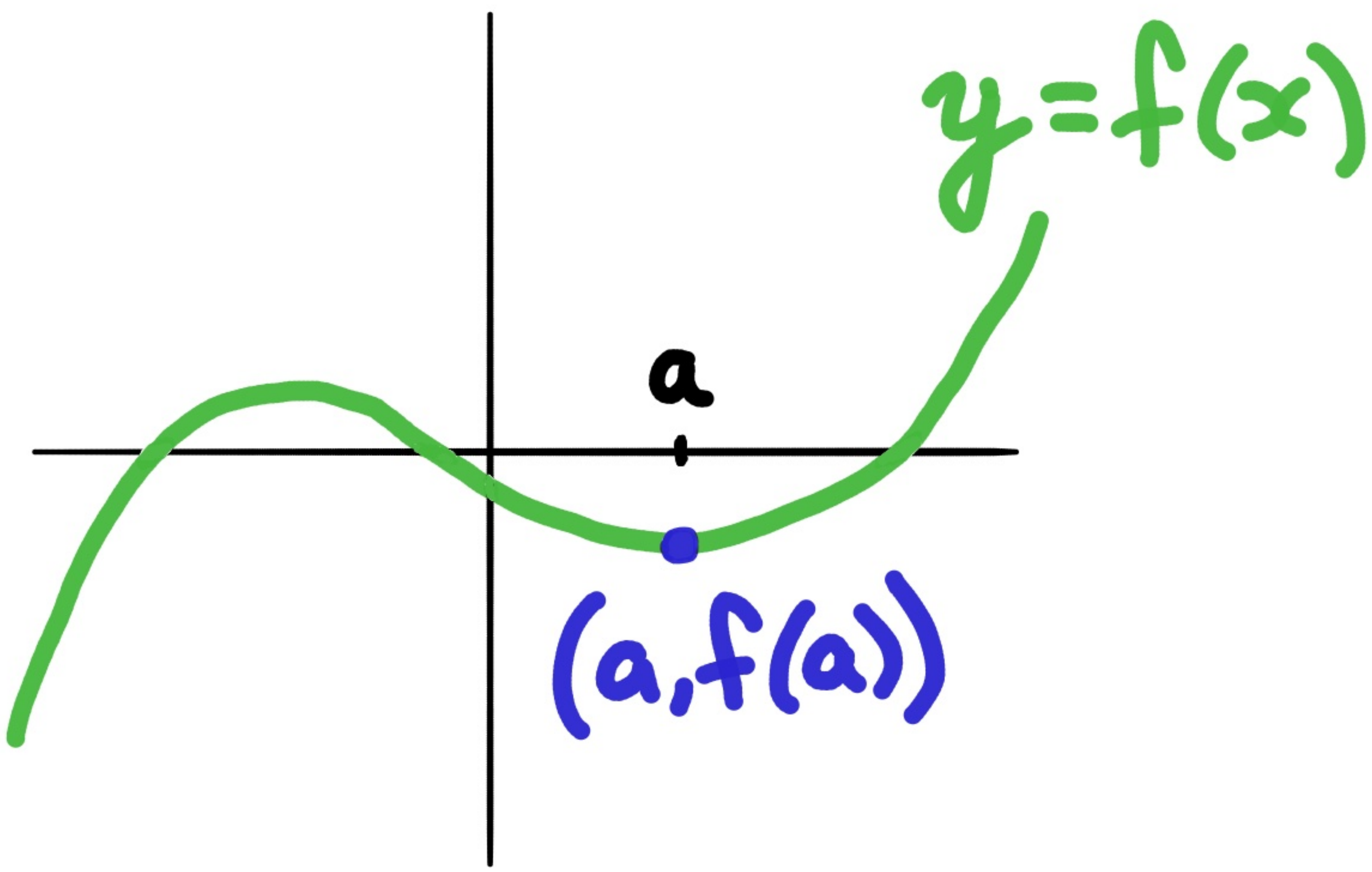
Curvature of graphs of functions

If $f:\mathbb{R}\to\mathbb{R}$, then the graph of $y=f(x)$ is a curve.



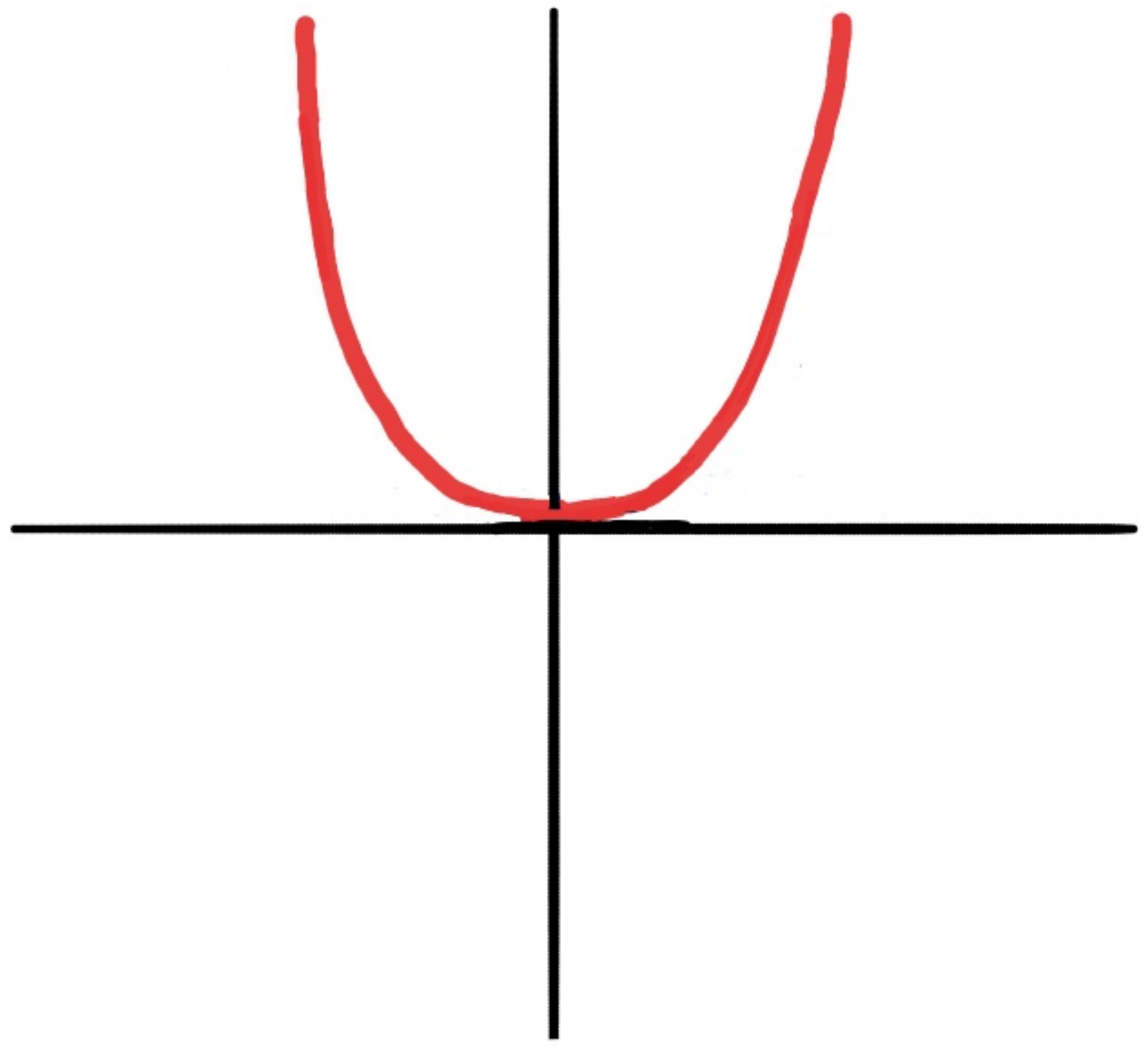
$\vec{r}(t)=\langle t, f(t)\rangle$

Curvature of graphs of functions



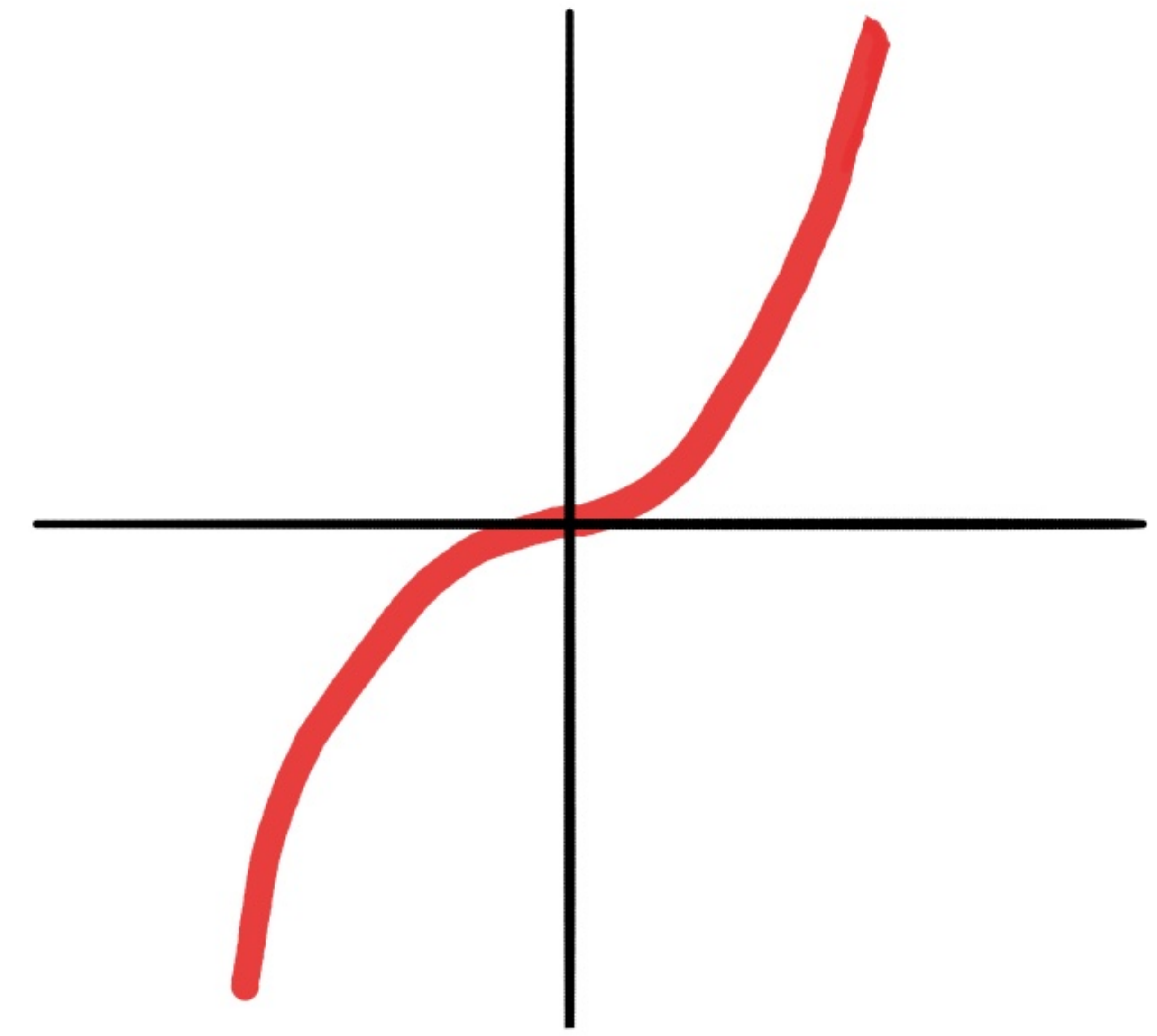
If $f'(a)=0$, then the curvature of the graph at $(a,f(a))$ is $|f''(a)|$.

Find the curvature of the following graphs at the specified points.



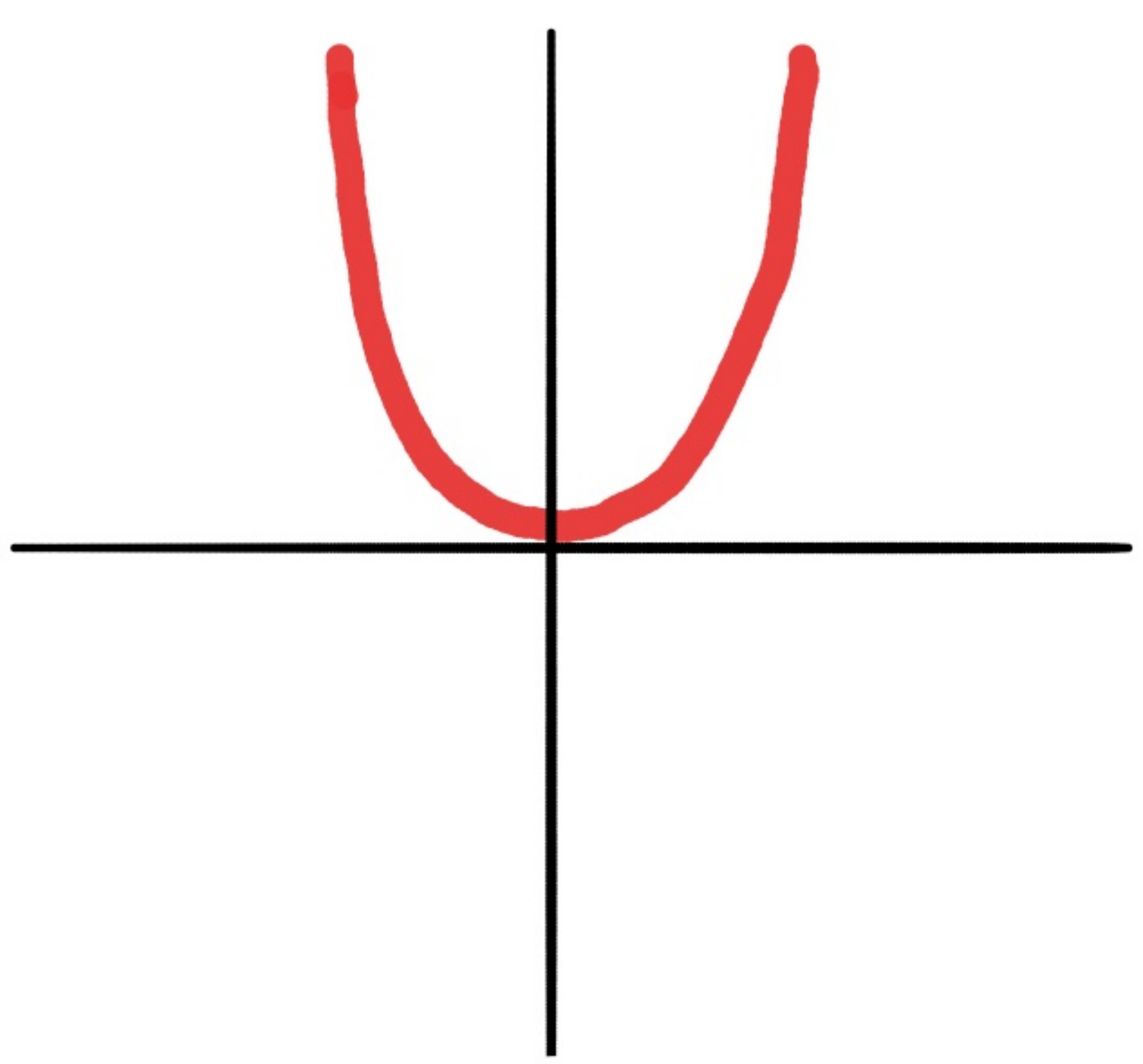
$y=x^2$

$(0,0)$



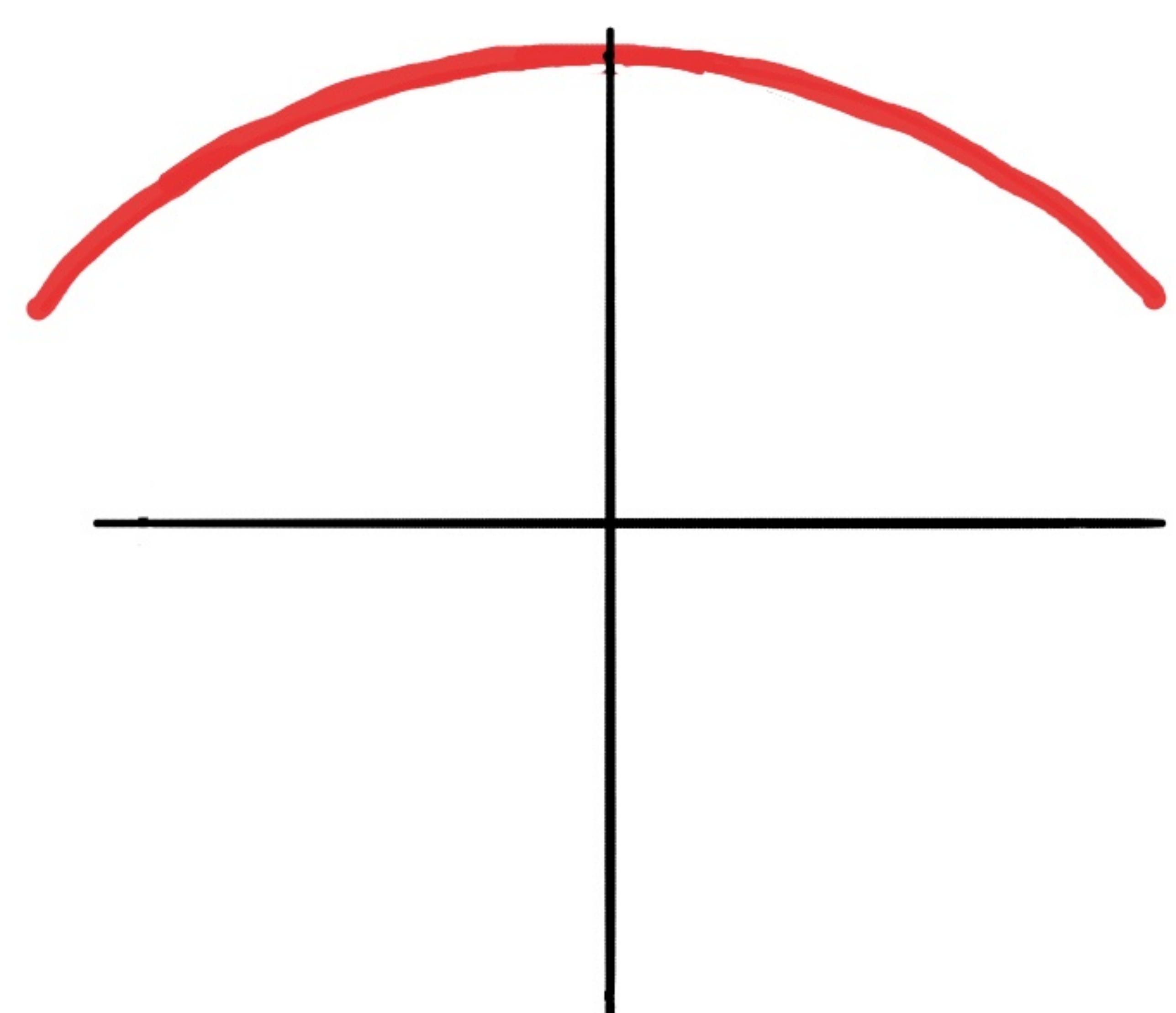
$y=x^3$

$(0,0)$



$y=x^4$

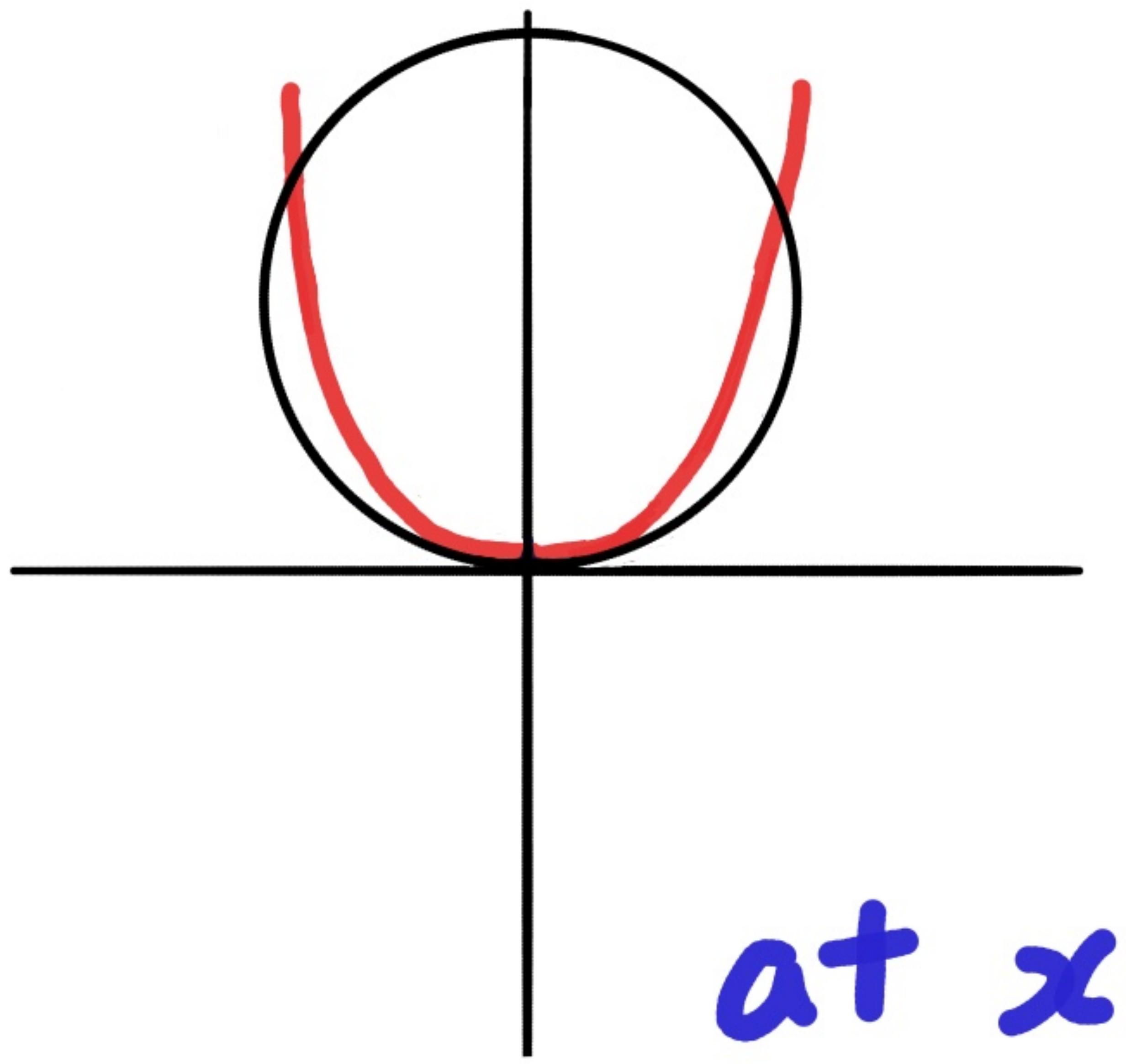
$(0,0)$



$y=\cos x$

$(0,1)$

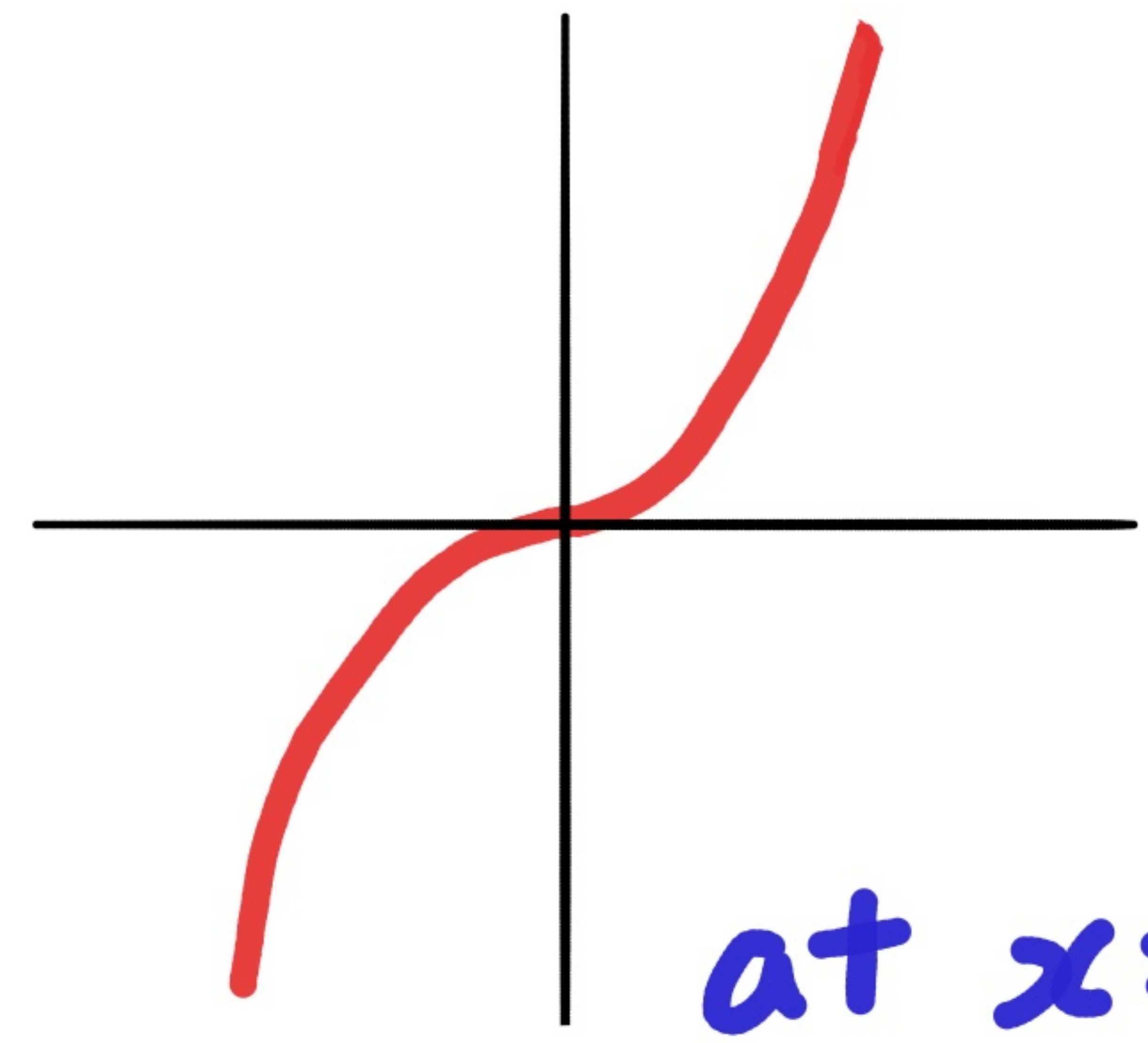
Find the curvature of the following graphs at the specified points.



$y=x^2$

$(0,0)$

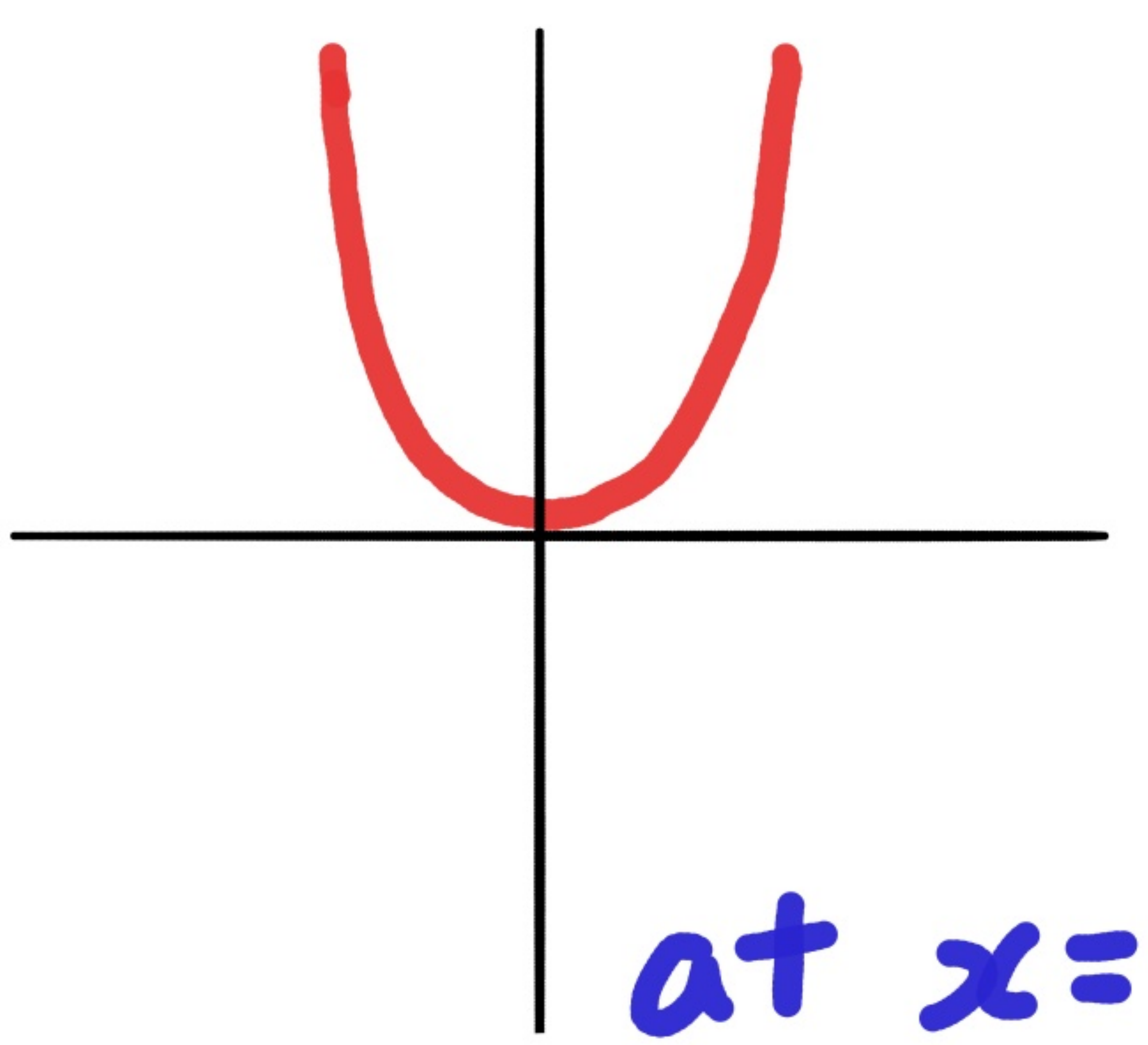
at $x=0$, $|y''|=2$



$y=x^3$

$(0,0)$

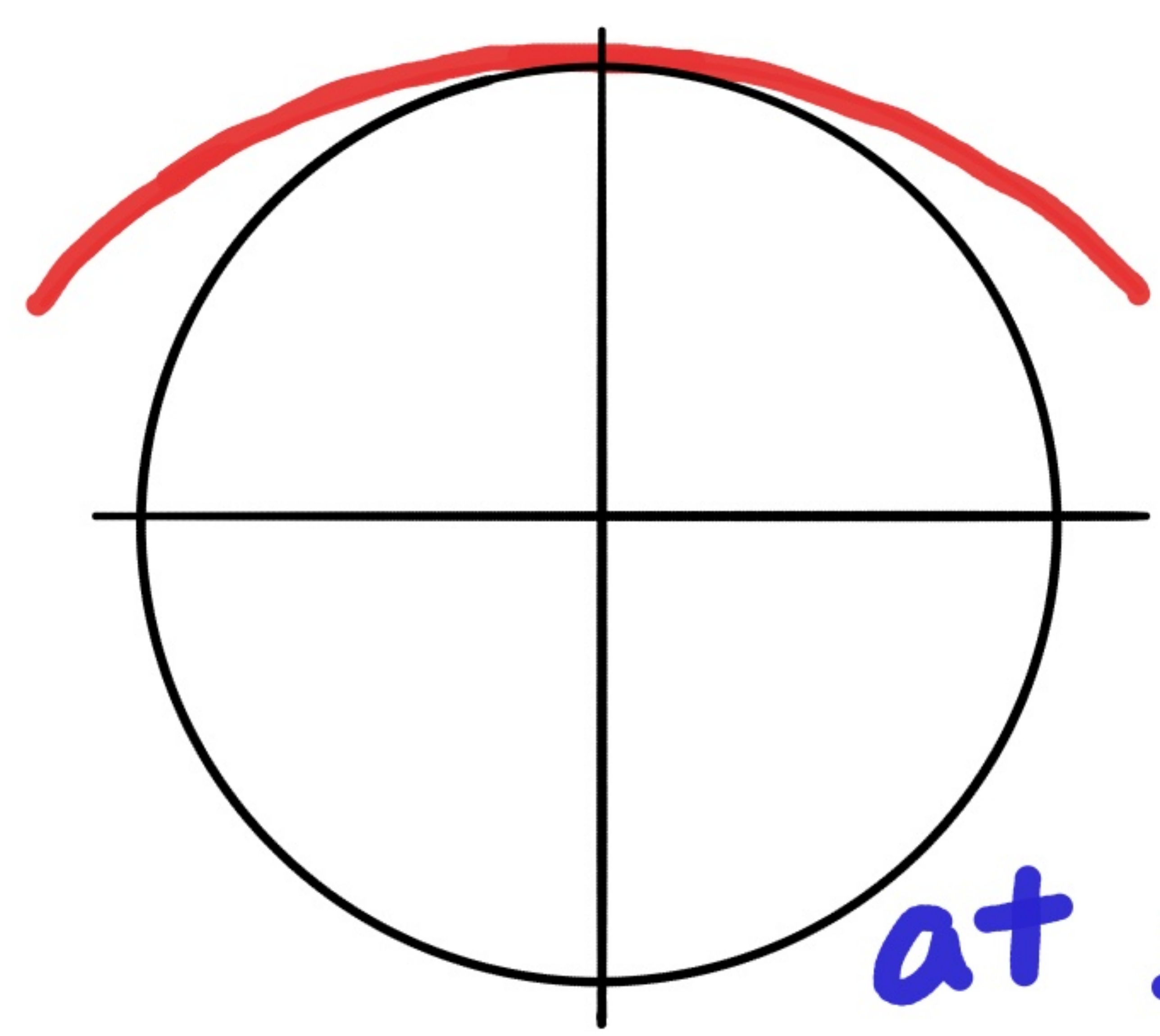
at $x=0$, $|y''|=0$



$y=x^4$

$(0,0)$

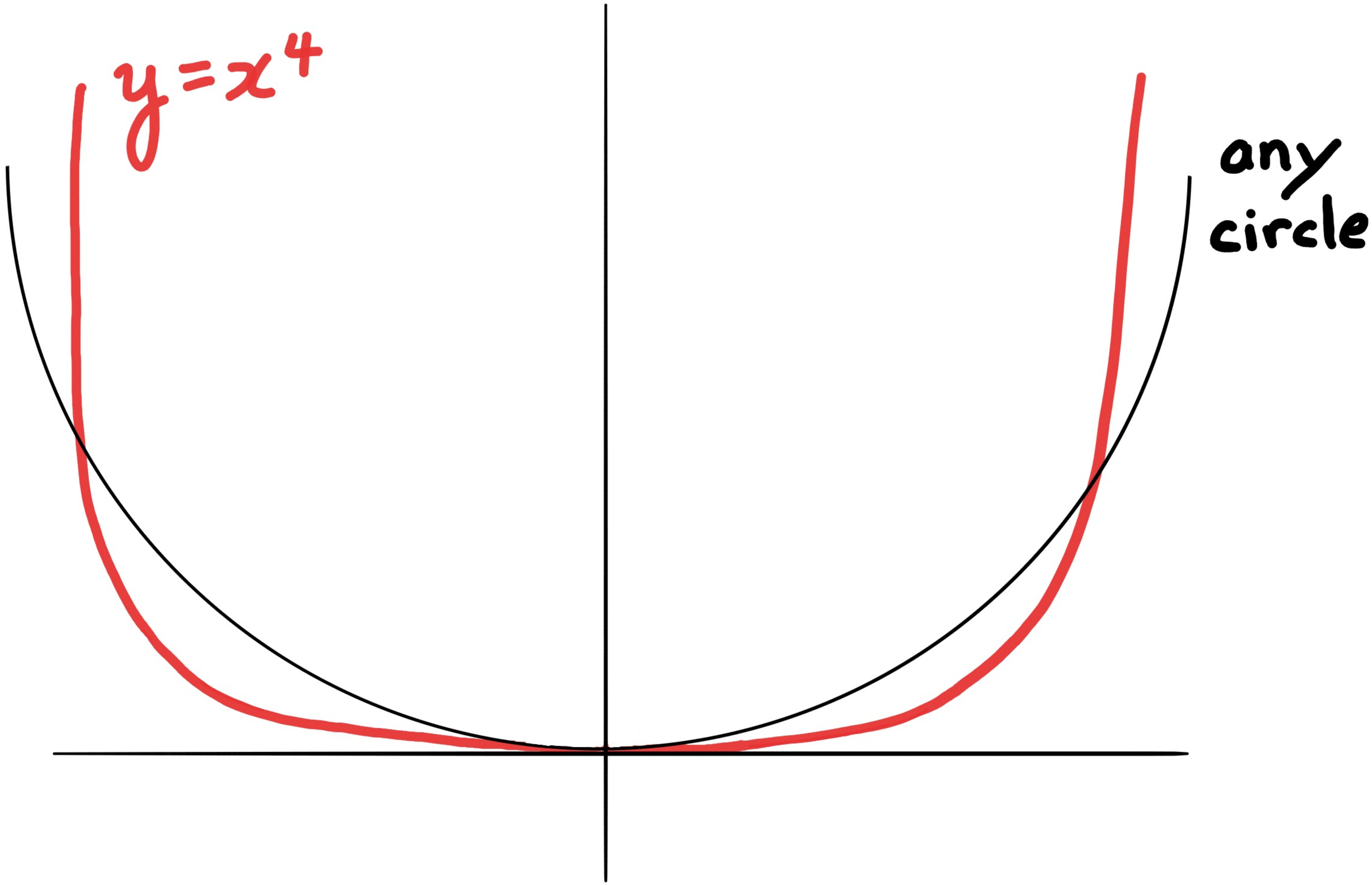
at $x=0$, $|y''|=0$



$y=\cos x$

$(0,1)$

at $x=0$, $|y''|=1$



$y=x^4$
any circle