Four

Ⅰ $\vec{f} : \mathbb{R} \to \mathbb{R}^n$

Ⅱ Parametrized smooth curves

Ⅲ Derivatives

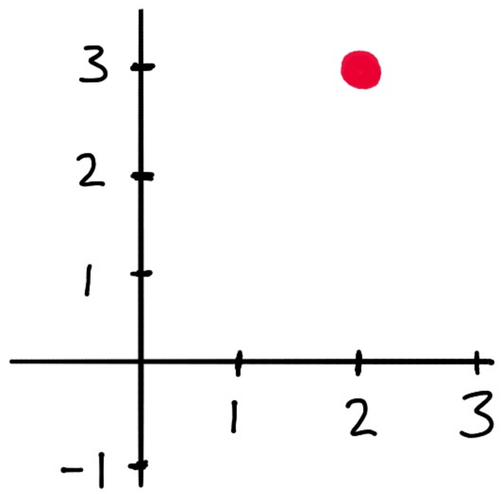
Ⅳ Arc length

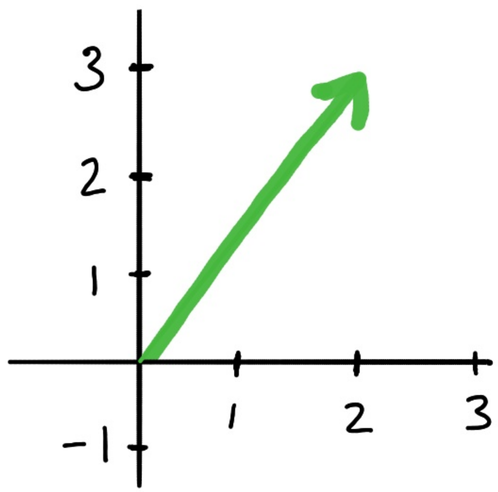
① $\vec{f} : \mathbb{R} \longrightarrow \mathbb{R}^n$

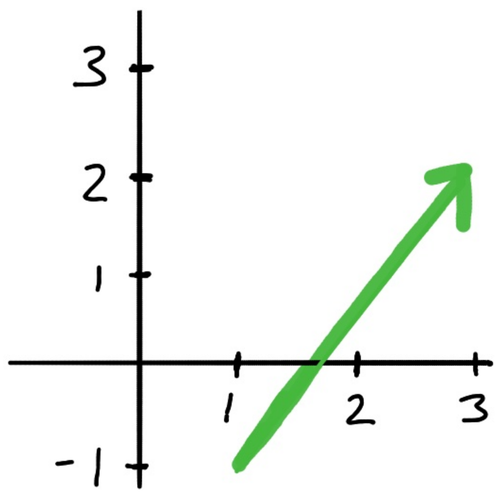
a a A

$p=(2,3)$

$\vec{v}=\langle 2,3\rangle$





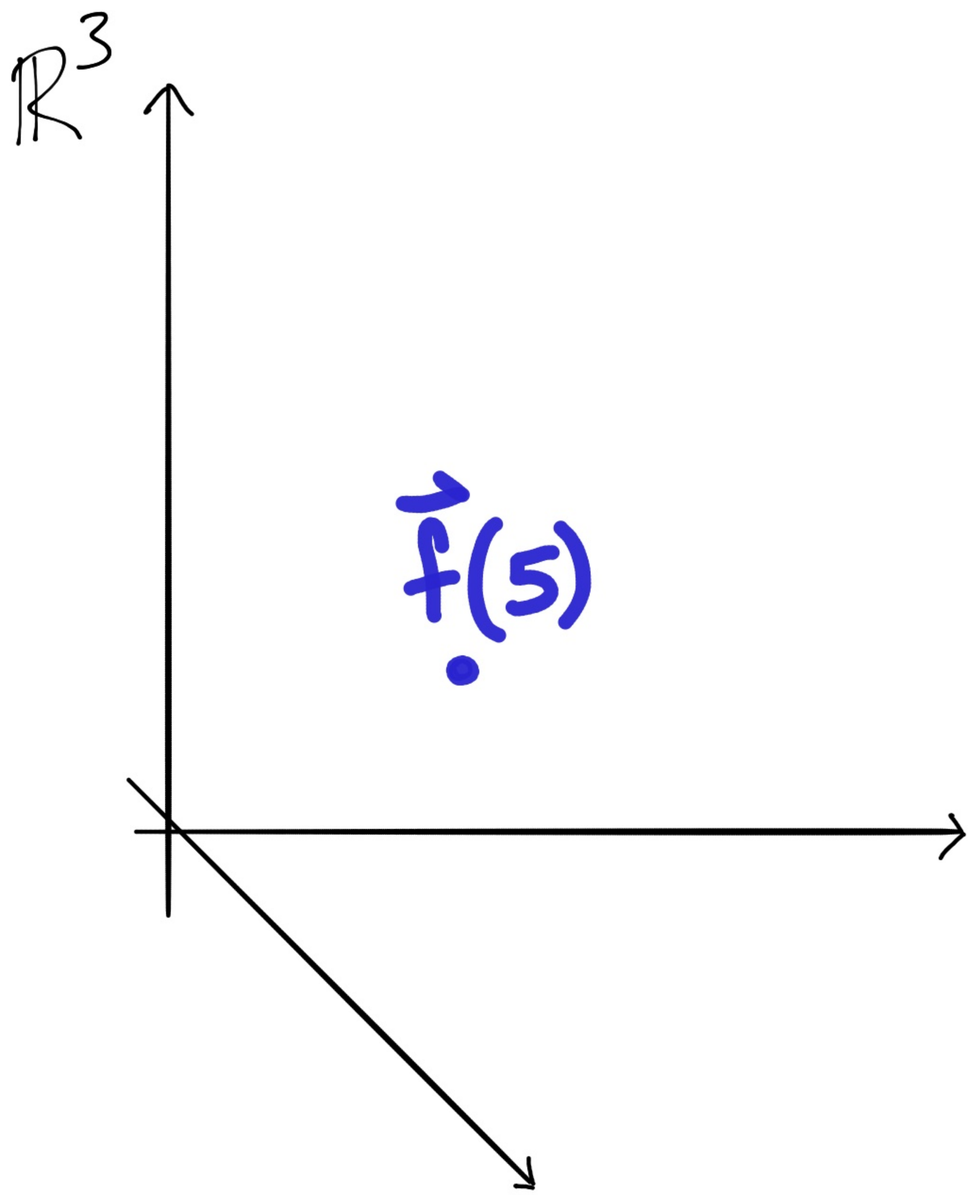


$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

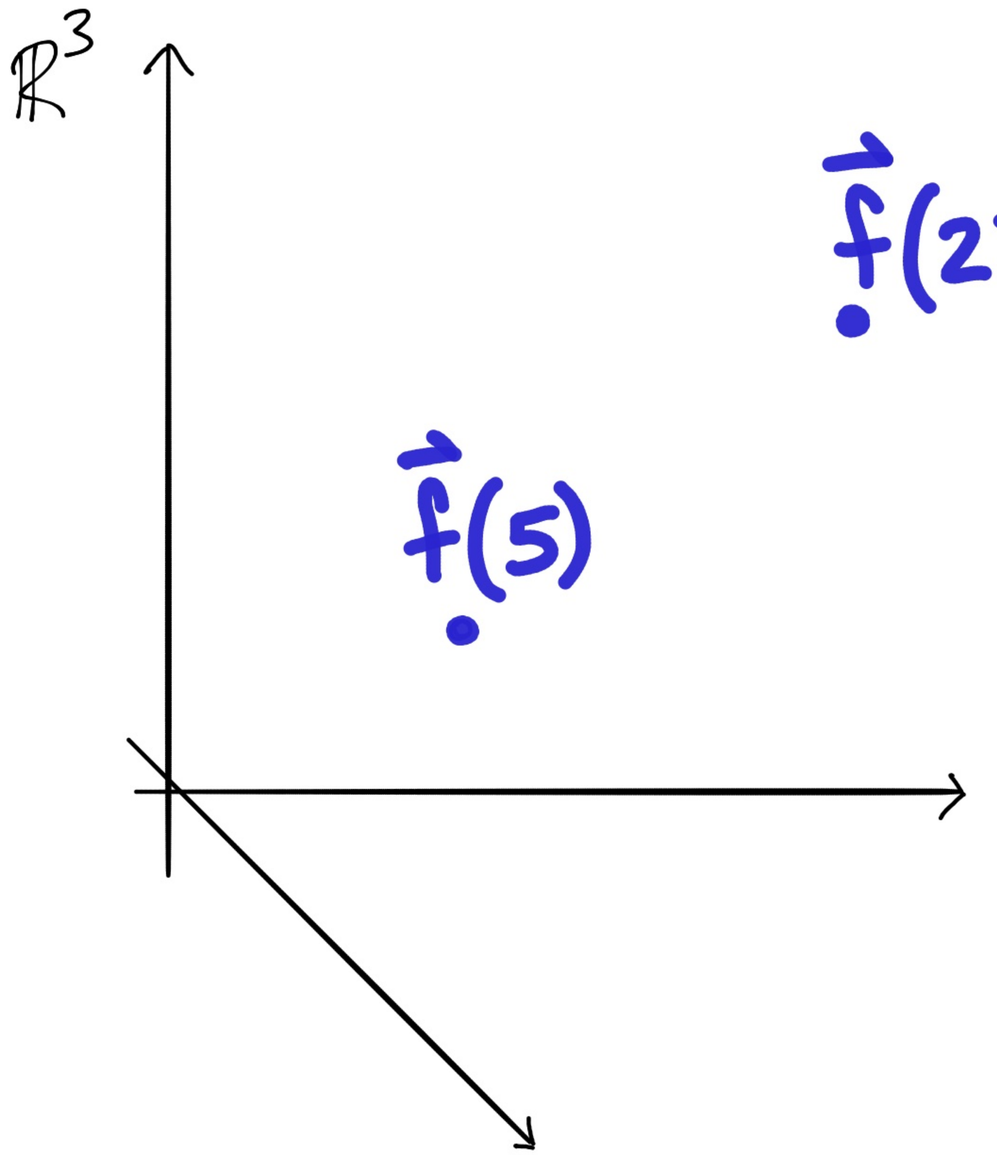
$\langle 2,3\rangle$

$(2,3)$

$$\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^n$$



$$\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^n$$



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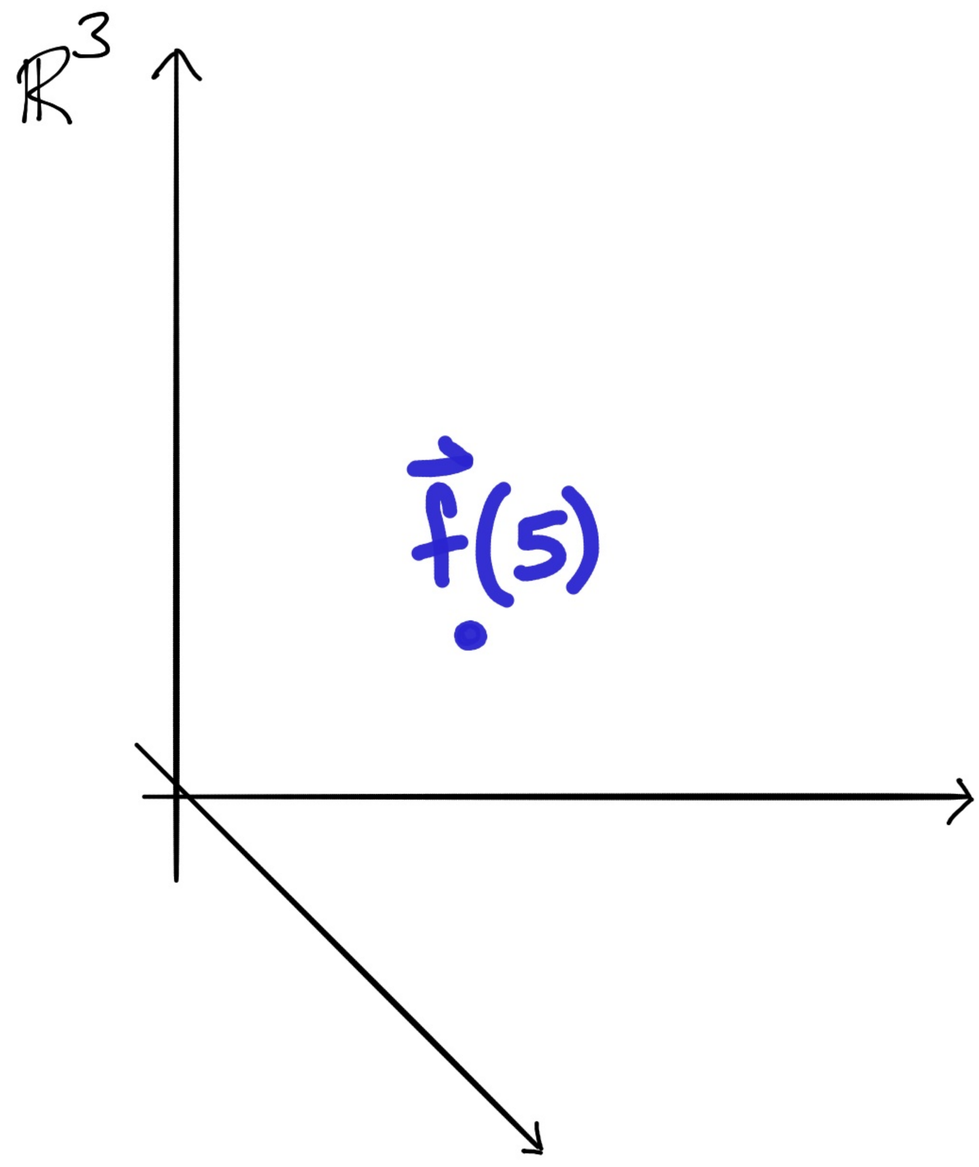
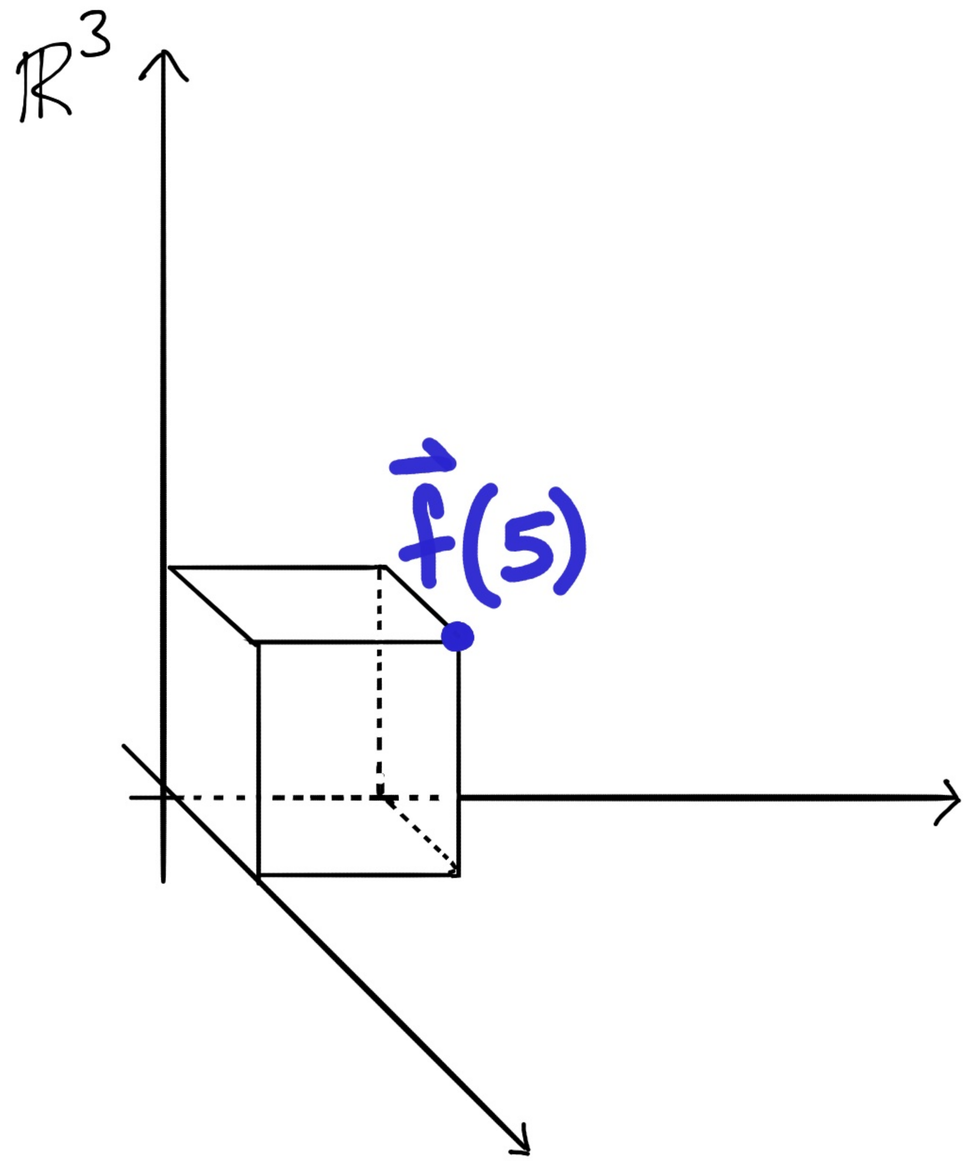


$$\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^n$$



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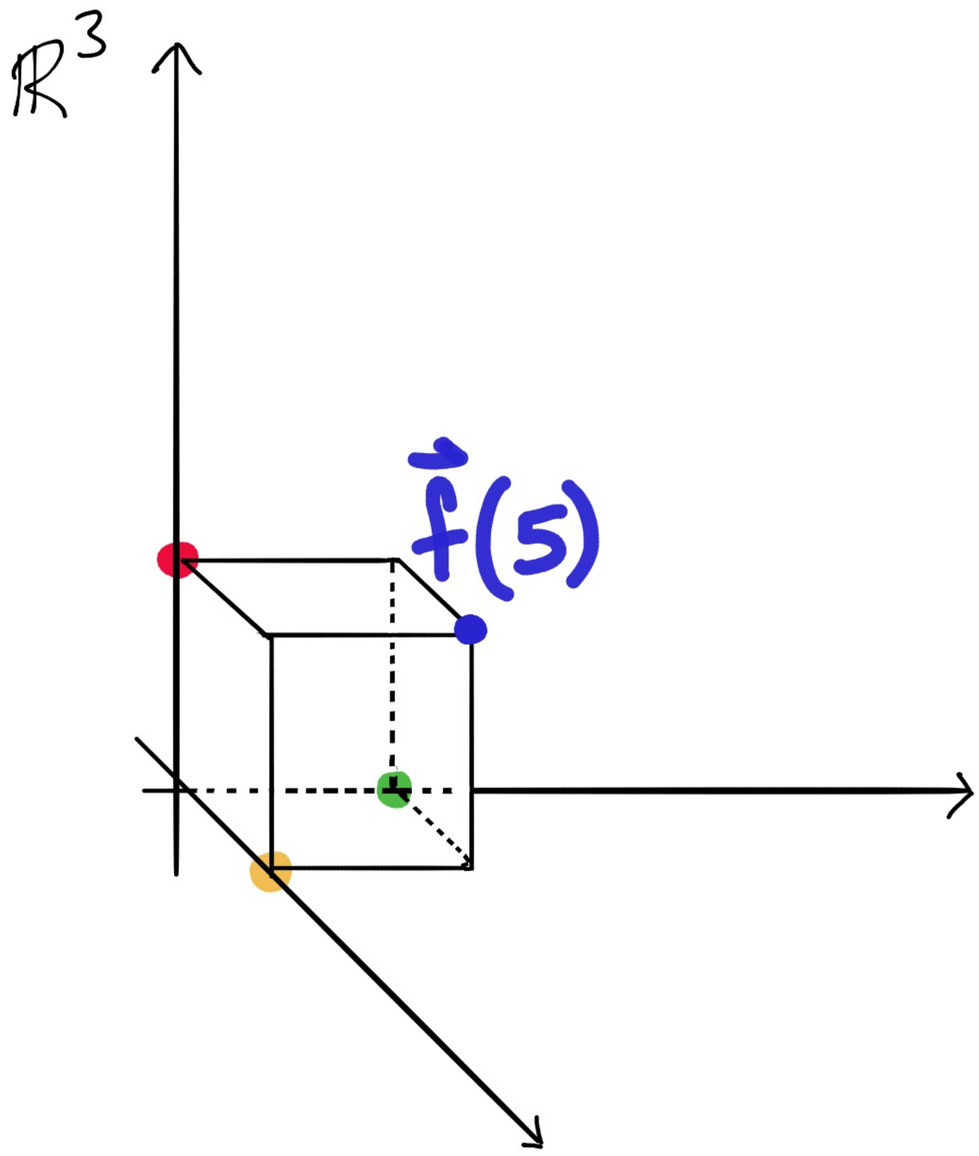

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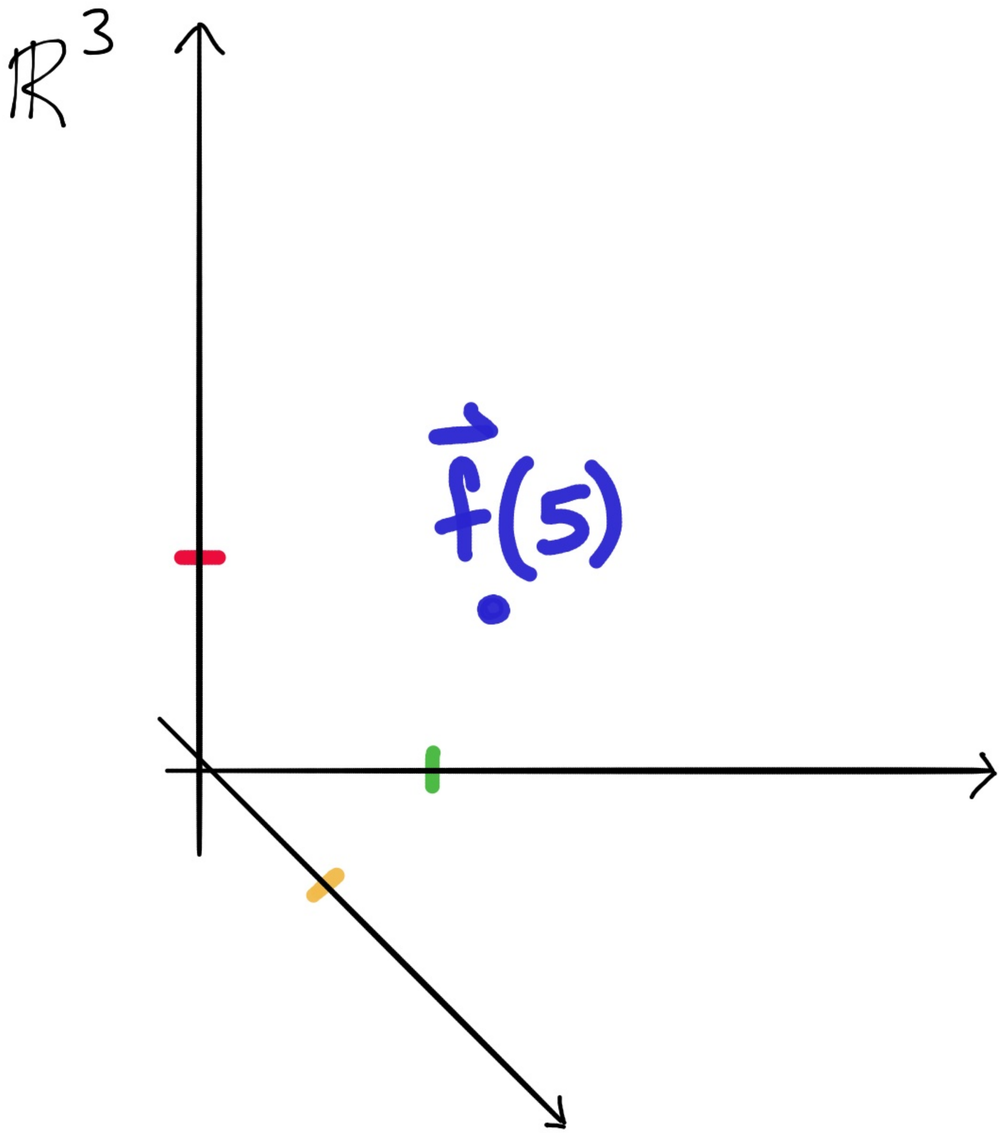
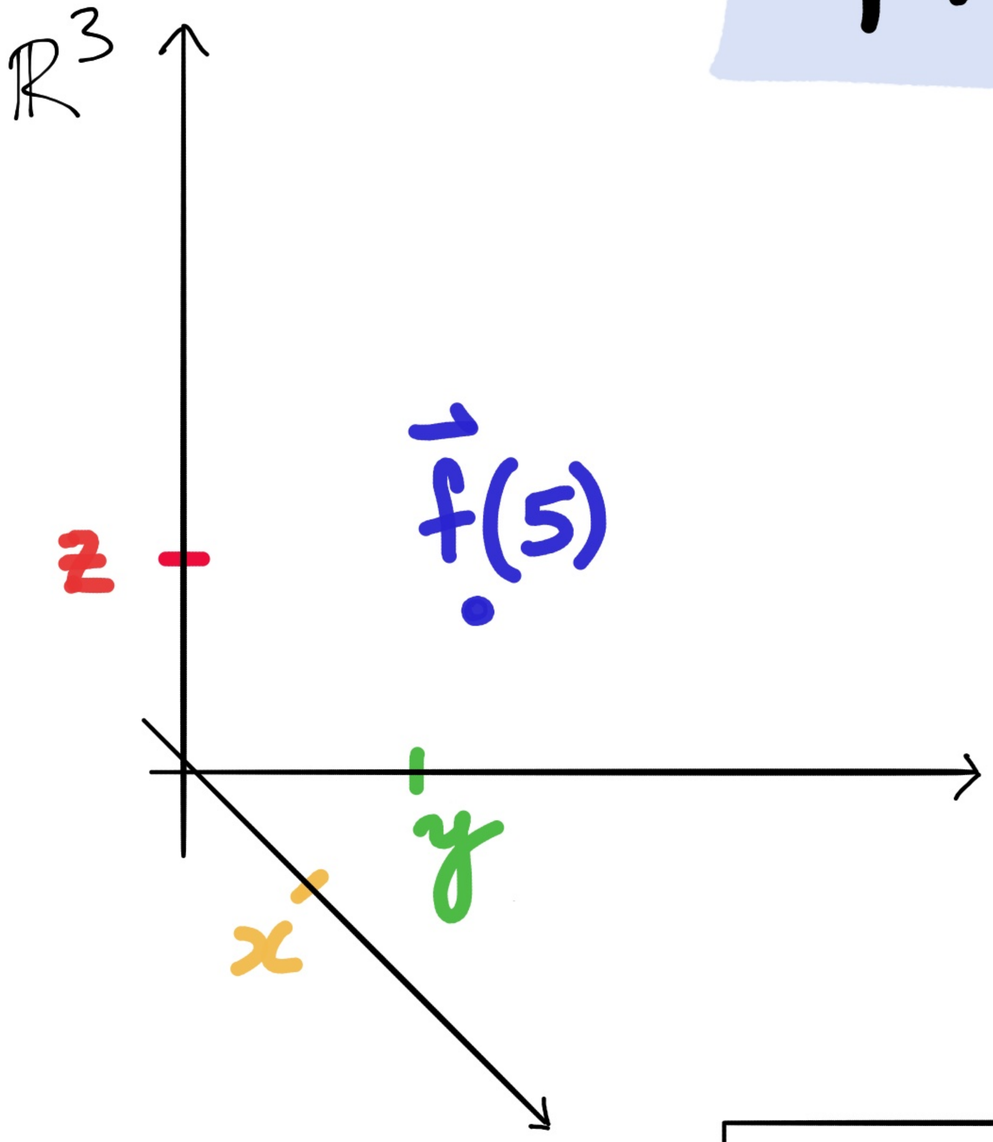
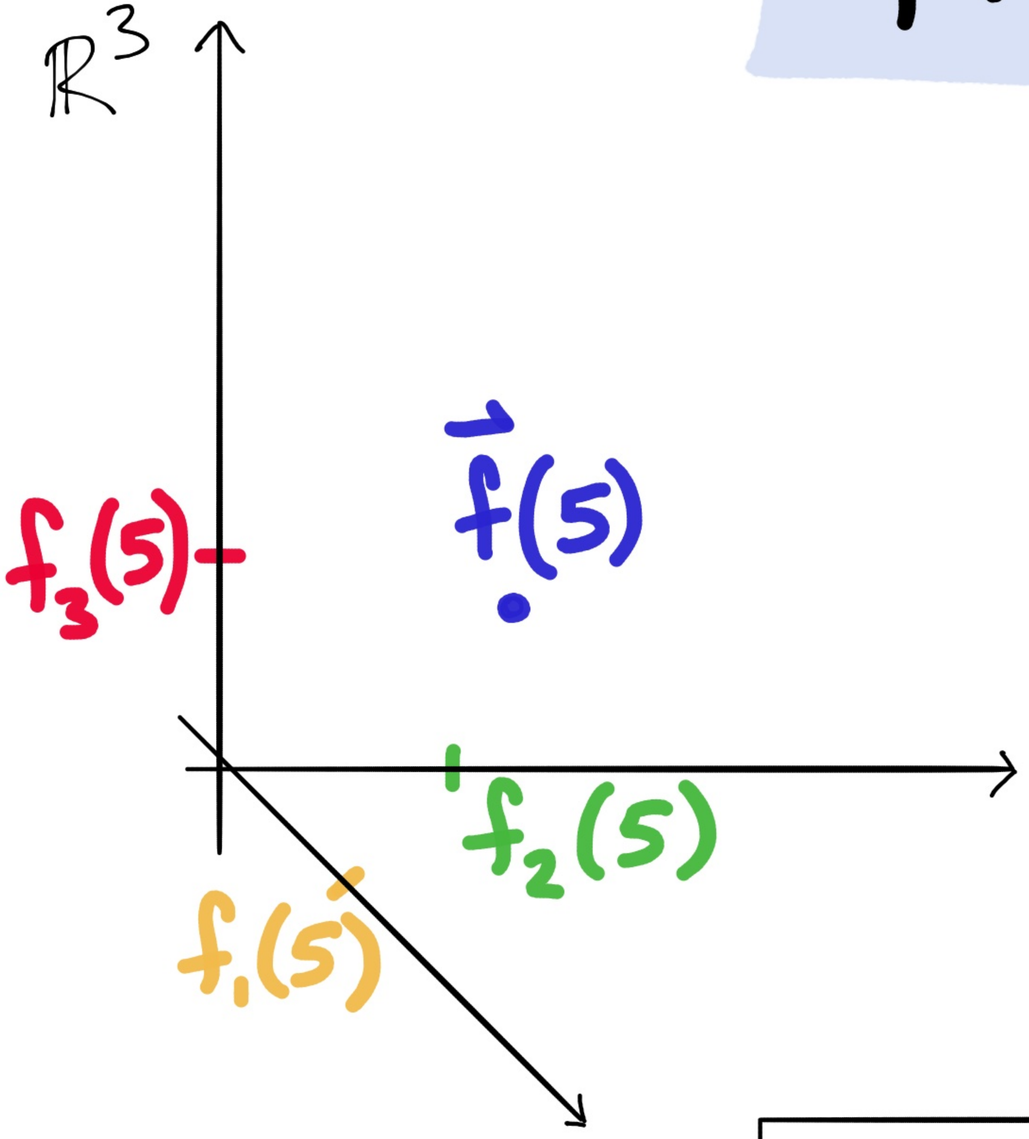


$$\vec{f} : \mathbb{R} \longrightarrow \mathbb{R}^n$$



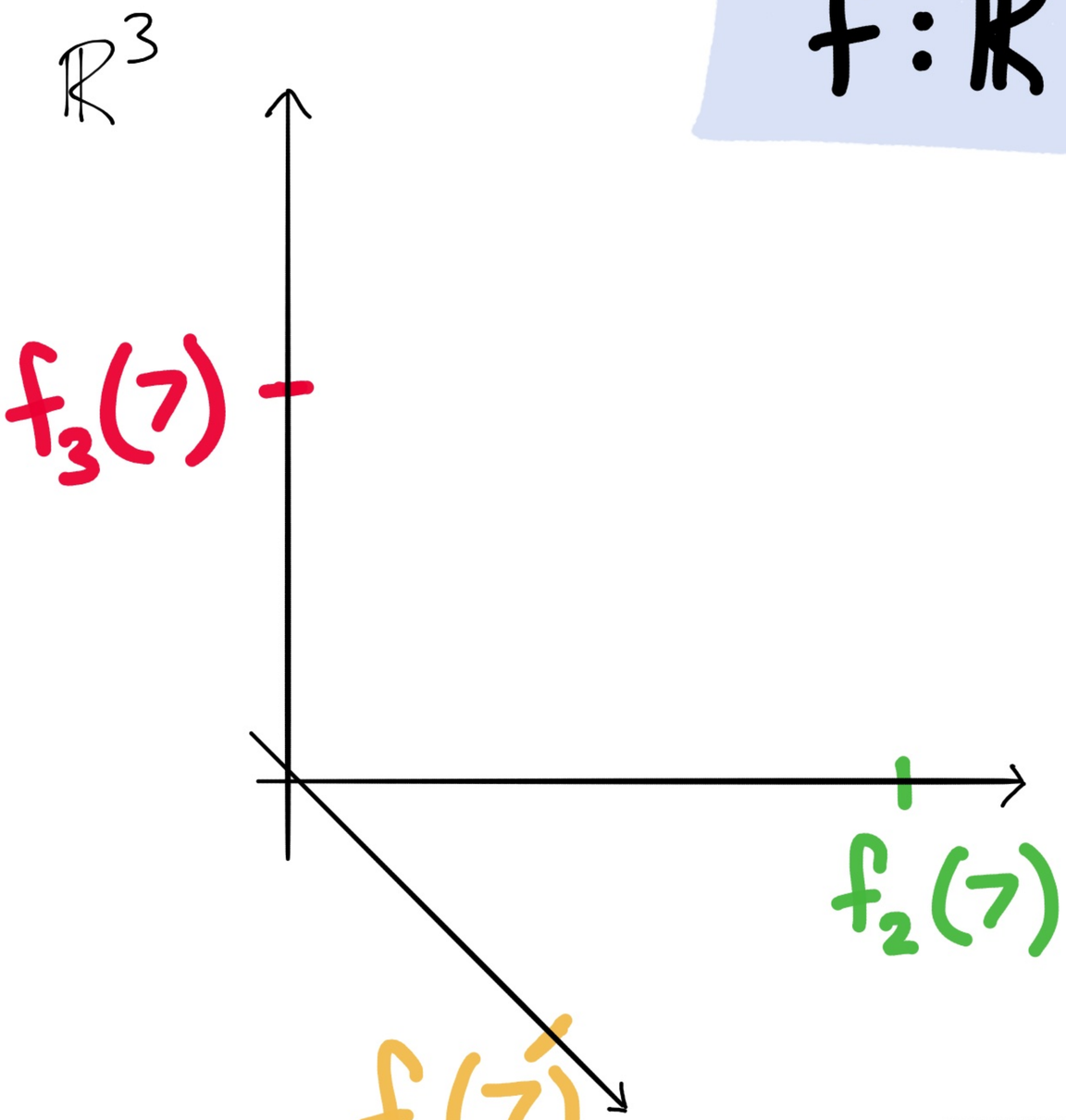
$$\vec{f}(5) = \langle x , y , z \rangle$$

$$\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^n$$



$$\vec{f}(5) = \langle f_1(5), f_2(5), f_3(5) \rangle$$

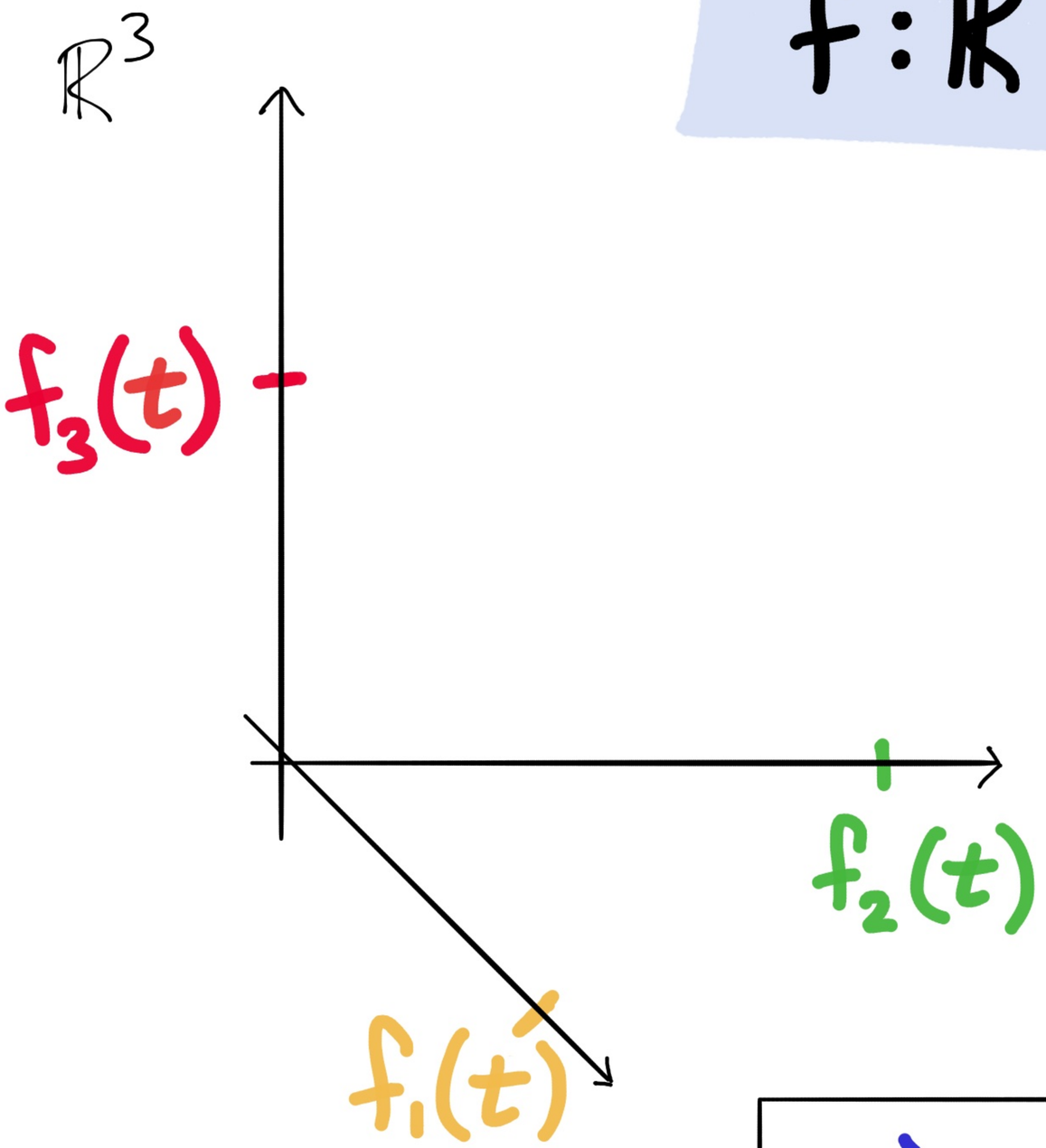
$$\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^n$$



$\vec{f}(7)$

$$\vec{f}(7) = \langle f_1(7), f_2(7), f_3(7) \rangle$$

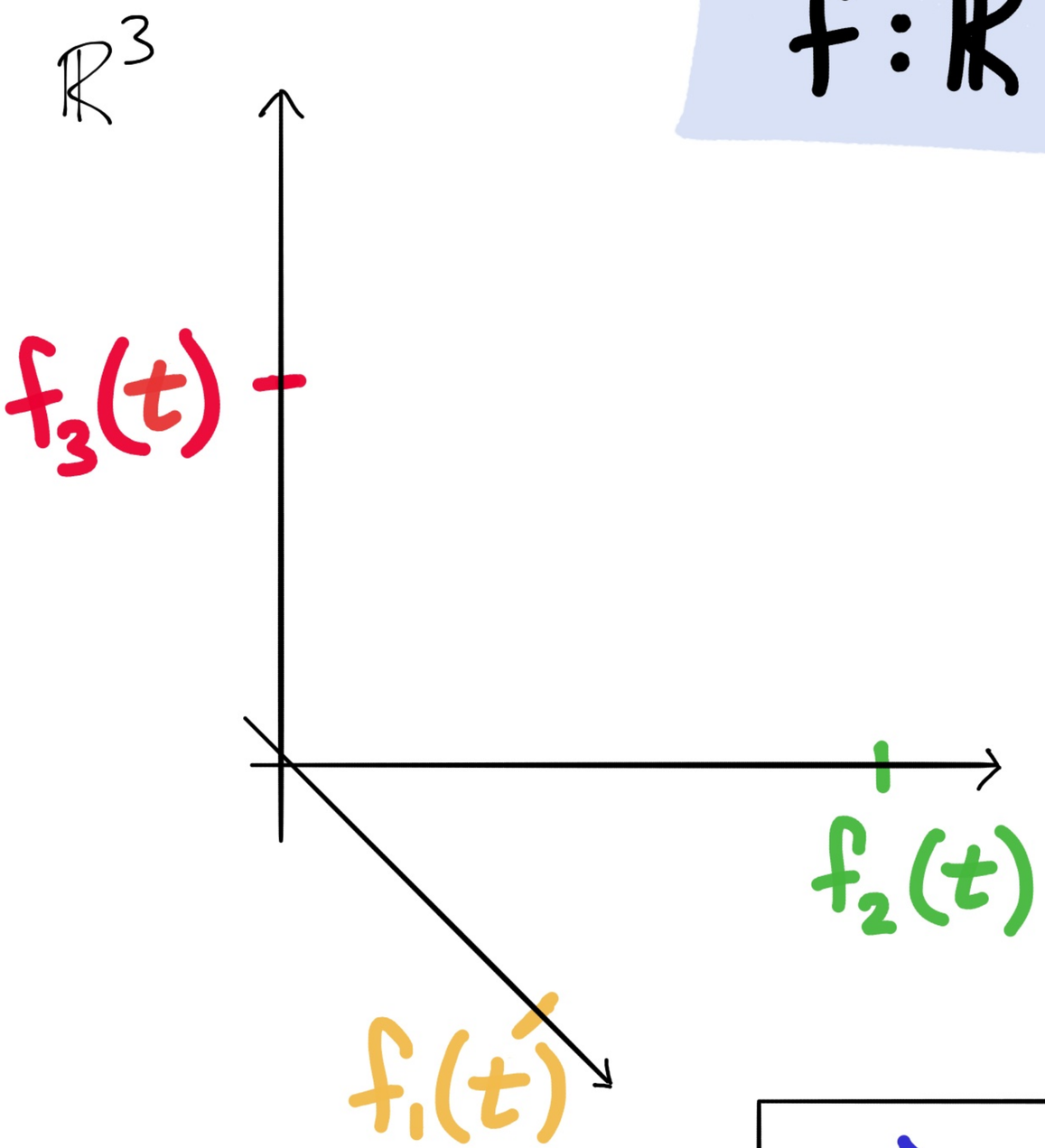
$$\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^n$$



$\vec{f}(t)$

$$\vec{f}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$$

$$\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^n$$



$$f_1: \mathbb{R} \longrightarrow \mathbb{R}$$
$$f_2: \mathbb{R} \longrightarrow \mathbb{R}$$
$$f_3: \mathbb{R} \longrightarrow \mathbb{R}$$

$$\vec{f}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$$

$$\vec{f}: \mathbb{R} \to \mathbb{R}^n$$

Any function $\vec{f}: \mathbb{R} \to \mathbb{R}^3$ can be described as

$$\vec{f}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$$

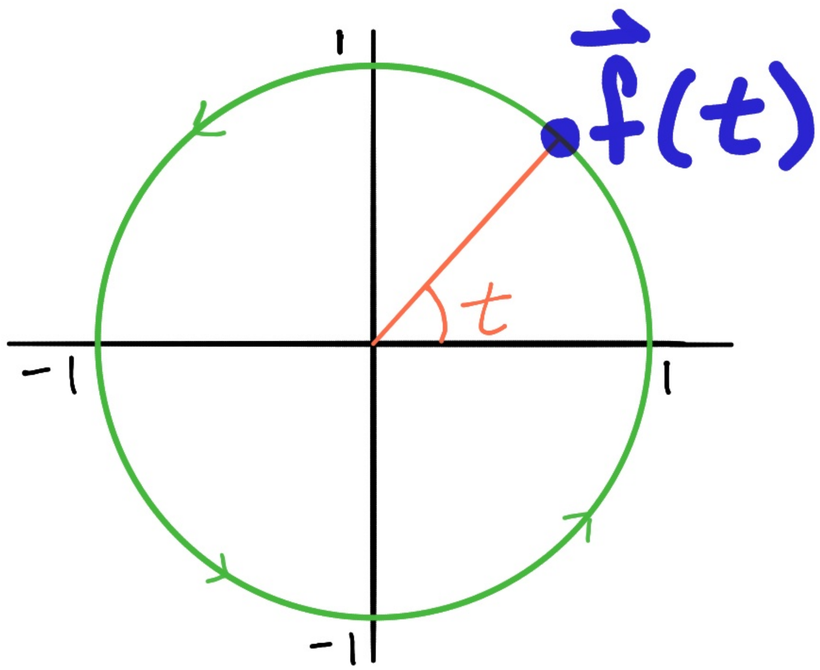
where $f_i: \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, 3$.

$$\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^n$$

Examples:

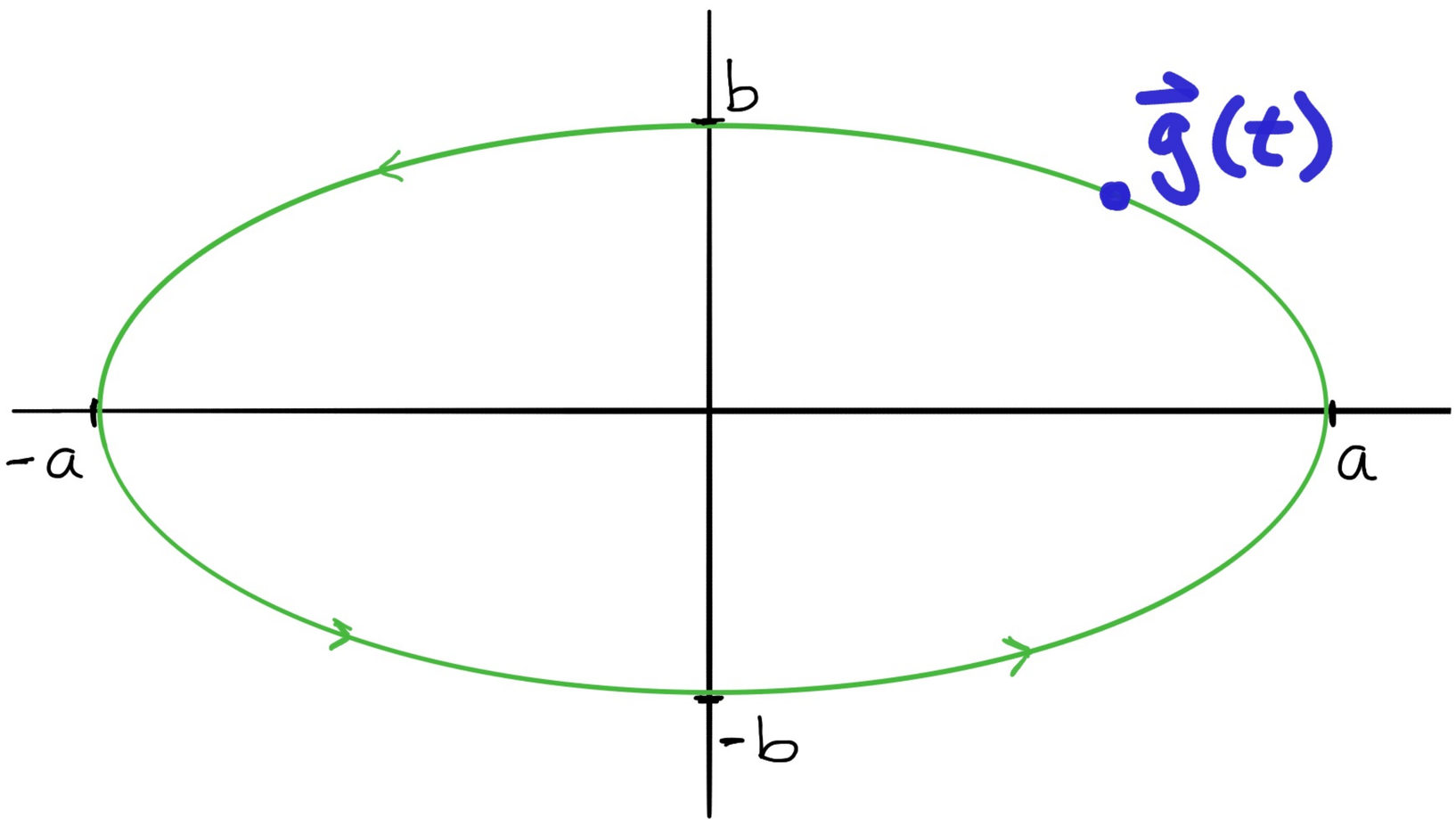
1.) $\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^2$, $\vec{f}(t) = \langle \cos t, \sin t \rangle$

(so $f_1(t) = \cos t$ and $f_2(t) = \sin t$)

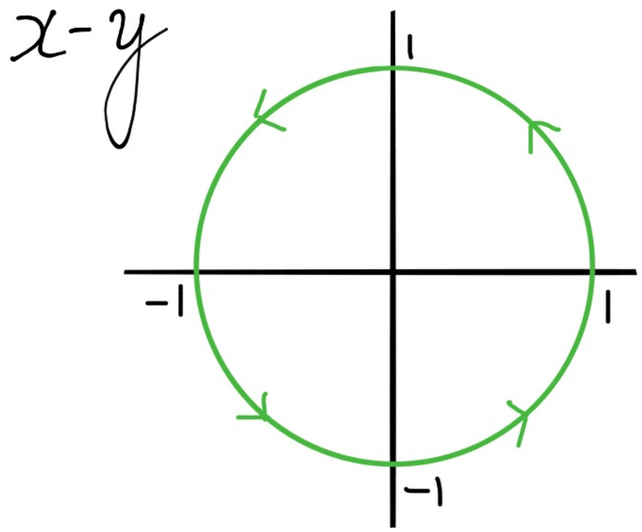


2.) Fix $a, b > 0$.

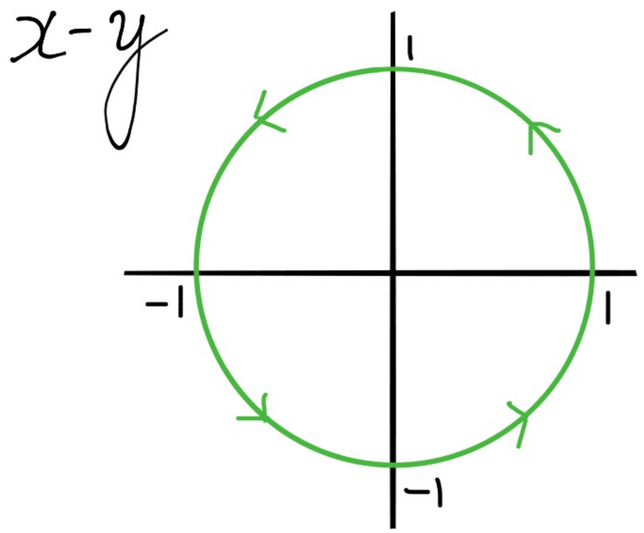
$$\vec{g}: \mathbb{R} \to \mathbb{R}^2, \qquad \vec{g}(t) = \langle a\cos t, b\sin t \rangle$$

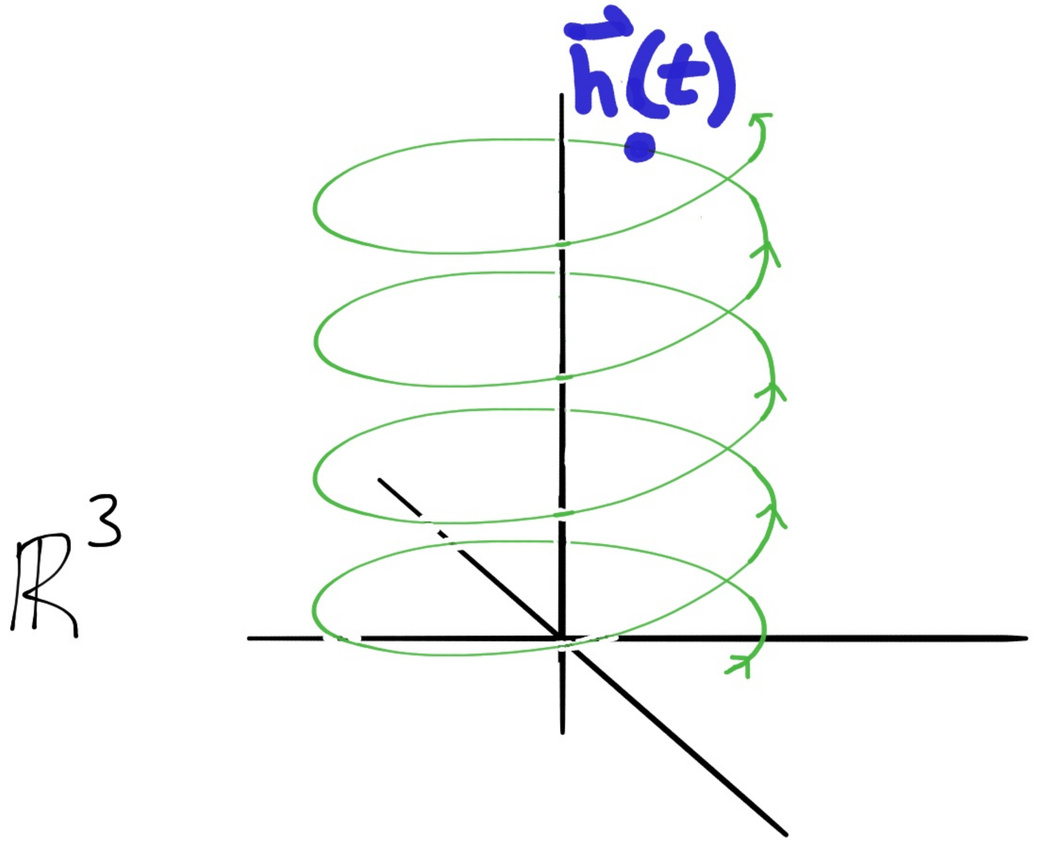


3.) $\vec{h} : \mathbb{R} \longrightarrow \mathbb{R}^3$, $\vec{h}(t) = \langle \cos t, \sin t, t \rangle$



3.) $\vec{h} : \mathbb{R} \longrightarrow \mathbb{R}^3$, $\vec{h}(t) = \langle \cos t, \sin t, t \rangle$





II Parametrized smooth curves

A **smooth parametrized curve** C in $\mathbb{R}^n$ is a collection of points in $\mathbb{R}^n$ that is the range of some **one-to-one** $\vec{f}:\mathbb{R}\to\mathbb{R}^n$ where each $f_i'(t)$ is continuous and not all $f_i'(t)$ equal zero simultaneously.

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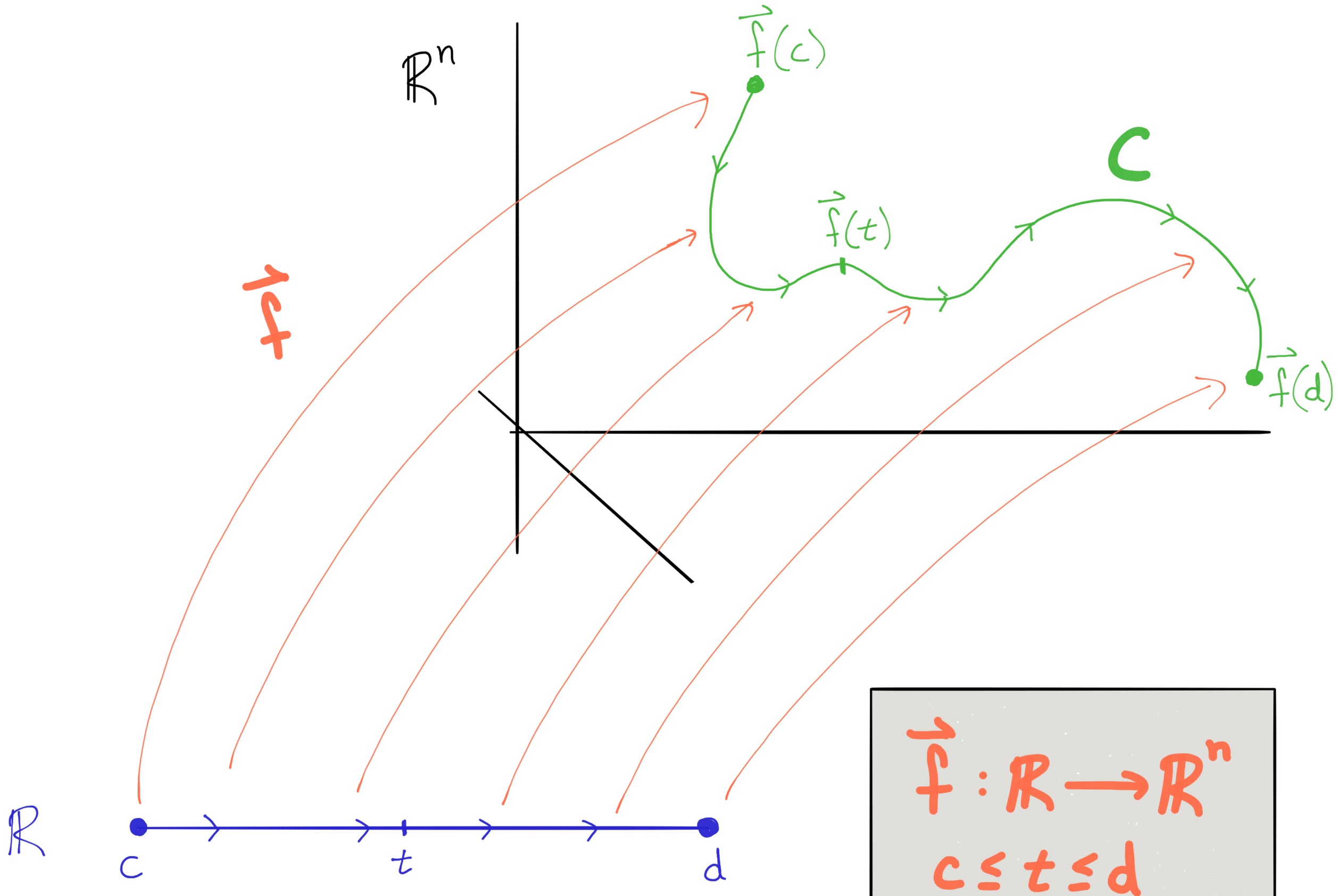
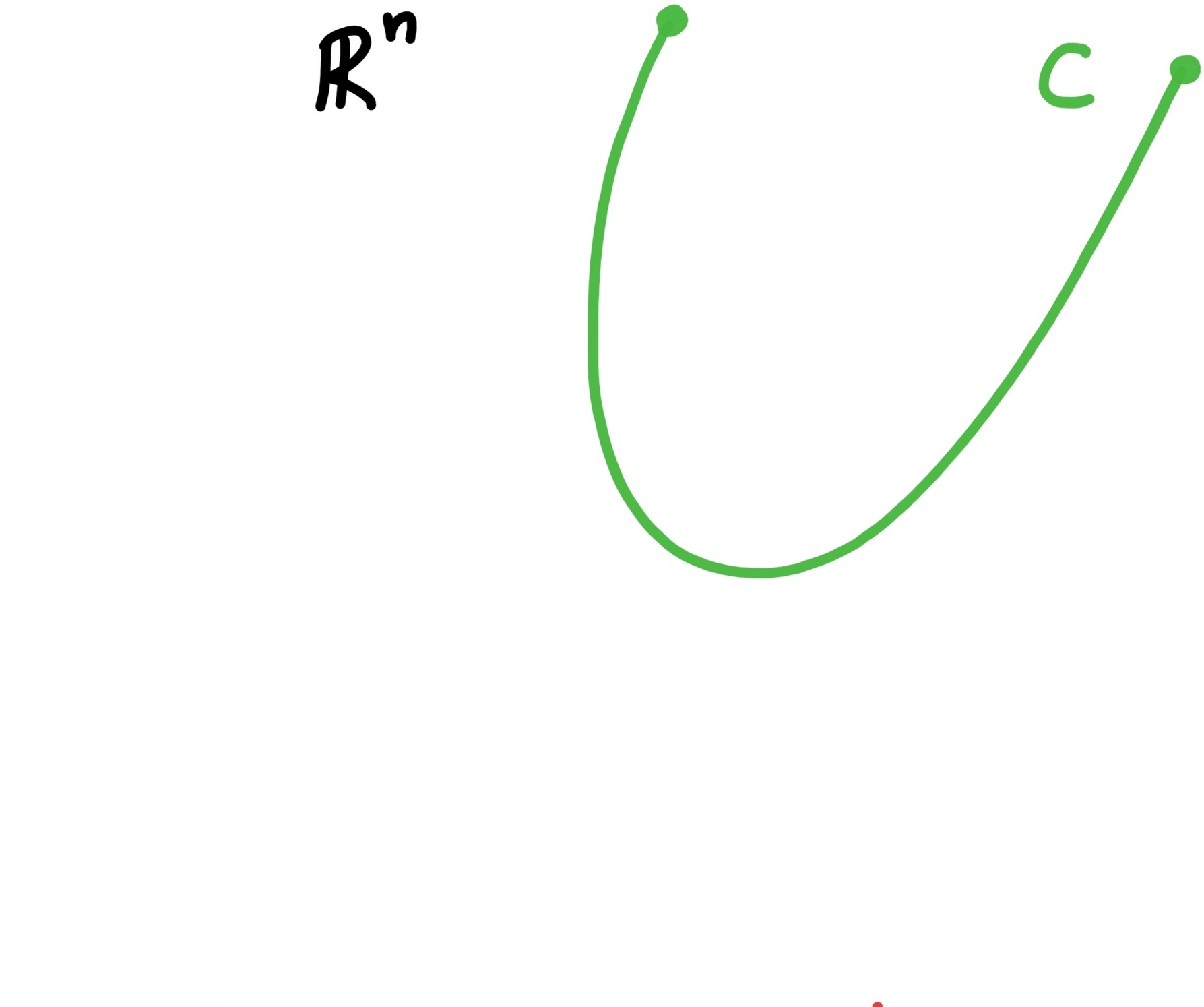
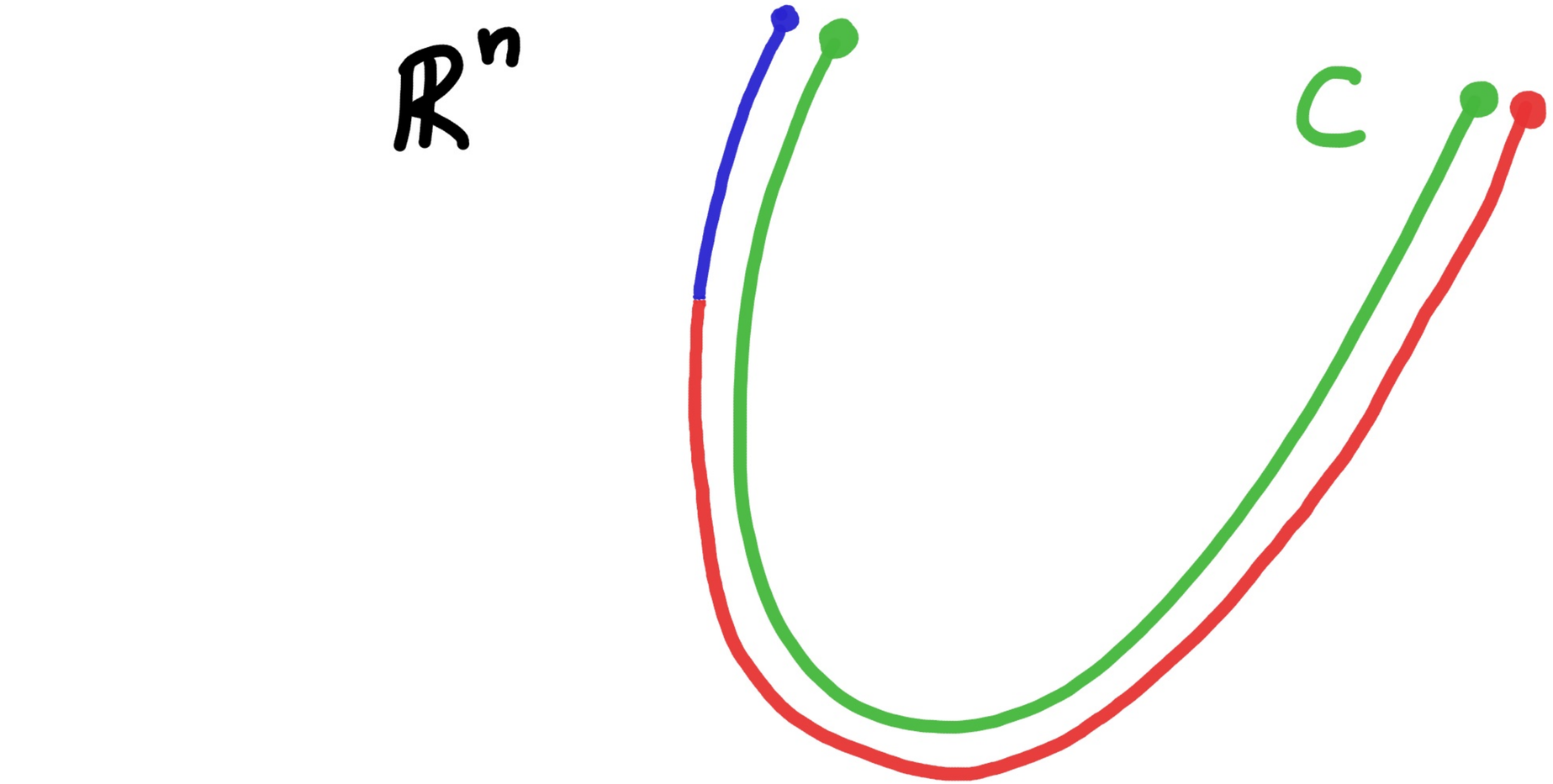
Example:

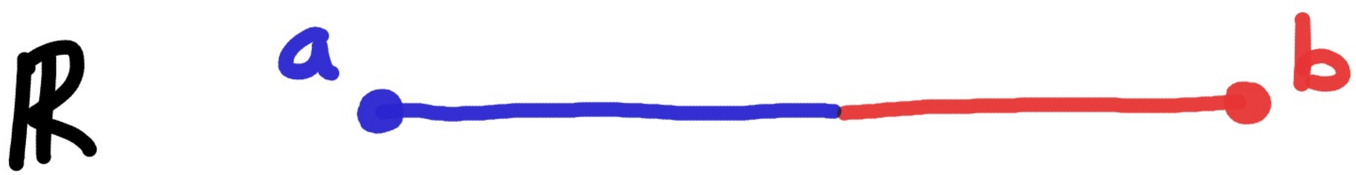
The unit circle in $\mathbb{R}^2$ is parametrized by $\vec{f}(t)=\langle \cos t, \sin t\rangle$ for $0\le t<2\pi$ since $\cos' t=-\sin t$ and $\sin' t=\cos t$ are continuous and not both zero for a common t.

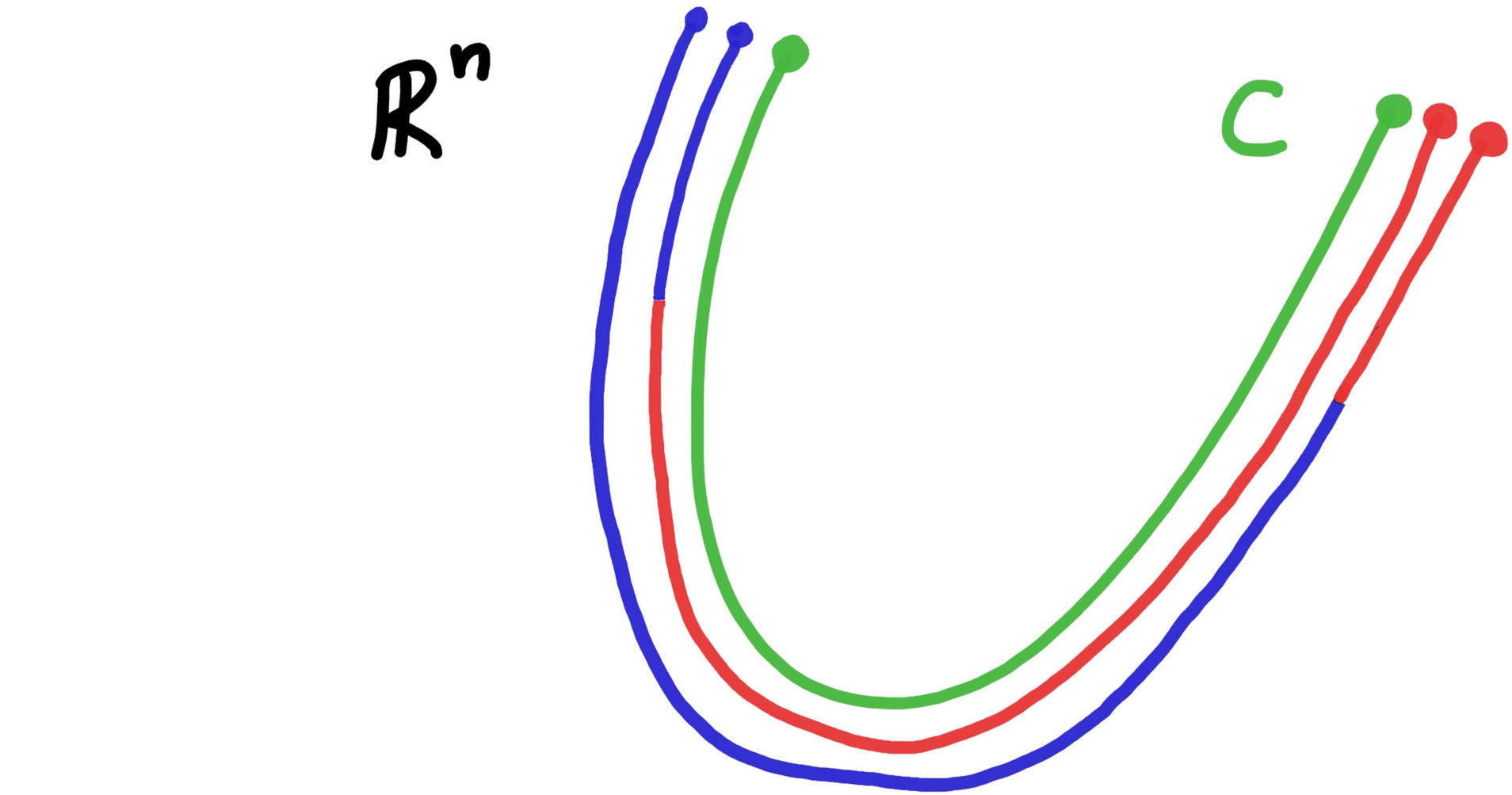


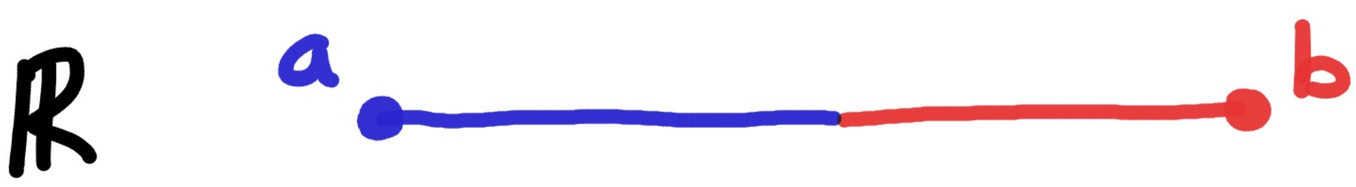
$$\vec{f} : \mathbb{R} \longrightarrow \mathbb{R}^n$$
$$c \le t \le d$$

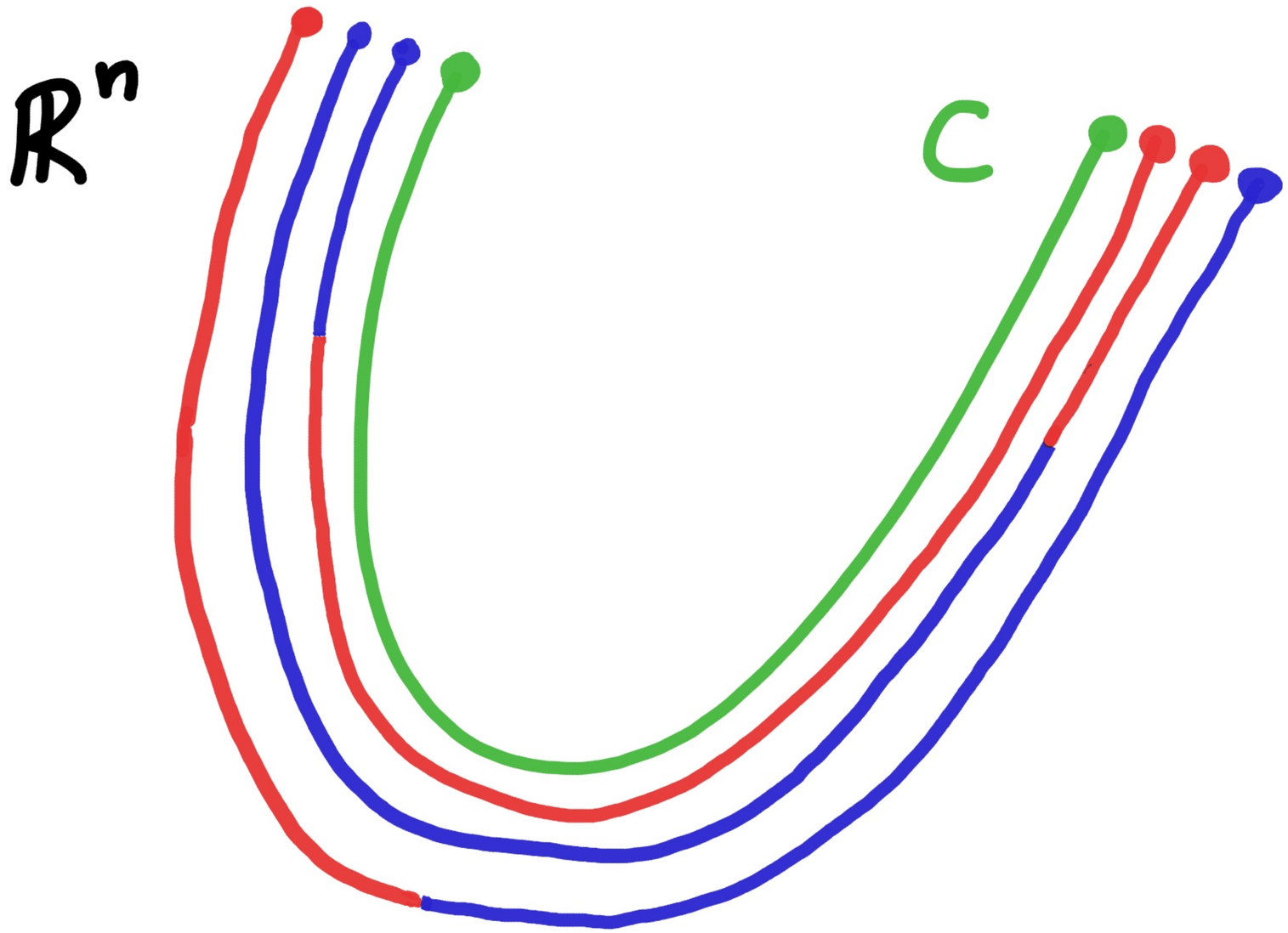

$\mathbb{R}^n$
C
$\mathbb{R}$
a
b

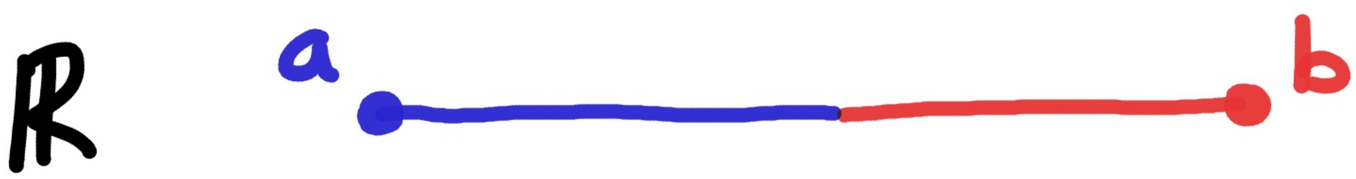

$\mathbb{R}^n$
C


$\mathbb{R}$
a
b

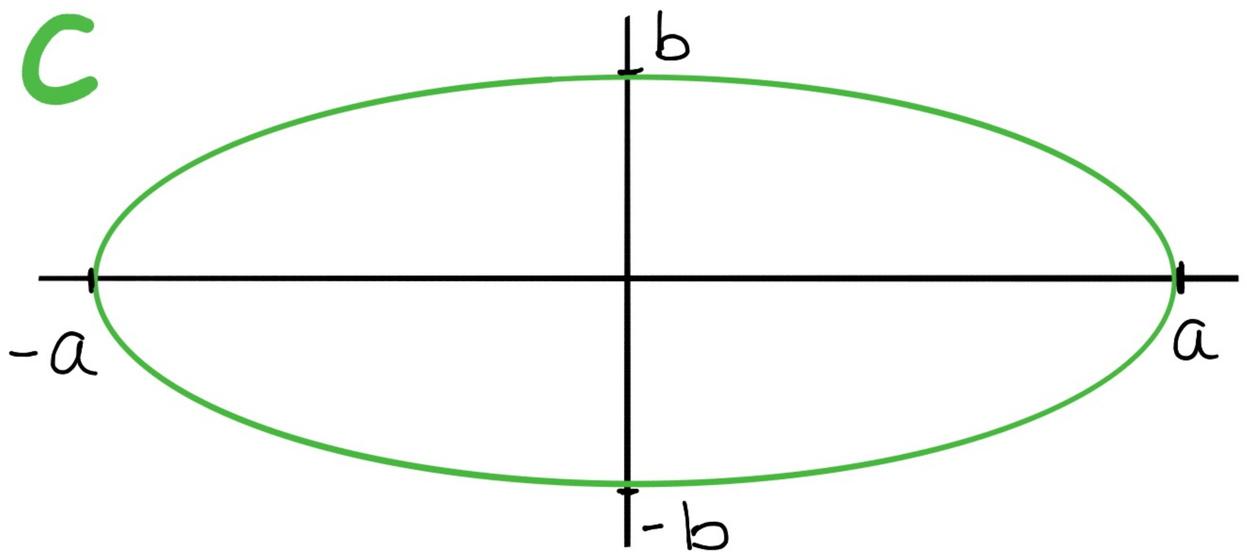

$\mathbb{R}^n$
C


$\mathbb{R}$
a
b


$\mathbb{R}^n$
C


$\mathbb{R}$
a
b

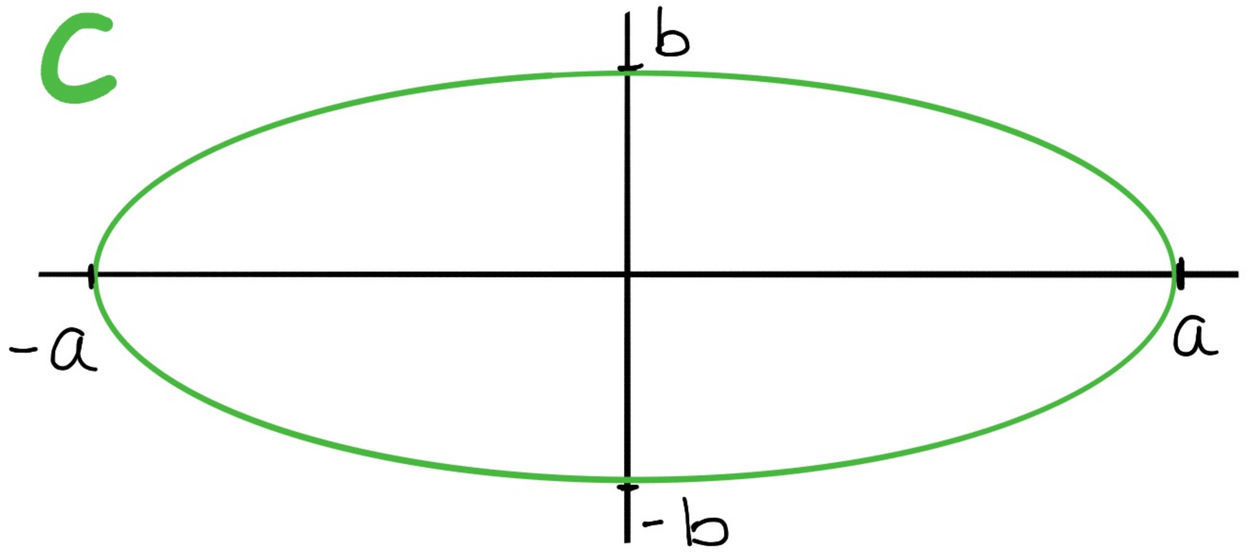
Example:



Fix $a, b > 0$.

Let C be the curve in $\mathbb{R}^2$ of all $(x,y) \in \mathbb{R}^2$ such that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Example:



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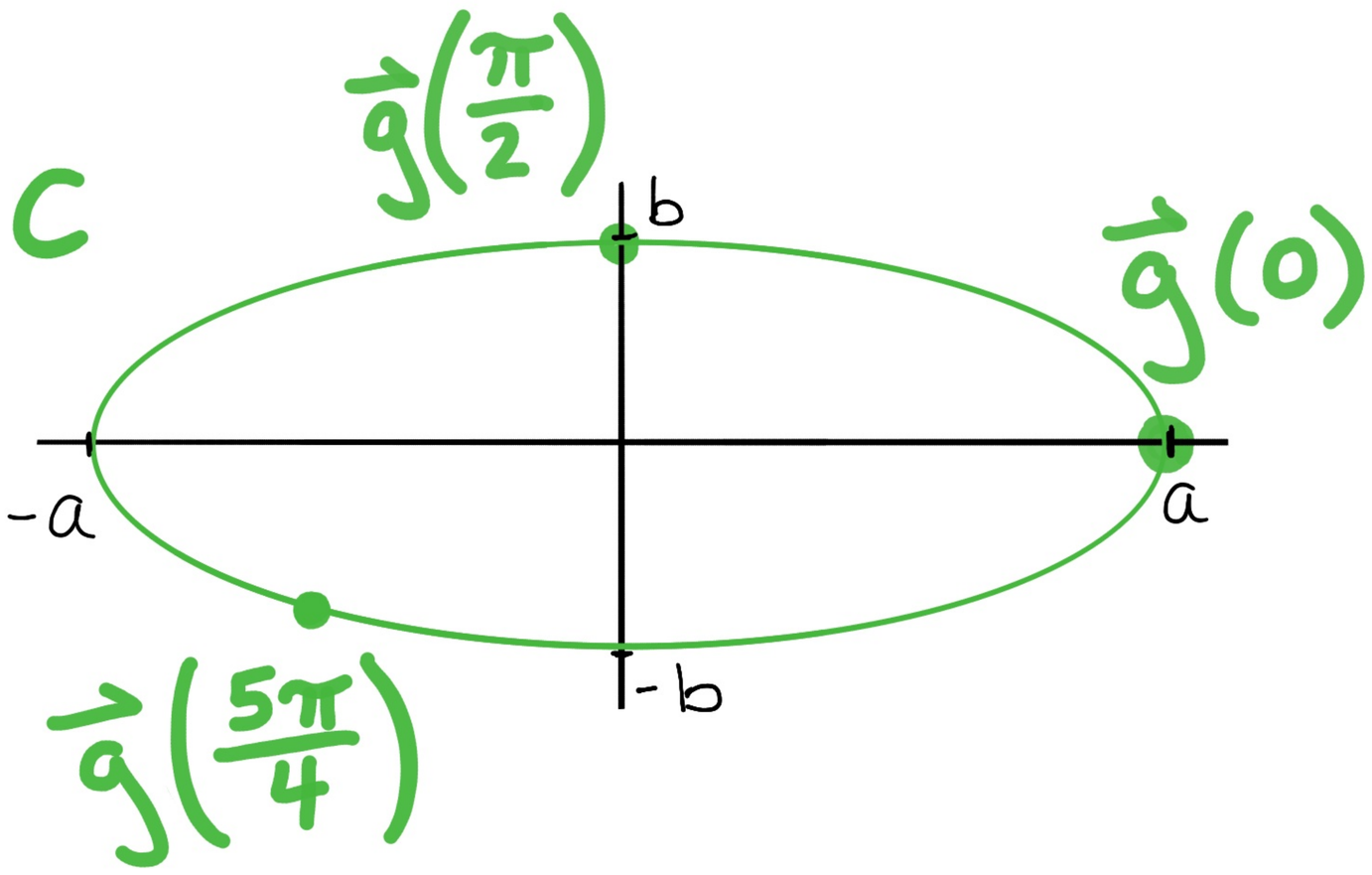
C is parametrized by

$\vec{g} : \mathbb{R} \to \mathbb{R}^2$, $\vec{g}(t) = \langle a\cos t, b\sin t \rangle$, $0 \le t < 2\pi$.

Example:



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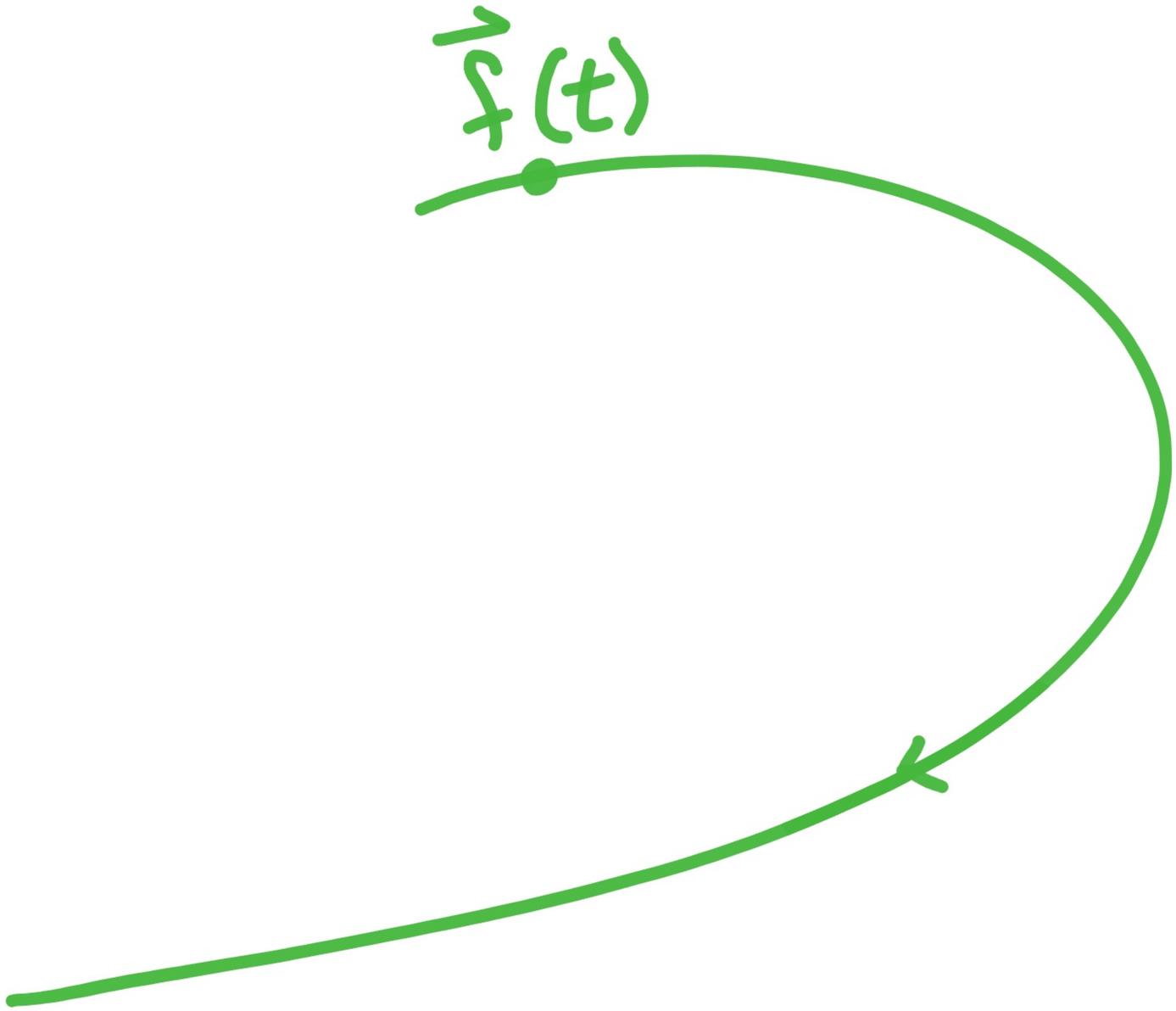
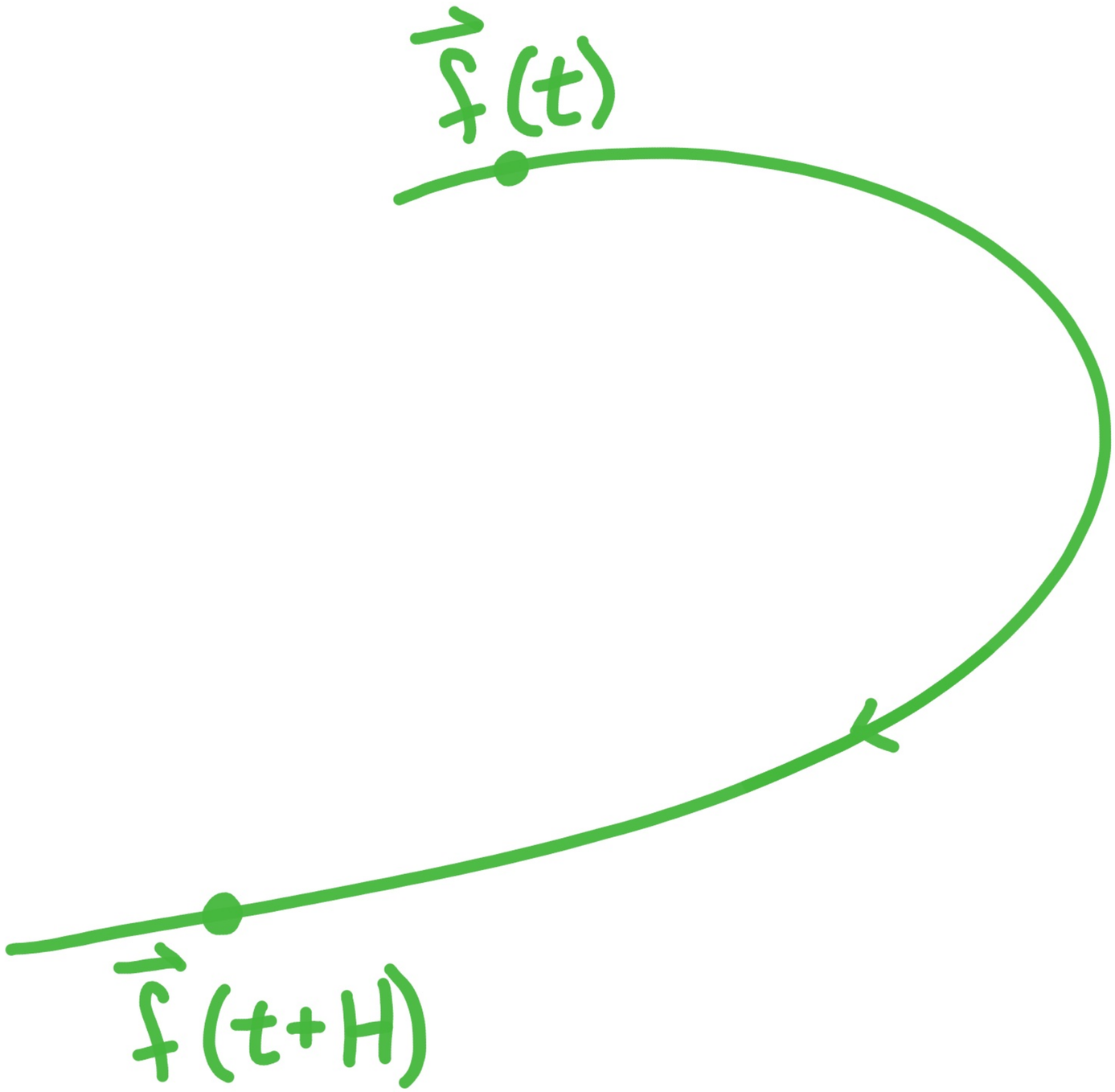
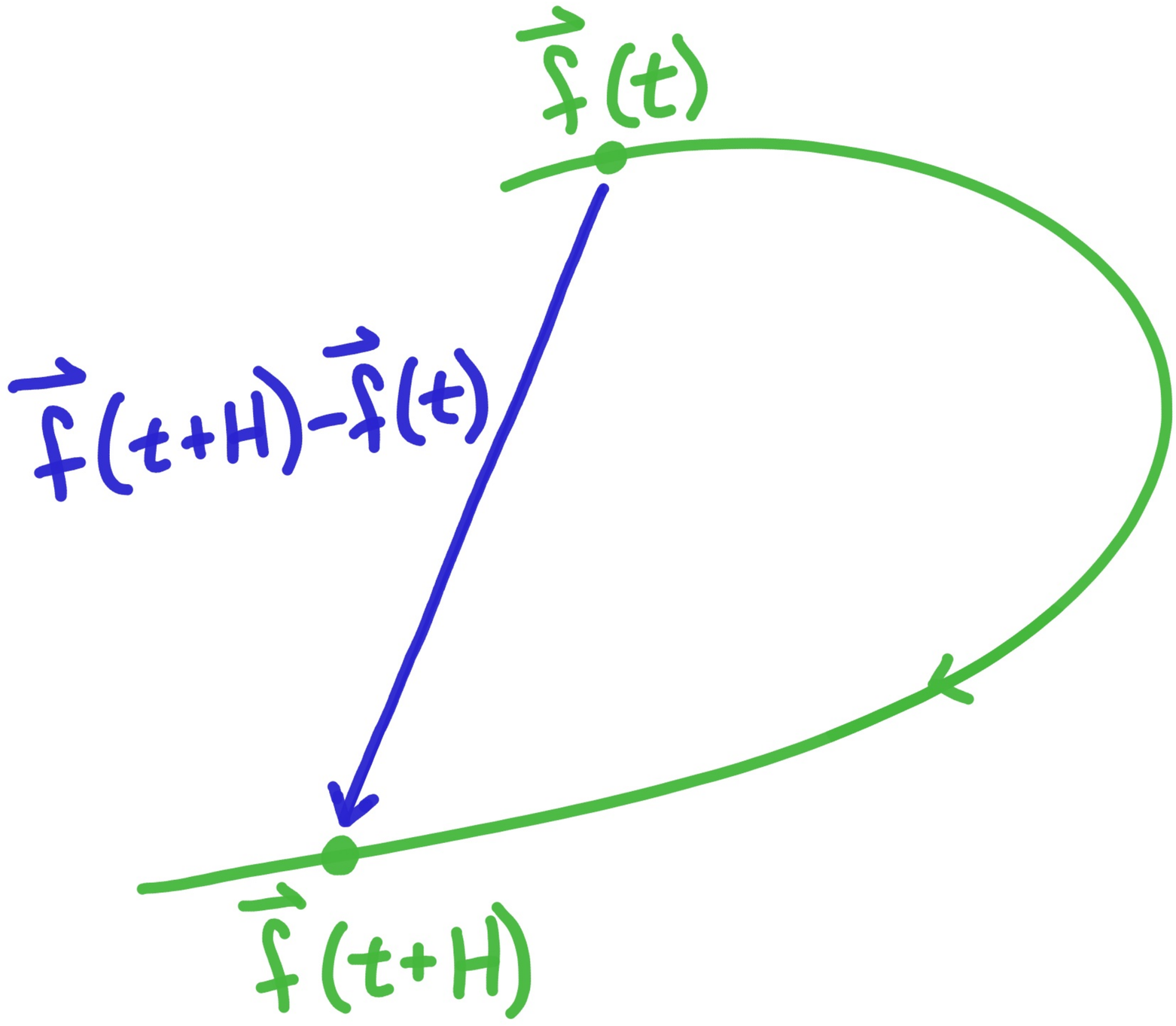
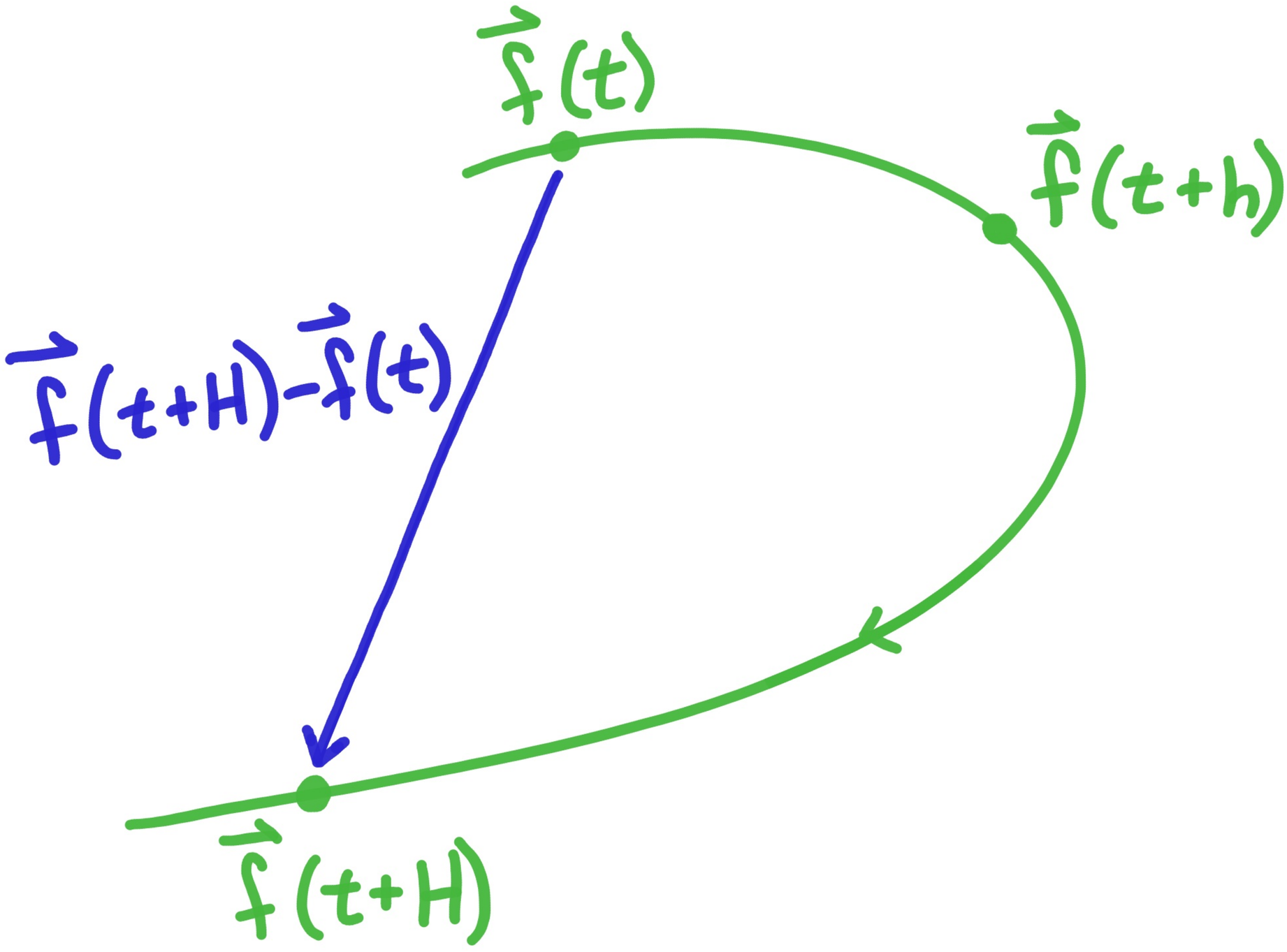
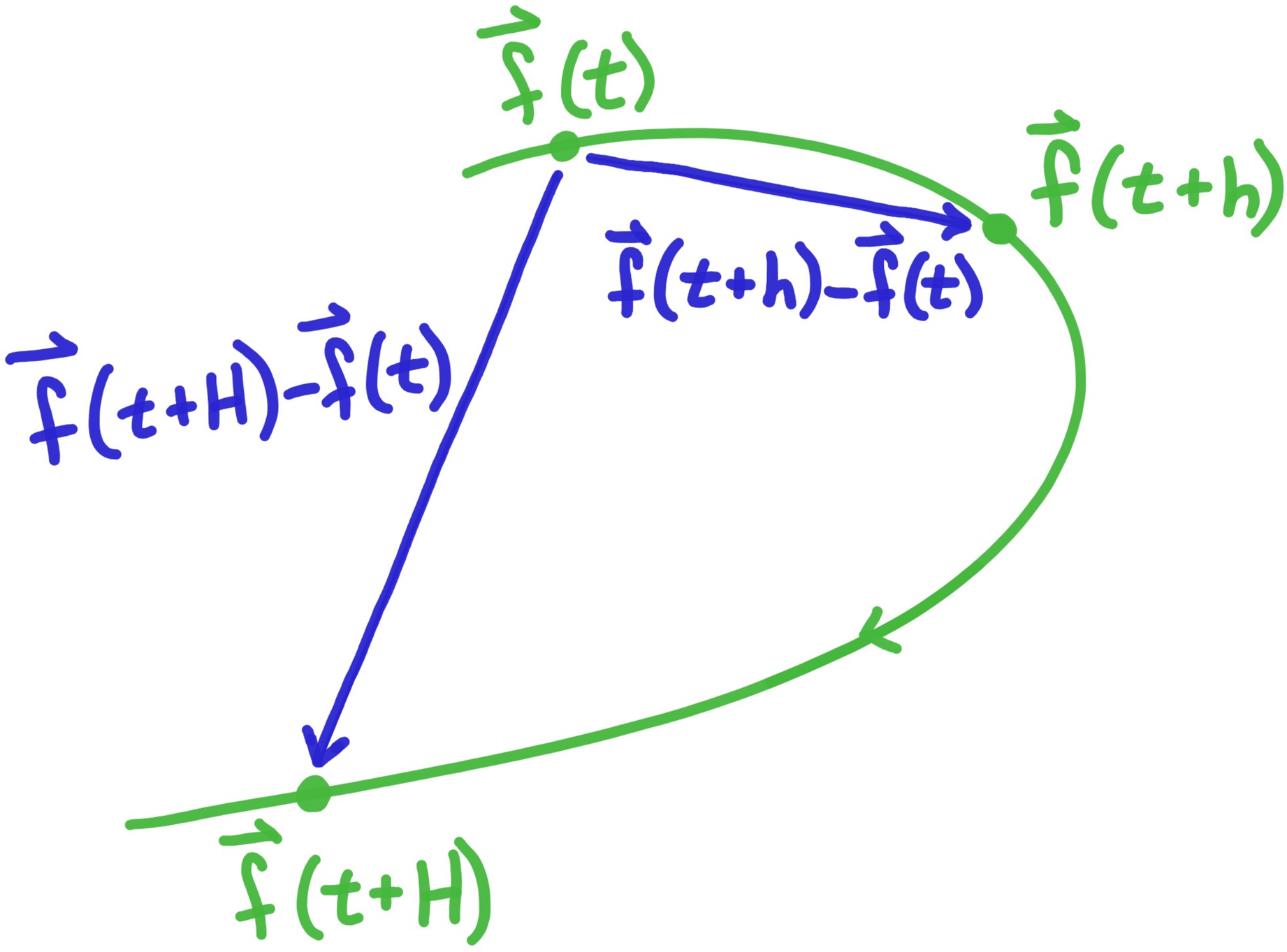
Derivatives


$\vec{r}(t)$


$\vec{f}(t)$
$\vec{f}(t+H)$


$\vec{f}(t)$
$\vec{f}(t+H)-\vec{f}(t)$
$\vec{f}(t+H)$


$\vec{f}(t)$
$\vec{f}(t+h)$
$\vec{f}(t+H)-\vec{f}(t)$
$\vec{f}(t+H)$


$\vec{f}(t)$
$\vec{f}(t+h)$
$\vec{f}(t+h)-\vec{f}(t)$
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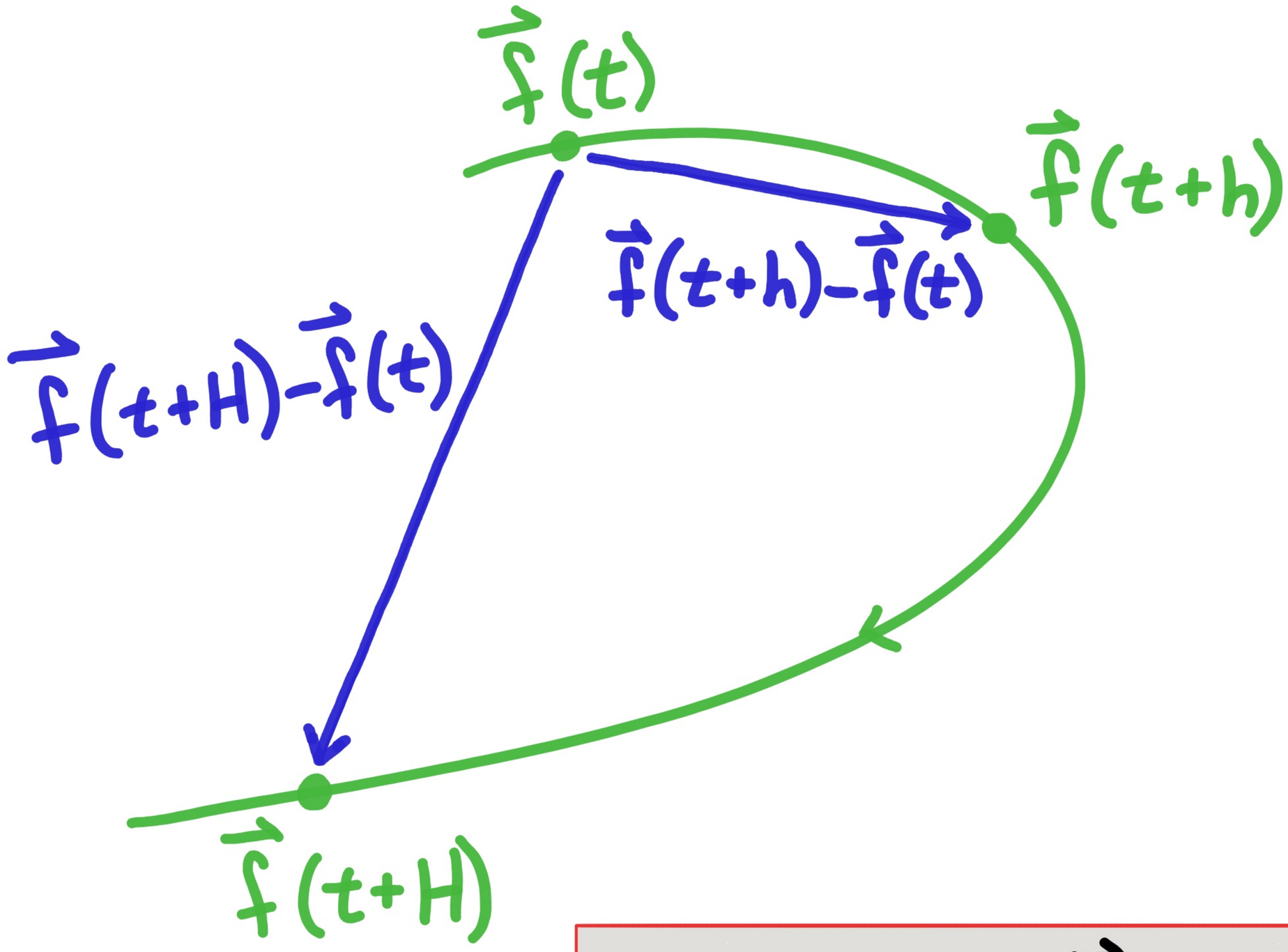
$\vec{f}(t+H)$



$$\vec{f}\,'(t) = \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}$$

$$\vec{f}'(t) = \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}$$

Proposition: If $\vec{f}(t) = \langle f_1(t), f_2(t) \rangle$, then $\vec{f}'(t) = \langle f_1'(t), f_2'(t) \rangle$.

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proof:

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proof:

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$$= \lim_{h \to 0} \frac{\langle f_1(t+h), f_2(t+h) \rangle - \langle f_1(t), f_2(t) \rangle}{h}$$

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$$= \lim_{h \to 0} \frac{\langle f_1(t+h) - f_1(t), f_2(t+h) - f(t) \rangle}{h}$$

$$= \lim_{h \to 0} \left\langle \frac{f_1(t+h) - f_1(t)}{h}, \frac{f_2(t+h) - f_2(t)}{h} \right\rangle$$

proof:

$$\vec{f}'(t) = \lim_{h\to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}$$

$$= \lim_{h\to 0} \frac{\langle f_1(t+h), f_2(t+h)\rangle - \langle f_1(t), f_2(t)\rangle}{h}$$

$$= \lim_{h\to 0} \frac{\langle f_1(t+h) - f_1(t), f_2(t+h) - f(t)}{h}$$

$$= \lim_{h\to 0} \left\langle \frac{f_1(t+h) - f_1(t)}{h} , \frac{f_2(t+h) - f_2(t)}{h} \right\rangle$$

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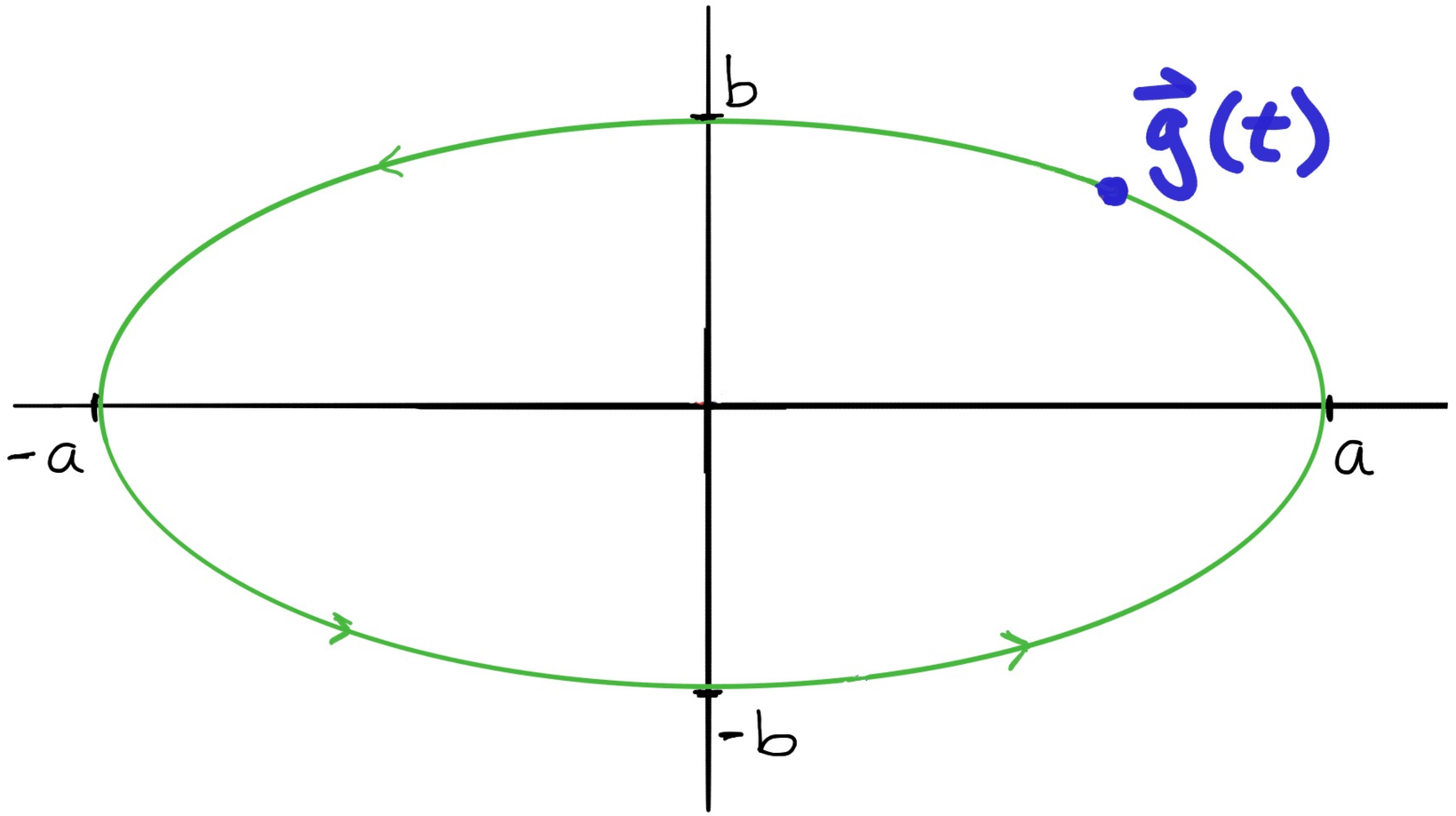
$$= \langle f_1'(t),\ f_2'(t)\rangle.$$

END

Example: $\vec{g}: \mathbb{R} \to \mathbb{R}^2$, $\vec{g}(t) = \langle a\cos t, b\sin t \rangle$



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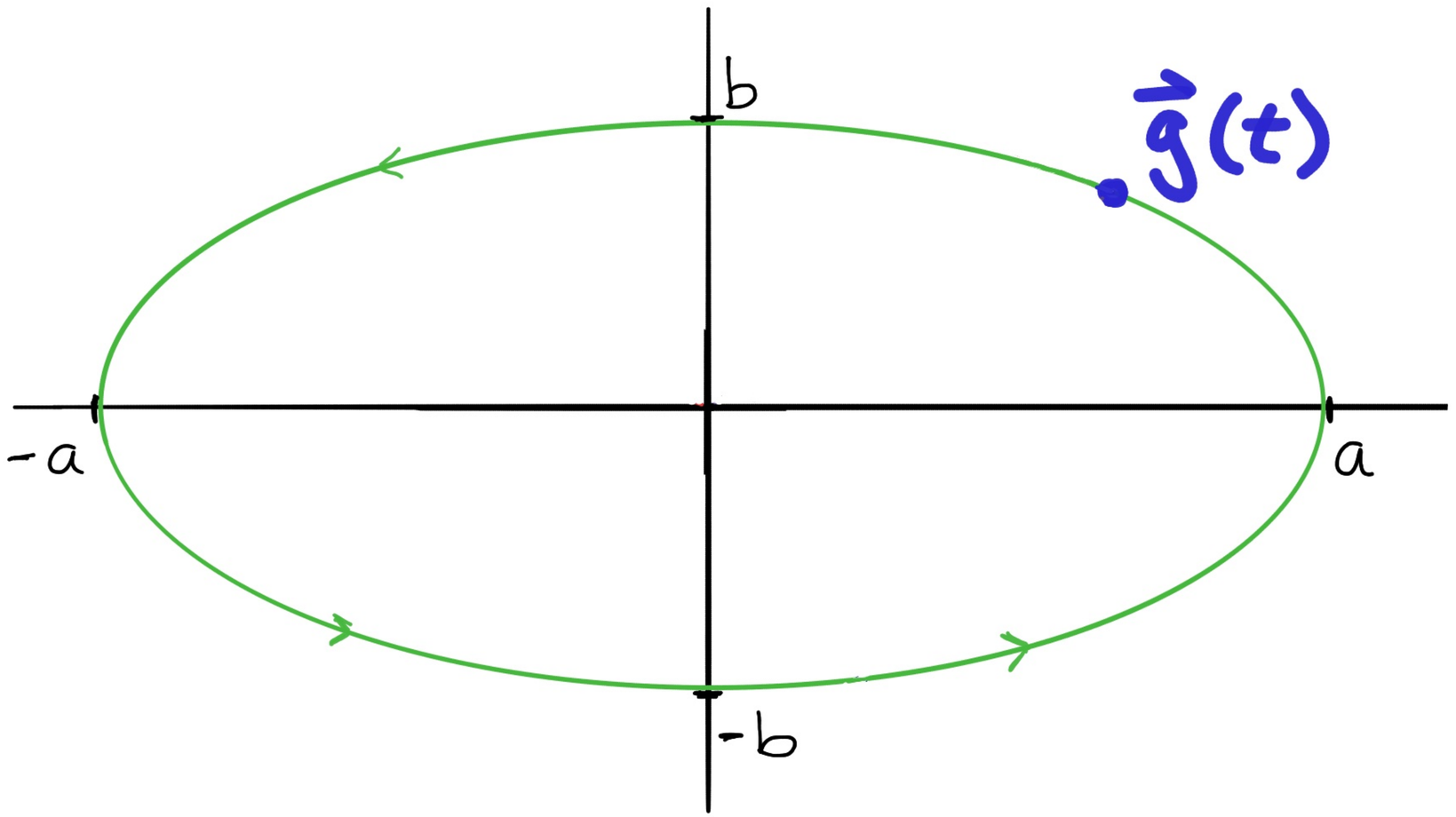

$\vec{g}'(t) = \langle -a\sin t, b\cos t \rangle$

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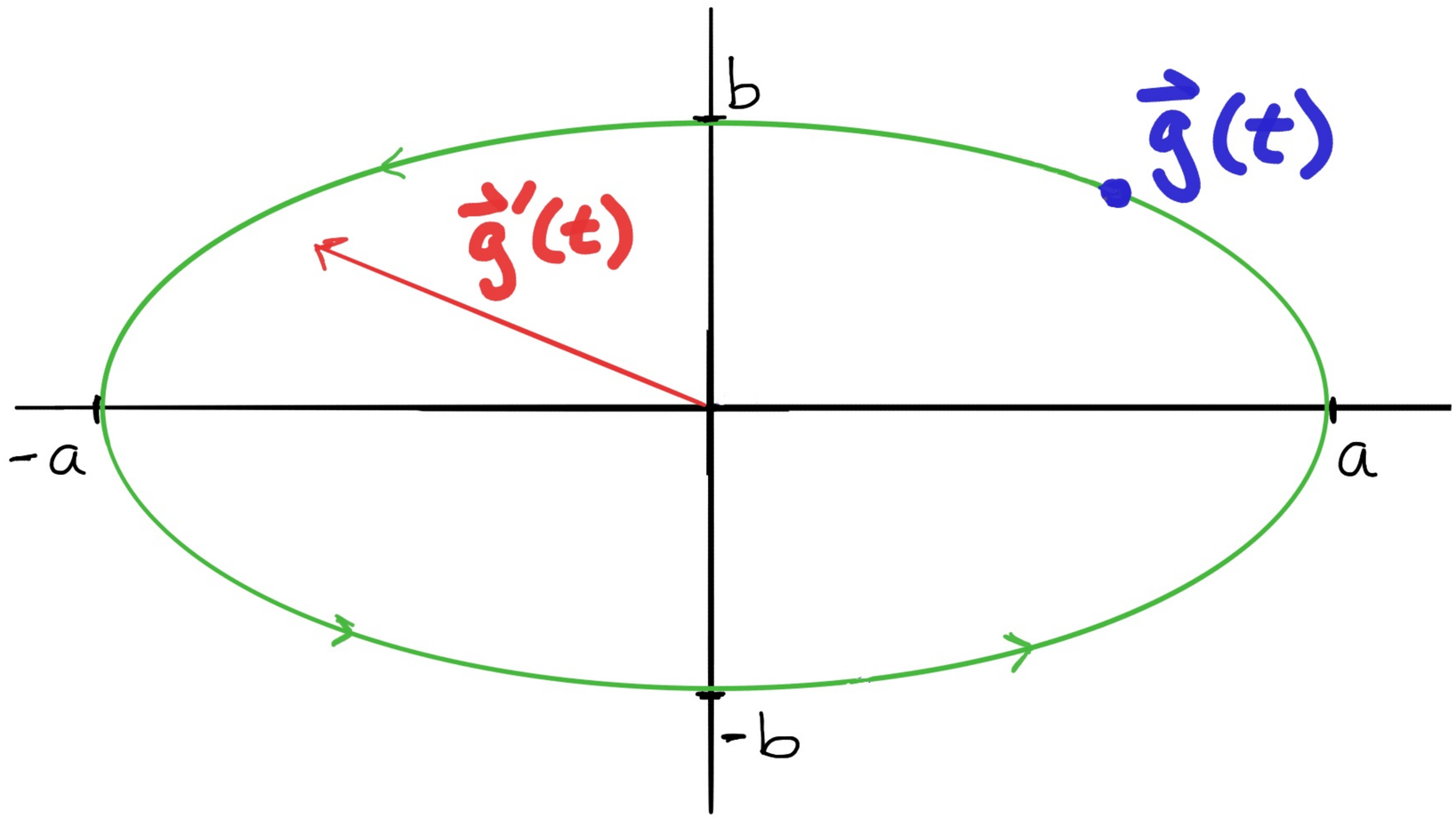

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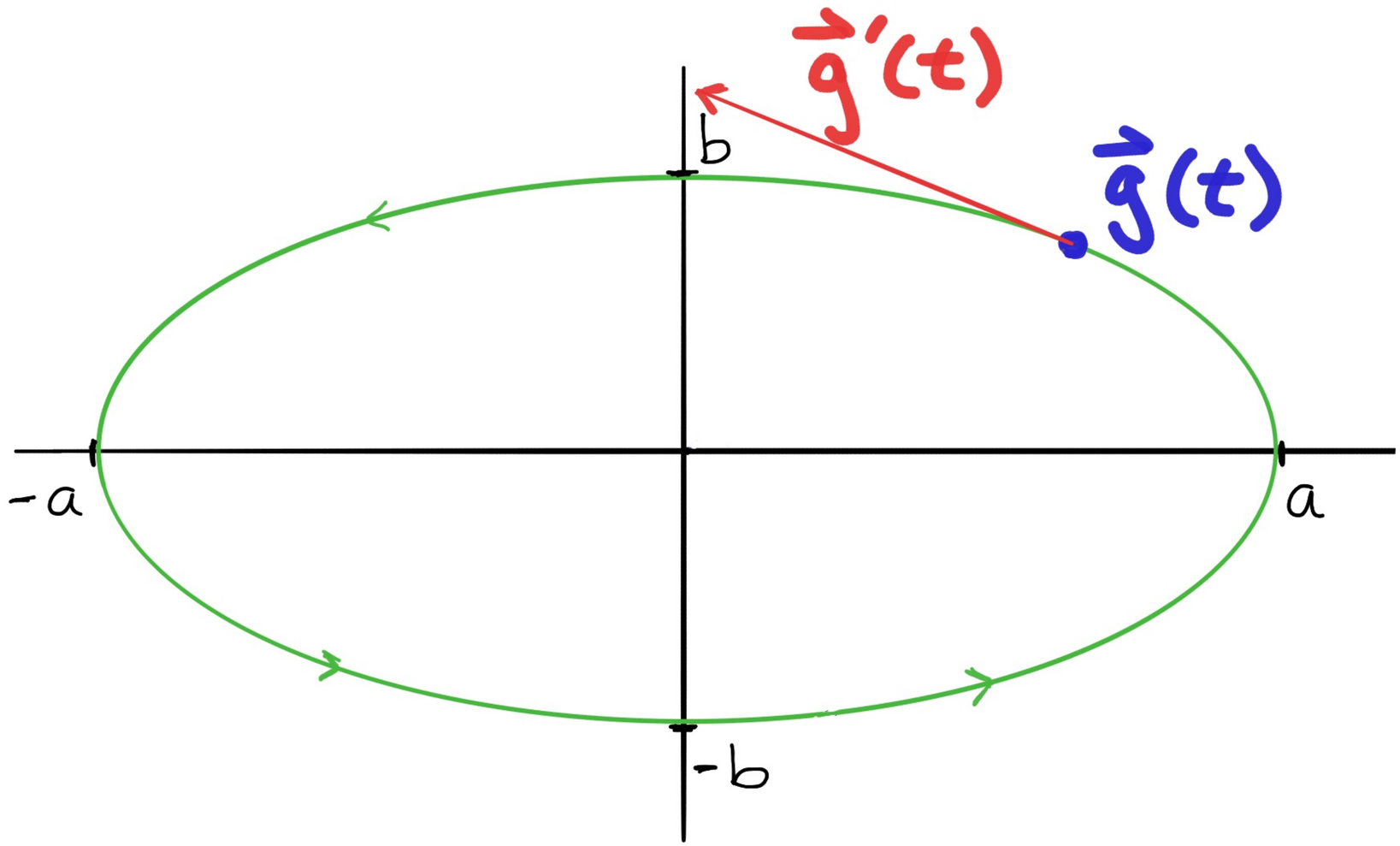

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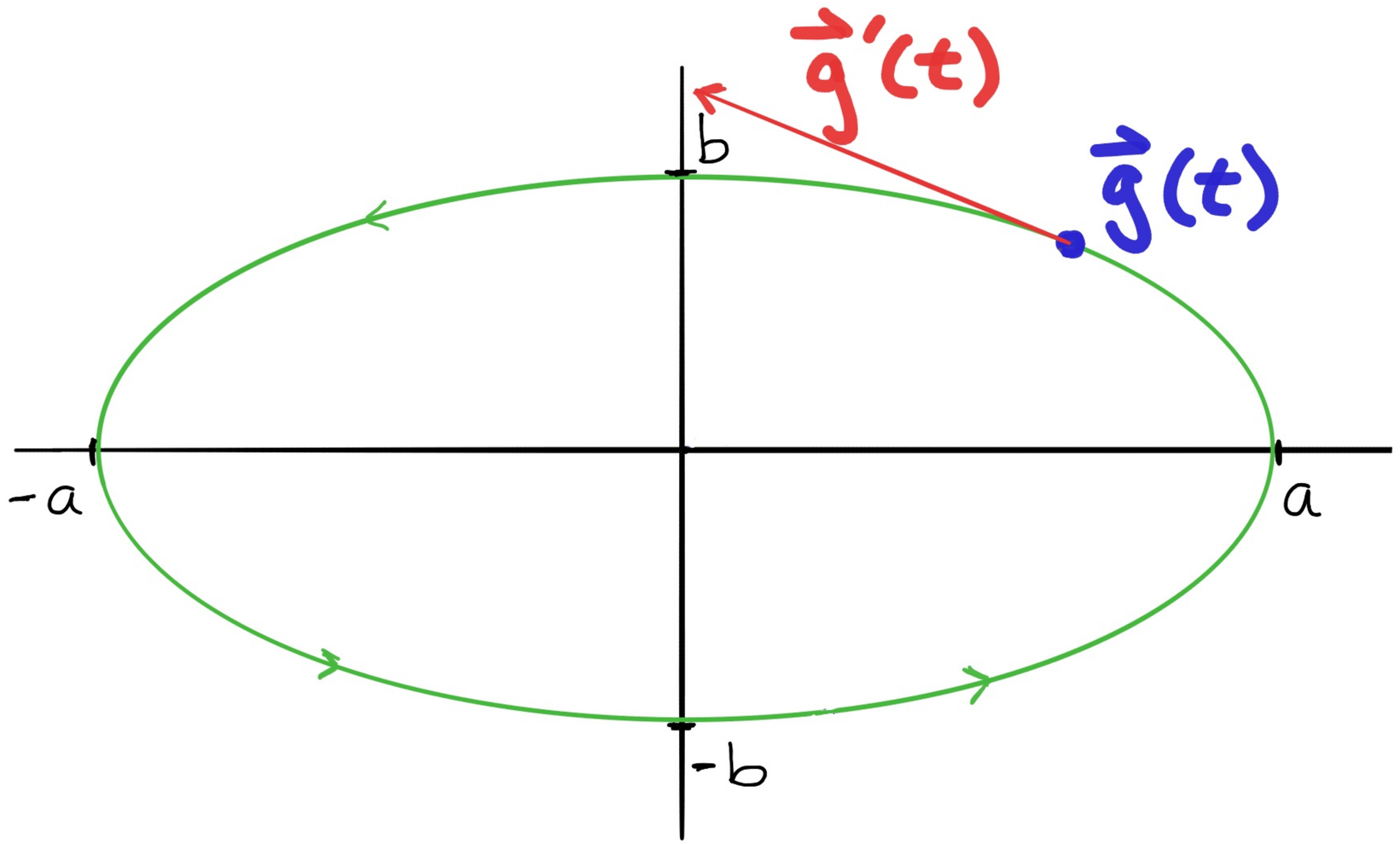
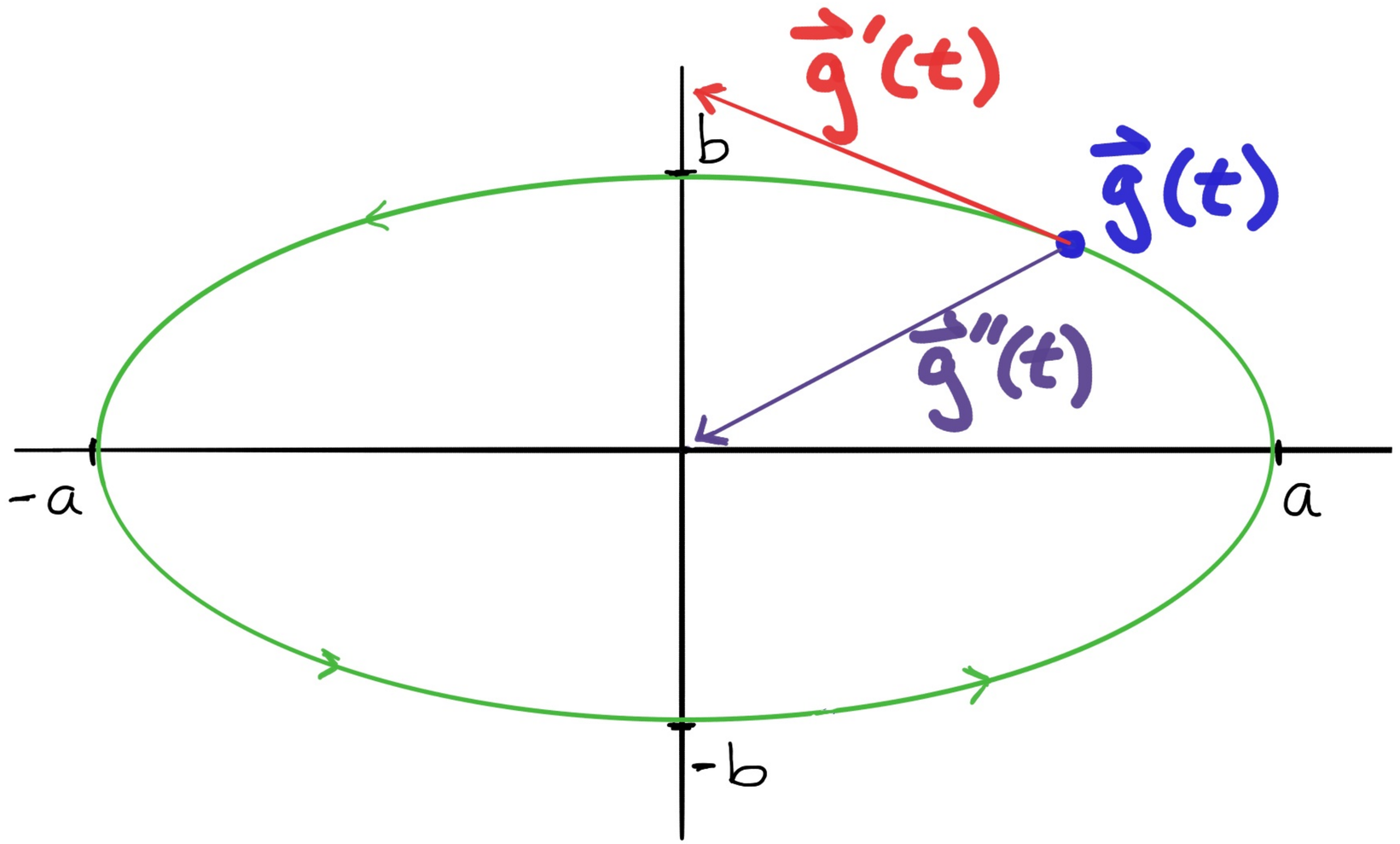


$$\vec{g}'(t) = \langle -a\sin t, b\cos t \rangle$$

$$\vec{g}''(t) = \langle -a\cos t, -b\sin t \rangle = -\vec{g}(t)$$

Example: $\vec{g}: \mathbb{R} \to \mathbb{R}^2$, $\vec{g}(t) = \langle a\cos t, b\sin t \rangle$



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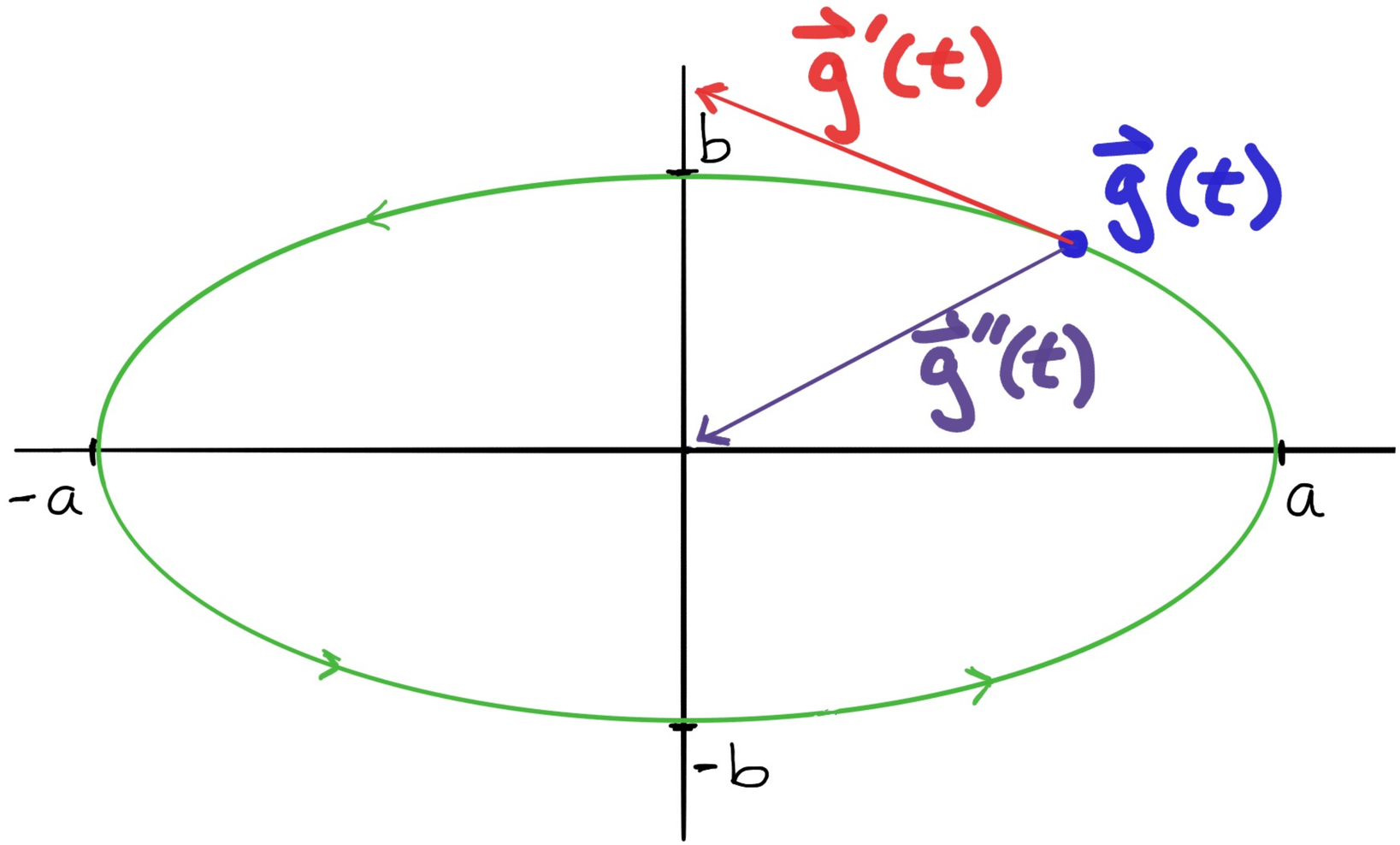
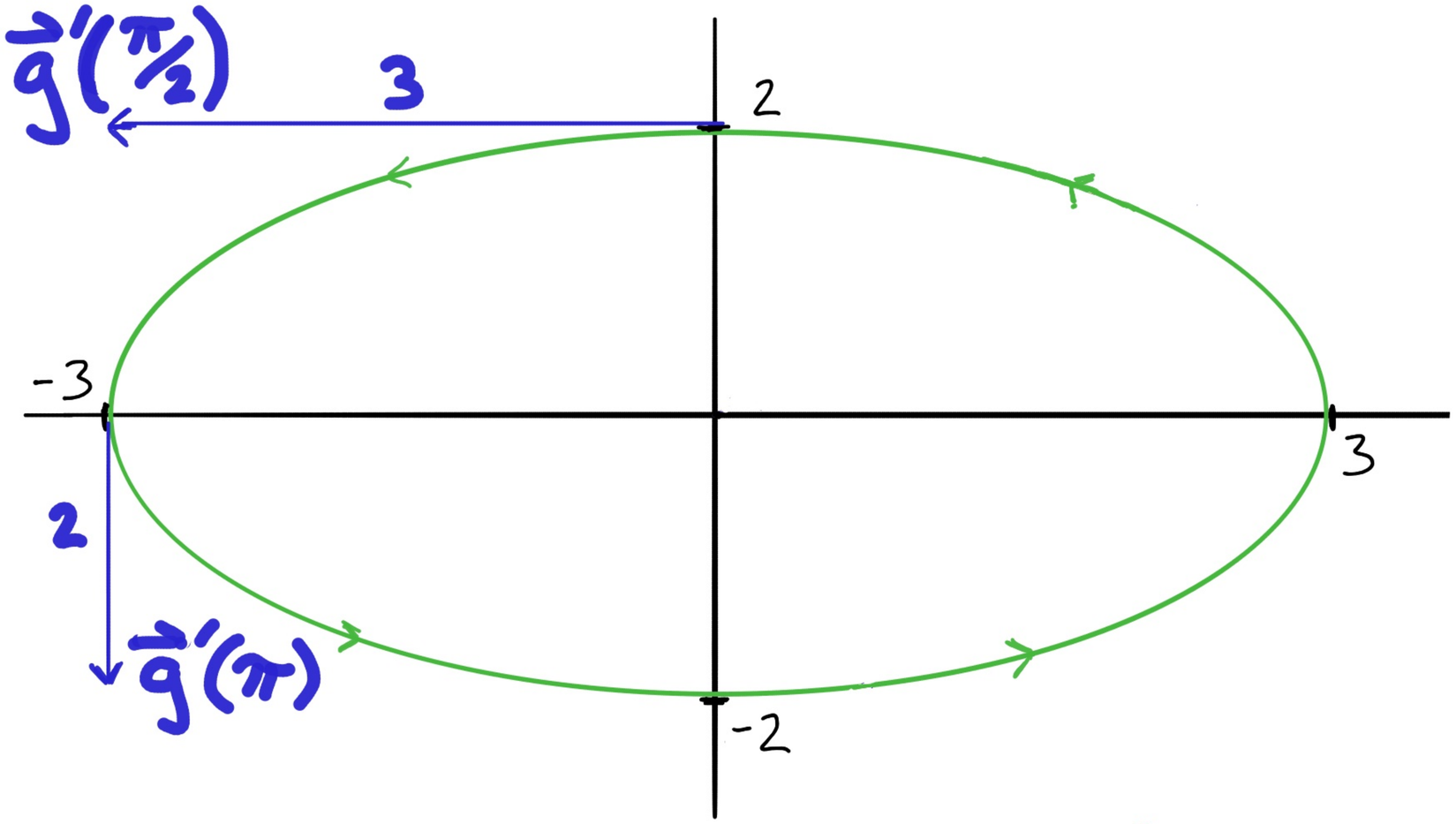


Let $a=3$ and $b=2$.
Find $\|\vec{g}'(t)\|$ for $t=\pi/2$ and $t=\pi$.

$$\vec{g}'(t) = \langle -a\sin t, b\cos t \rangle$$

$$\vec{g}''(t) = \langle -a\cos t, -b\sin t \rangle = -\vec{g}(t)$$

Example: $\vec{q}:\mathbb{R}\to\mathbb{R}^2$, $\vec{q}(t)=\langle a\cos t, b\sin t\rangle$



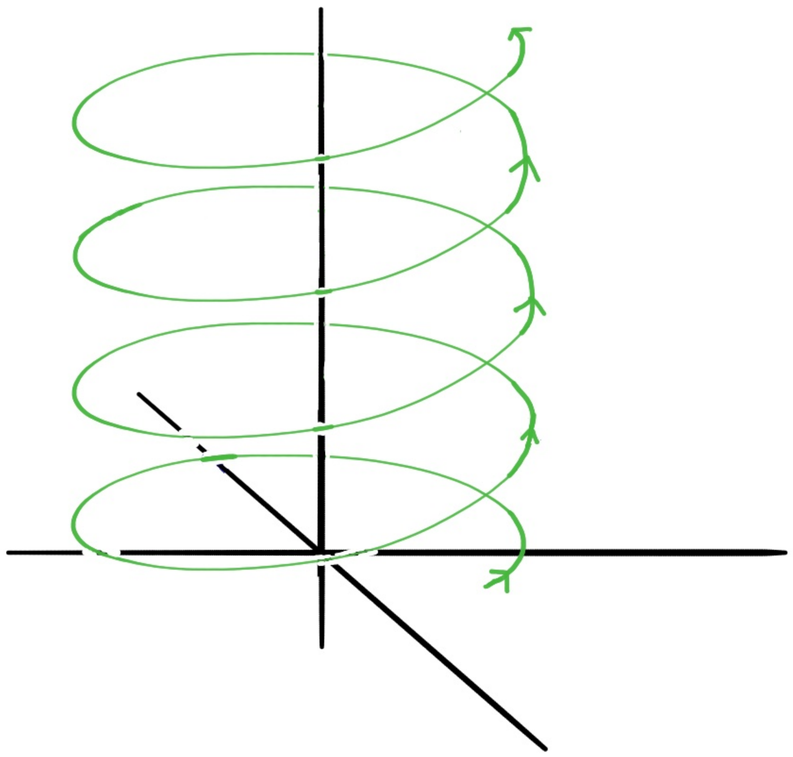
Let $a=3$ and $b=2$. Find $\|\vec{q}'(t)\|$ for $t=\pi/2$ and $t=\pi$.

$$\vec{q}'(t)=\langle -a\sin t, b\cos t\rangle$$

$$\|\vec{q}'(\pi/2)\| = \left\|\left\langle -3\sin\tfrac{\pi}{2}, 2\cos\tfrac{\pi}{2}\right\rangle\right\| = \|\langle -3,0\rangle\| = 3$$

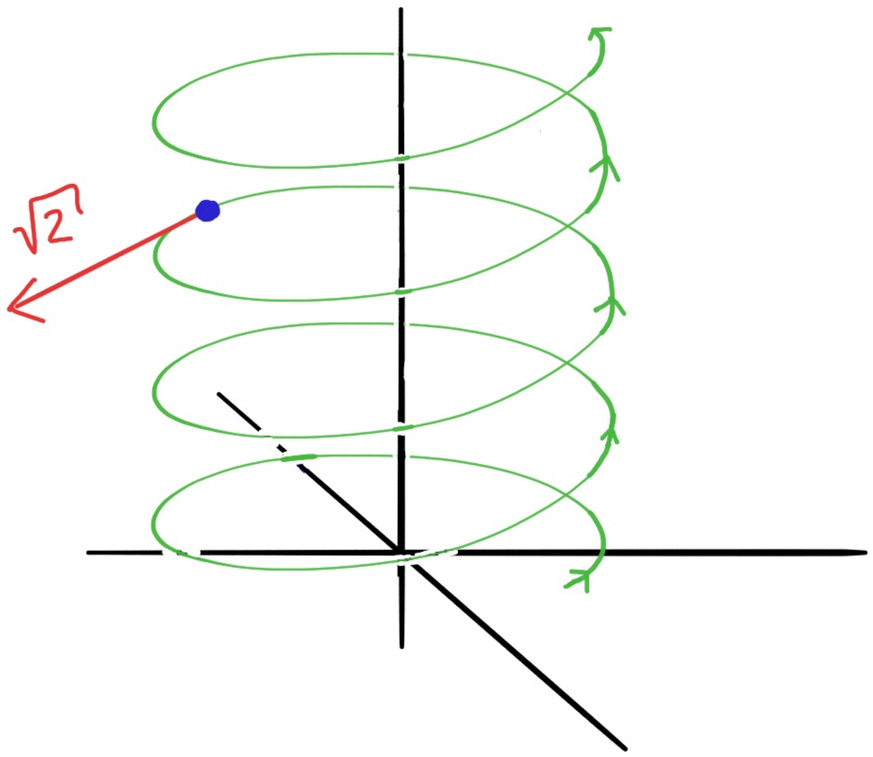
$$\|\vec{q}'(\pi)\| = \|\langle -3\sin\pi, 2\cos\pi\rangle\| = \|\langle 0,-2\rangle\| = 2$$

Example: $\vec{h} : \mathbb{R} \longrightarrow \mathbb{R}^3$, $\vec{h}(t) = \langle \cos t, \sin t, t \rangle$



Find $\|\vec{h}\,'(t)\|$.

Example: $\vec{h} : \mathbb{R} \longrightarrow \mathbb{R}^3$, $\vec{h}(t) = \langle \cos t, \sin t, t \rangle$



Find $\|\vec{h}\,'(t)\|$.

$\vec{h}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$, so

$$\|\vec{h}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{1+1} = \sqrt{2}$$

$$\vec{f}'(t) = \lim_{h \to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}$$

If $h \approx 0$,

then $\vec{f}'(t) \approx \dfrac{\vec{f}(t+h) - \vec{f}(t)}{h}$.

That is, $\vec{f}(t+h) - \vec{f}(t) \approx h\,\vec{f}'(t)$

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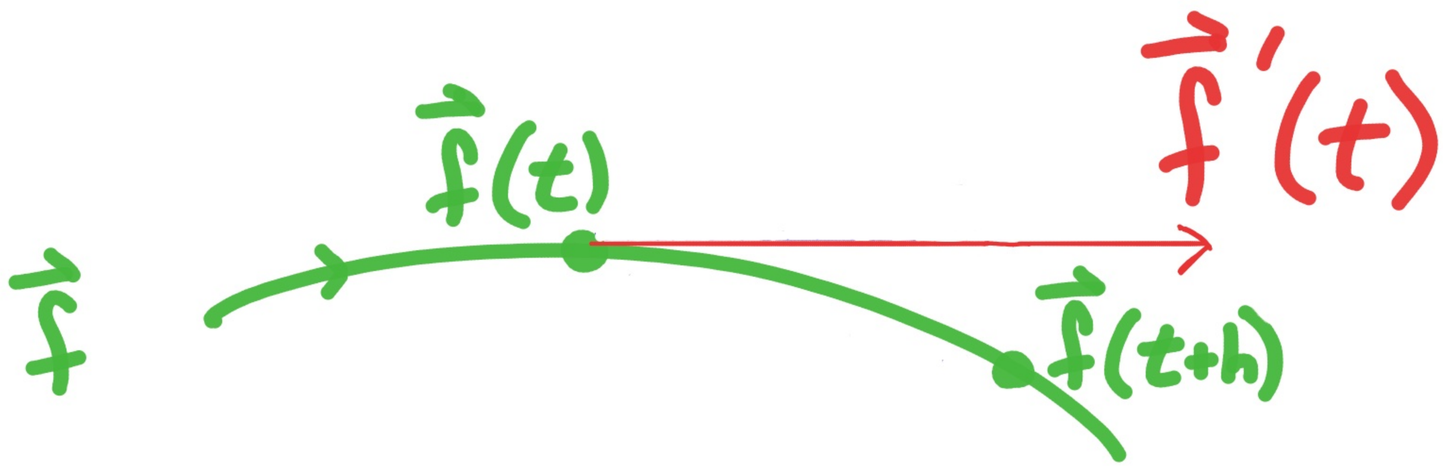

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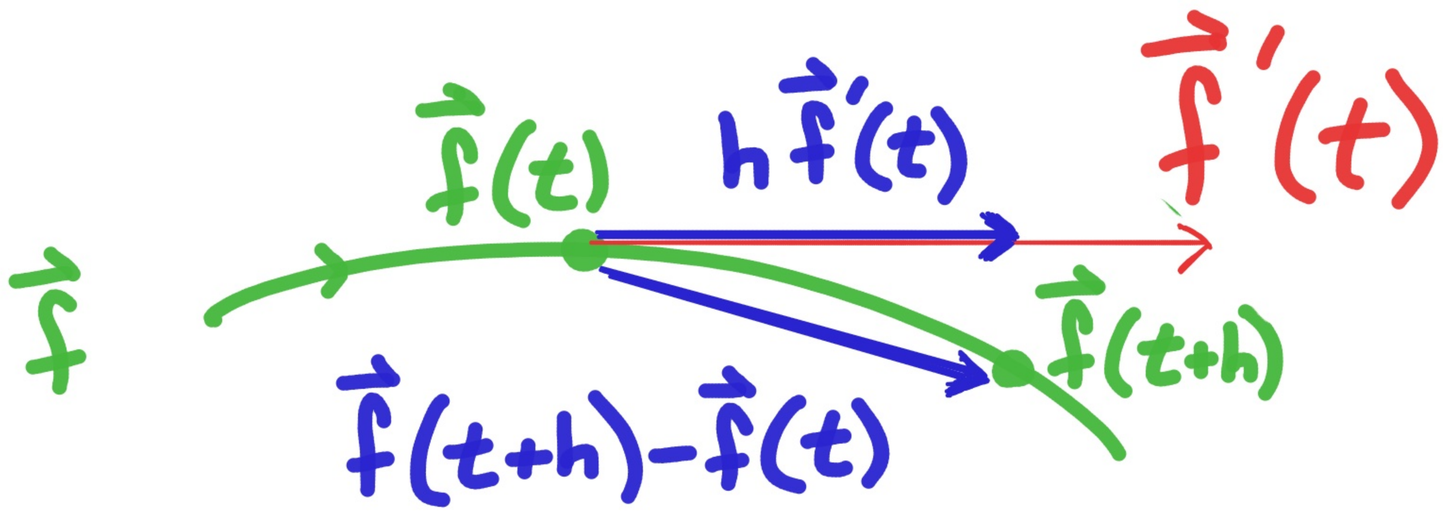
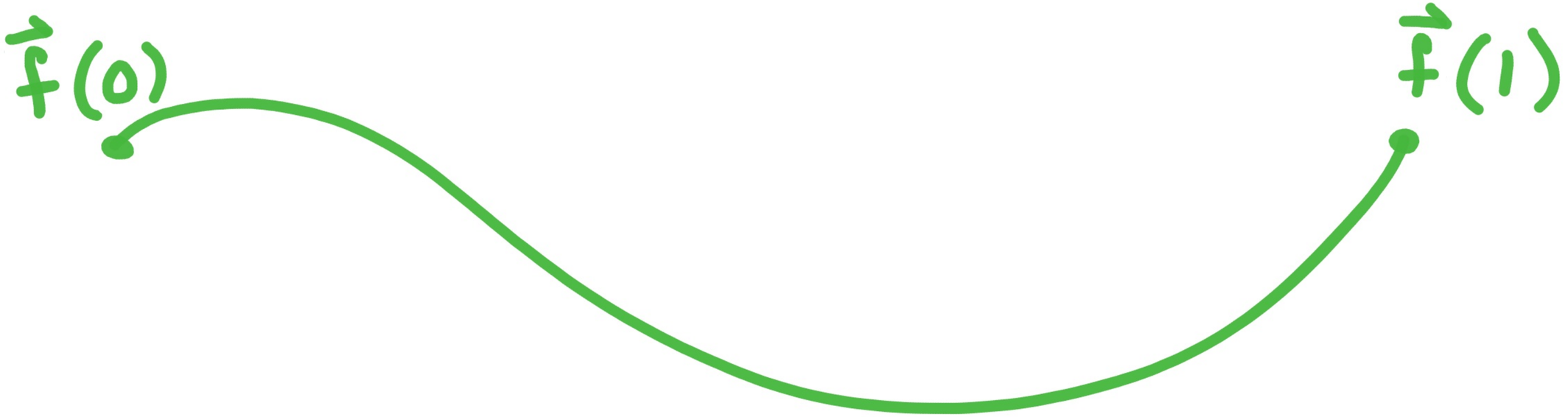


Ⅳ Arc length

Approximating lengths

C is a smooth curve parametrized by $\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^n$.



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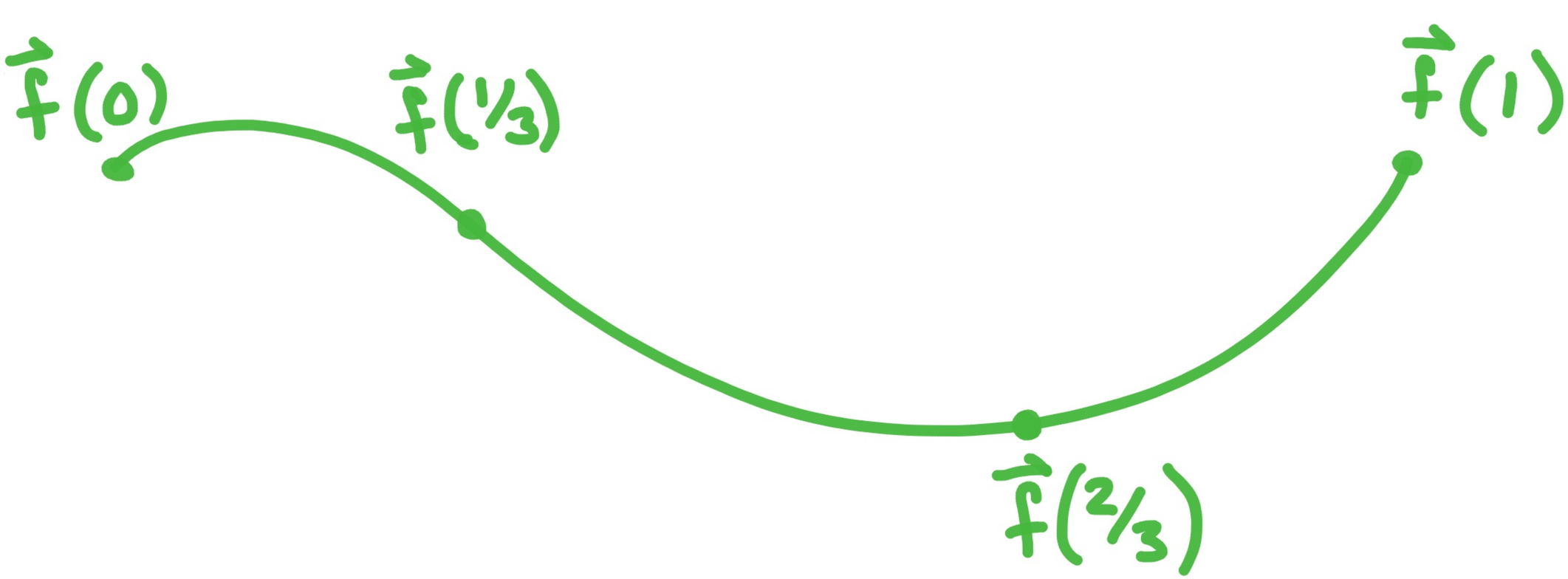

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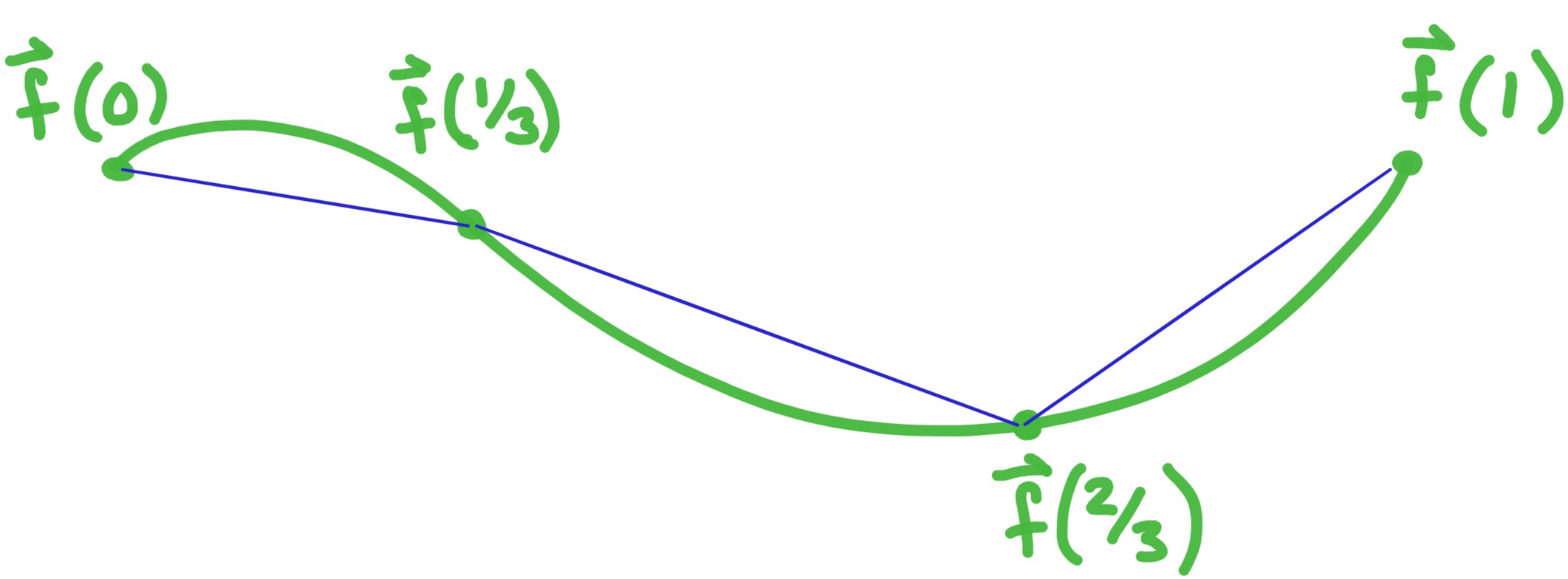

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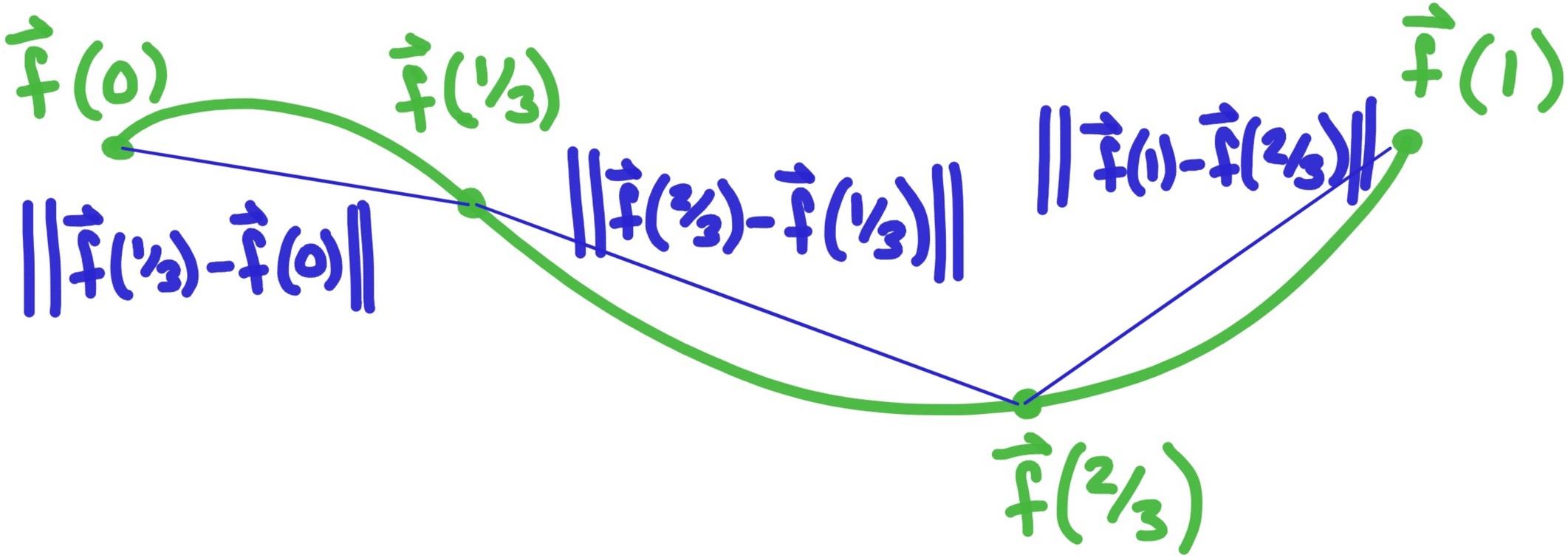

Arc length of C from $\vec{f}(0)$ to $\vec{f}(1)$

$$\approx \|\vec{f}(1/3) - \vec{f}(0)\| + \|\vec{f}(2/3) - \vec{f}(1/3)\| + \|\vec{f}(1) - \vec{f}(2/3)\|$$

$$\vec{f}(t+h) - \vec{f}(t) \approx h\,\vec{f}'(t)$$

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$$= \|\vec{f}(0+1/3) - \vec{f}(0)\| + \|\vec{f}(1/3+1/3) - \vec{f}(1/3)\| + \|\vec{f}(2/3+1/3) - \vec{f}(2/3)\|$$

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$$\approx \|1/3\,\vec{f}'(0)\| + \|1/3\,\vec{f}'(1/3)\| + \|1/3\,\vec{f}'(2/3)\|$$

$$\vec{f}(t+h) - \vec{f}(t) \approx h\,\vec{f}'(t)$$

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$$\approx \|\vec{f}(1/3) - \vec{f}(0)\| + \|\vec{f}(2/3) - \vec{f}(1/3)\| + \|\vec{f}(1) - \vec{f}(2/3)\|$$

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$$\approx \|1/3\,\vec{f}'(0)\| + \|1/3\,\vec{f}'(1/3)\| + \|1/3\,\vec{f}'(2/3)\|$$

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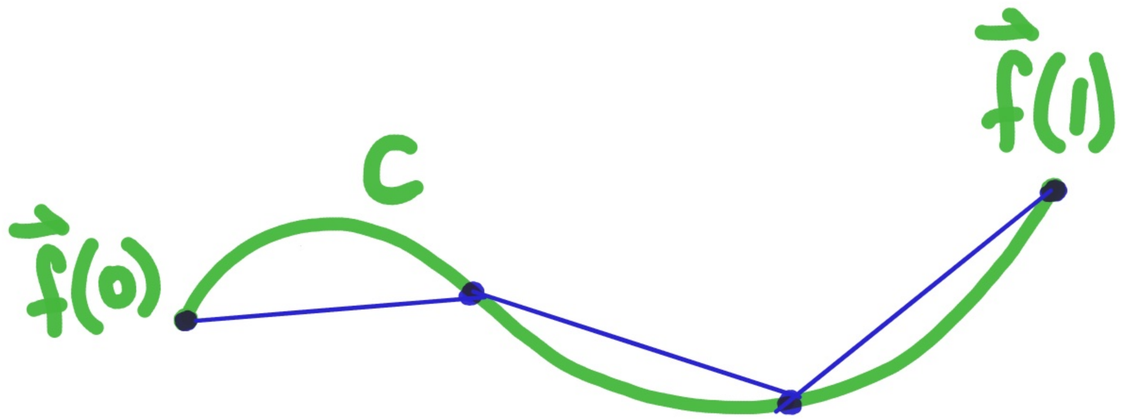
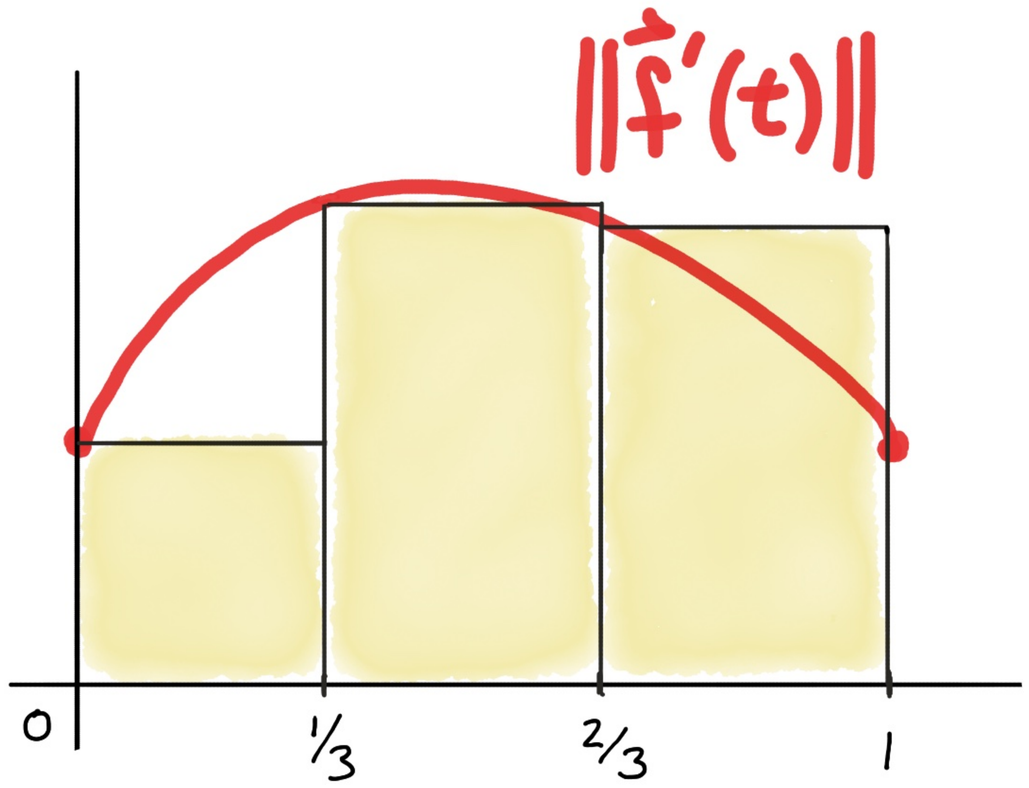
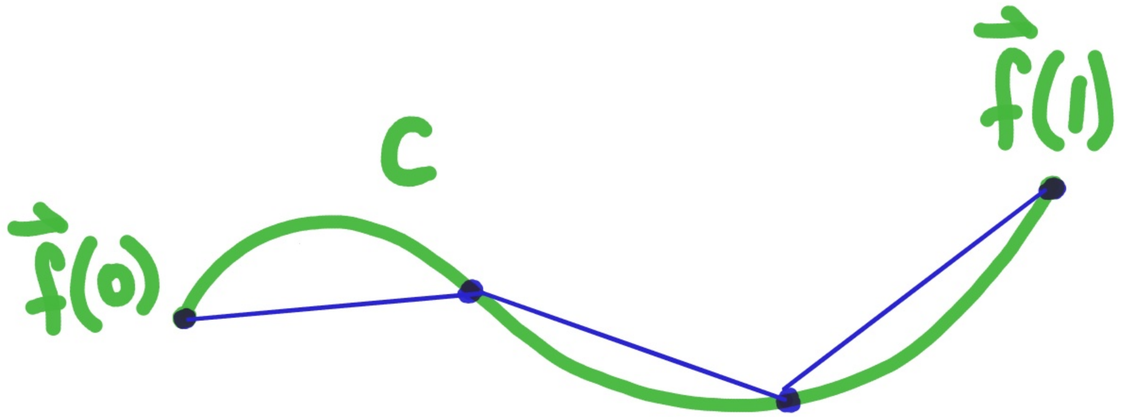
$$\vec{f}(t+h) - \vec{f}(t) \approx h\,\vec{f}'(t)$$

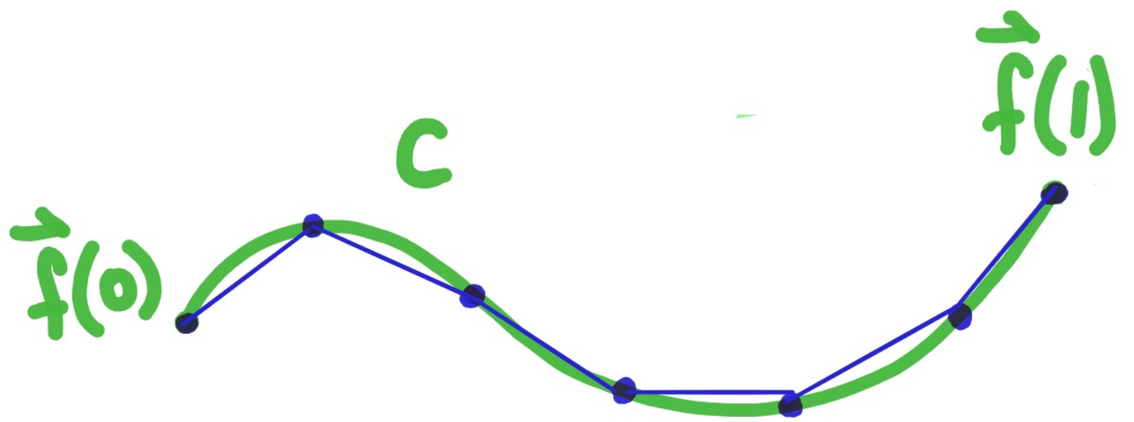
Arc length of C from $\vec{f}(0)$ to $\vec{f}(1)$

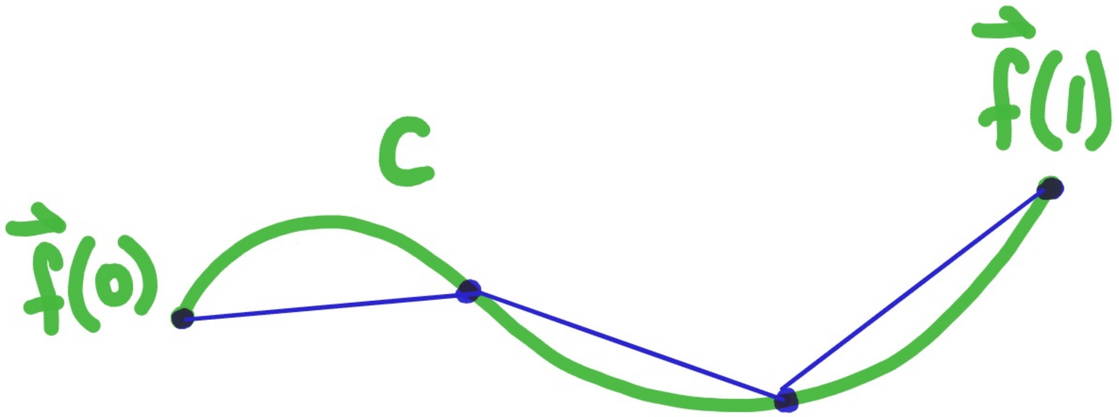
$$\approx \tfrac{1}{3}\|\vec{f}'(0)\| + \tfrac{1}{3}\|\vec{f}'(\tfrac{1}{3})\| + \tfrac{1}{3}\|\vec{f}'(\tfrac{2}{3})\|$$

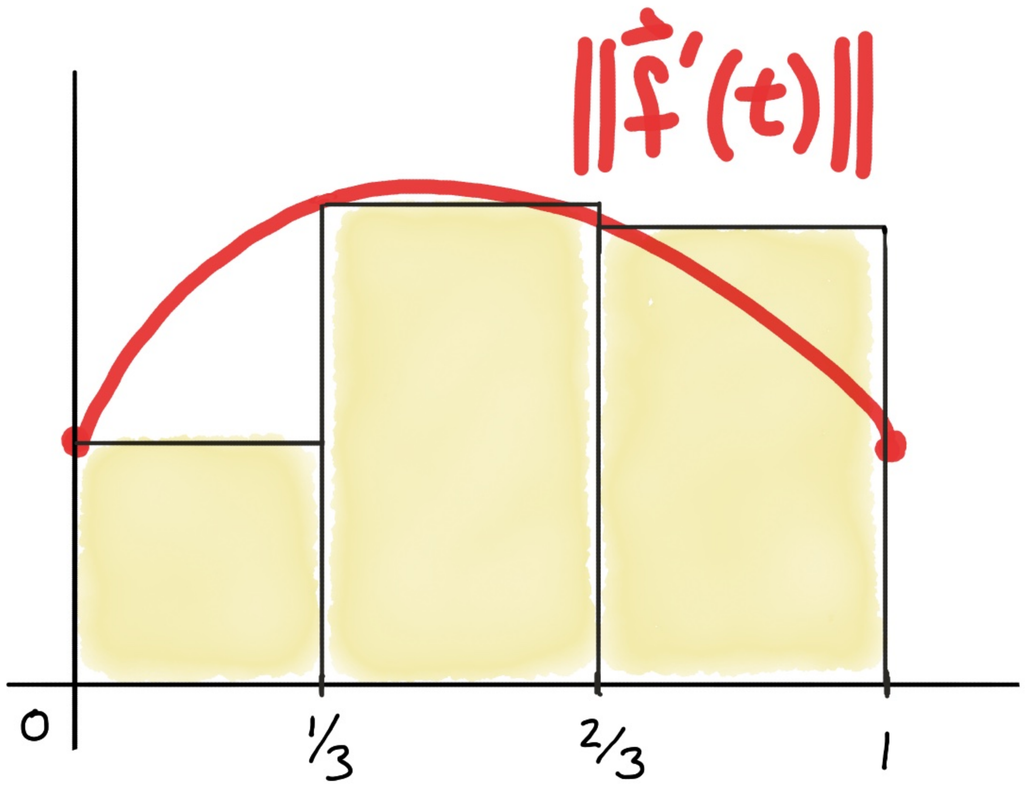


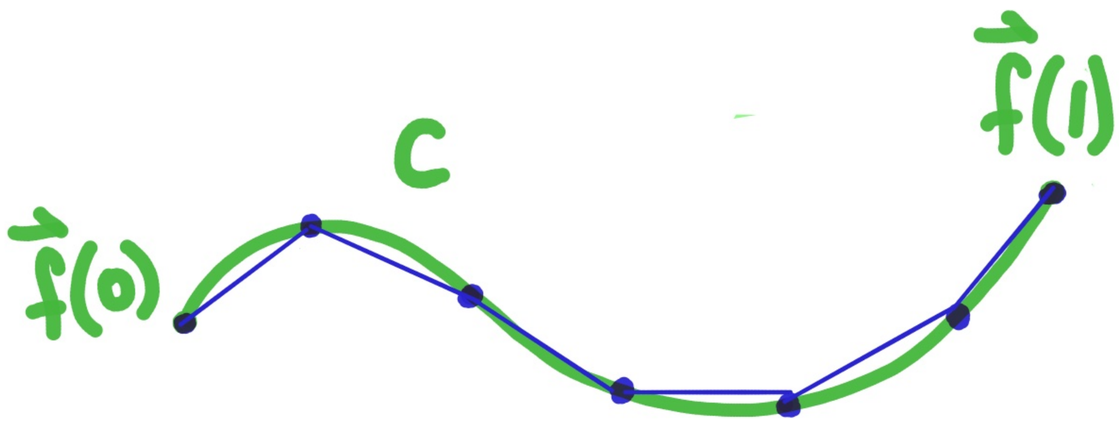


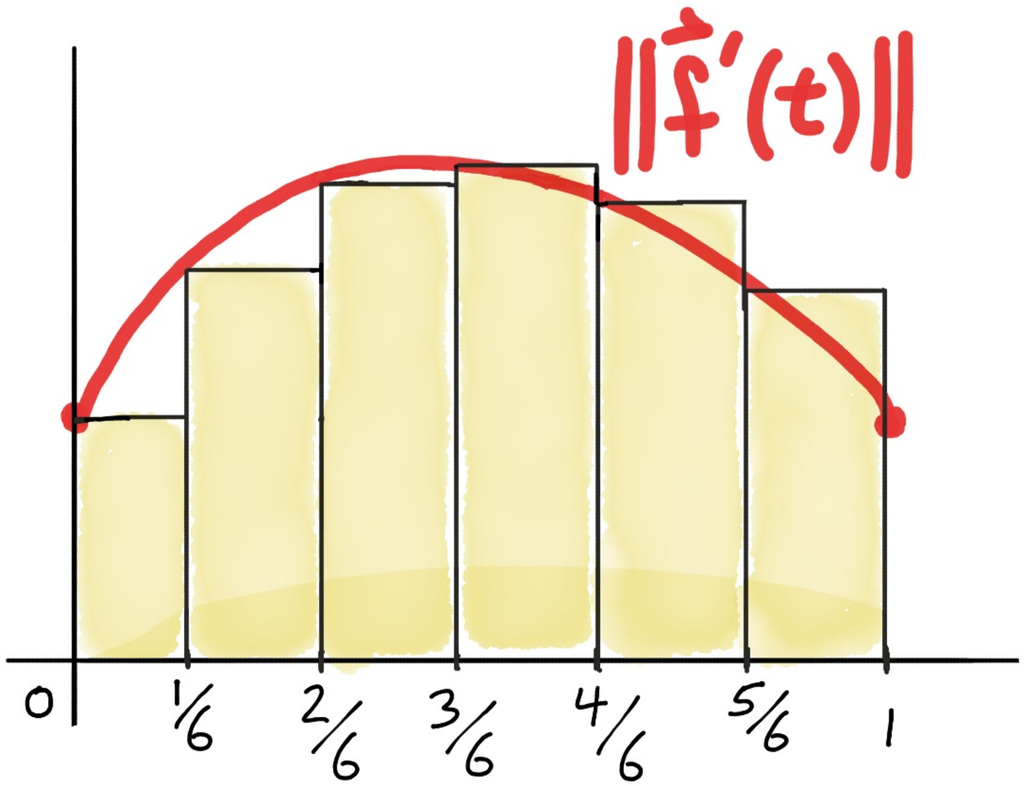

$\vec{f}(1)$
C
$\vec{f}(0)$

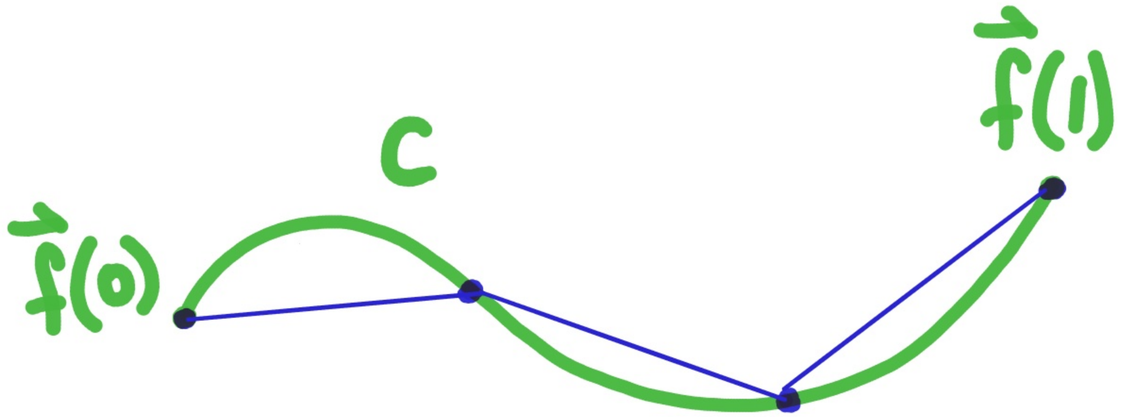

$\vec{f}(1)$
C
$\vec{f}(0)$

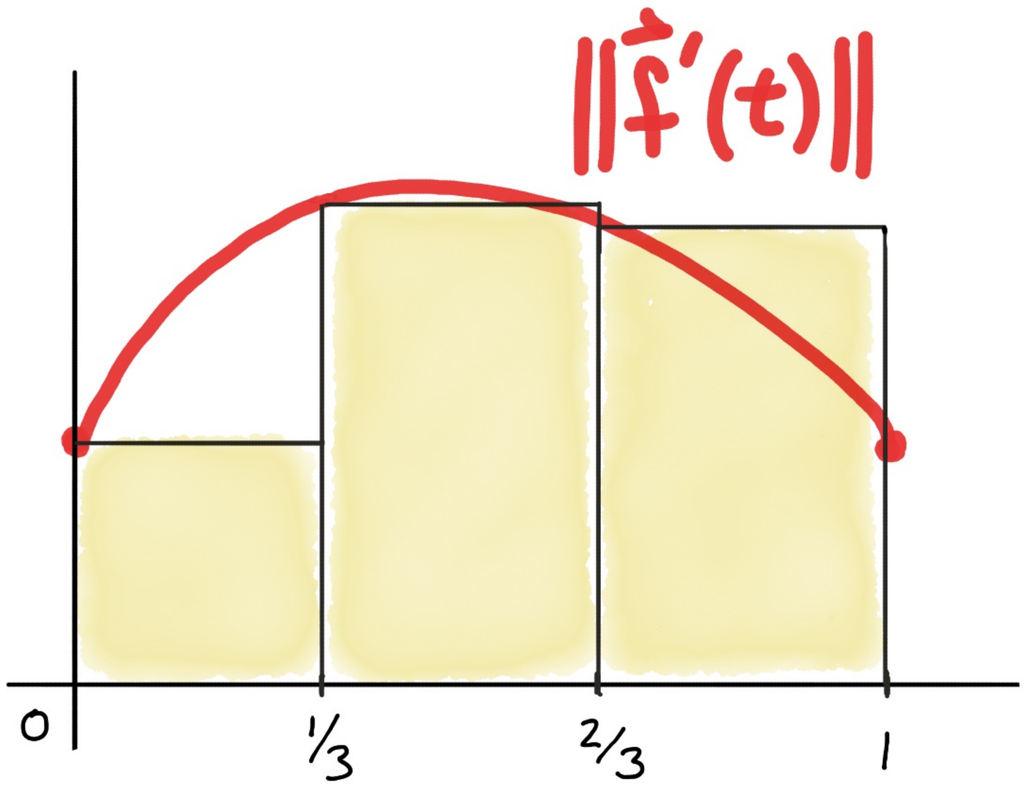

$\vec{f}(1)$
C
$\vec{f}(0)$

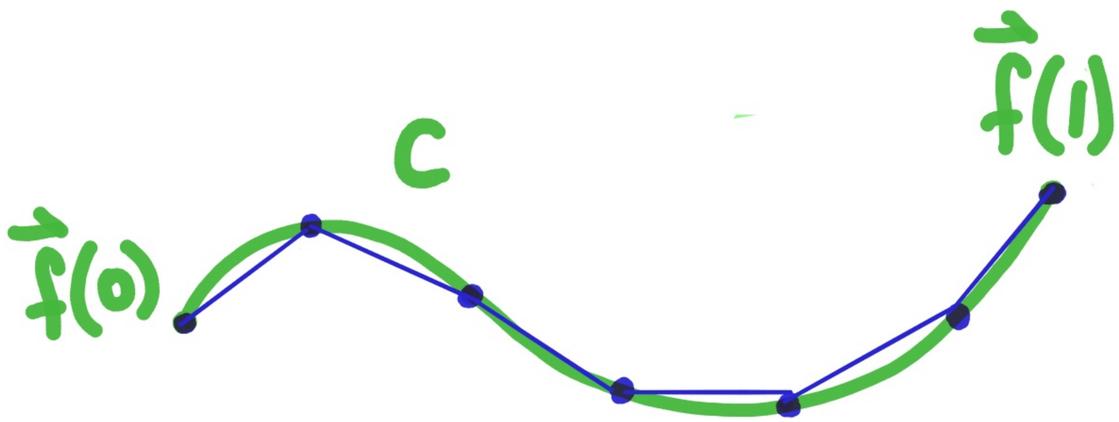

$\|\vec{f}'(t)\|$
0
1/3
2/3
1

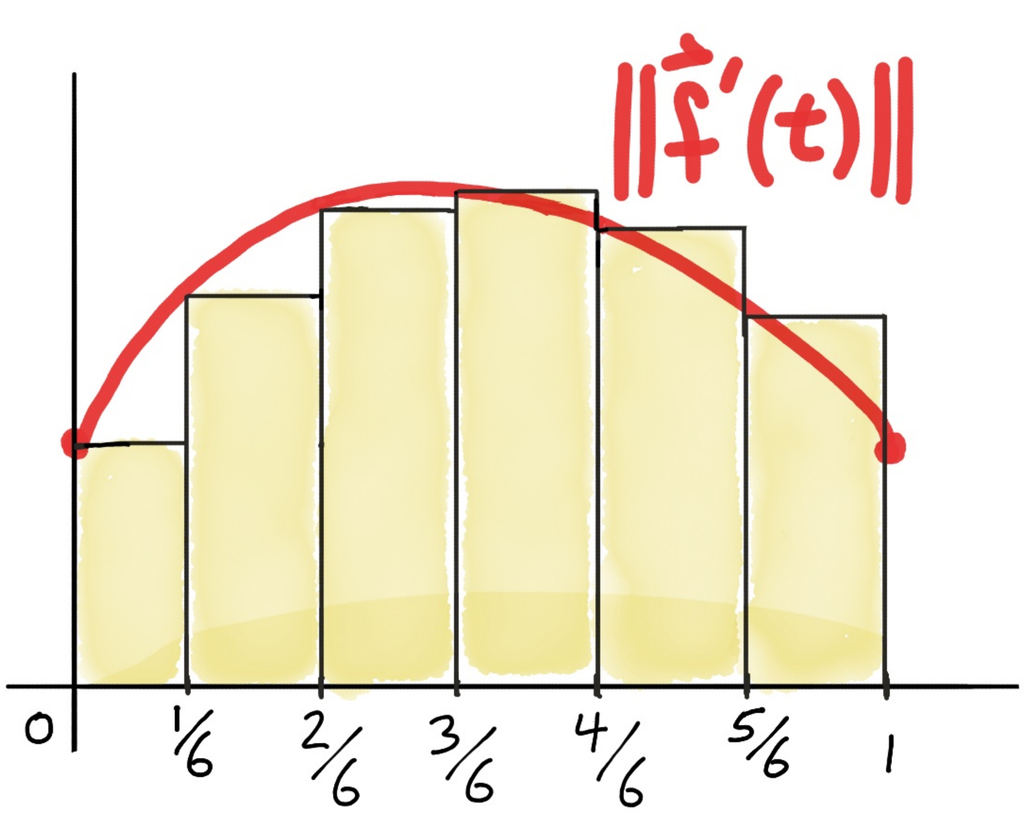

$\vec{f}(1)$
C
$\vec{f}(0)$


$\|\vec{f}'(t)\|$
0
1/6
2/6
3/6
4/6
5/6
1





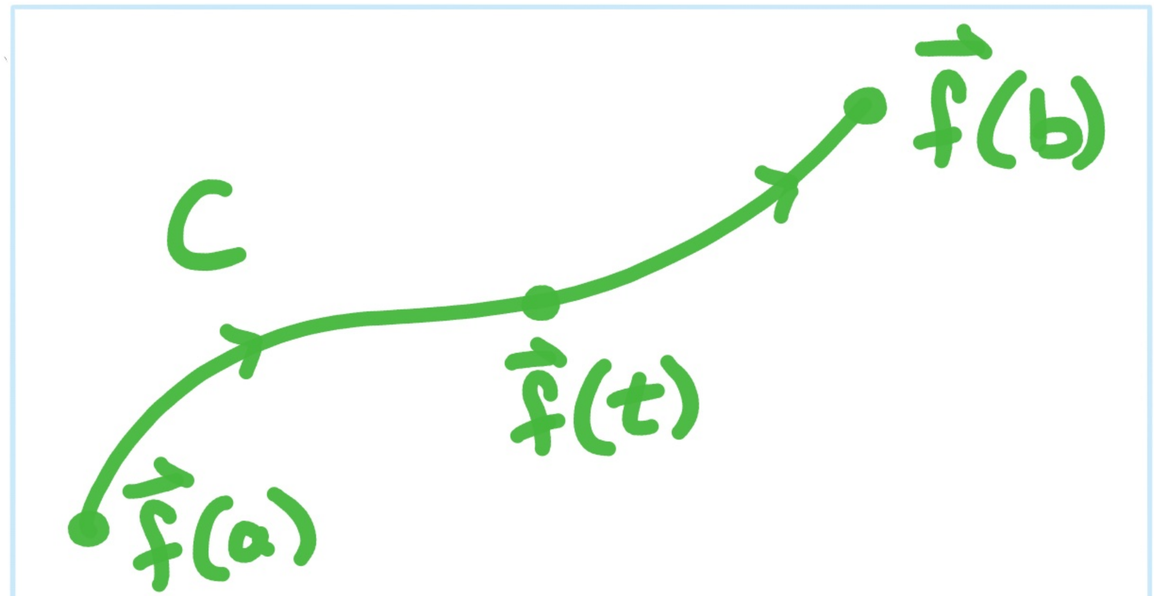




In the limit, arc length of C from $\vec{f}(0)$ to $\vec{f}(1)$ is $\int_0^1 \|\vec{f}'(t)\|\,dt$.

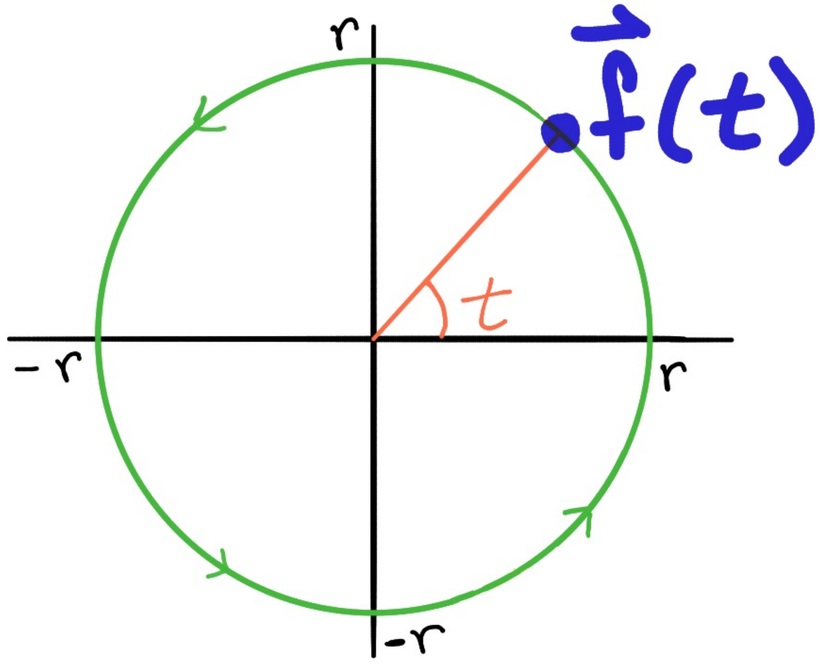
Definition: The arc length of a smooth curve C, parametrized by $\vec{f}$, from $\vec{f}(a)$ to $\vec{f}(b)$ is

$$\int_a^b \|\vec{f}\,'(t)\|\,dt$$



Arc length (circumference) for circle of radius r

$$\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^2, \quad \vec{f}(t) = \langle r\cos t, r\sin t \rangle, \; 0 \le t < 2\pi$$



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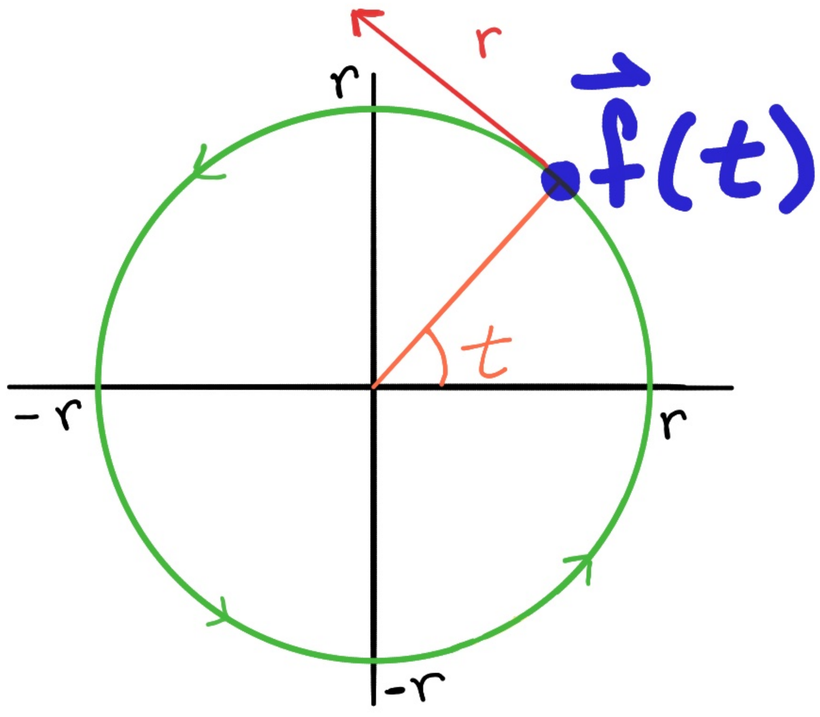


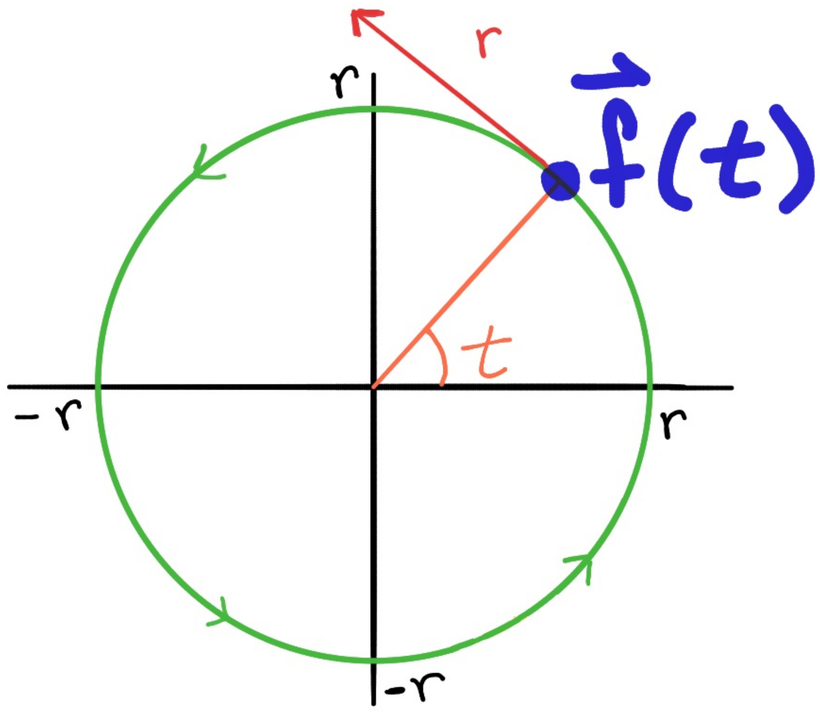
$$\vec{f}'(t) = \langle -r\sin t, r\cos t \rangle$$

$$\|\vec{f}'(t)\| = \sqrt{r^2\sin^2 t + r^2\cos^2 t}$$

$$= \sqrt{r^2 \cdot 1} = r$$

Arc length (circumference) for circle of radius r

$$\vec{f}: \mathbb{R} \longrightarrow \mathbb{R}^2, \quad \vec{f}(t) = \langle r\cos t, r\sin t \rangle, \; 0 \le t < 2\pi$$



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$$\|\vec{f}'(t)\| = \sqrt{r^2\sin^2 t + r^2\cos^2 t}$$

$$= \sqrt{r^2 \cdot 1} = r$$

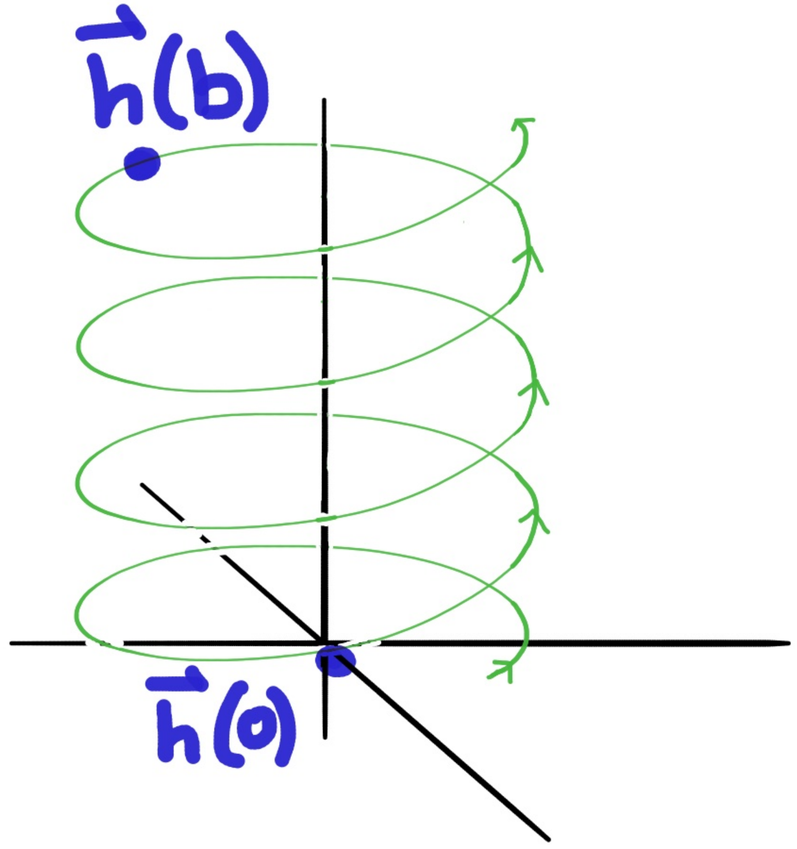
Arc length: $\displaystyle\int_0^{2\pi} \|\vec{f}'(t)\|\,dt = \int_0^{2\pi} r\,dt = 2\pi r$

Arc length for helix.

$\vec{h} : \mathbb{R} \longrightarrow \mathbb{R}^3$, $\vec{h}(t) = \langle \cos t, \sin t, t \rangle$

Arclength from $\vec{h}(0)$ to $\vec{h}(b)$ is

$$\int_0^b \|\vec{h}\,'(t)\|\, dt =$$



Arc length for helix.

$$\vec{h} : \mathbb{R} \longrightarrow \mathbb{R}^3 \quad , \quad \vec{h}(t) = \langle \cos t, \sin t, t \rangle$$

Arclength from $\vec{h}(0)$ to $\vec{h}(b)$ is

$$\int_0^b \| \vec{h}\,'(t) \| \, dt = \int_0^b \sqrt{2} \, dt = \sqrt{2}\, b$$