Thirty-four

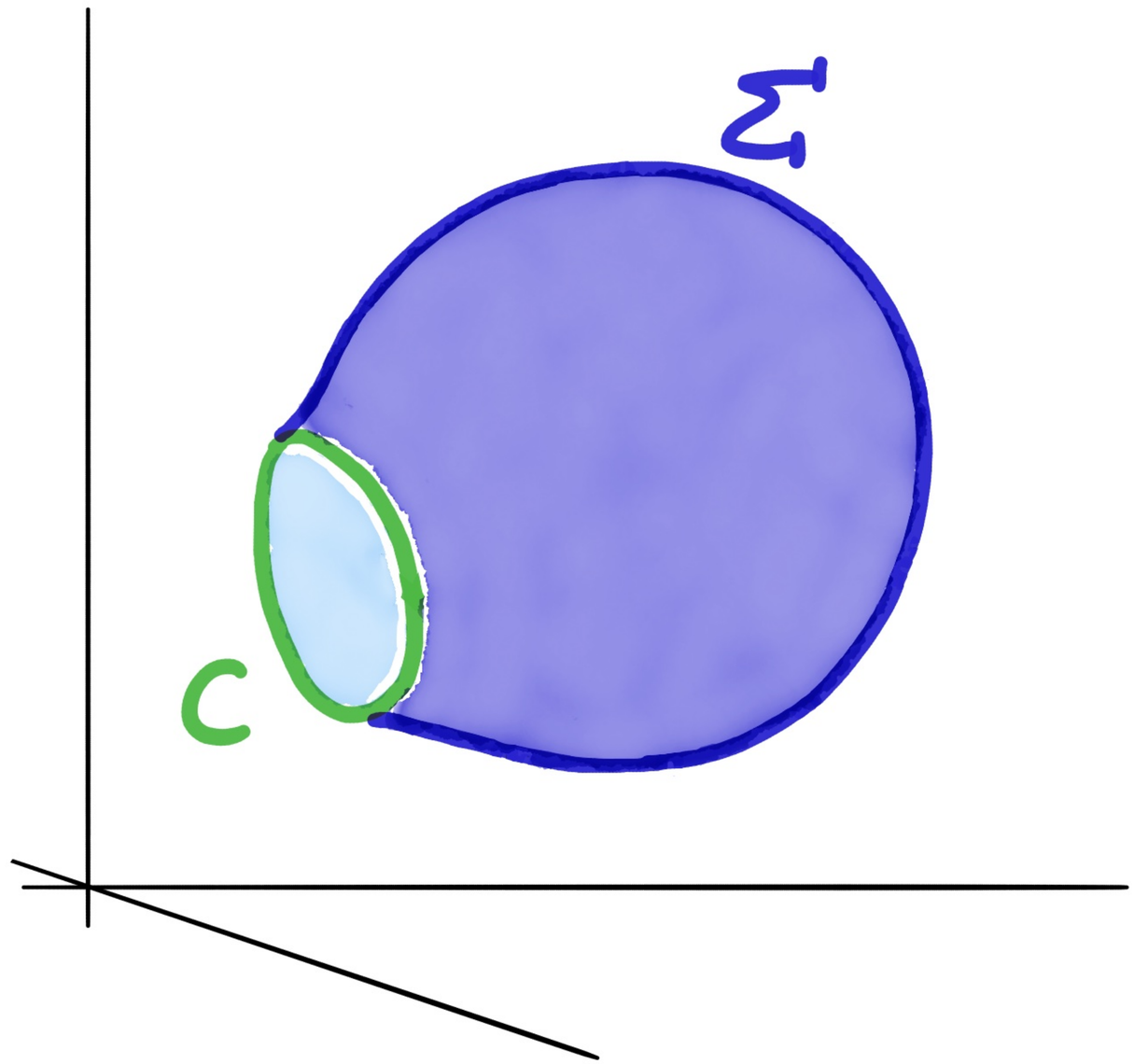
Ⅰ Stokes's Theorem

Ⅱ Last look at Central Theorem

Ⅲ Stokes's Theorem in context

Ⓘ Stokes's Theorem

Σ a closed, bounded, two-sided surface in $\mathbb{R}^3$ with simple closed boundary curve C.



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For every $(x,y,z) \in \Sigma$, let $\vec{n}(x,y,z) \in \mathbb{R}^3$ be a continuously varying unit vector that's orthoganal to Σ.

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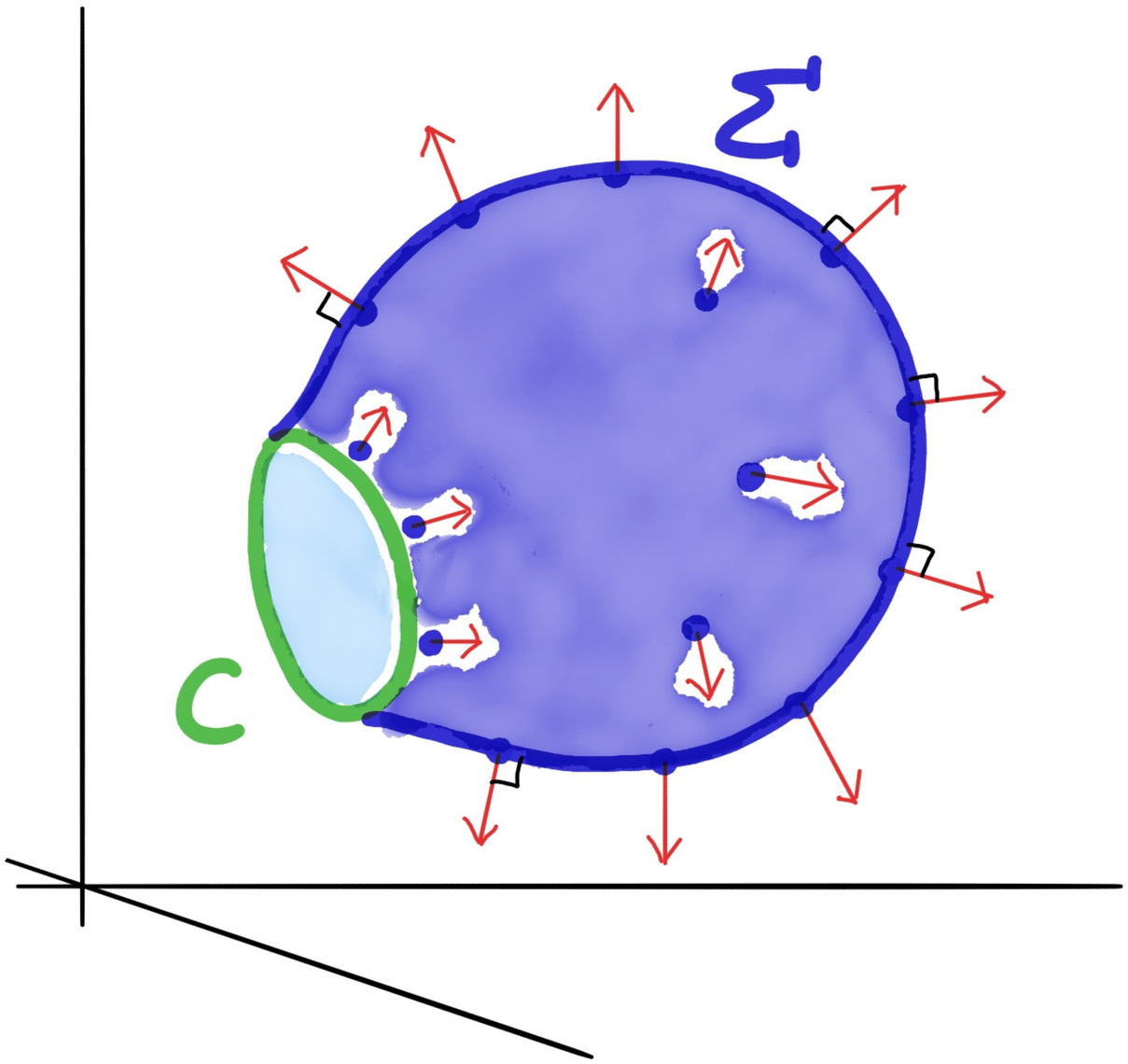

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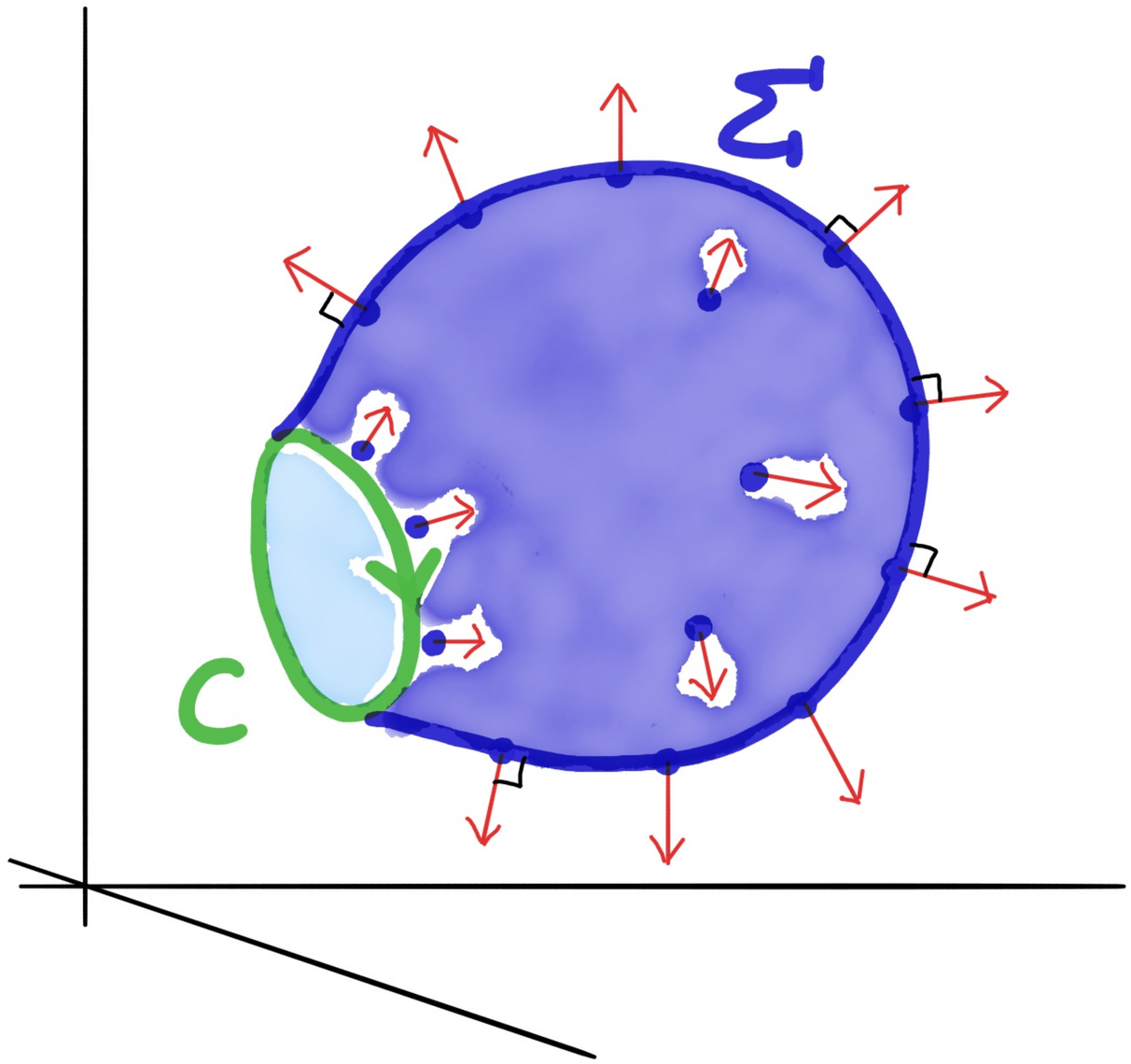

Orient C so that if your head is at $\vec{n}$, and Σ is on your left, then C goes forward.

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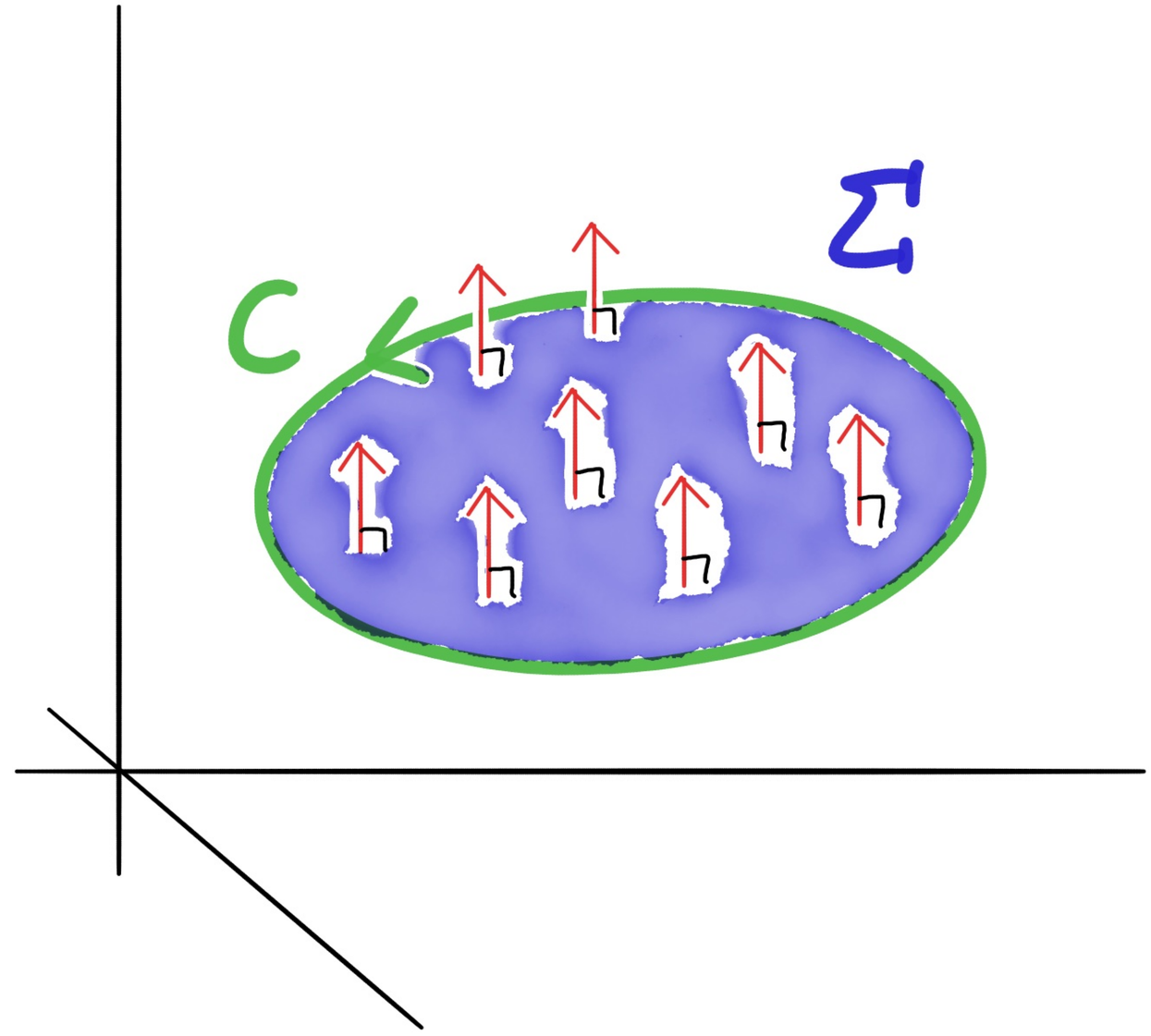

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Stokes's Theorem:

Let $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F} = (F_1, F_2, F_3)$ be a vector field.

If Σ, C, and $\vec{n}$ are as on the previous page, then

$$\oint_C F_1\,dx + F_2\,dy + F_3\,dz = \iint_{\Sigma} (\operatorname{curl} \vec{F}) \cdot \vec{n}\,dS$$

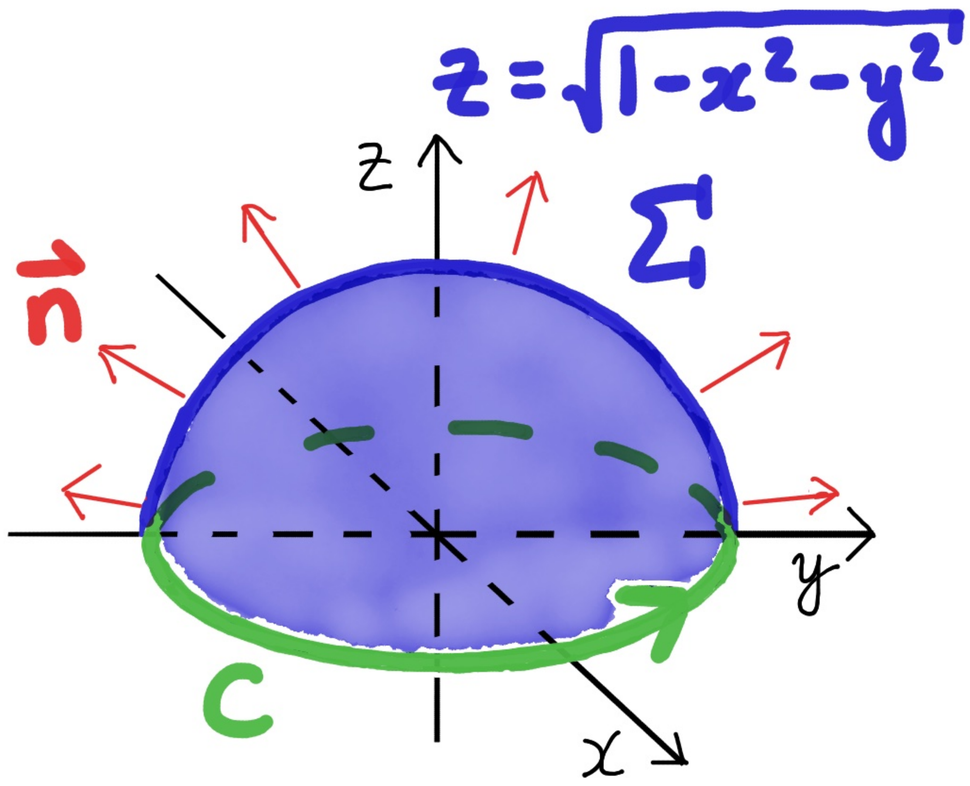
Example:

- Σ : $x^2+y^2+z^2=1$, $z\geq 0$. $\quad z=\sqrt{1-x^2-y^2}=f(x,y)$
- $\vec{n}$: upper normal , $\vec{n}(x,y,z)=(x,y,z)$.
- C : $x=\cos t$, $y=\sin t$, $z=0$, $0\leq t\leq 2\pi$.
- $\vec{F}(x,y,z)=(-y,x,xyz)$

Find $\iint_{\Sigma}(\operatorname{curl}\vec{F})\cdot\vec{n}\,dS$

① By hand.

② Using Stokes's Theorem.



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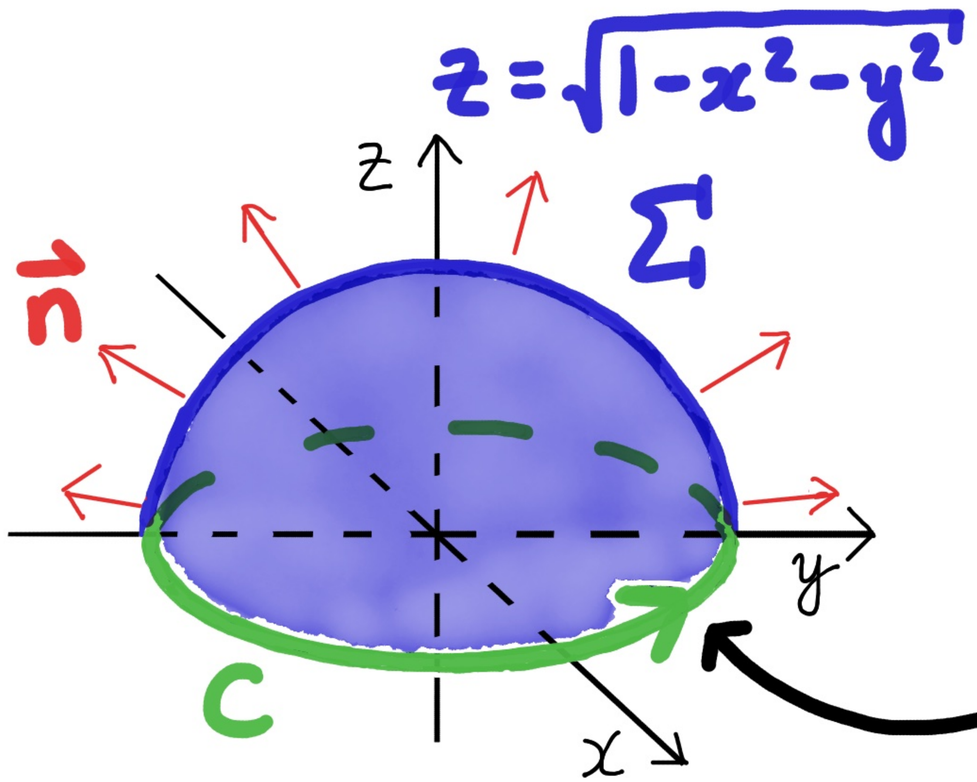

$R=\{(x,y): x^2+y^2\leq 1\}$

$=\{(r,\theta): 0\leq r\leq 1,\ 0\leq\theta\leq 2\pi\}$

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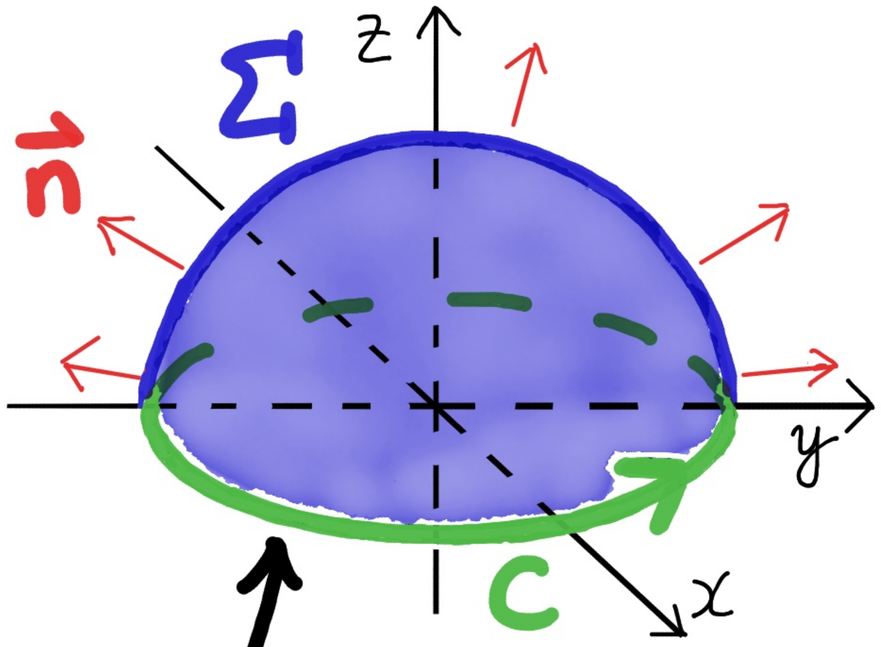

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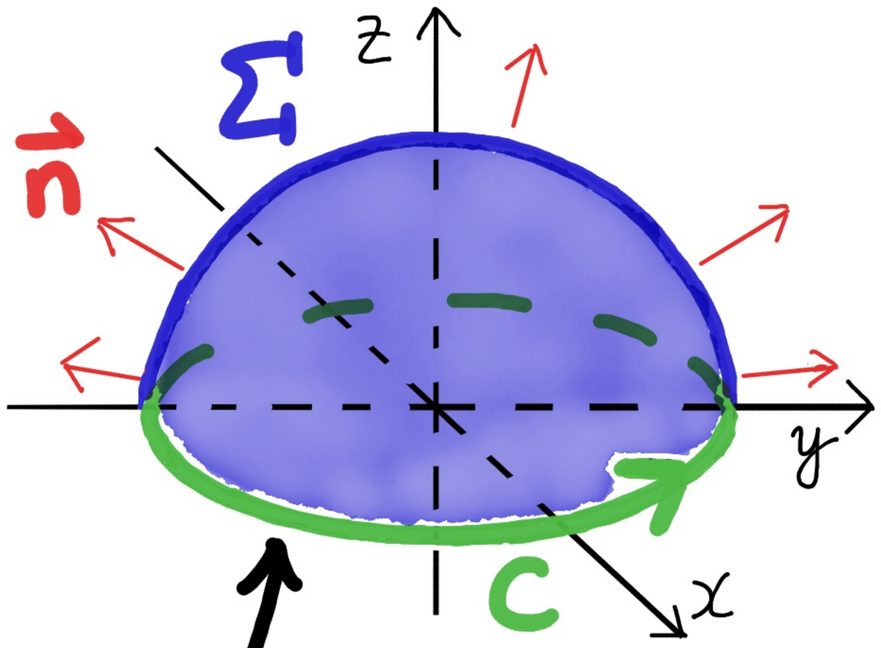

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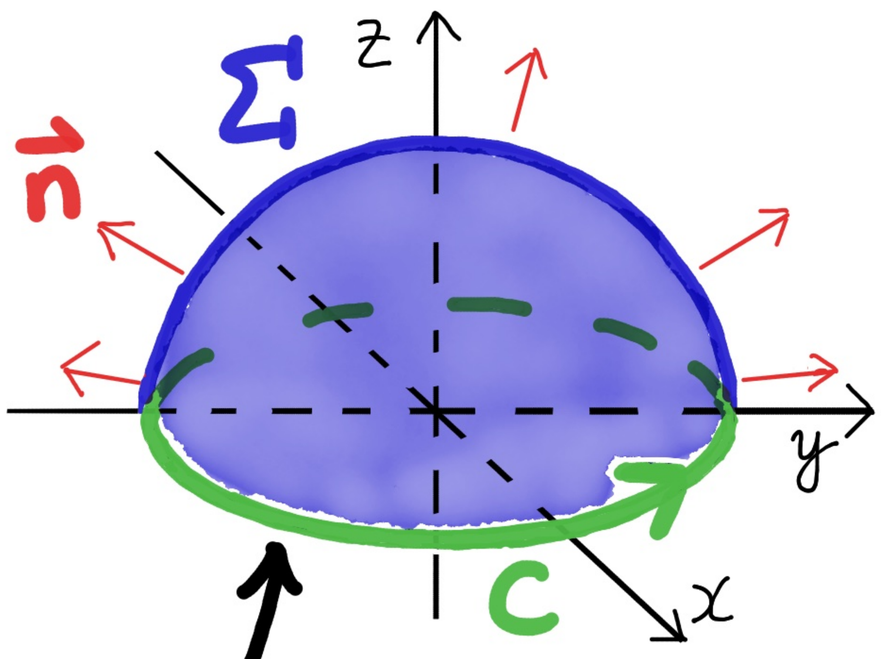

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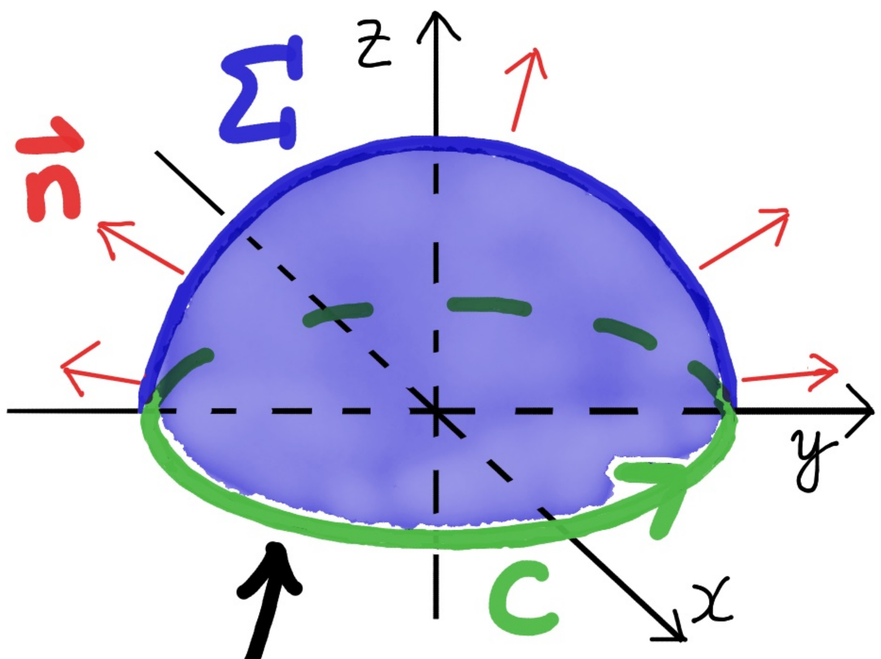

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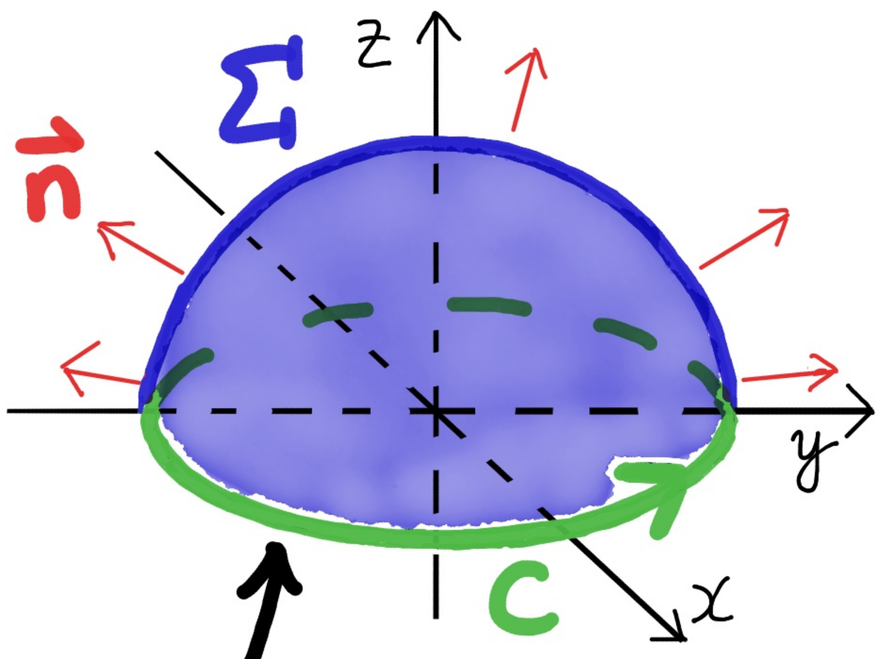

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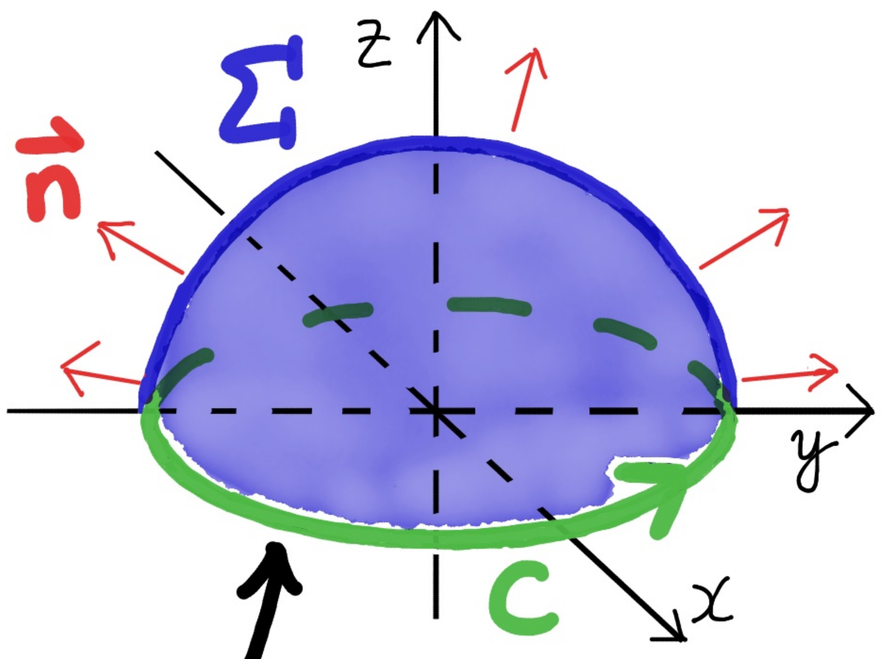

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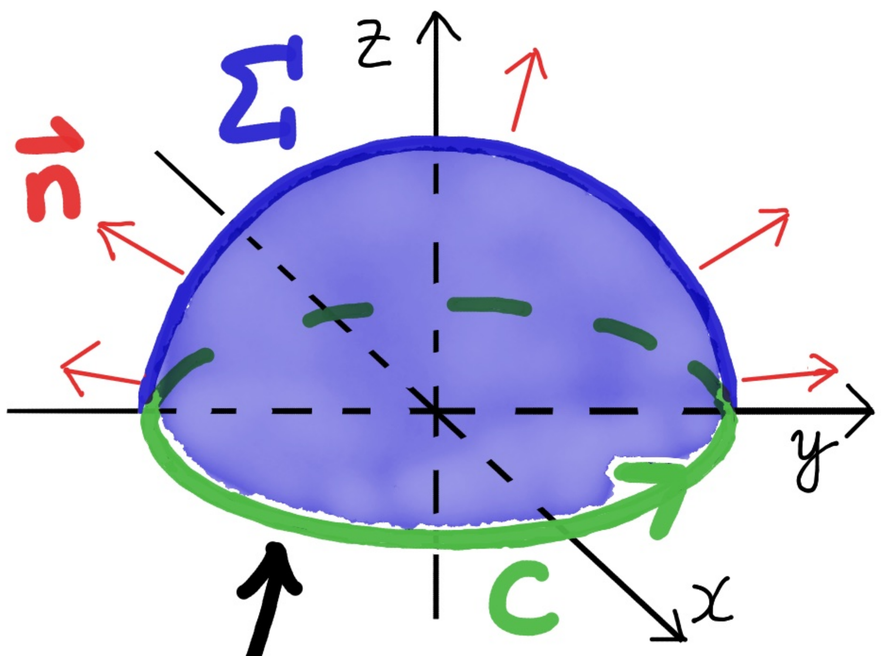

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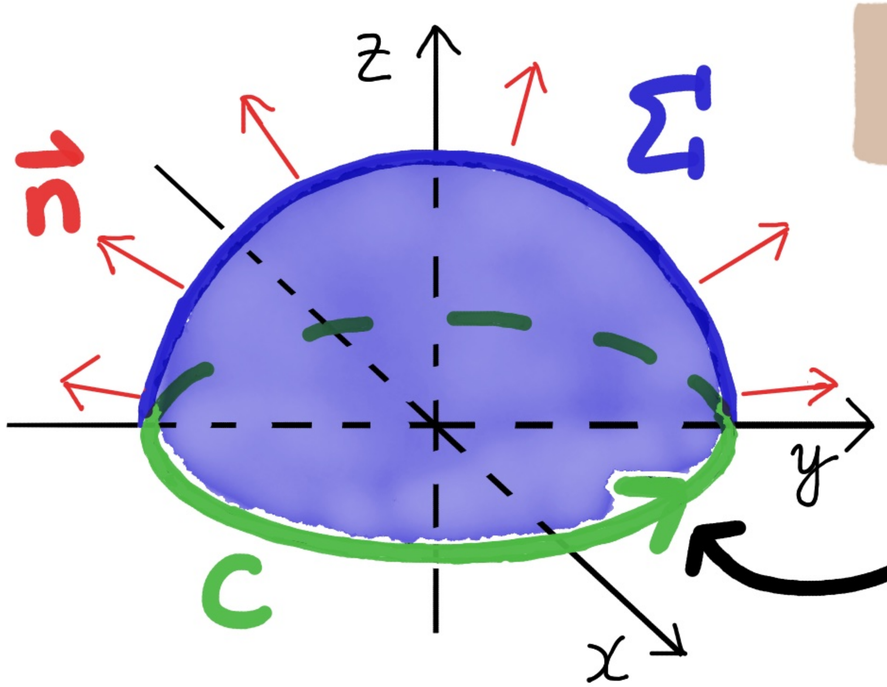

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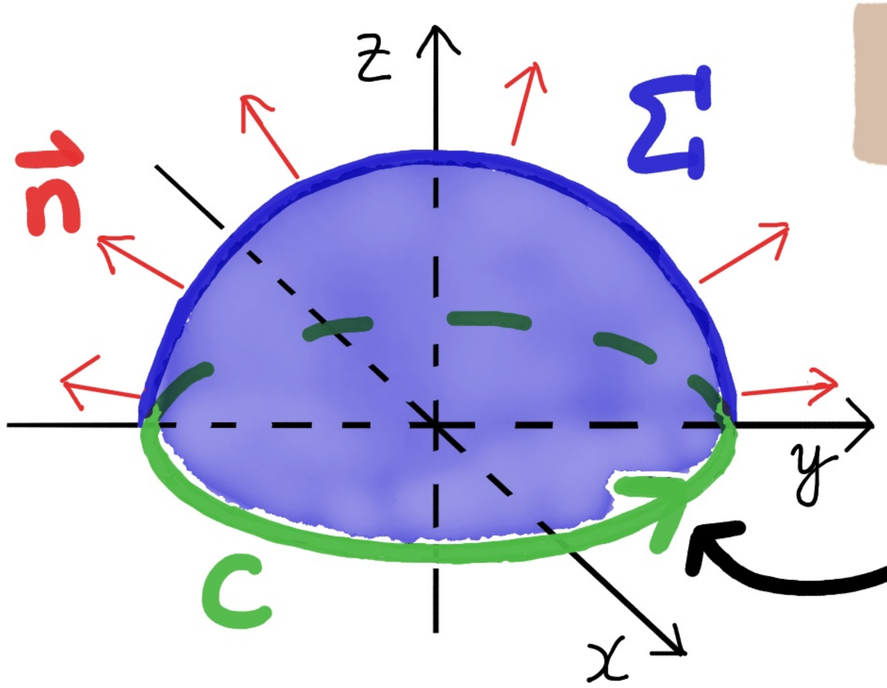

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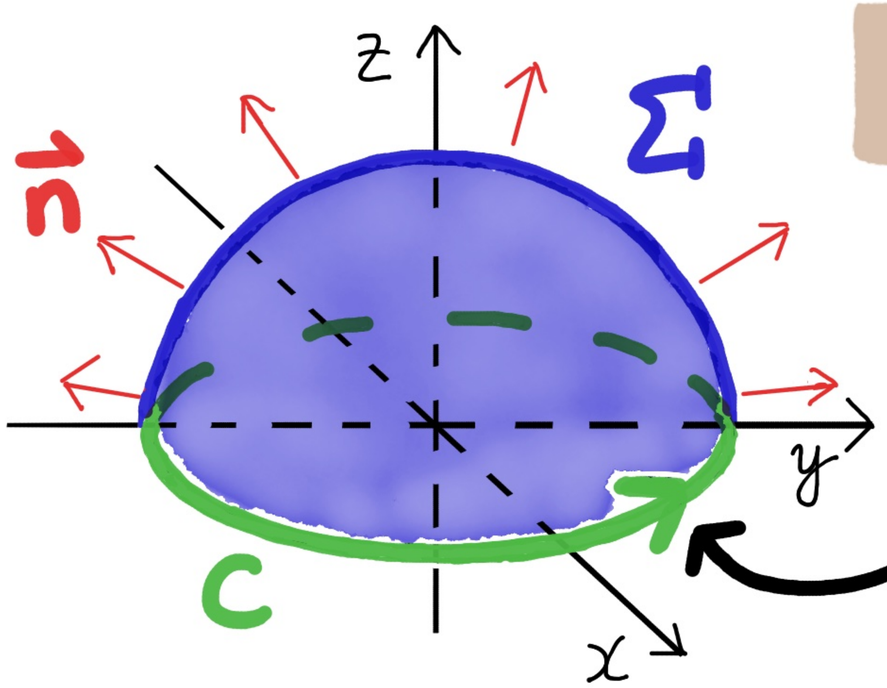

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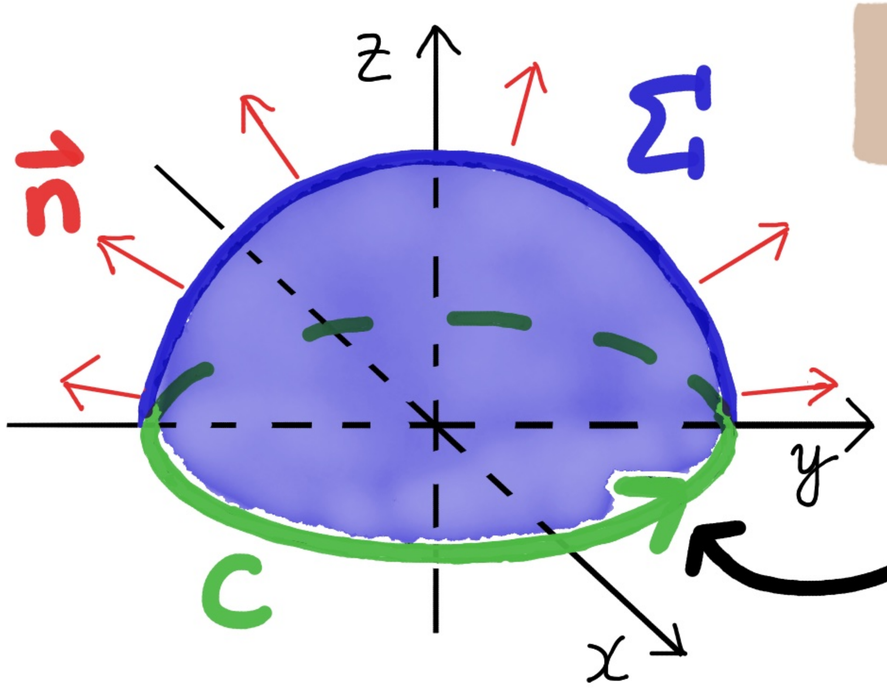

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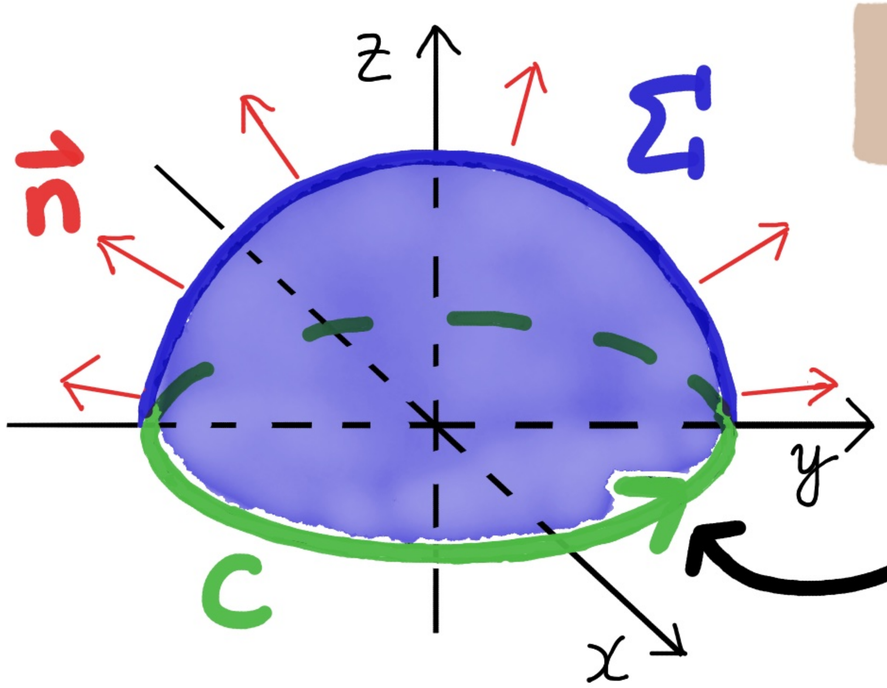

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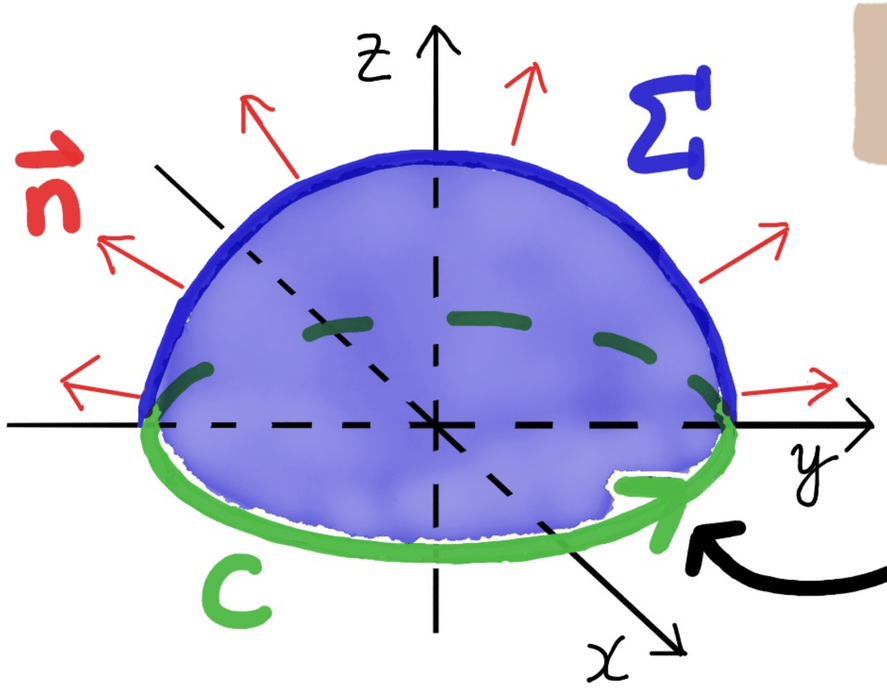

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$$R = \{(x,y) : x^2 + y^2 \le 1\}$$

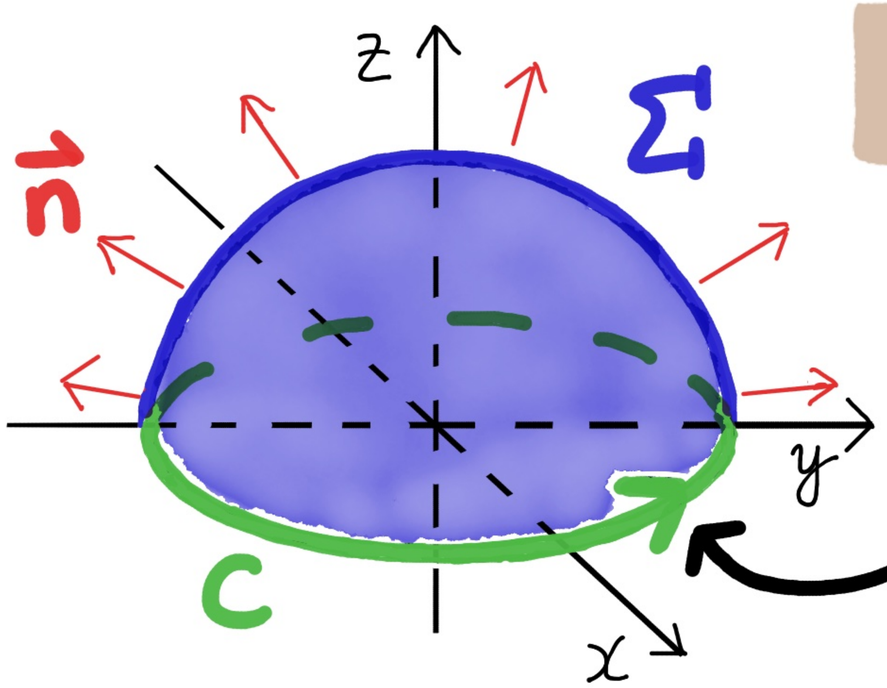
- $\vec{F}(x,y,z) = (-y, x, xyz)$

$$\iint_{\Sigma} (\text{curl}\,\vec{F}) \cdot \vec{n}\, dS = \oint_C F_1\,dx + F_2\,dy + F_3\,dz$$

$$= \oint_C -y\,dx + x\,dy + xyz\,dz$$

$$= \oint_C -y\,dx + x\,dy$$

$$= 2\,\text{Area}(R)$$



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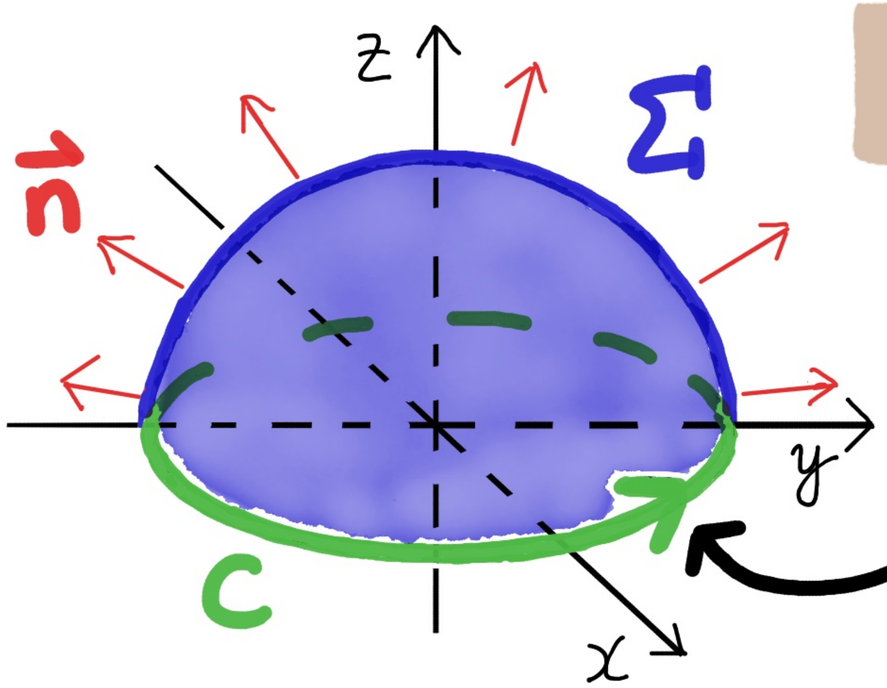

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END

Ⅱ Last look at Central Theorem

Theorem:

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$, $\vec{F} = (F_1, \dots, F_n)$.

The following are equivalent:

① $\vec{F} = \nabla f$.

② $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ for $1 \le i, j \le n$.

③ $\oint_C \sum_{i=1}^{n} F_i \, dx_i = 0$ for simple closed curves C.

④ $\int_C \sum_{i=1}^{n} F_i \, dx_i$ is independent of path.

$$\vec{F} = \nabla f$$

✓ ① ✓

$$\int_C \sum_{i=1}^{n} F_i \, dx_i$$ ④ ind. of path

② $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$

✓ ③ ✓ (for n=2)

$$\oint_C \sum_{i=1}^{n} F_i \, dx_i = 0$$

proof: (② ⇒ ③) (n=3)

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Suppose $\frac{\partial F_3}{\partial y}=\frac{\partial F_2}{\partial z}$, $\frac{\partial F_1}{\partial z}=\frac{\partial F_3}{\partial x}$, and $\frac{\partial F_2}{\partial x}=\frac{\partial F_1}{\partial y}$.

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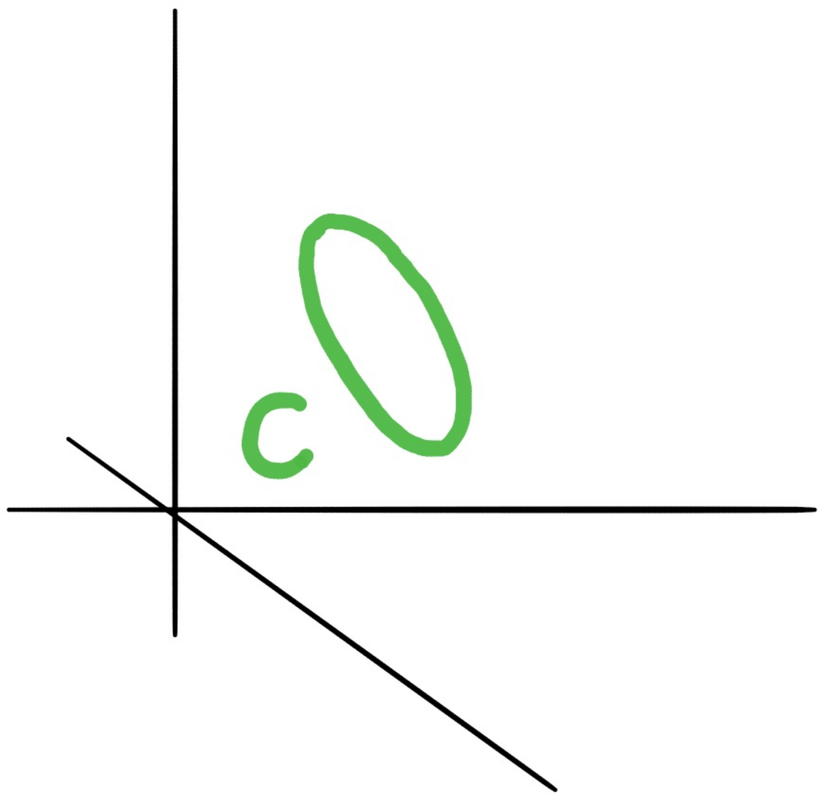
Let C be a simple closed curve in $\mathbb{R}^3$.



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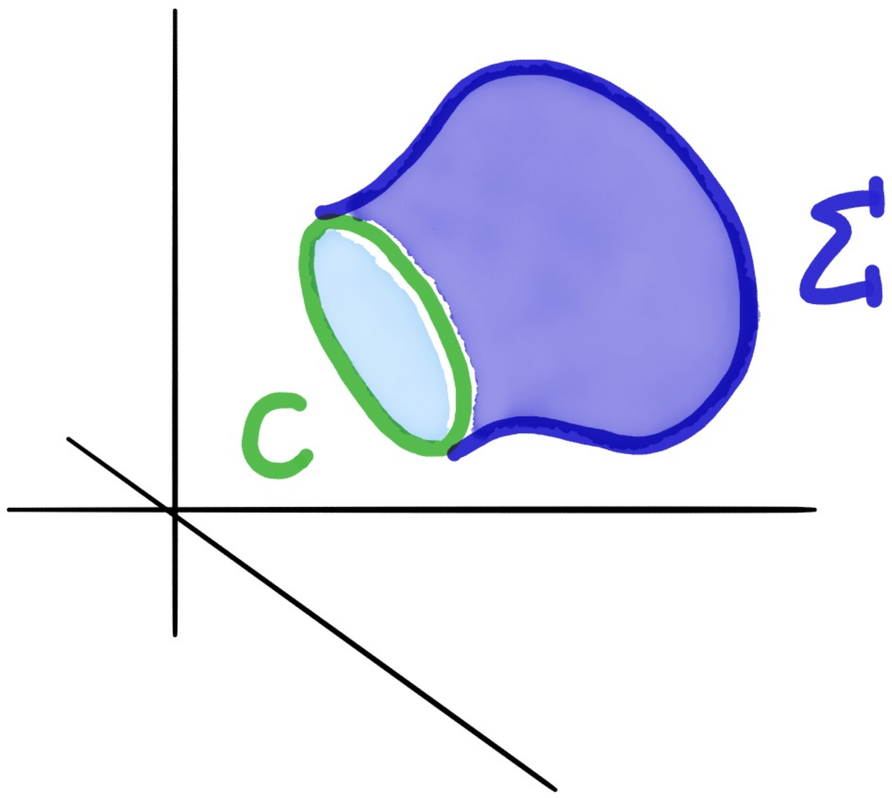
Let Σ be a surface in $\mathbb{R}^3$ with boundary C.



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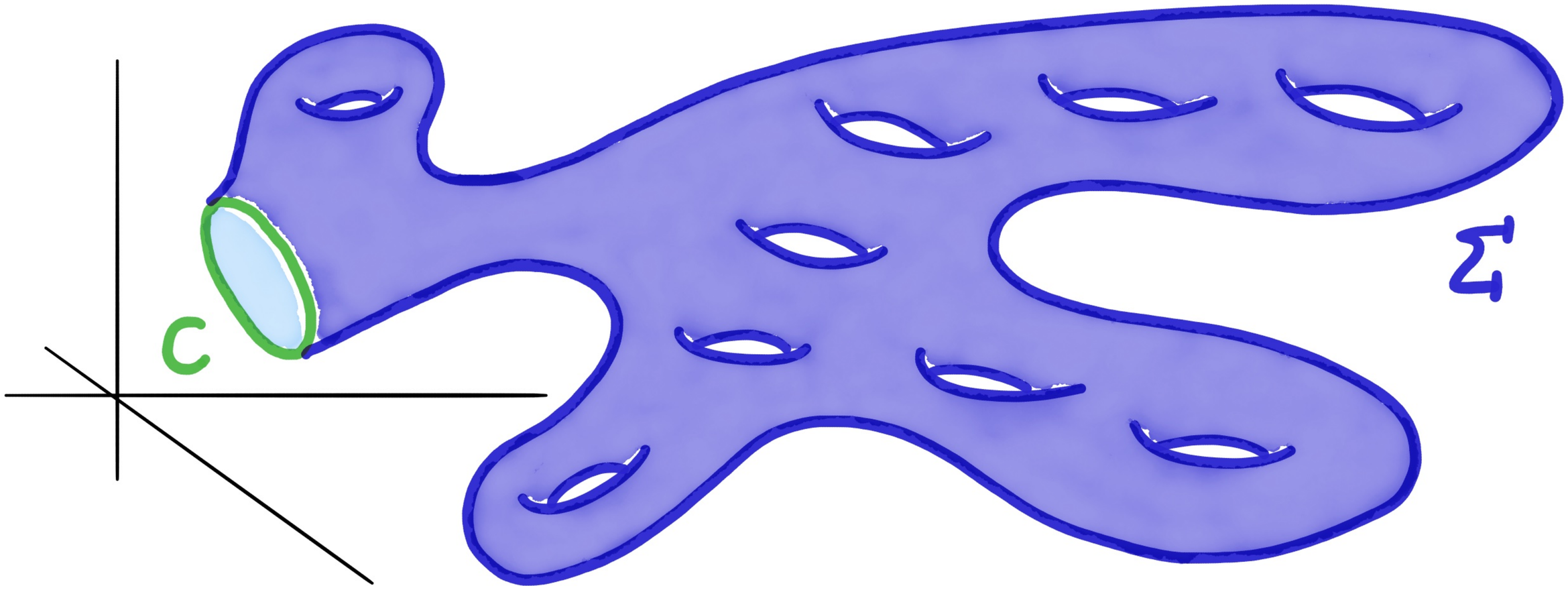

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Let C be a simple closed curve in $\mathbb{R}^3$.

Let Σ be a surface in $\mathbb{R}^3$ with boundary C.

Then,

$$\oint_C F_1\,dx + F_2\,dy + F_3\,dz =$$

proof: (② ⇒ ③) $(n=3)$

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$$\oint_C F_1\,dx + F_2\,dy + F_3\,dz = \iint_{\Sigma} (\operatorname{curl}\vec{F})\cdot\vec{n}\,dS$$

proof: (② ⇒ ③) $(n=3)$

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$$=\iint_\Sigma \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\cdot\vec{n}\,dS$$

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$$= \iint_\Sigma (0,0,0)\cdot\vec{n}\,dS$$

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Theorem:

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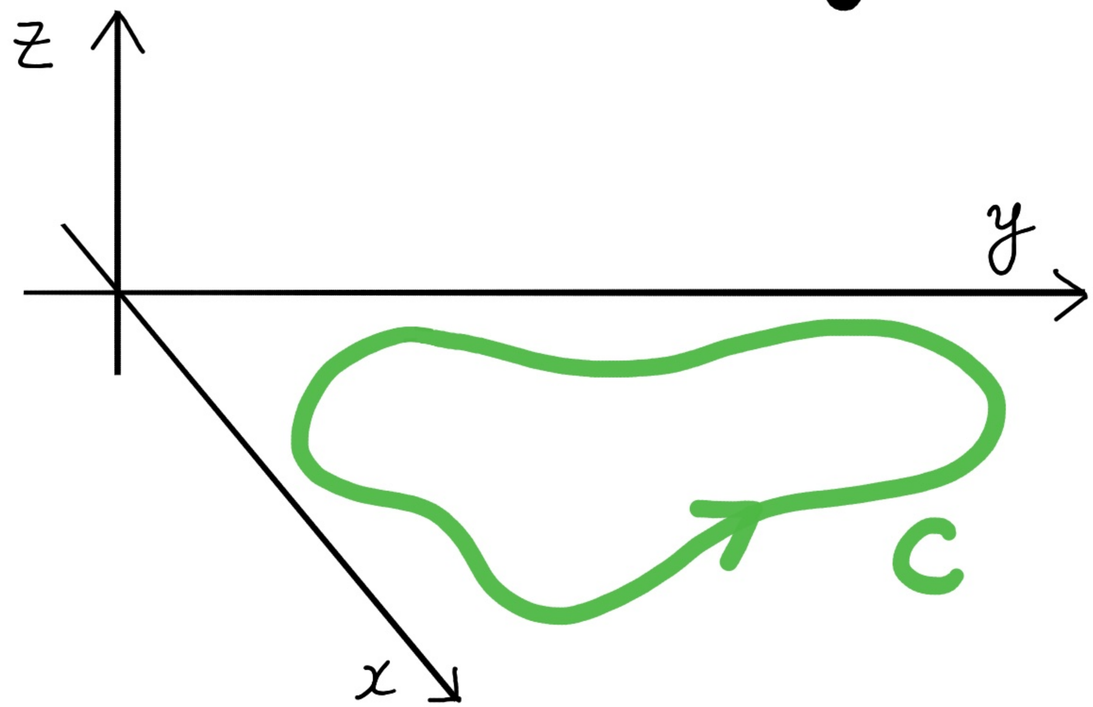
⑪ Stokes's Theorem in context

Green's from Stokes's

C, curve in xy-plane



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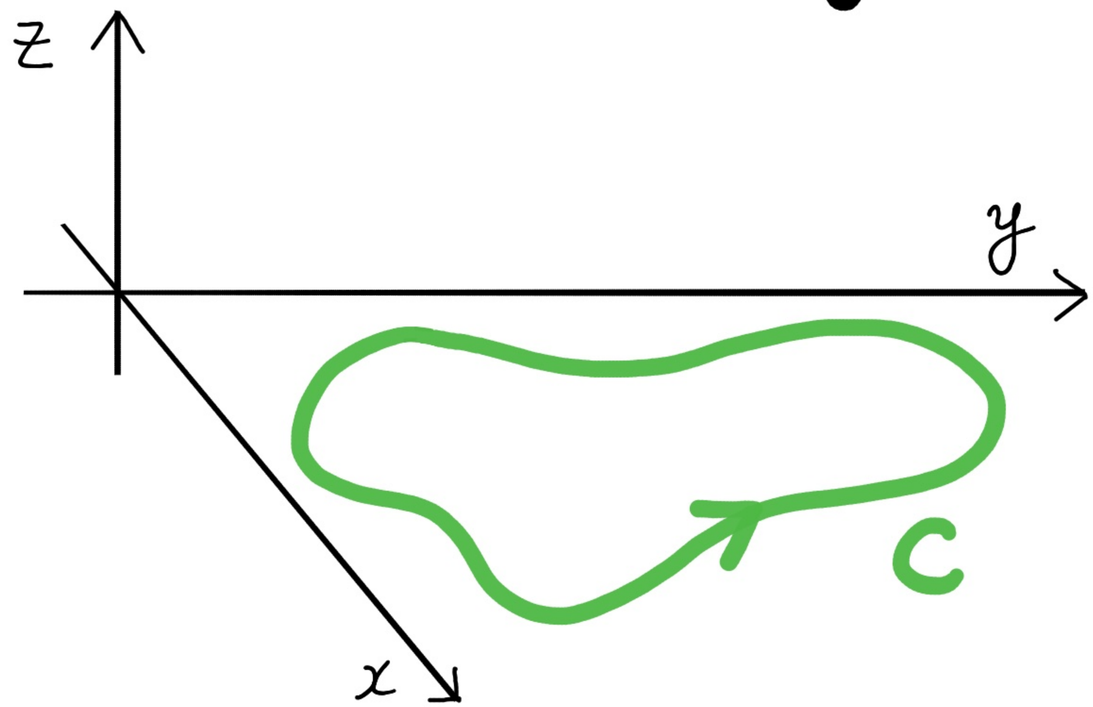
$$\oint_C F_1\,dx + F_2\,dy =$$

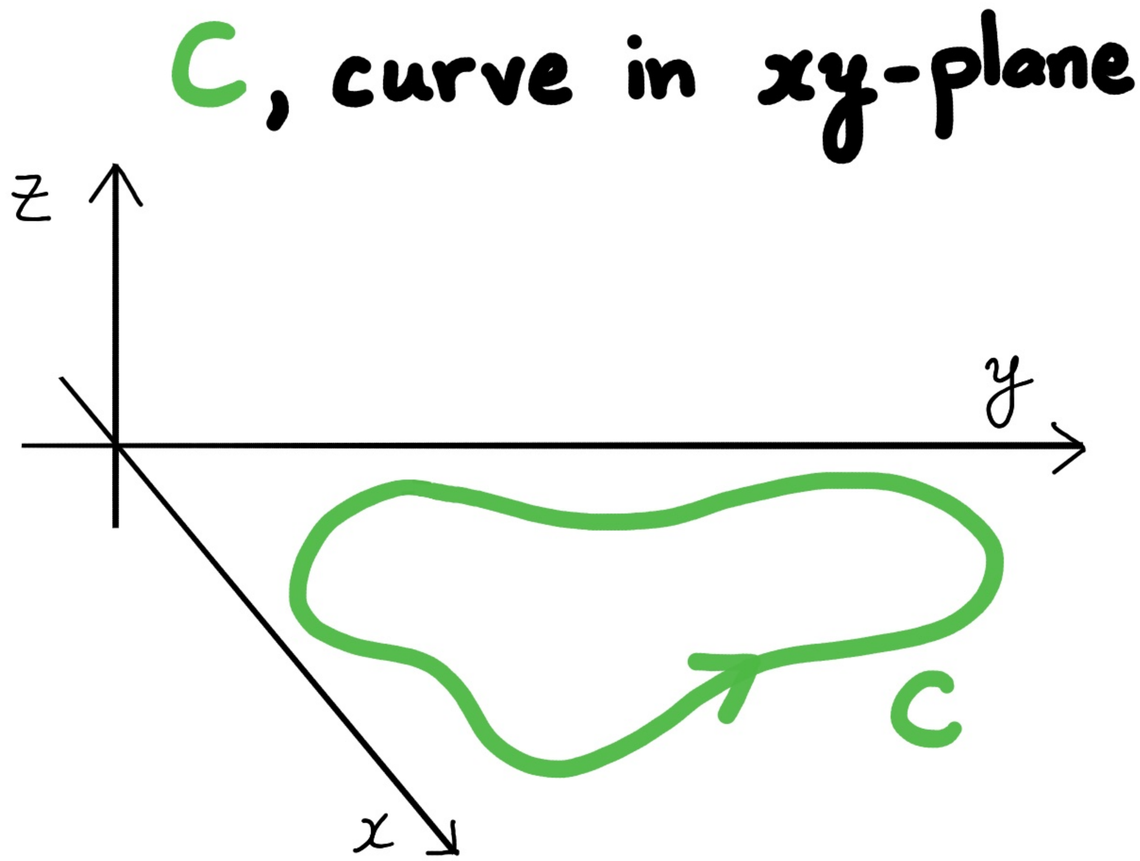
C, curve in xy-plane



Green's from Stokes's

$$\oint_C F_1\,dx + F_2\,dy =$$

$$= \oint_C F_1\,dx + F_2\,dy + 0\cdot dz$$



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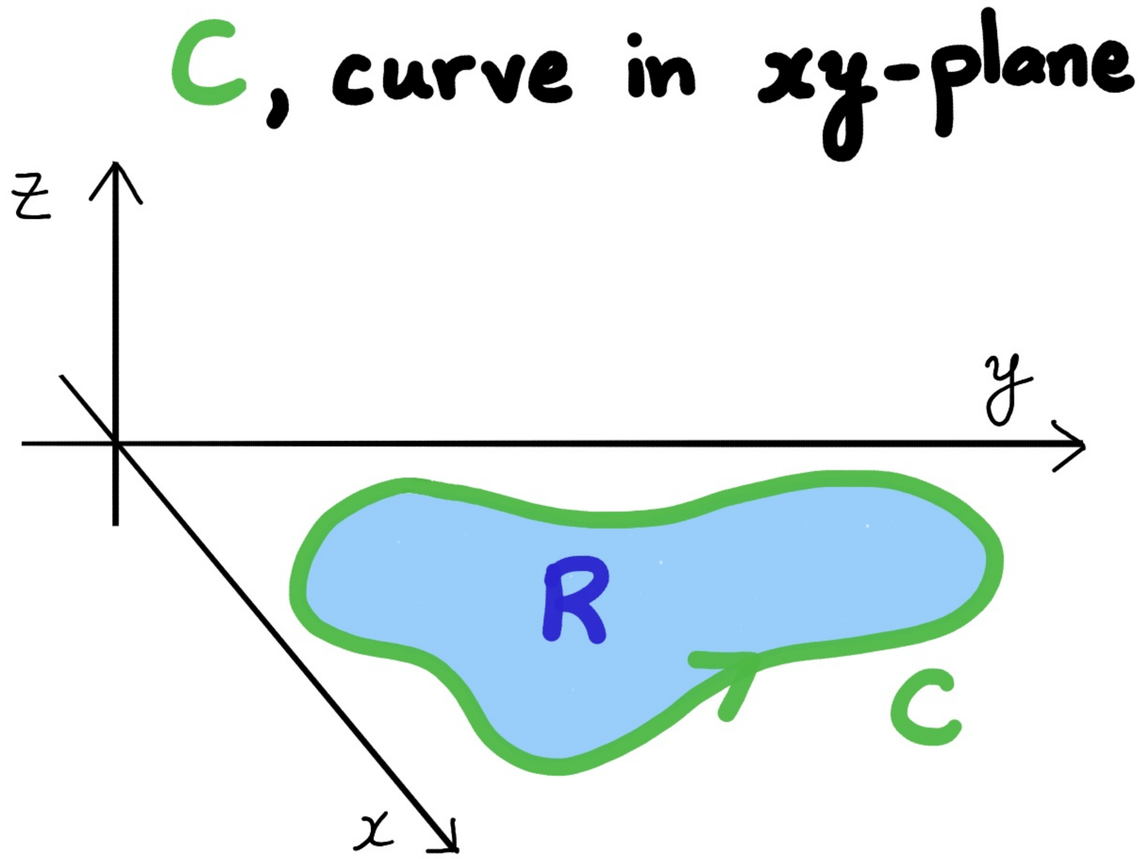

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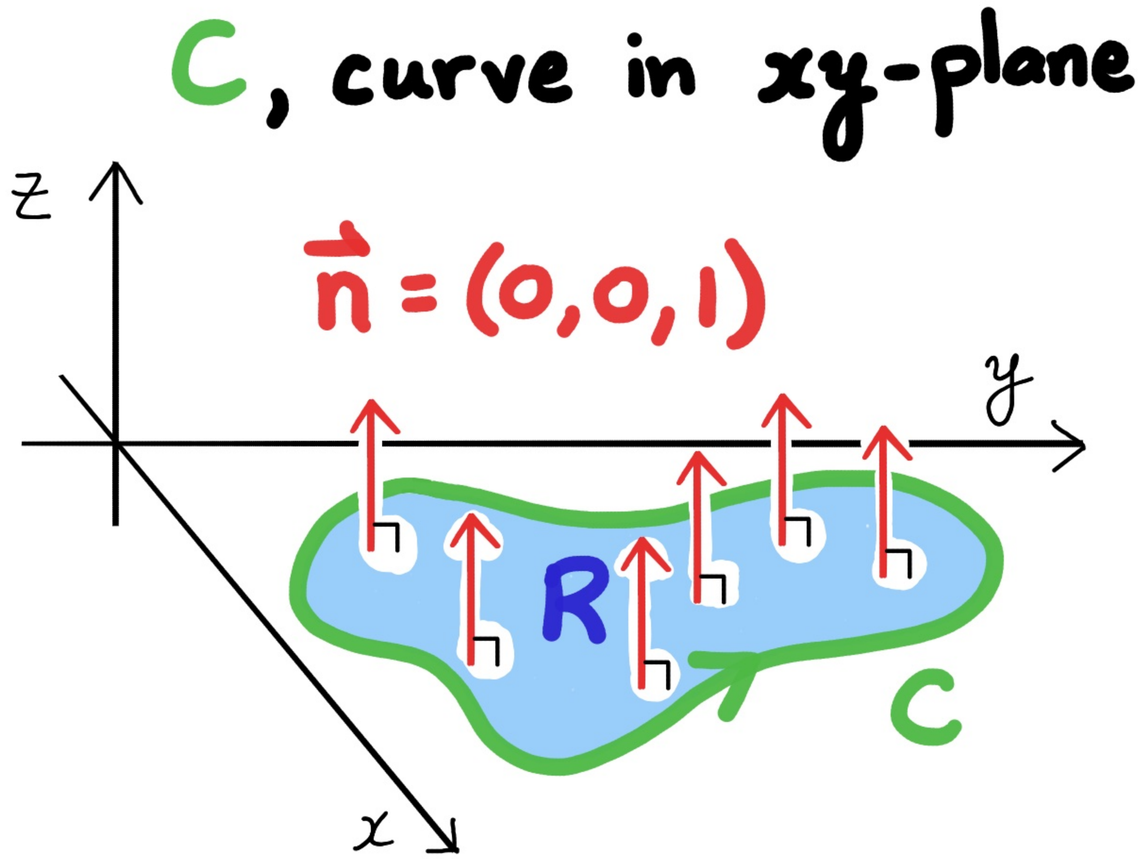

Green's from Stokes's

$$\oint_C F_1\,dx + F_2\,dy =$$

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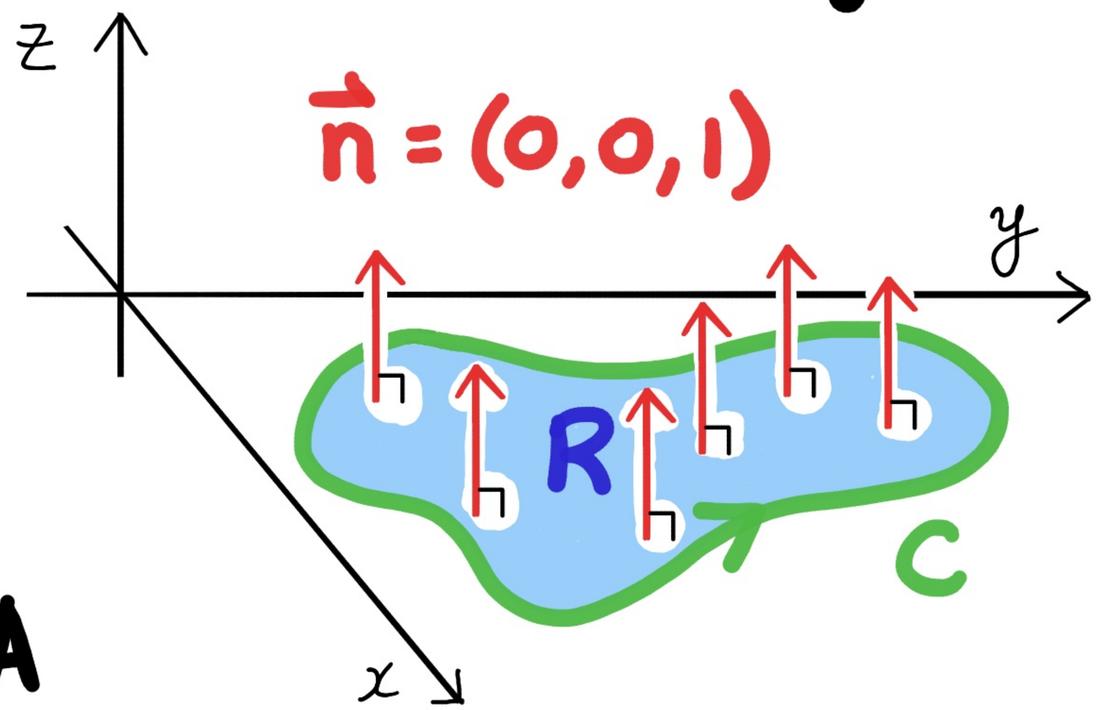
$$= \iint_R \operatorname{curl}(F_1, F_2, 0)\cdot \vec{n}\; dA$$

C, curve in xy-plane



Green's from Stokes's

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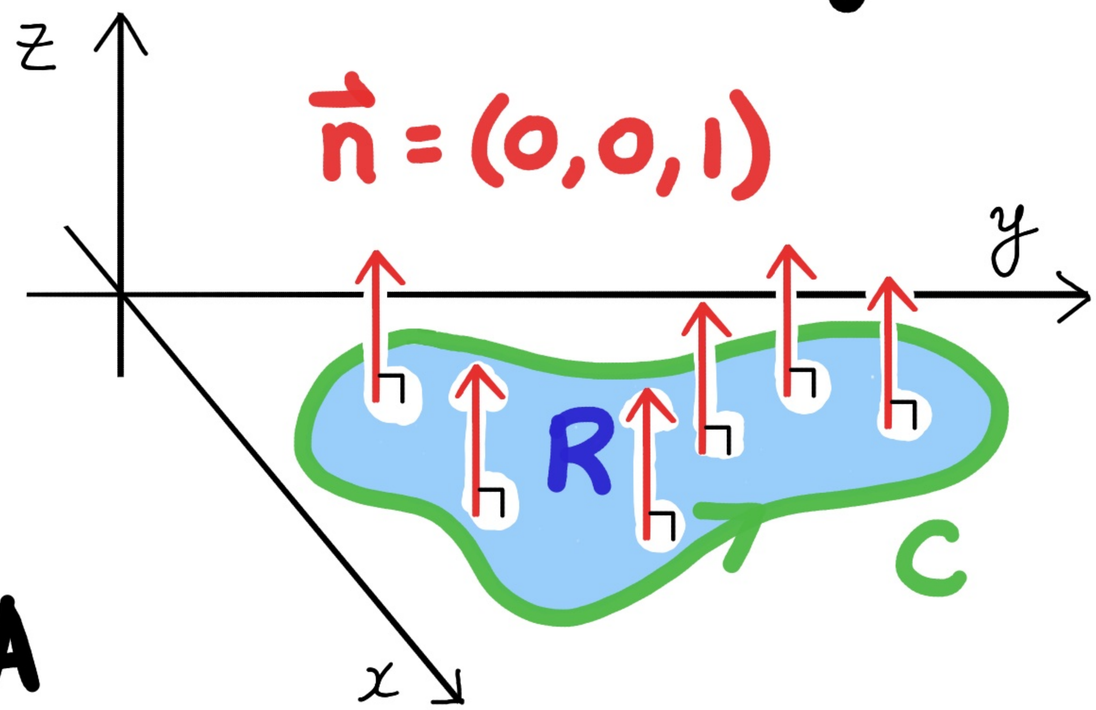

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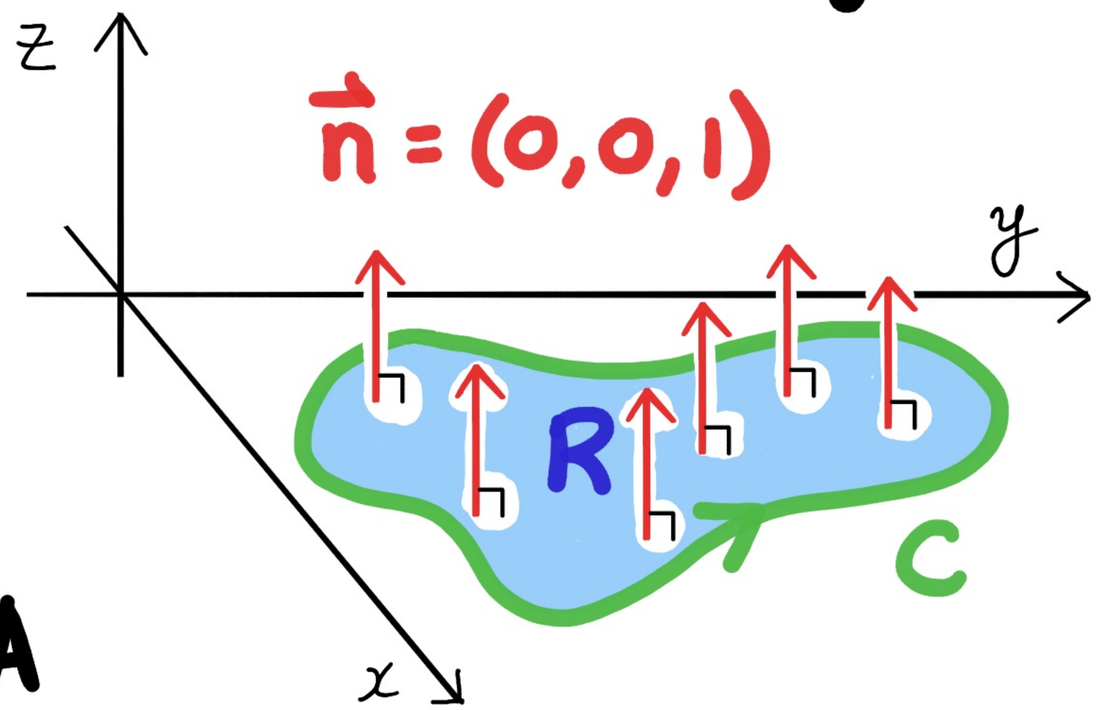
Green's from Stokes's

C, curve in xy-plane



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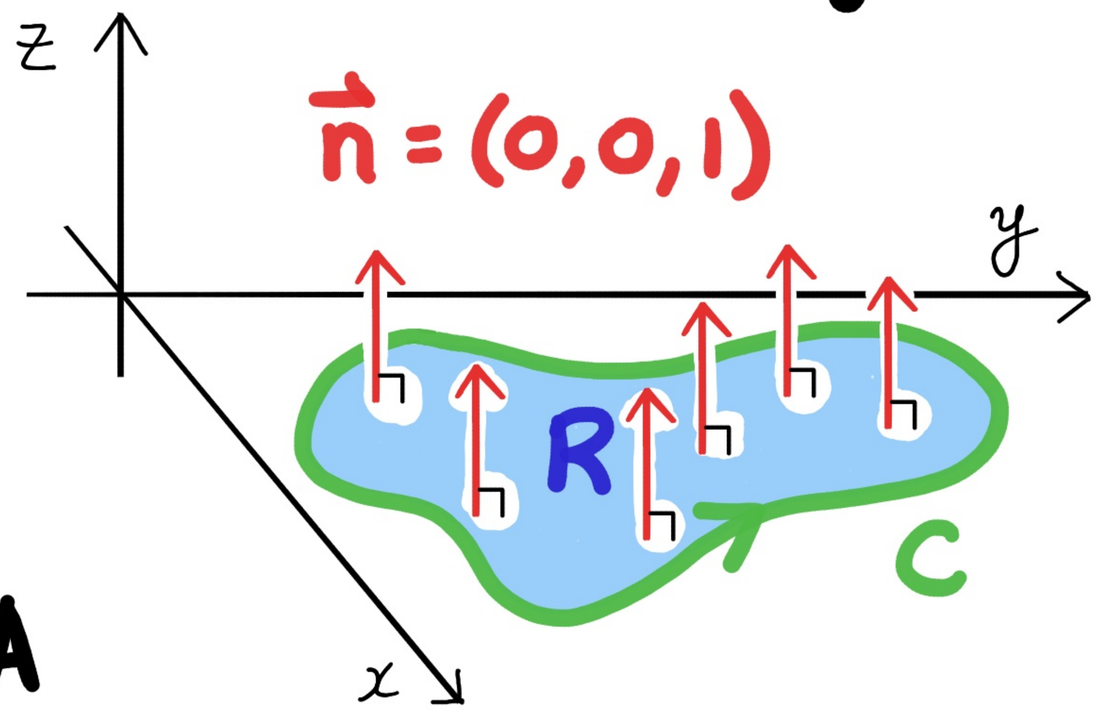
$$= \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\; dA\,.$$

Green's from Stokes's

C, curve in xy-plane



$$\oint_C F_1\,dx + F_2\,dy =$$

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Stokes's

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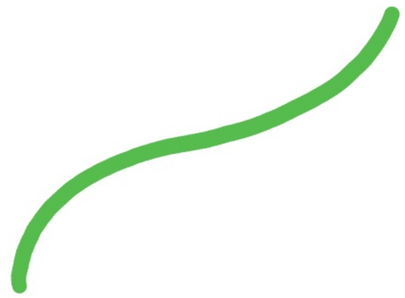
$$= \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\; dA\,.$$

0-dimensional

point

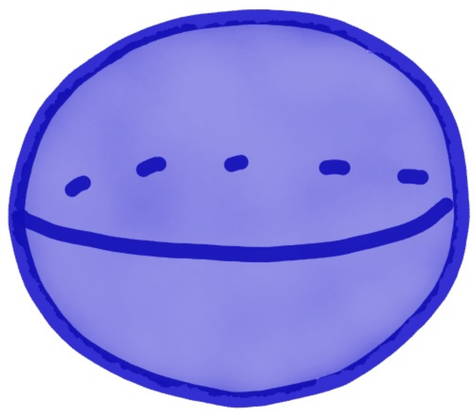
1-dimensional



curve

"road"

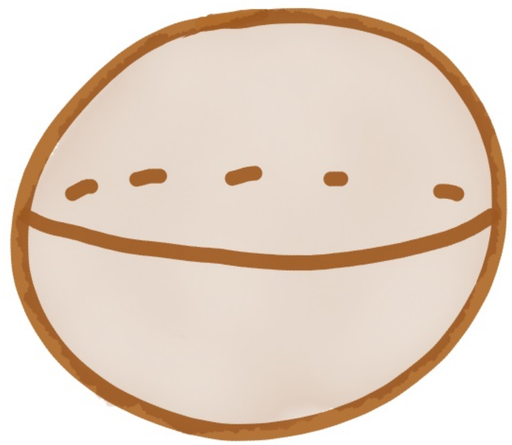
2-dimensional



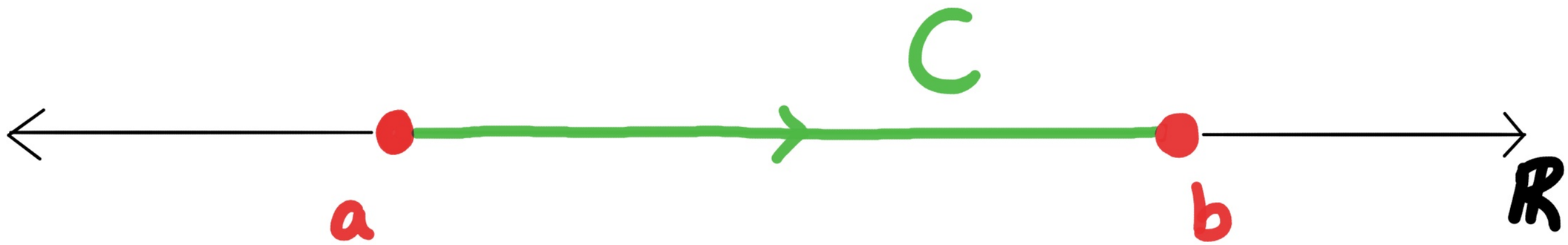
surface

"planet"

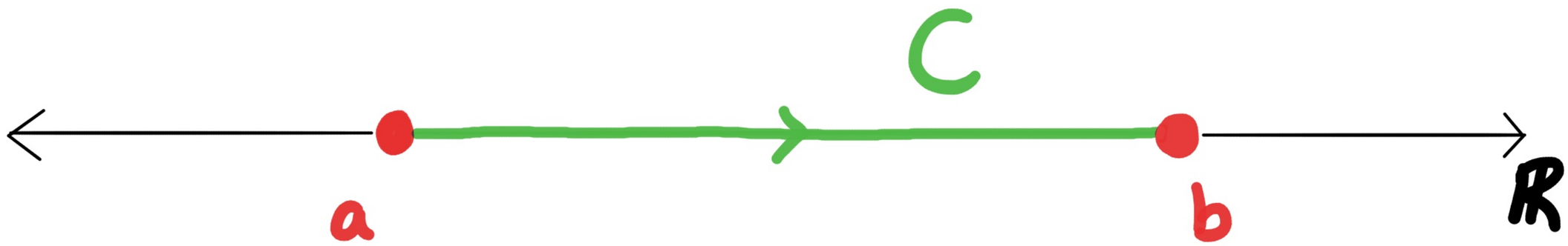
3-dimensional



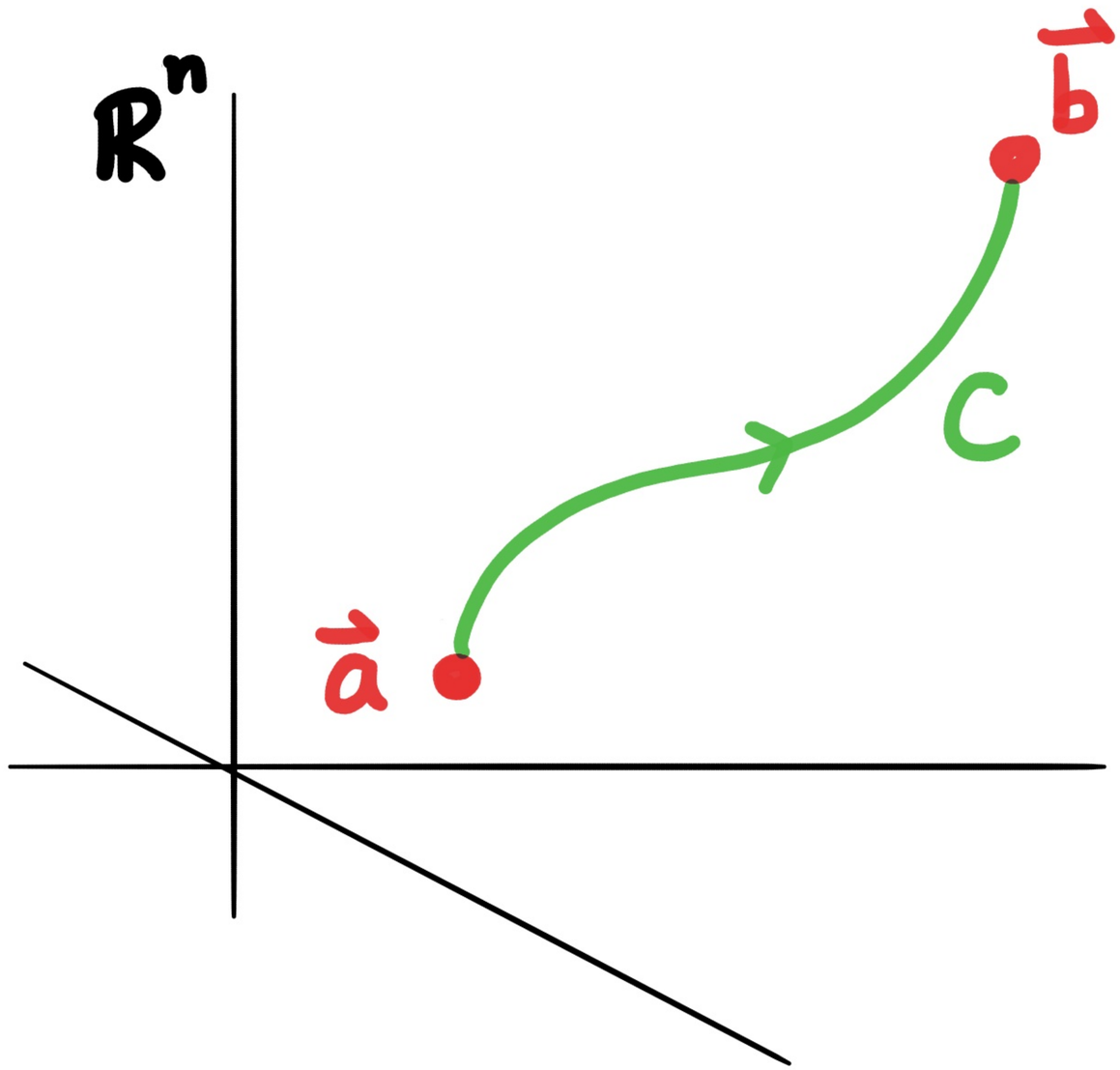
"universe"



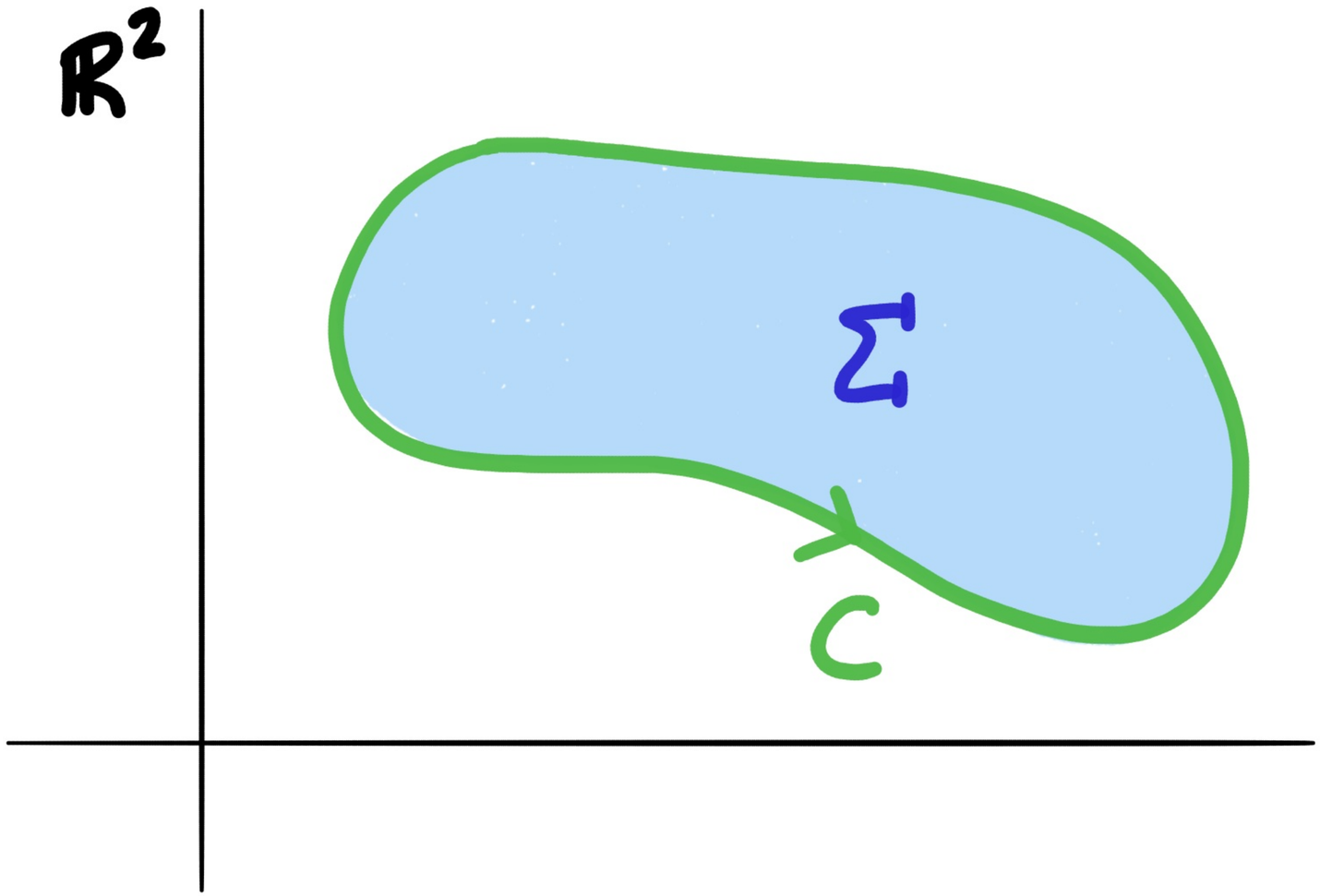
$$\left[f(x)\right]_a^b = \int_a^b f'(x)\,dx$$



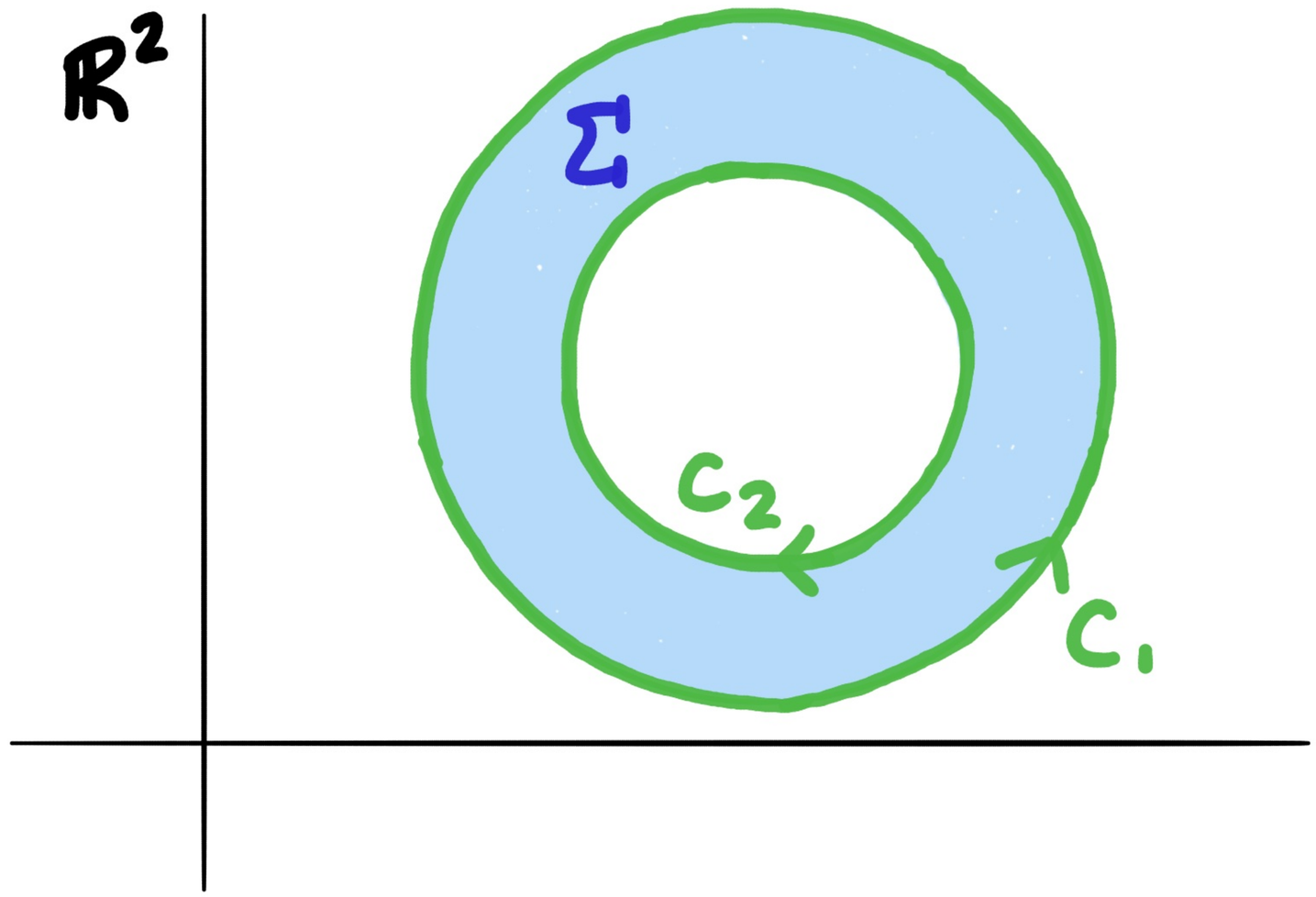
$$\left[f(x)\right]_a^b = \int_C f'(x)\,dx$$



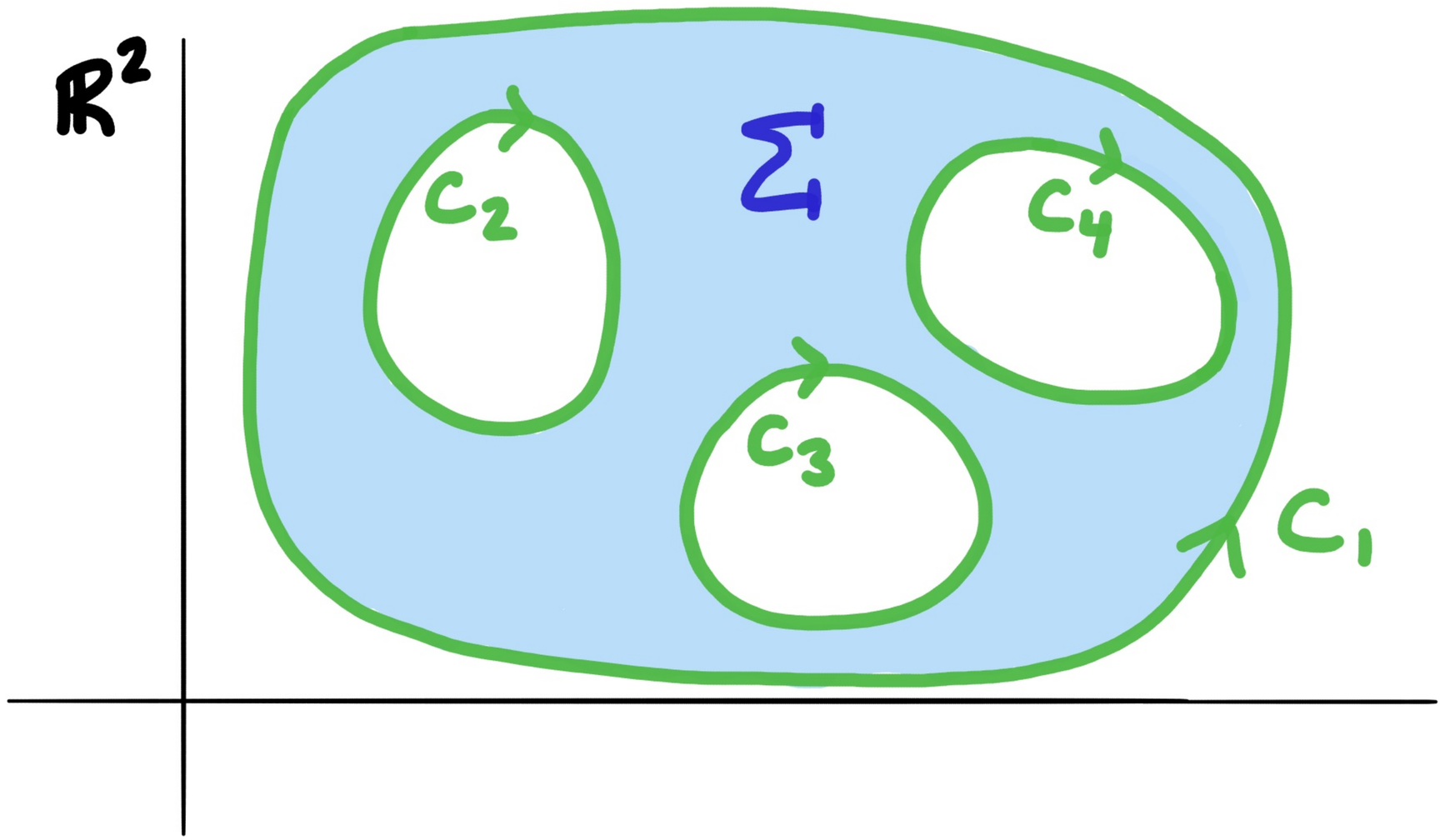
$$[f]_{\vec{a}}^{\vec{b}} = \int_C \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$



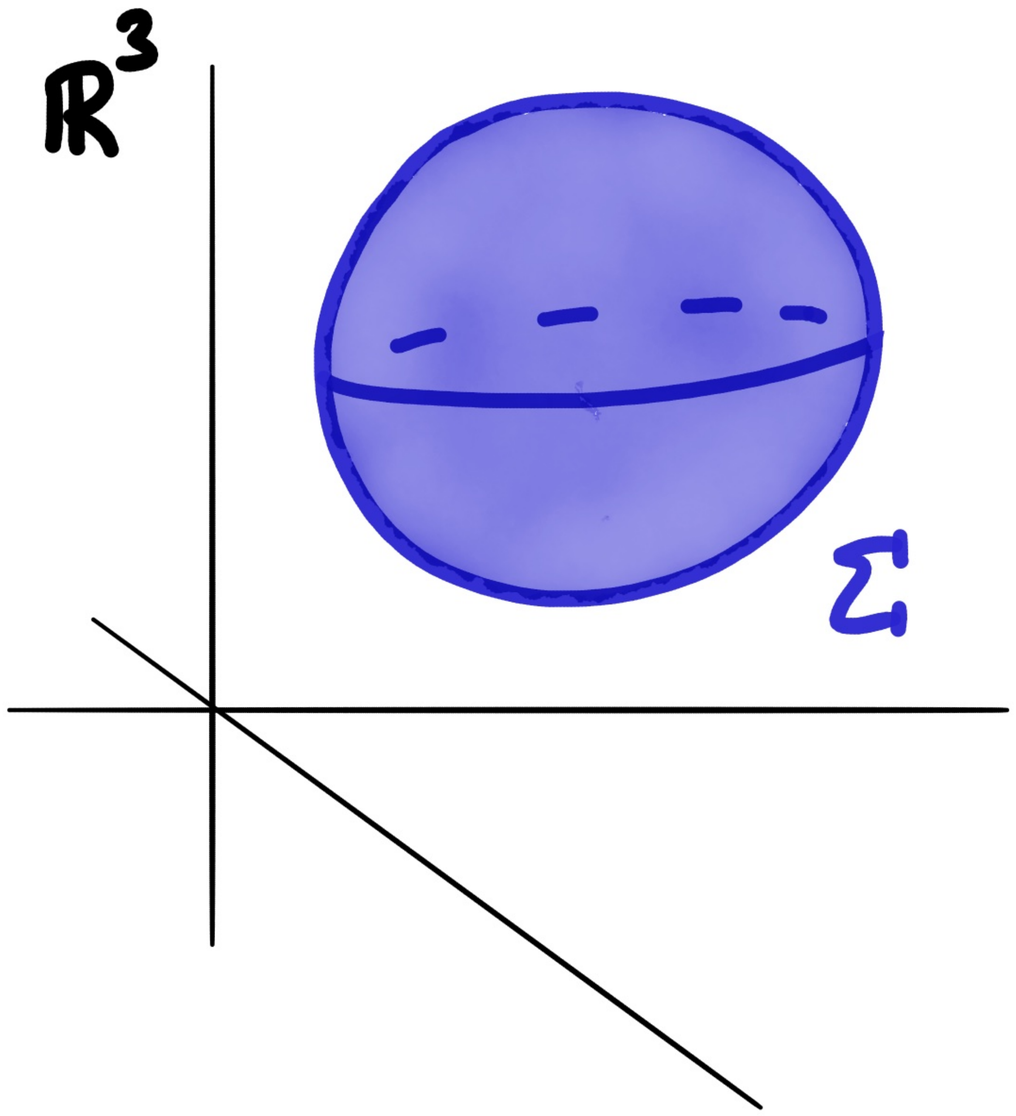
$$\int_C F_1\,dx + F_2\,dy = \iint_\Sigma \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$



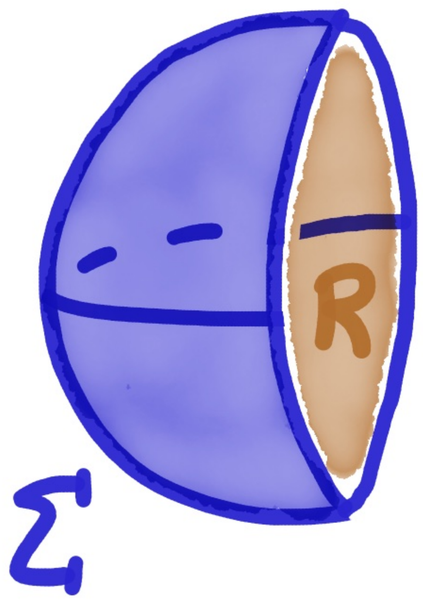
$$\int_{C_1 \cup C_2} F_1\,dx + F_2\,dy = \iint_{\Sigma} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$



$$\int_{C_1 \cup C_2 \cup C_3 \cup C_4} F_1\,dx + F_2\,dy = \iint_{\Sigma} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$



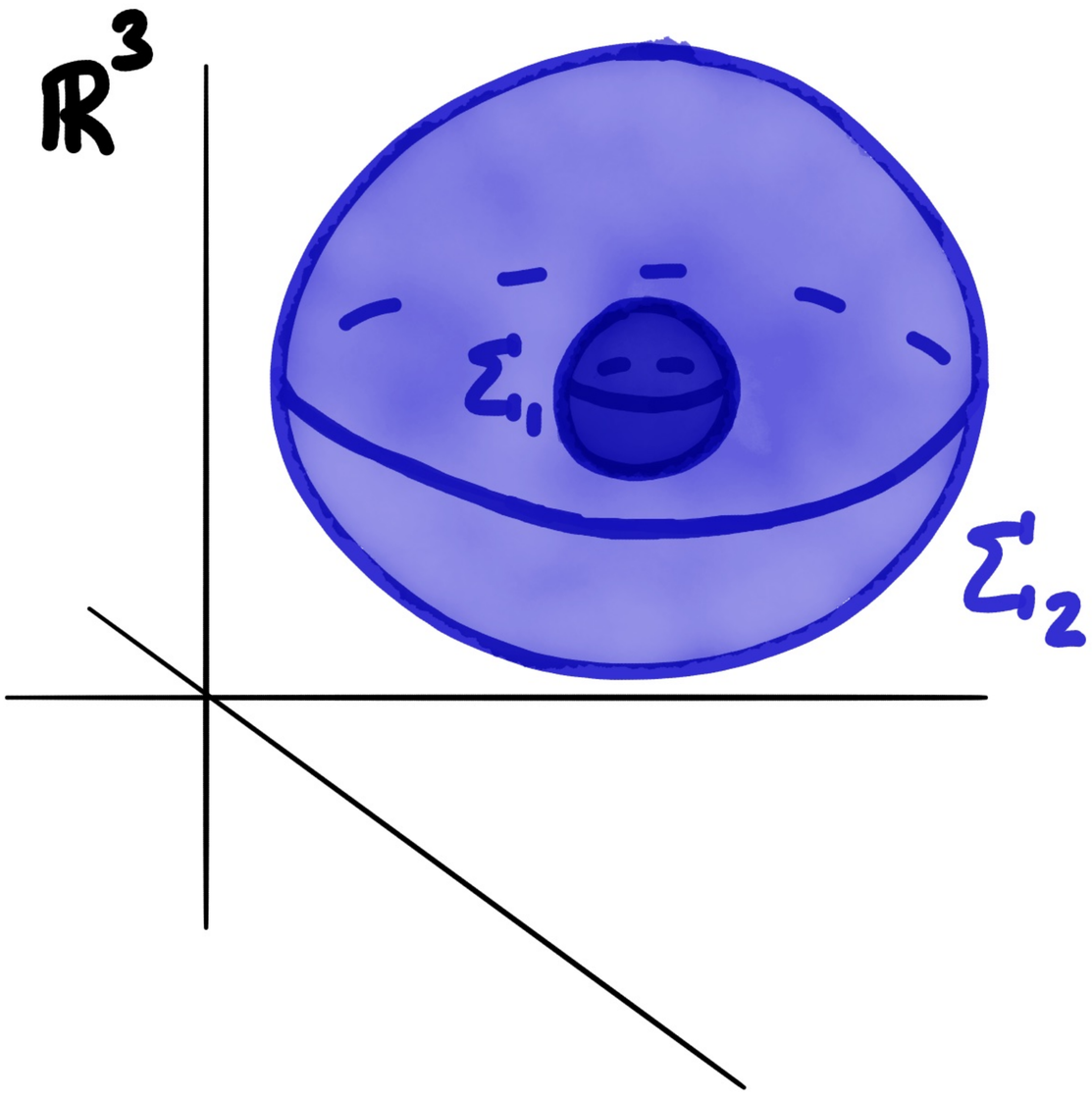
Σ bounds R

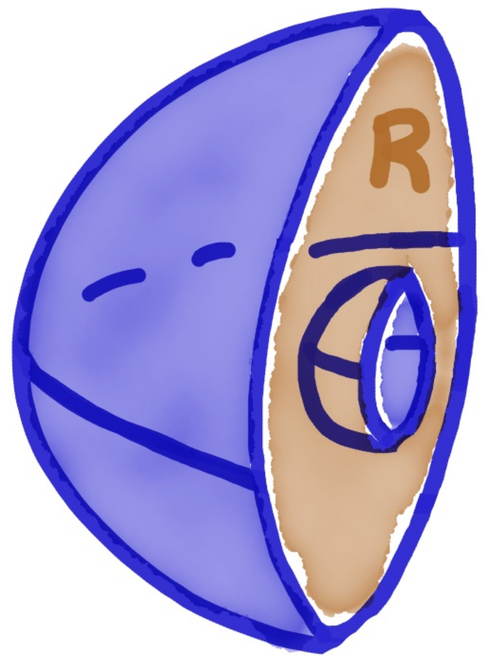


$$\iint_{\Sigma} \vec{F} \cdot \vec{n} \, dS = \iiint_{R} \operatorname{div} \vec{F} \, dV$$

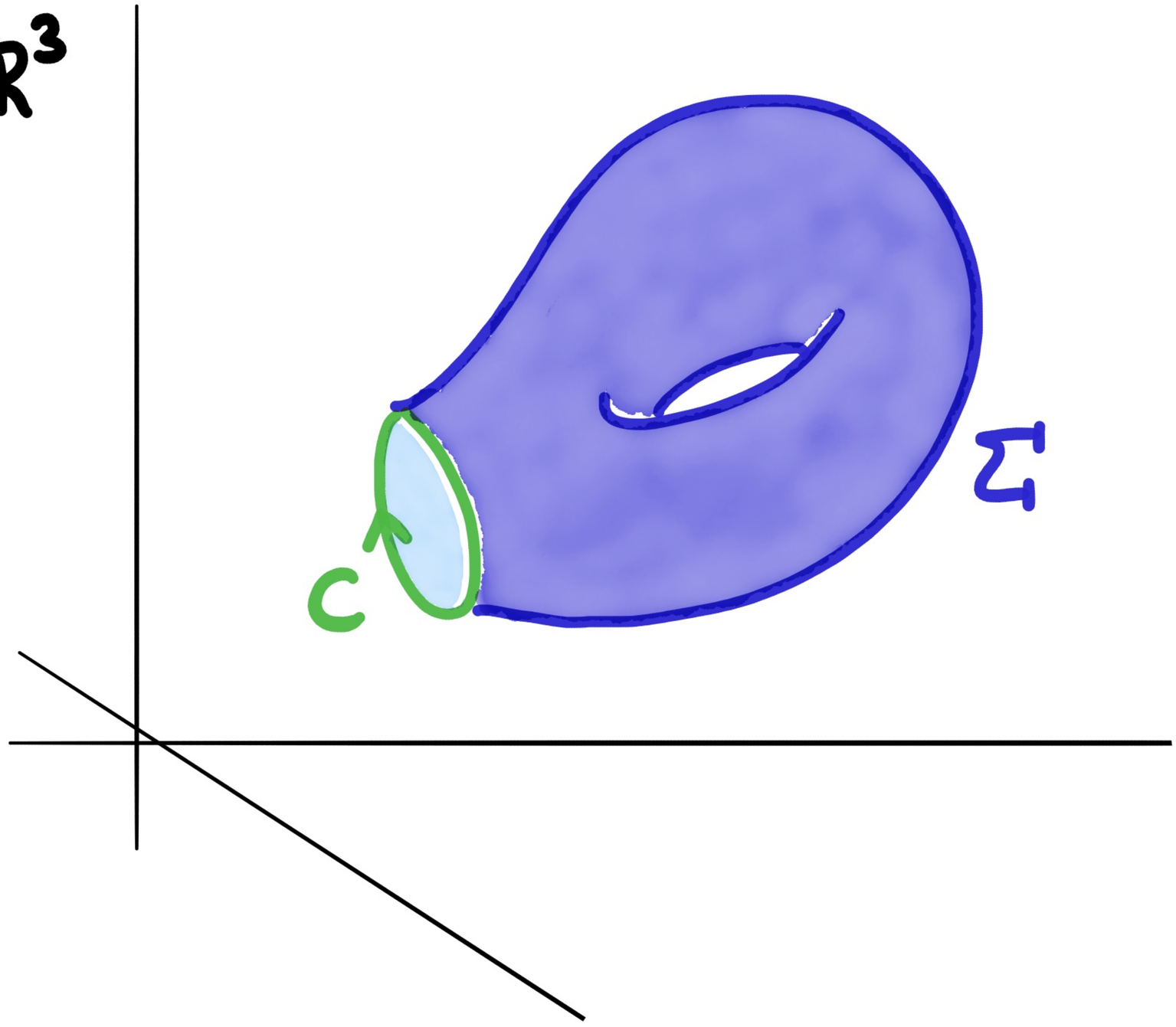
Σ_1 , Σ_2 concentric spheres

R : region between them





$$\iint_{\Sigma_1 \cup \Sigma_2} \vec{F} \cdot \vec{n} \, dS = \iiint_{R} \operatorname{div} \vec{F} \, dV$$



$$\int_C F_1\,dx + F_2\,dy + F_3\,dz = \iint_\Sigma (\operatorname{curl}\vec{F})\cdot\vec{n}\,dS$$

General version of Stokes's Theorem

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Then,

$$\int_B \omega = \int_M d\omega .$$