Thirty-three

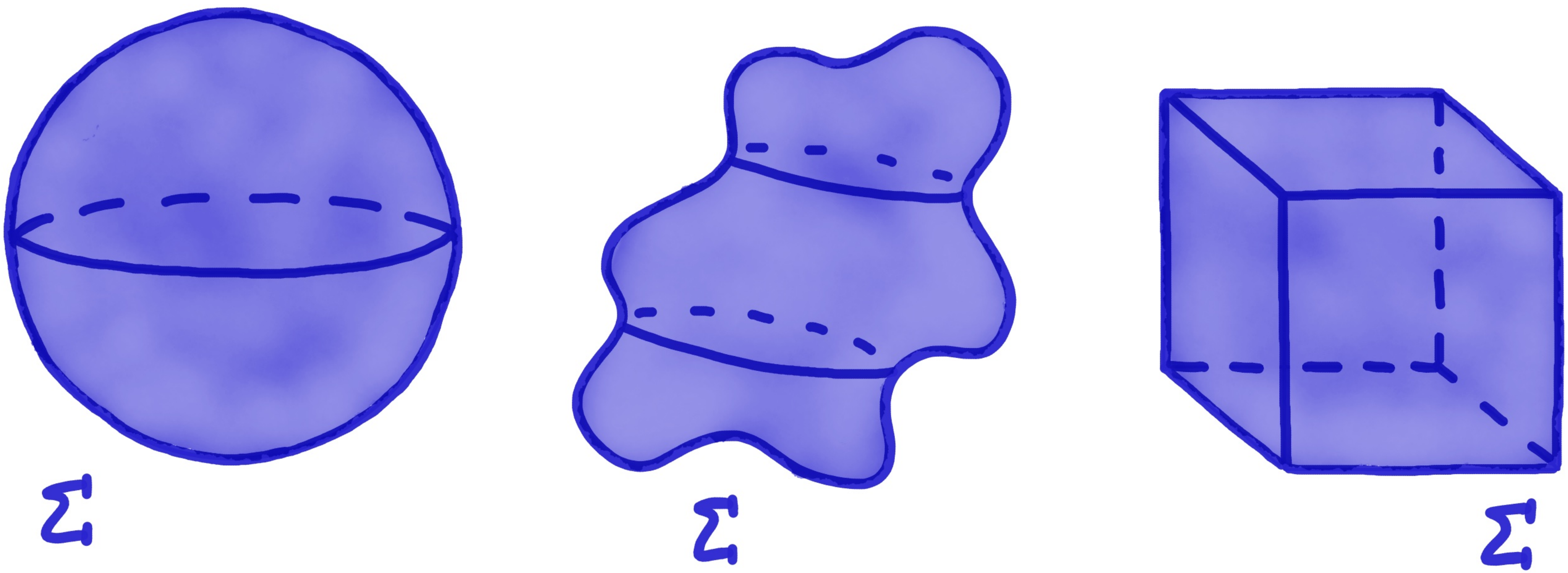
(I) Outer unit normal vectors

(II) Gauss's Divergence Theorem

(III) Two-sided surfaces

I Outer unit normal vectors

Let Σ be a (possibly lumpy) sphere in $\mathbb{R}^3$.



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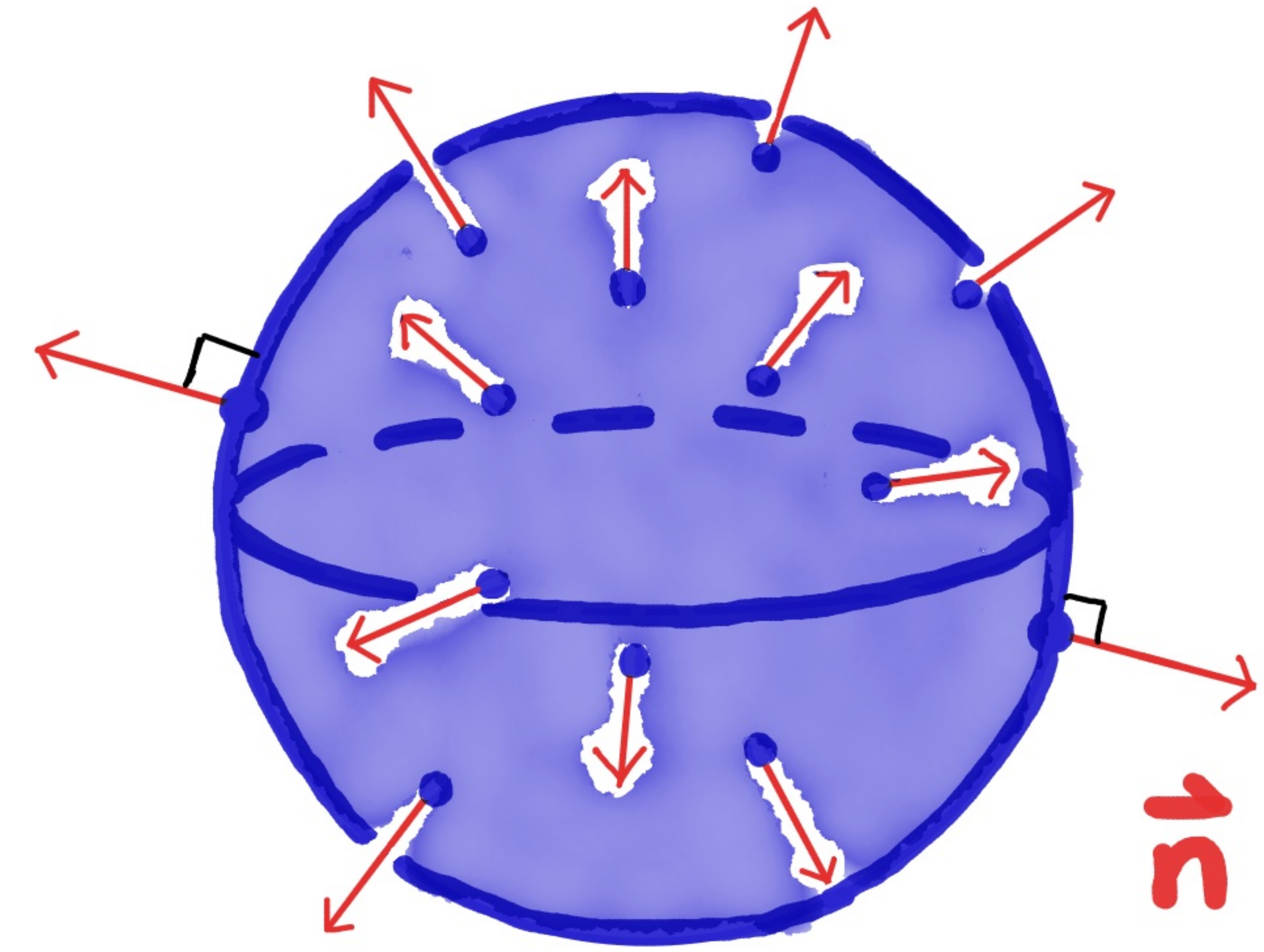

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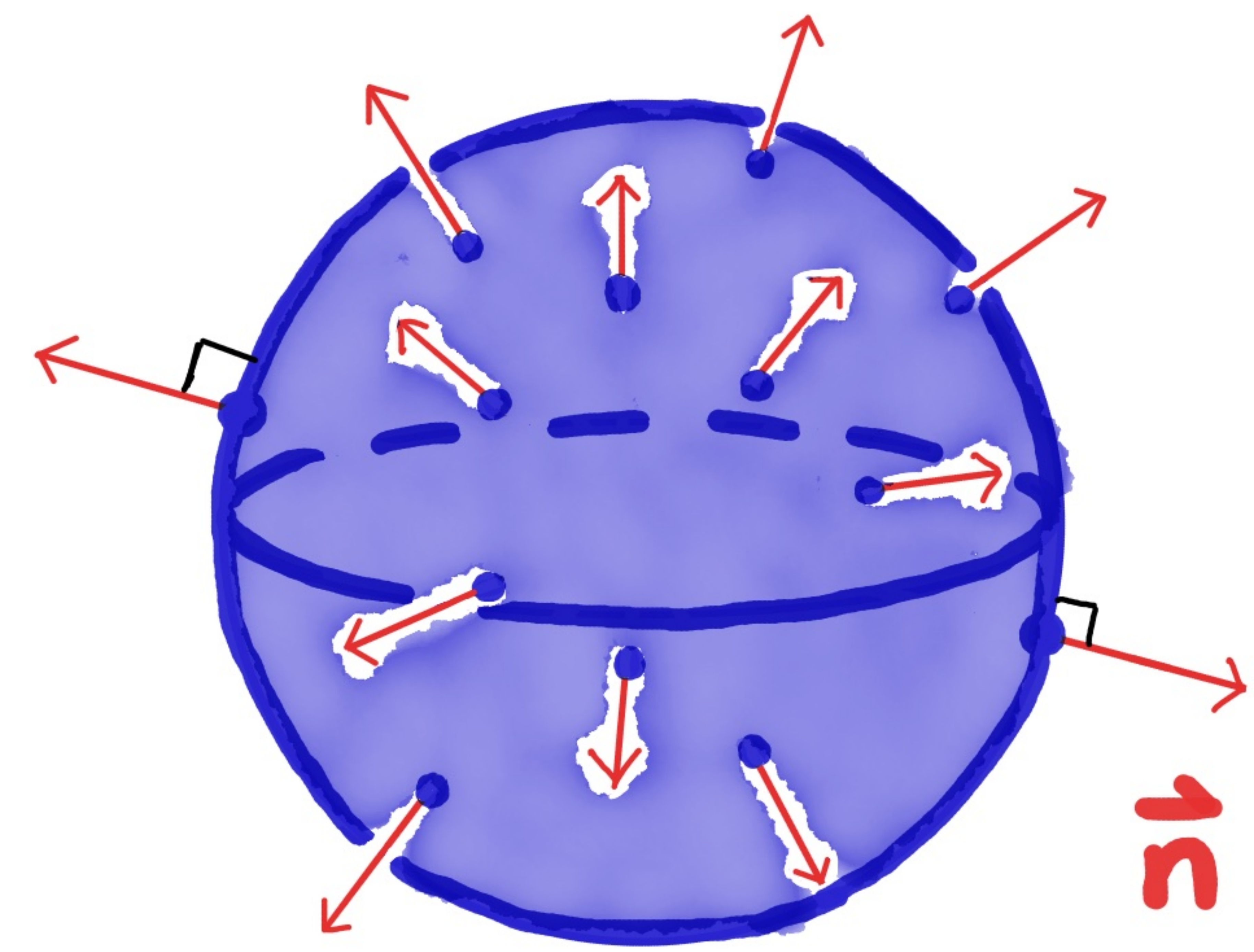

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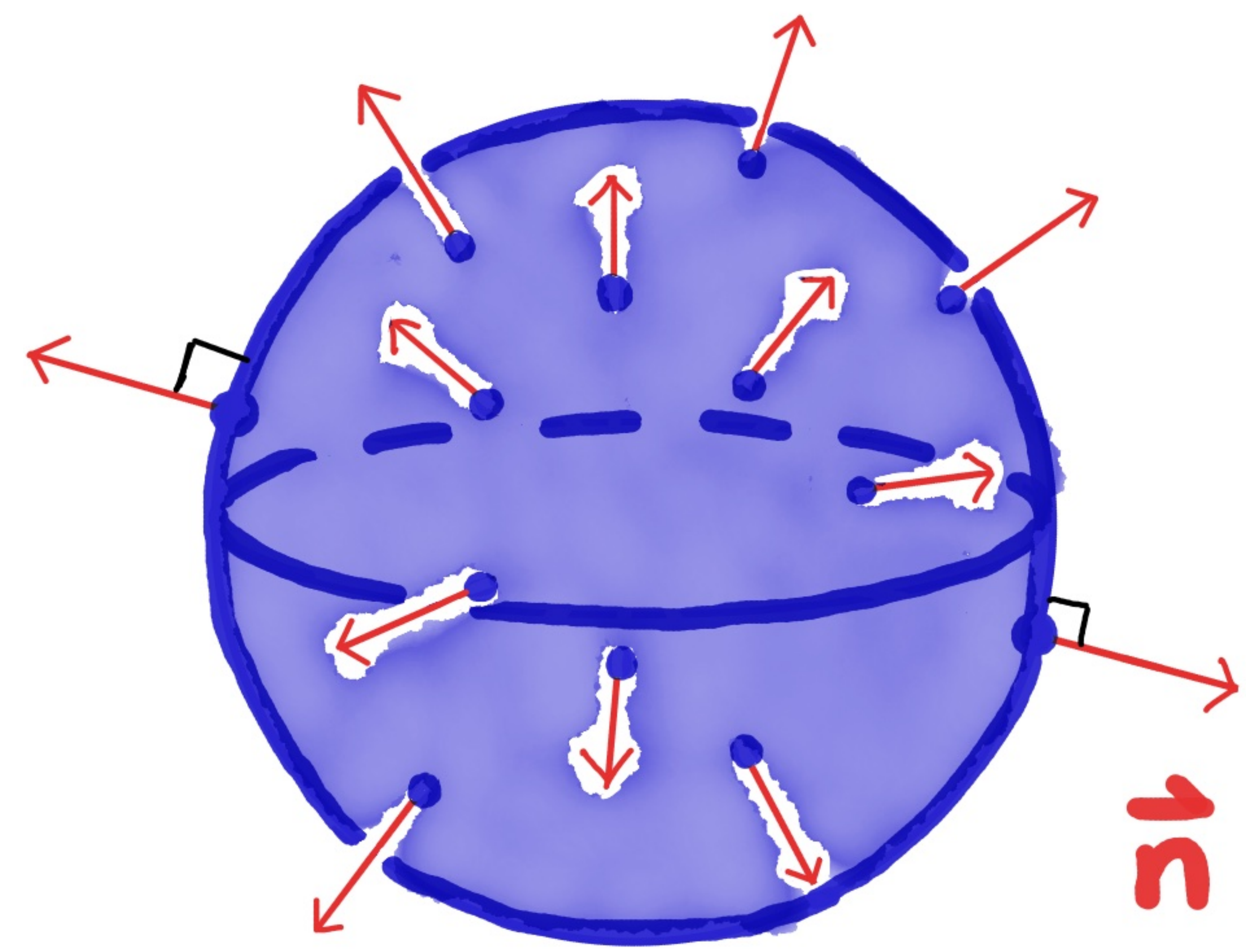

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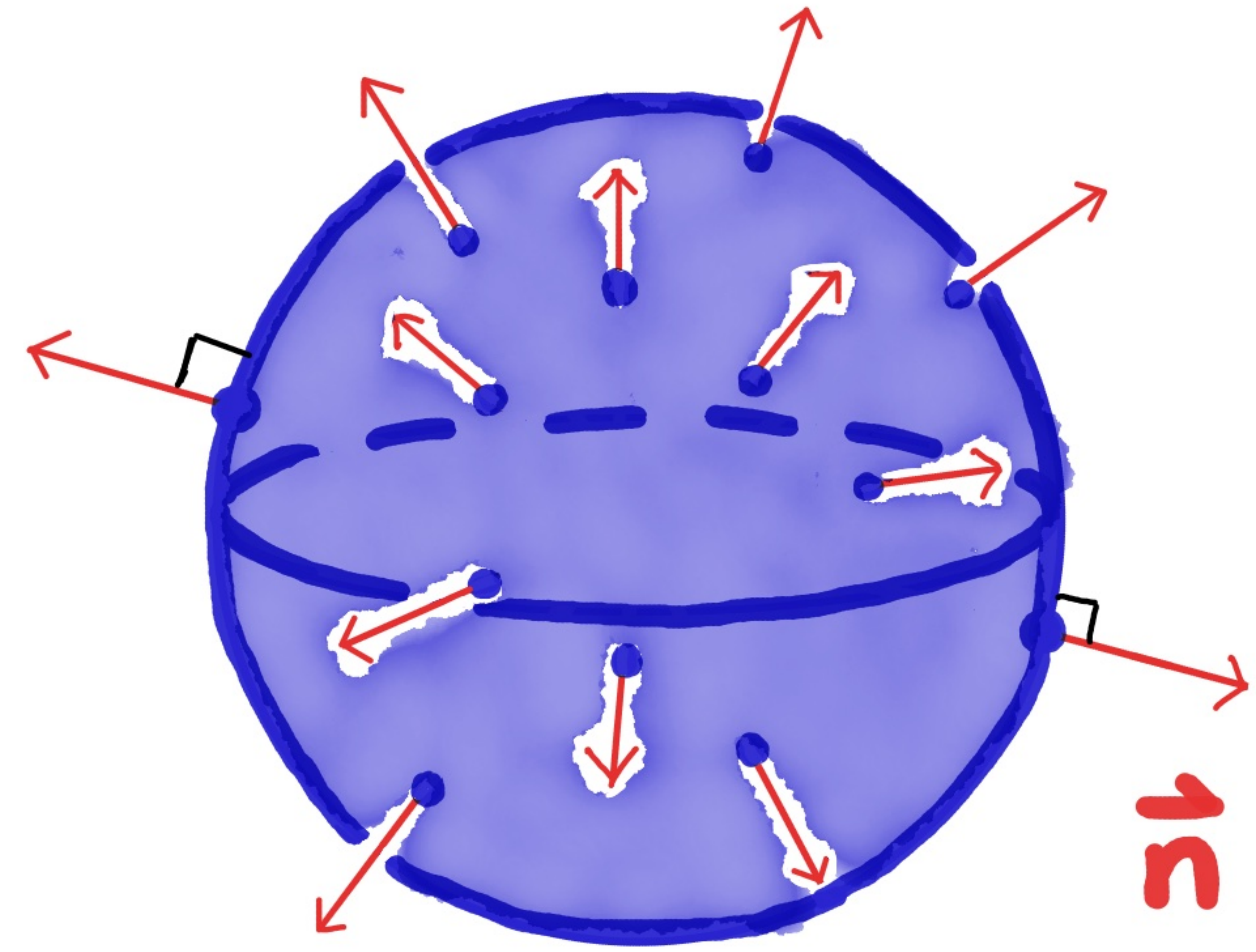

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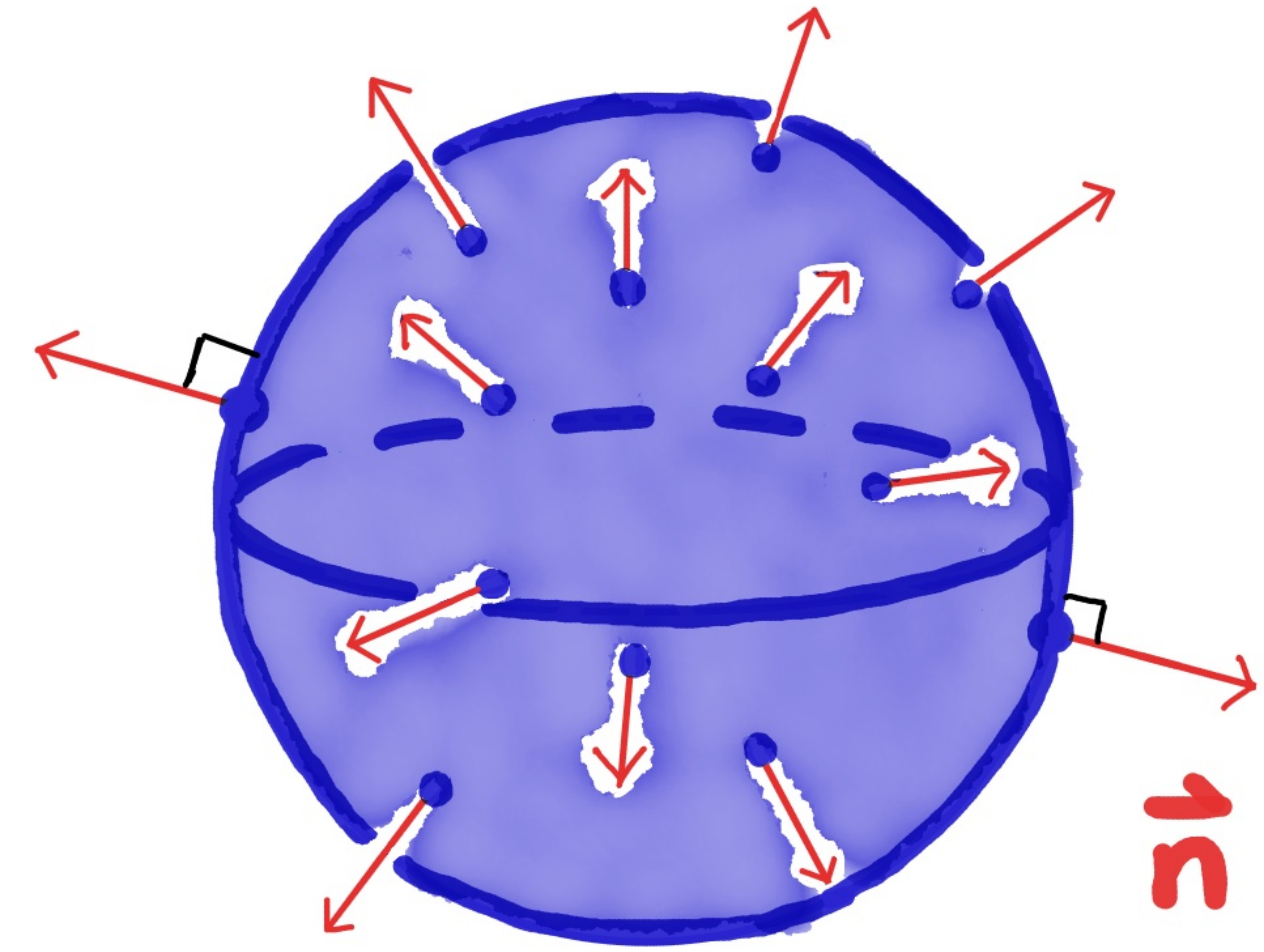

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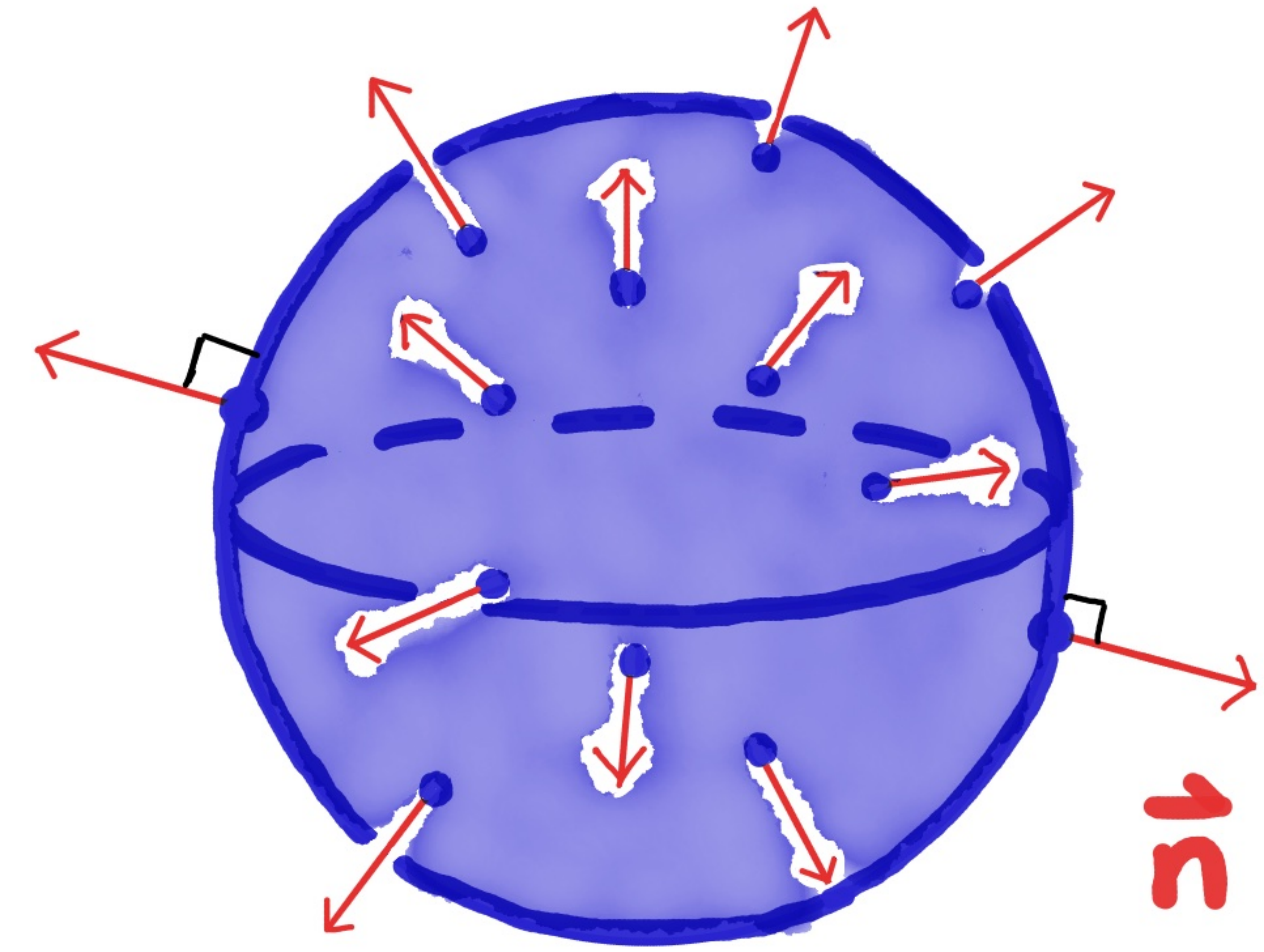

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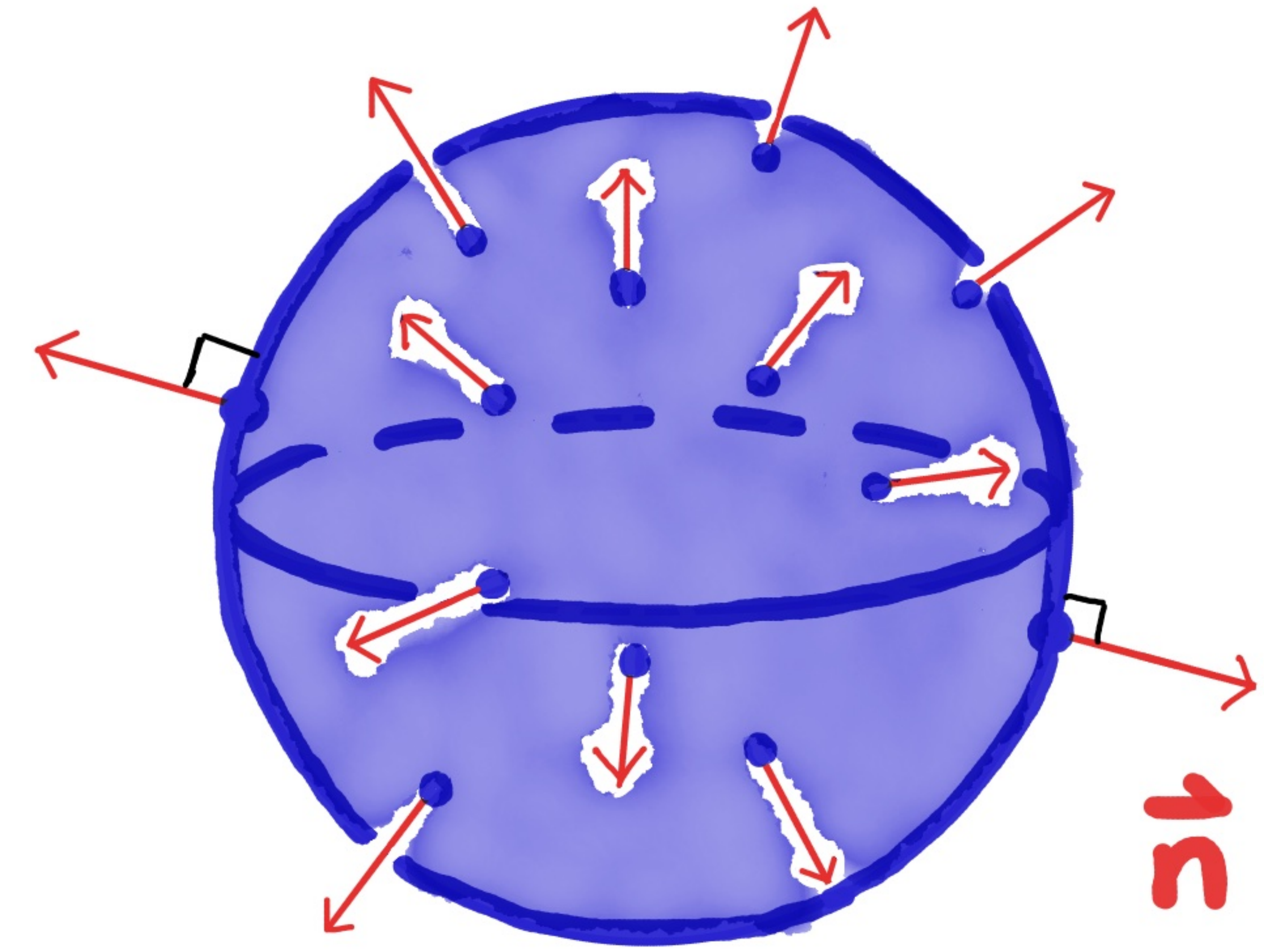

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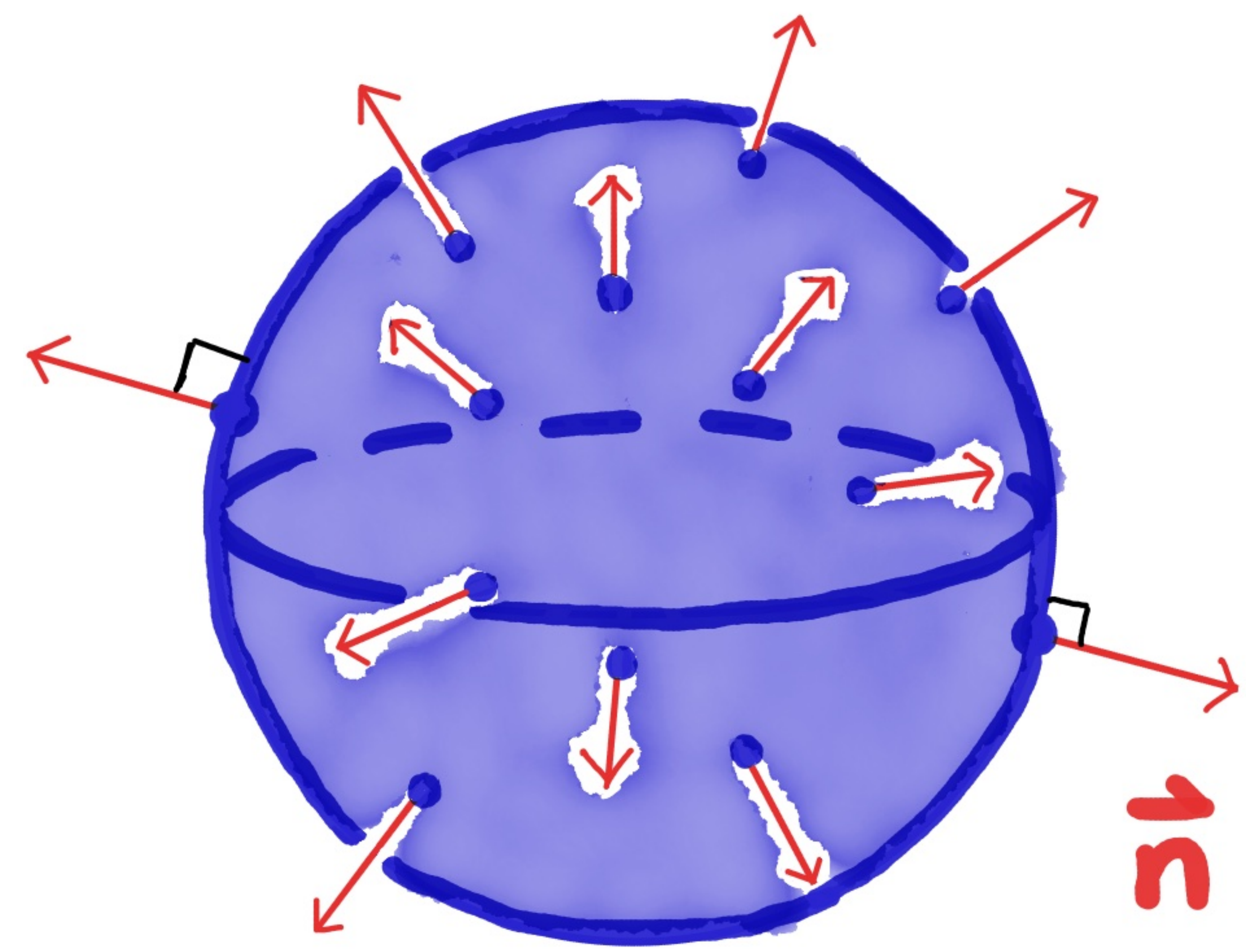

$\vec{n}$ is called the

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For every $(x,y,z) \in \Sigma$, let $\vec{n}(x,y,z) \in \mathbb{R}^3$ be the unit vector that's orthoganal to Σ and points out.

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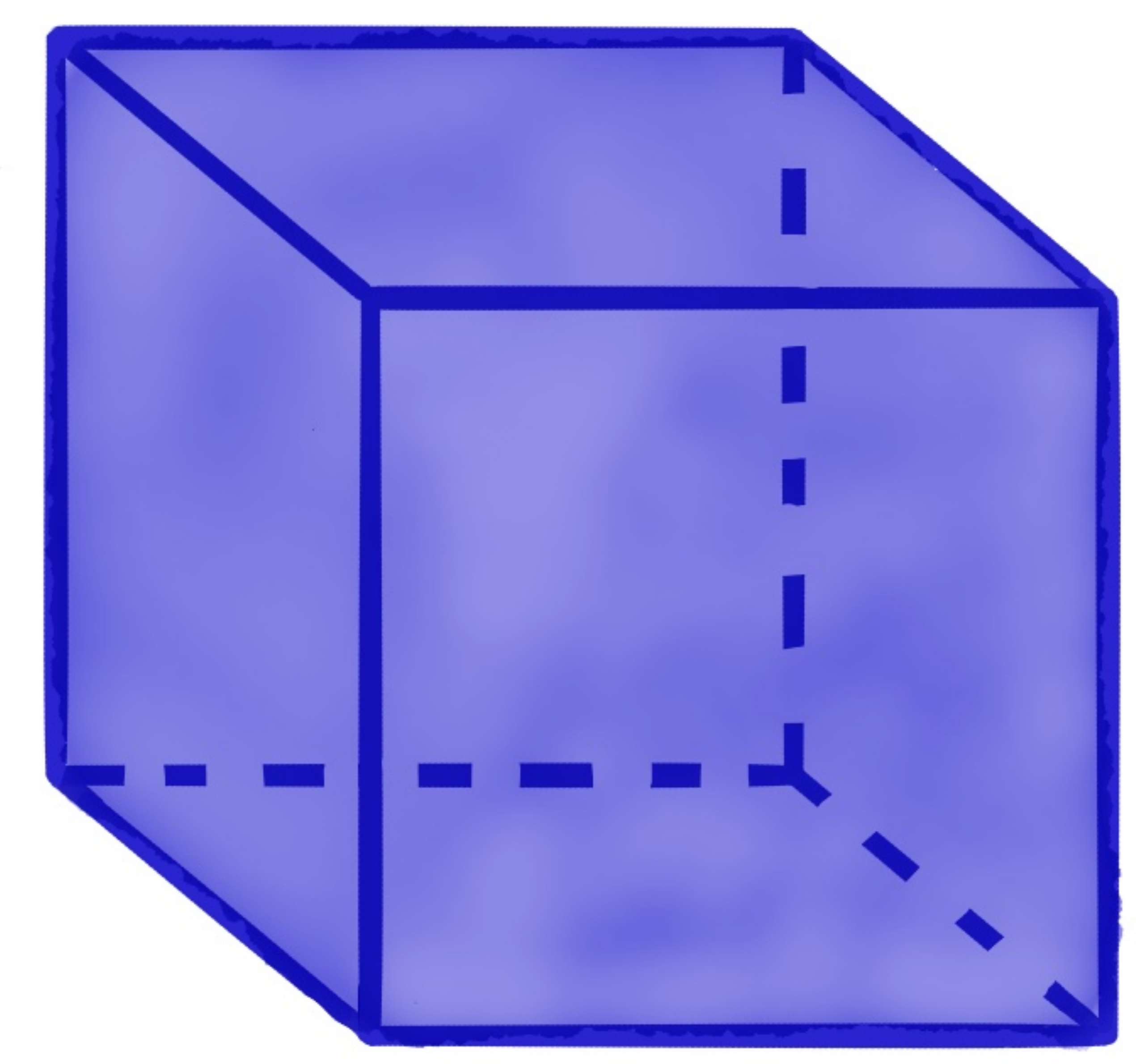

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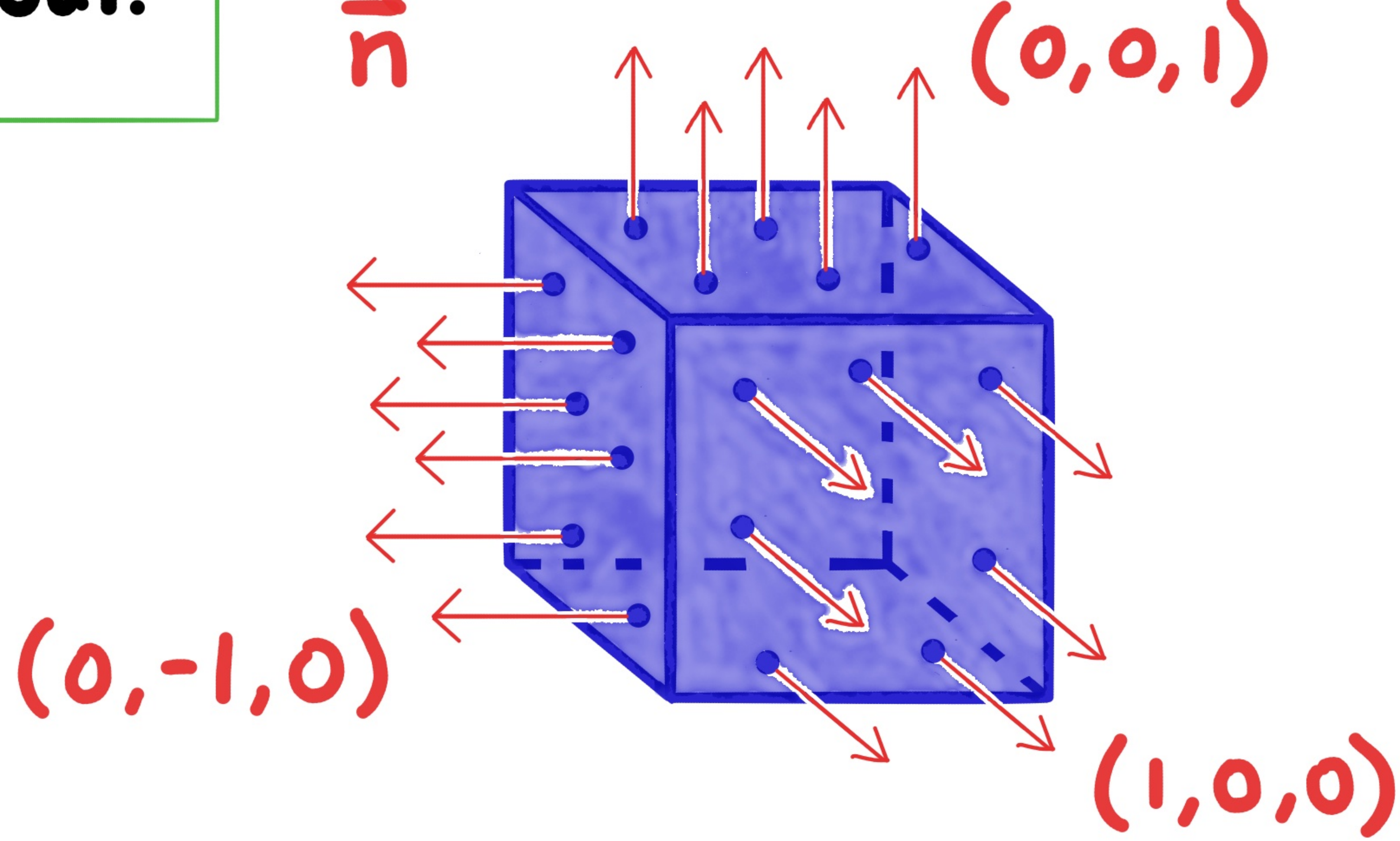

II Gauss's Divergence Theorem

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Then,

$$\boxed{\iint_{\Sigma} \vec{F}\cdot\vec{n}\, dS = \iiint_{R} \operatorname{div}\vec{F}\, dV}$$

Example:

- Σ the sphere of radius a centered at $\vec{O} \in \mathbb{R}^3$.
- $R = \{(x,y,z) : x^2+y^2+z^2 \leq a^2\}$
- $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{F}(x,y,z) = (xy, zy+x, x^2z)$
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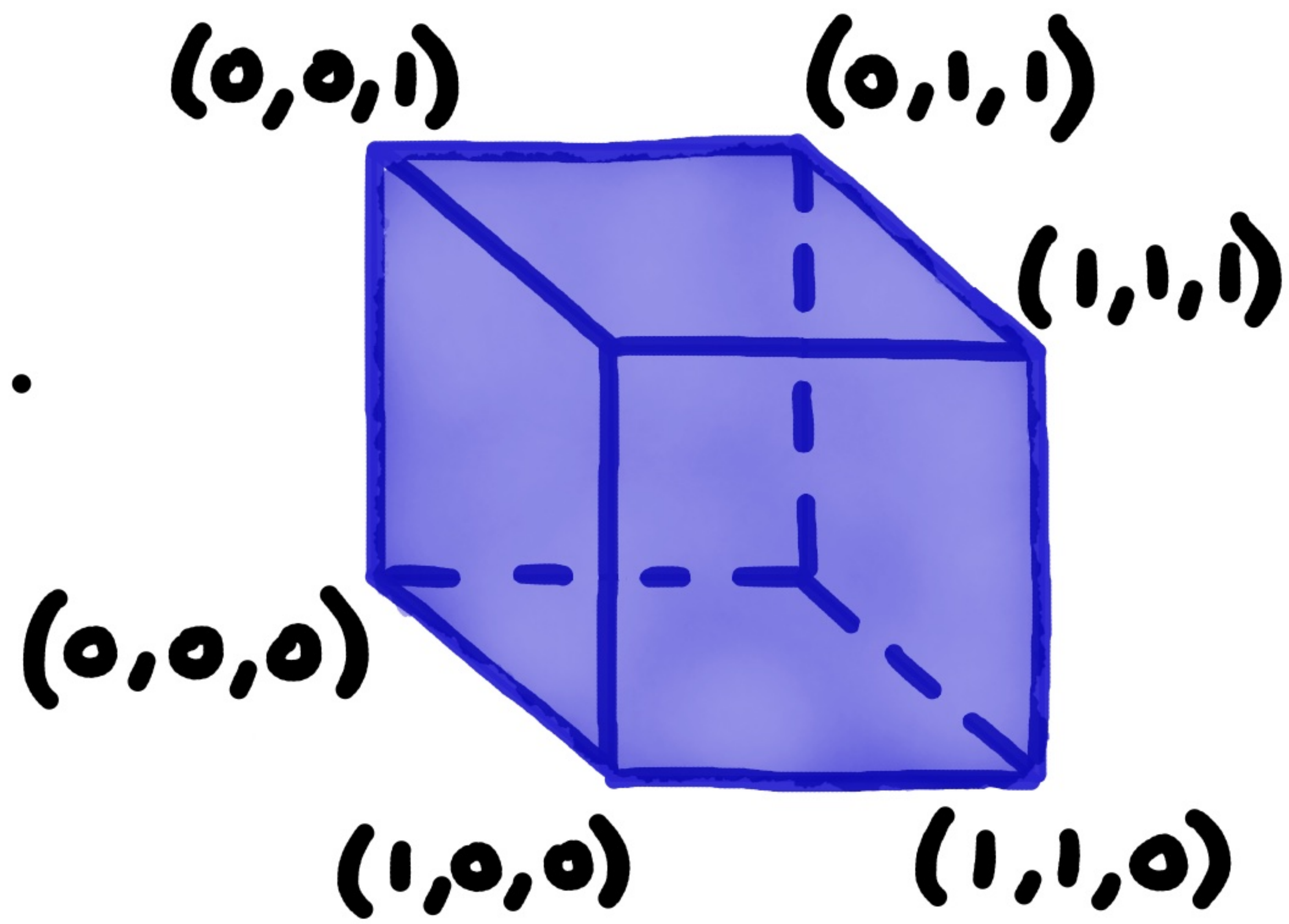
Equal by GDT.

Example:

- $R = \{(x,y,z) : 0 \le x,y,z \le 1\}$.
- Σ the exterior of the cube.
- $\vec{F}(x,y,z) = (5x, 3y, z)$.



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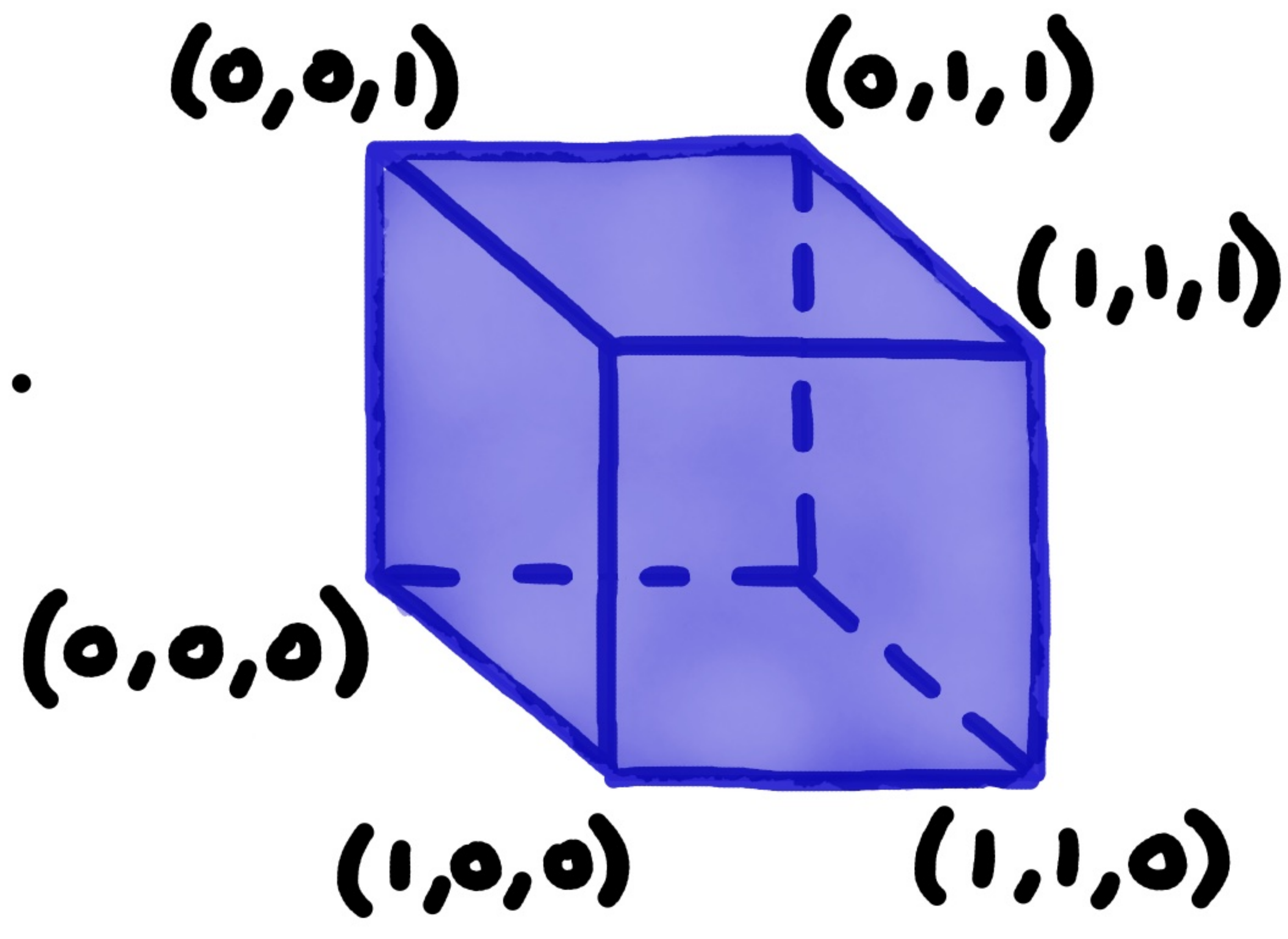

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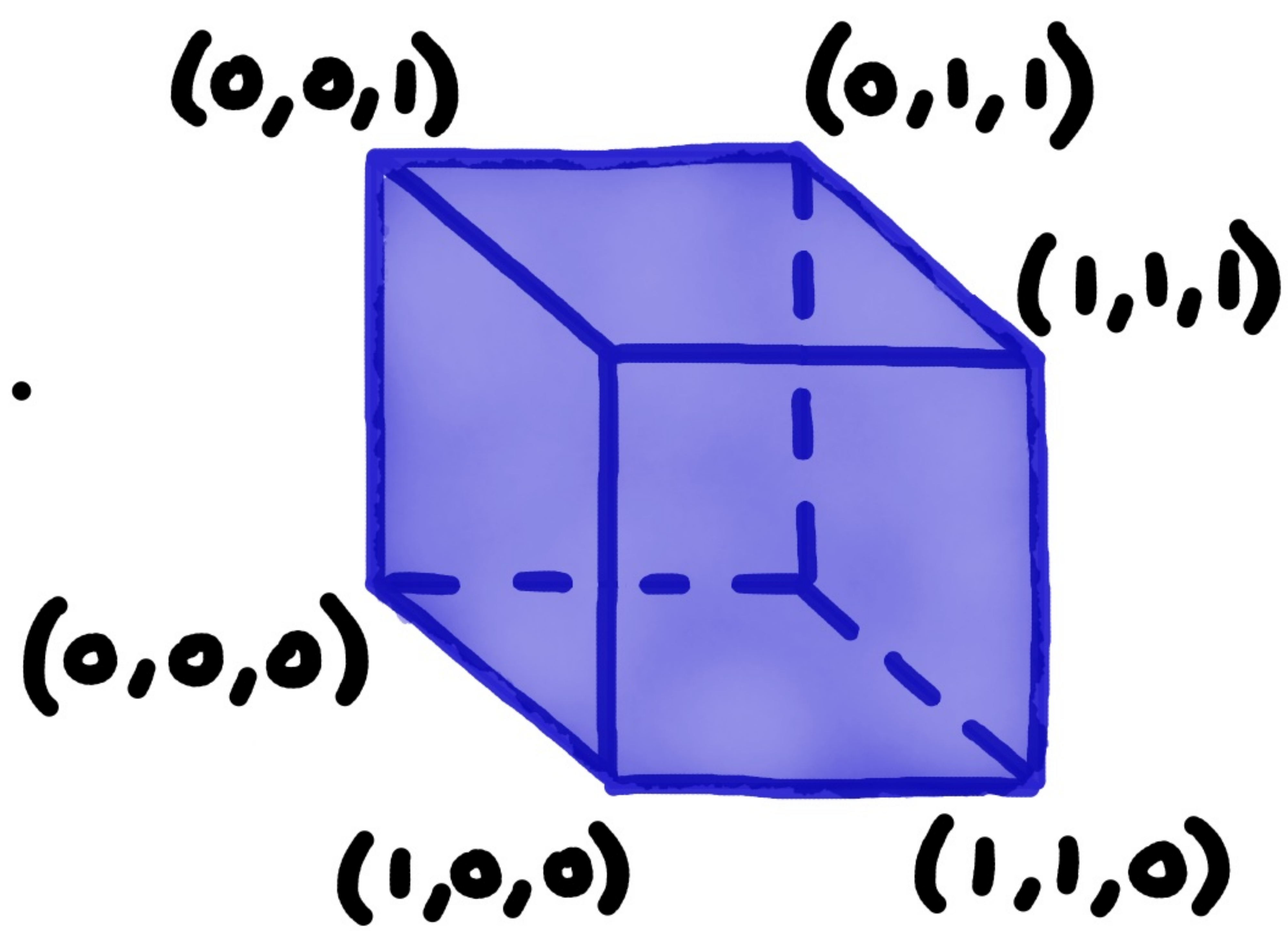

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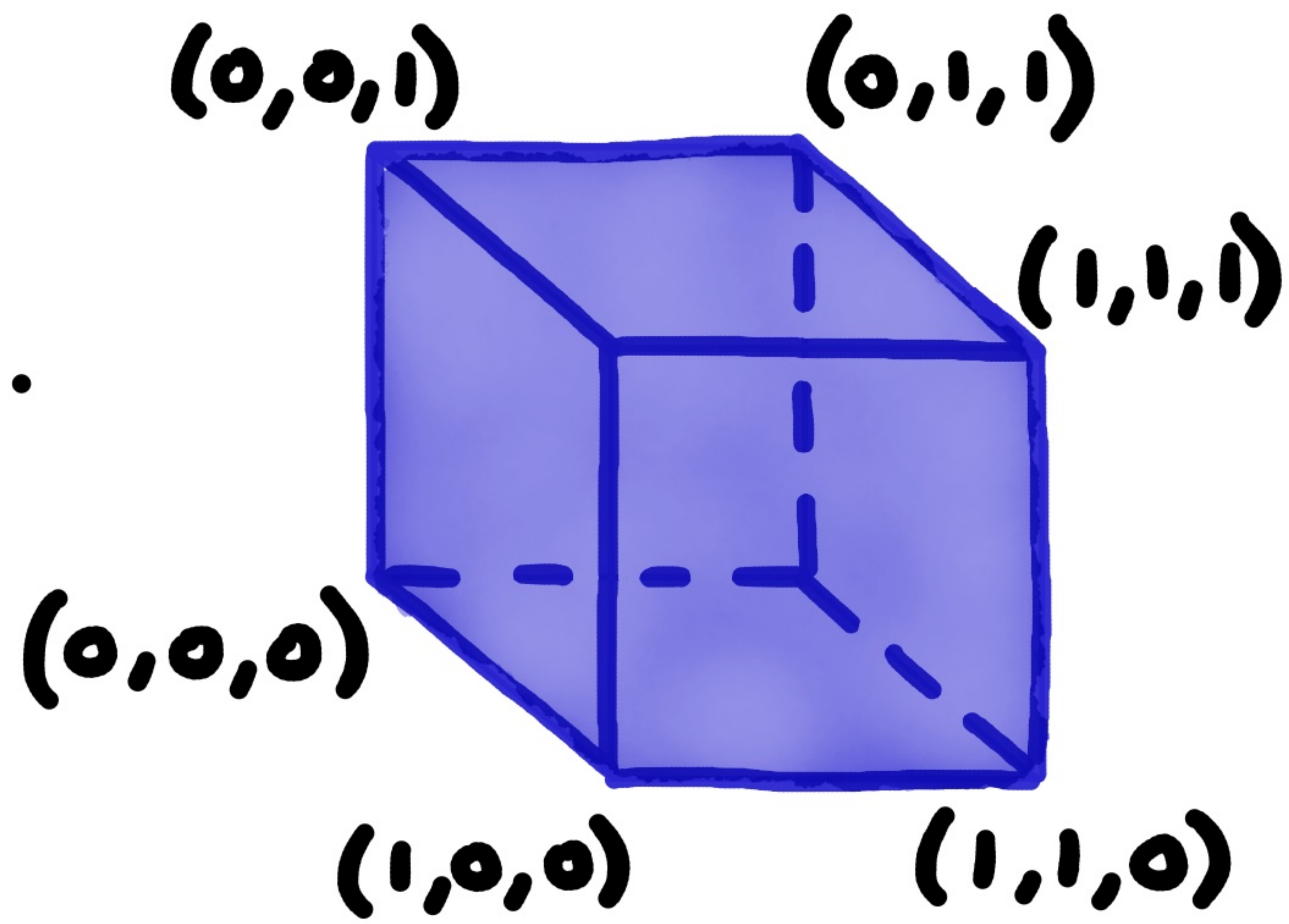

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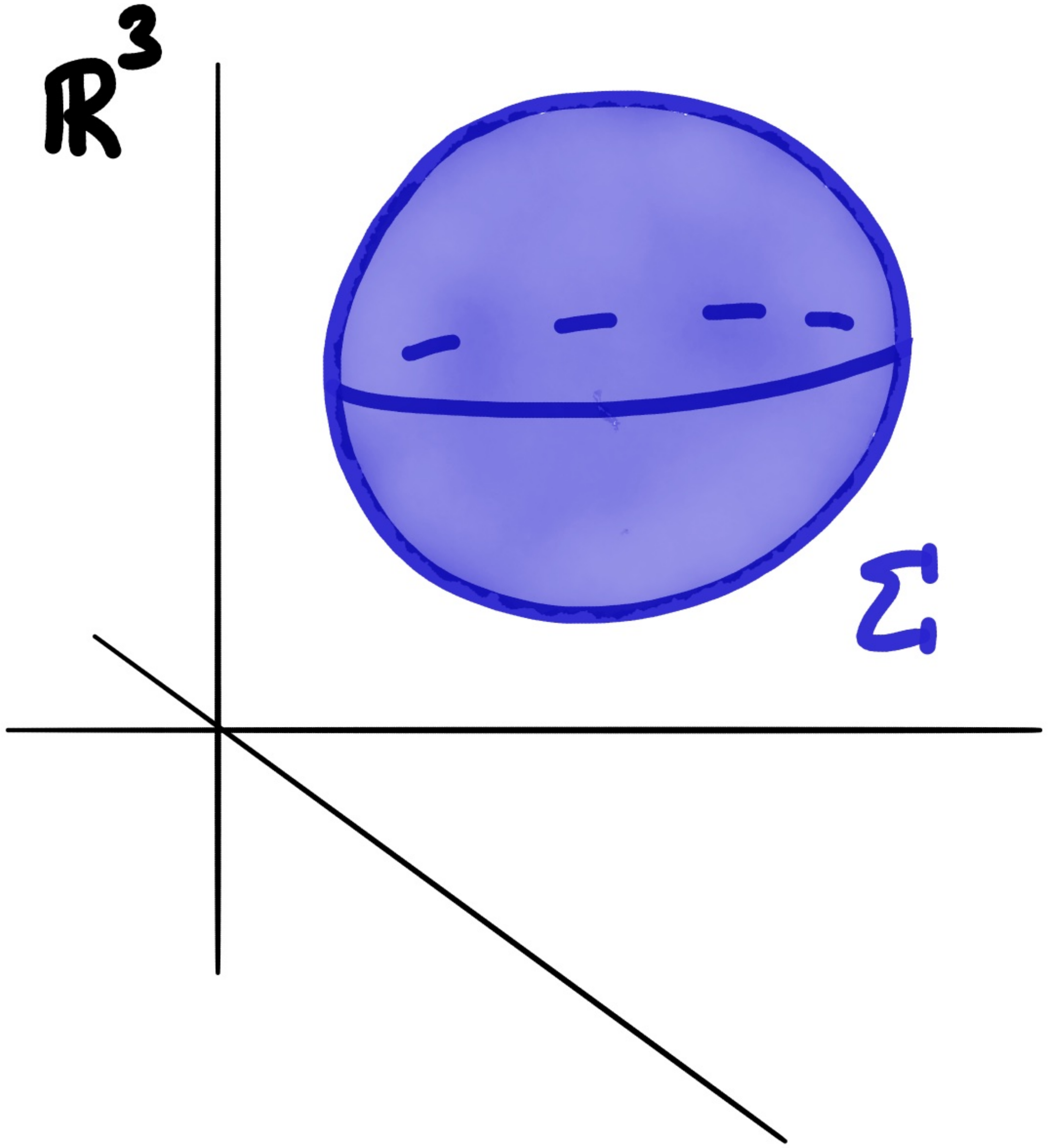
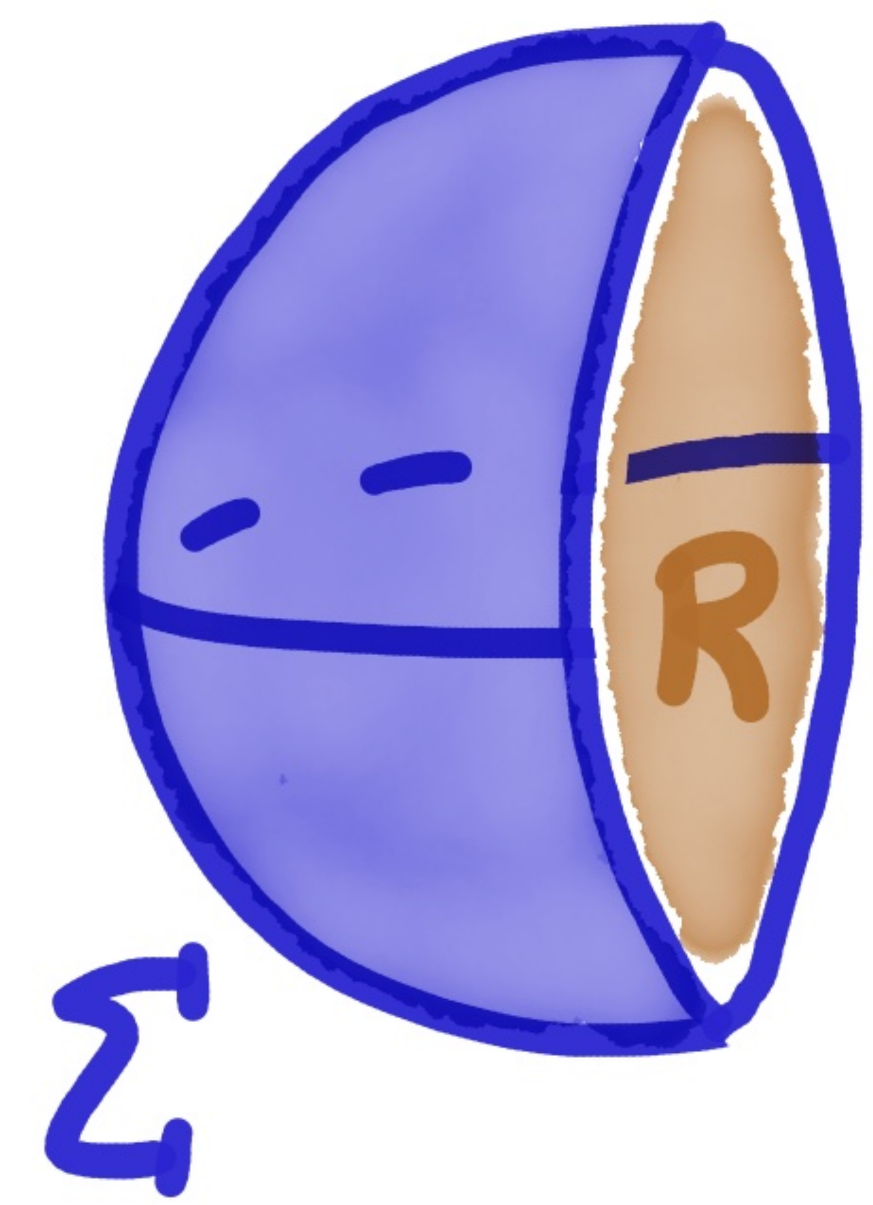
Σ bounds R





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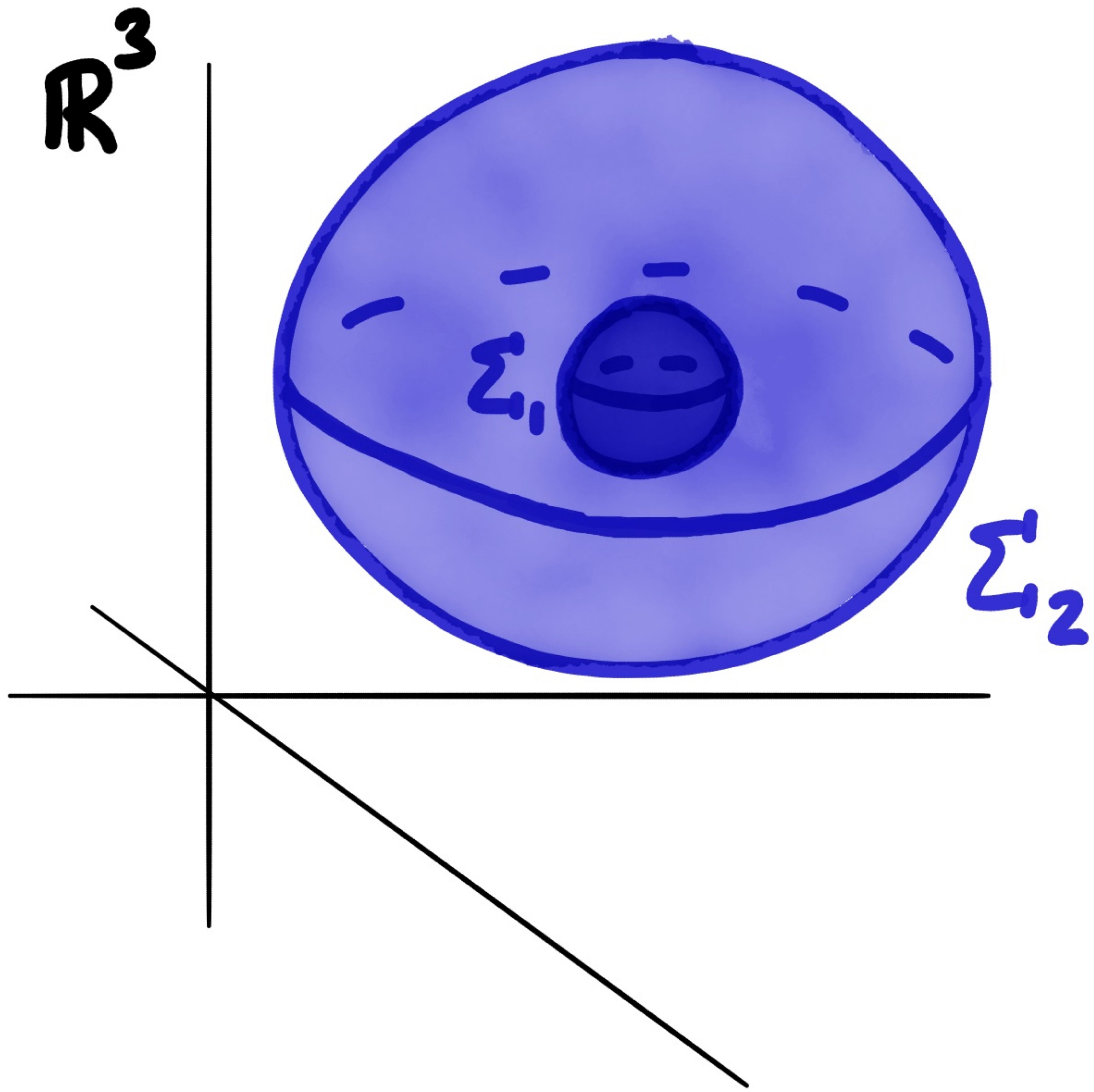
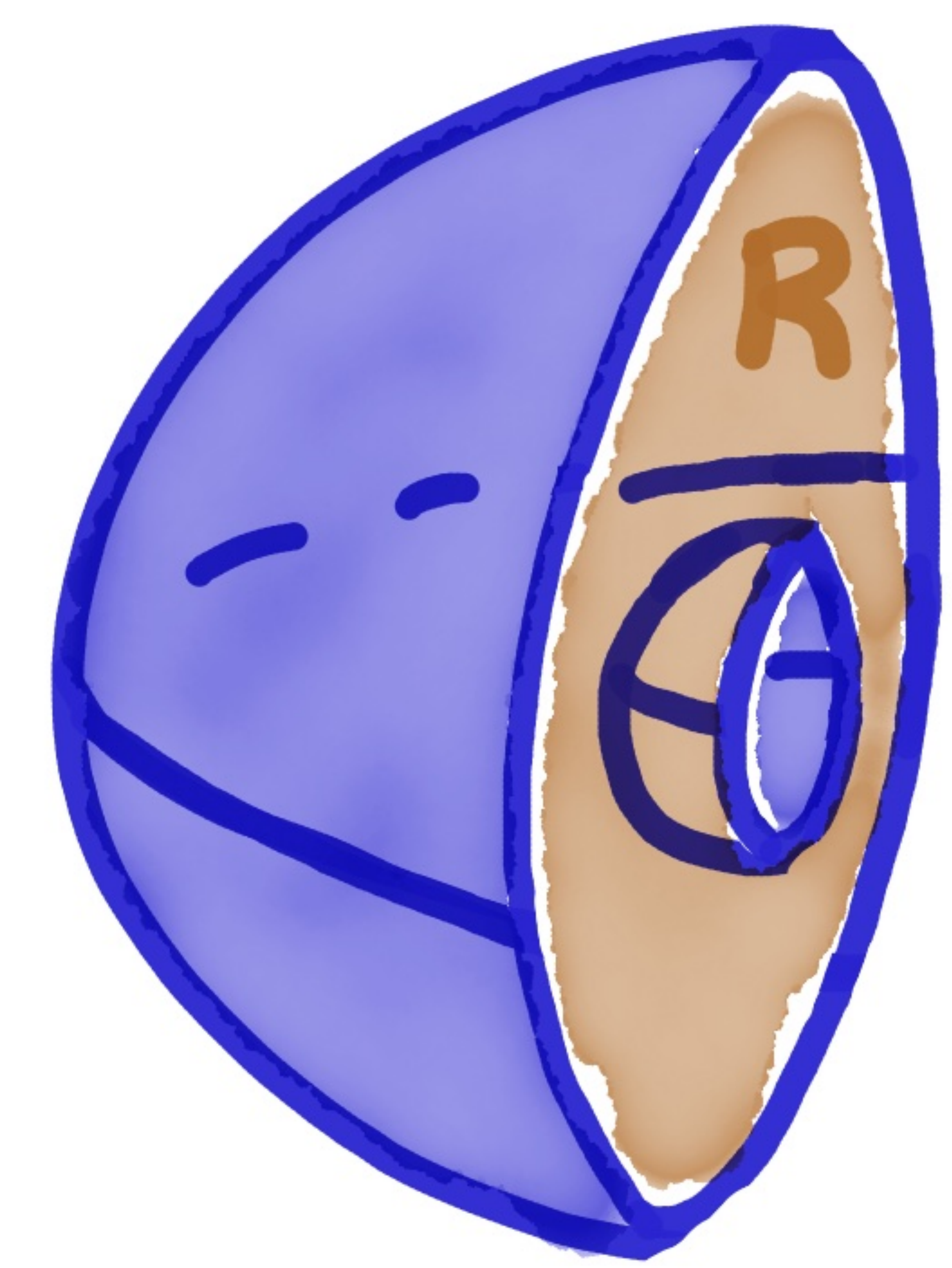
Σ_1, Σ_2 concentric spheres

R: region between them





$$\iint_{\Sigma_1 \cup \Sigma_2} \vec{F} \cdot \vec{n} \, dS = \iiint_R \operatorname{div} \vec{F} \, dV$$

Proof of GDT for cube:

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Let:

$R = \{(x,y,z) : 0 \le x,y,z \le 1\}$.

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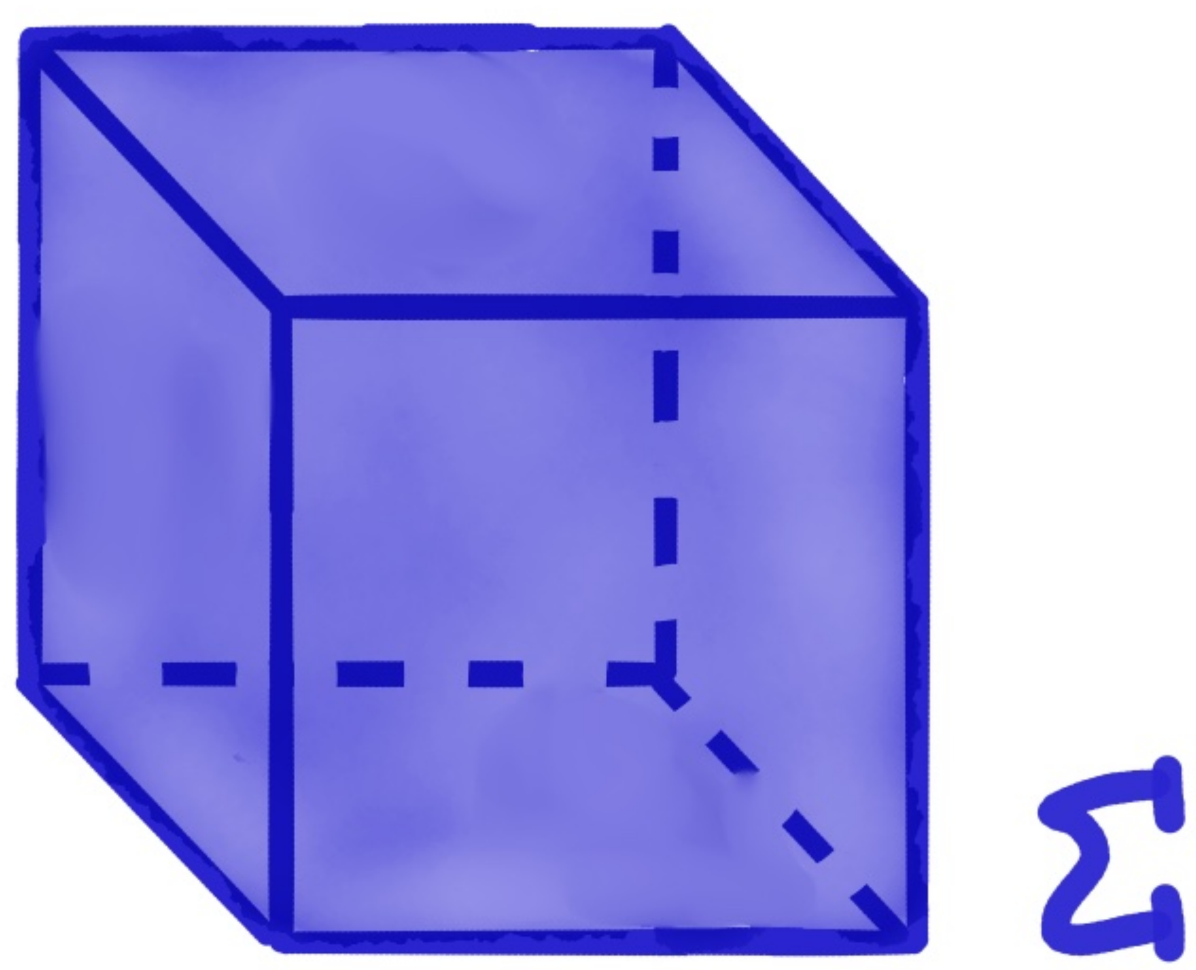

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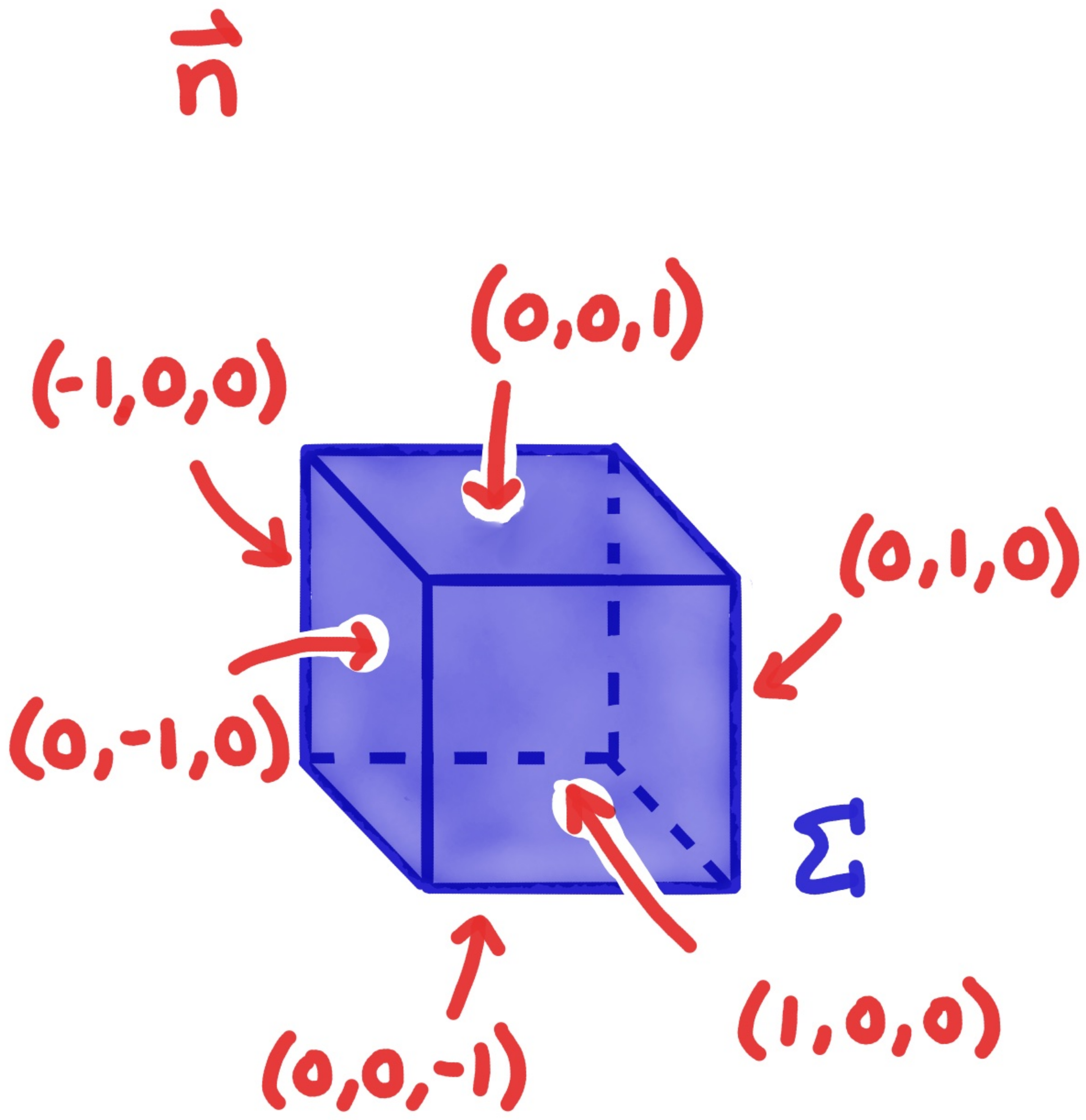

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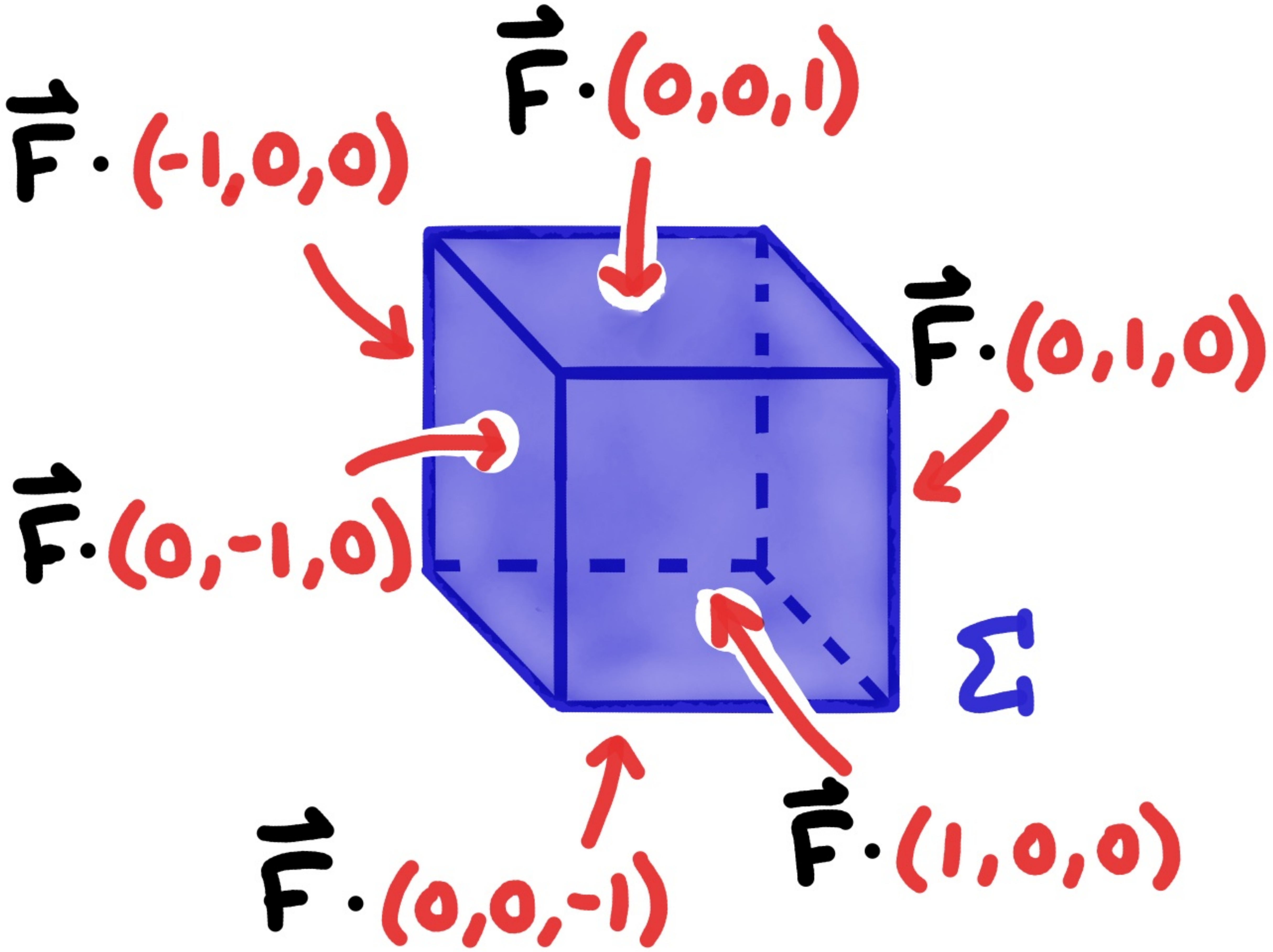

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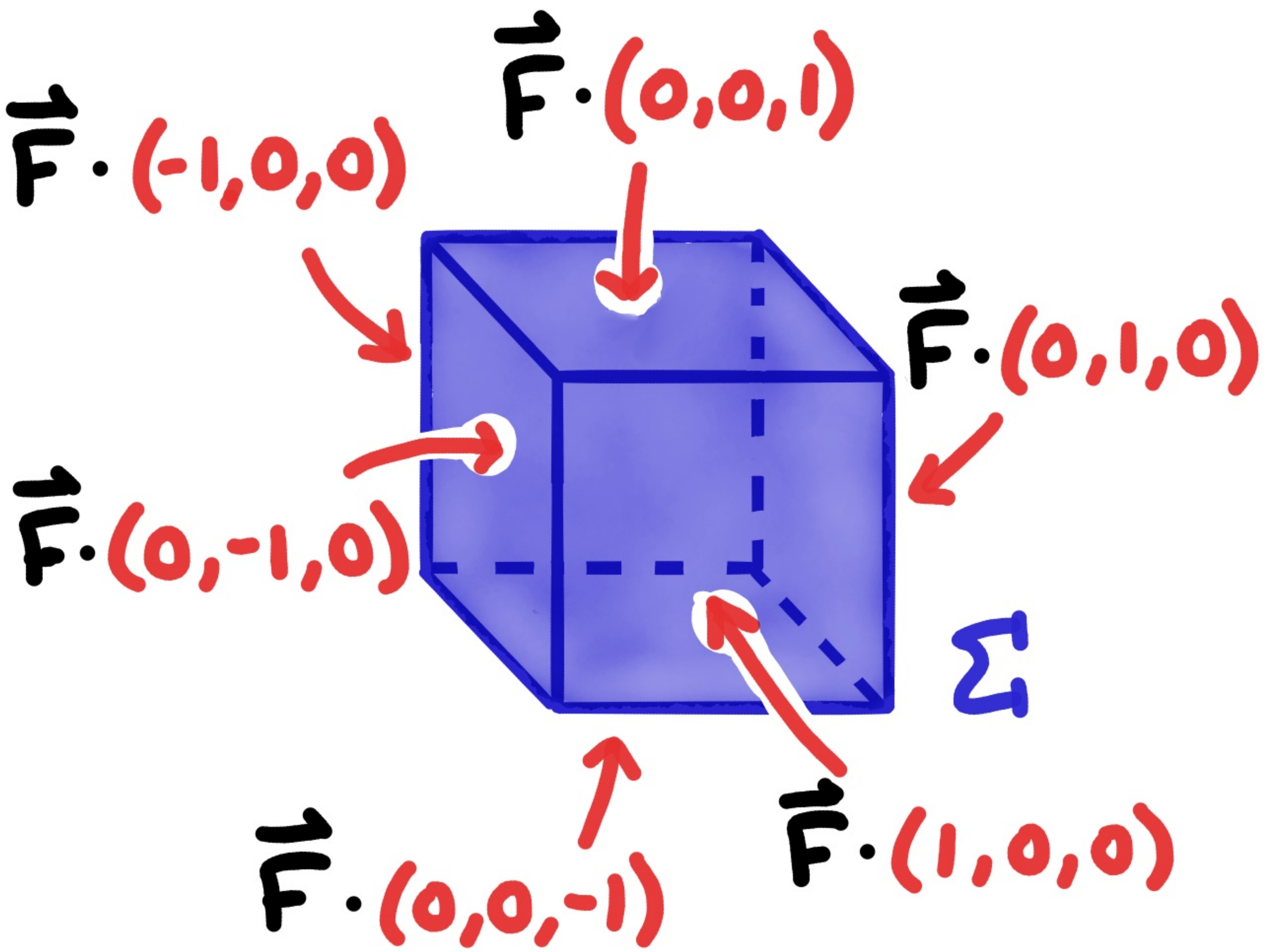

$$\vec{F} \cdot (0,0,1) = (F_1, F_2, F_3) \cdot (0,0,1) = F_3$$

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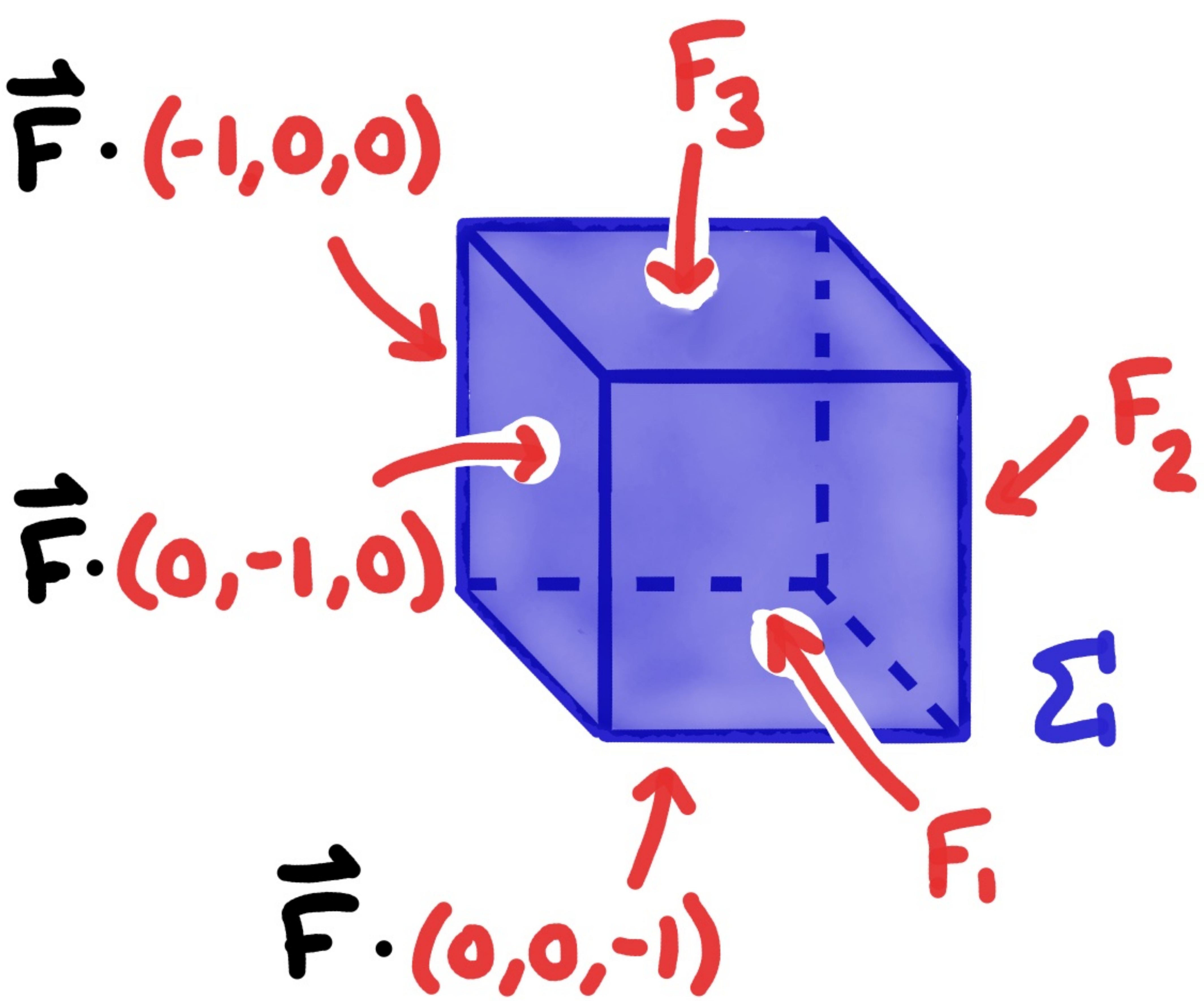

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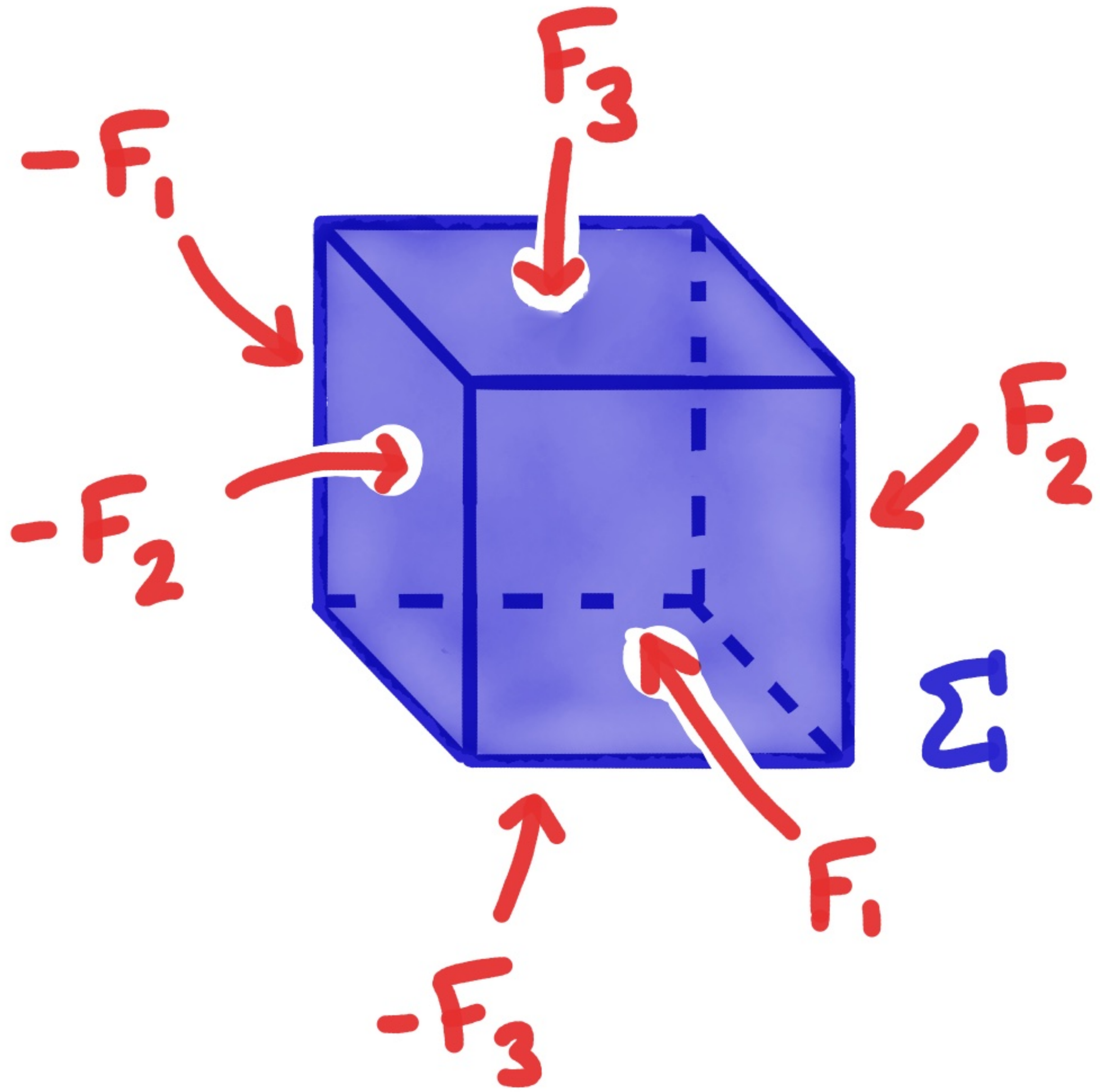

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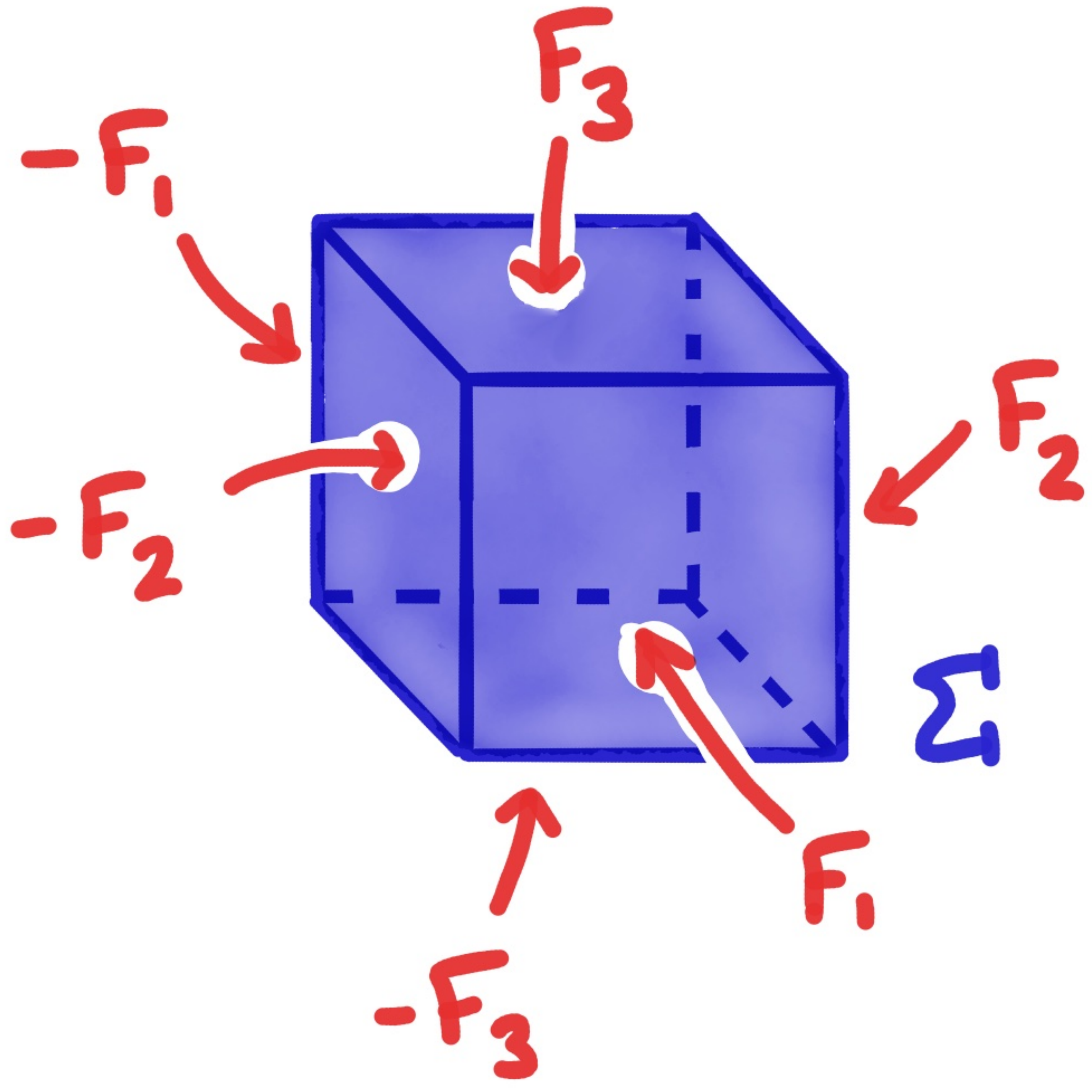

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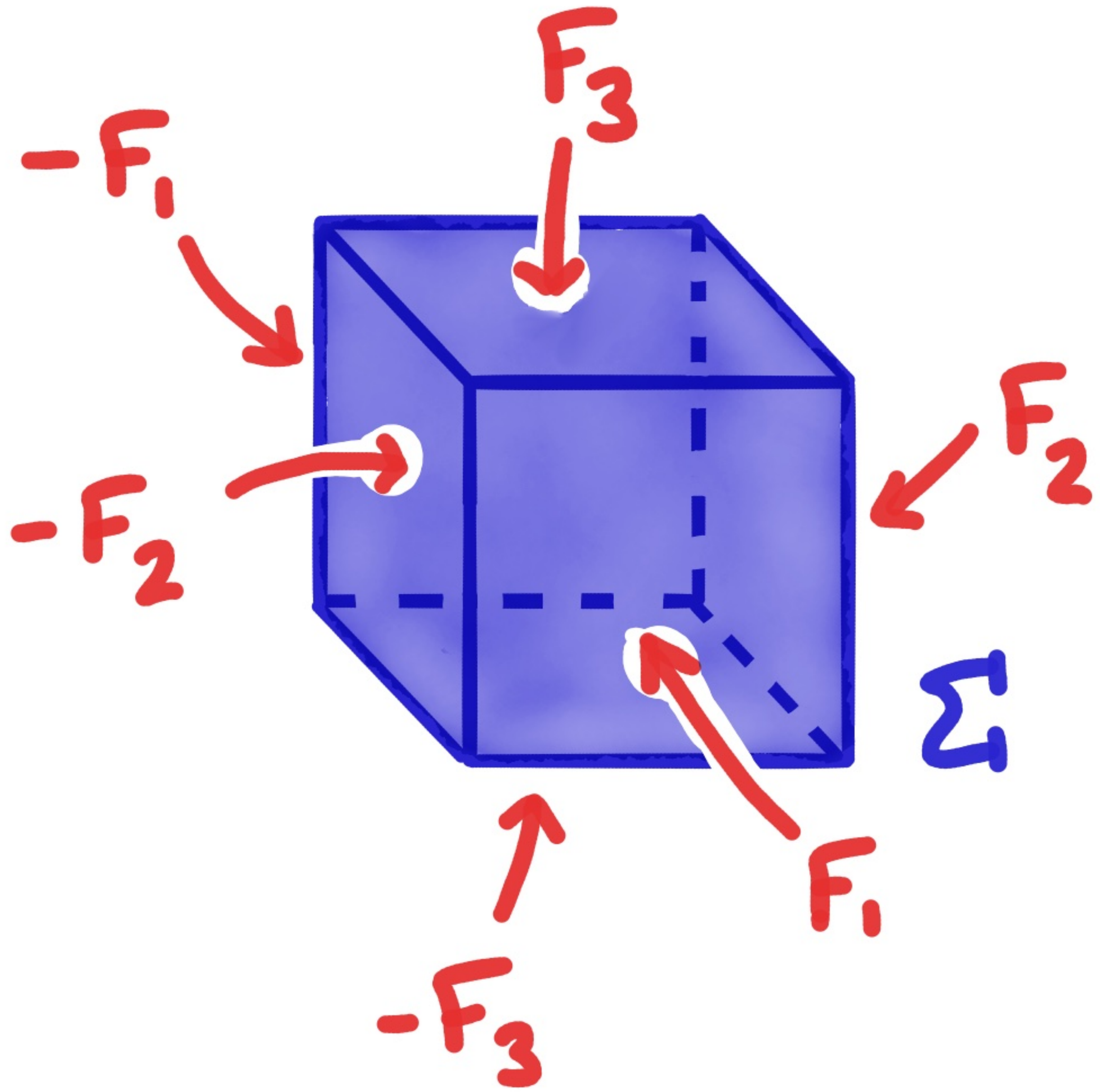

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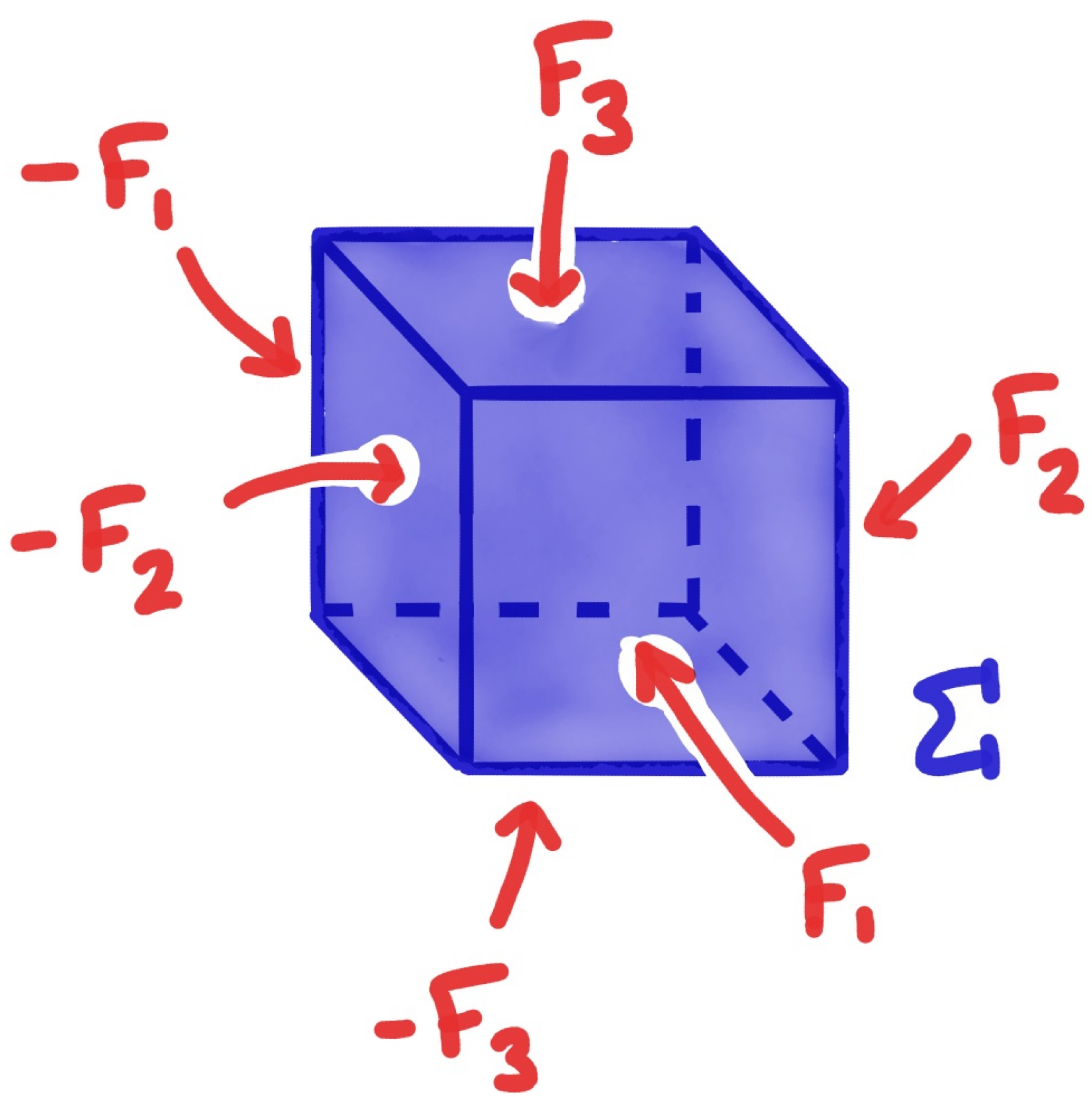


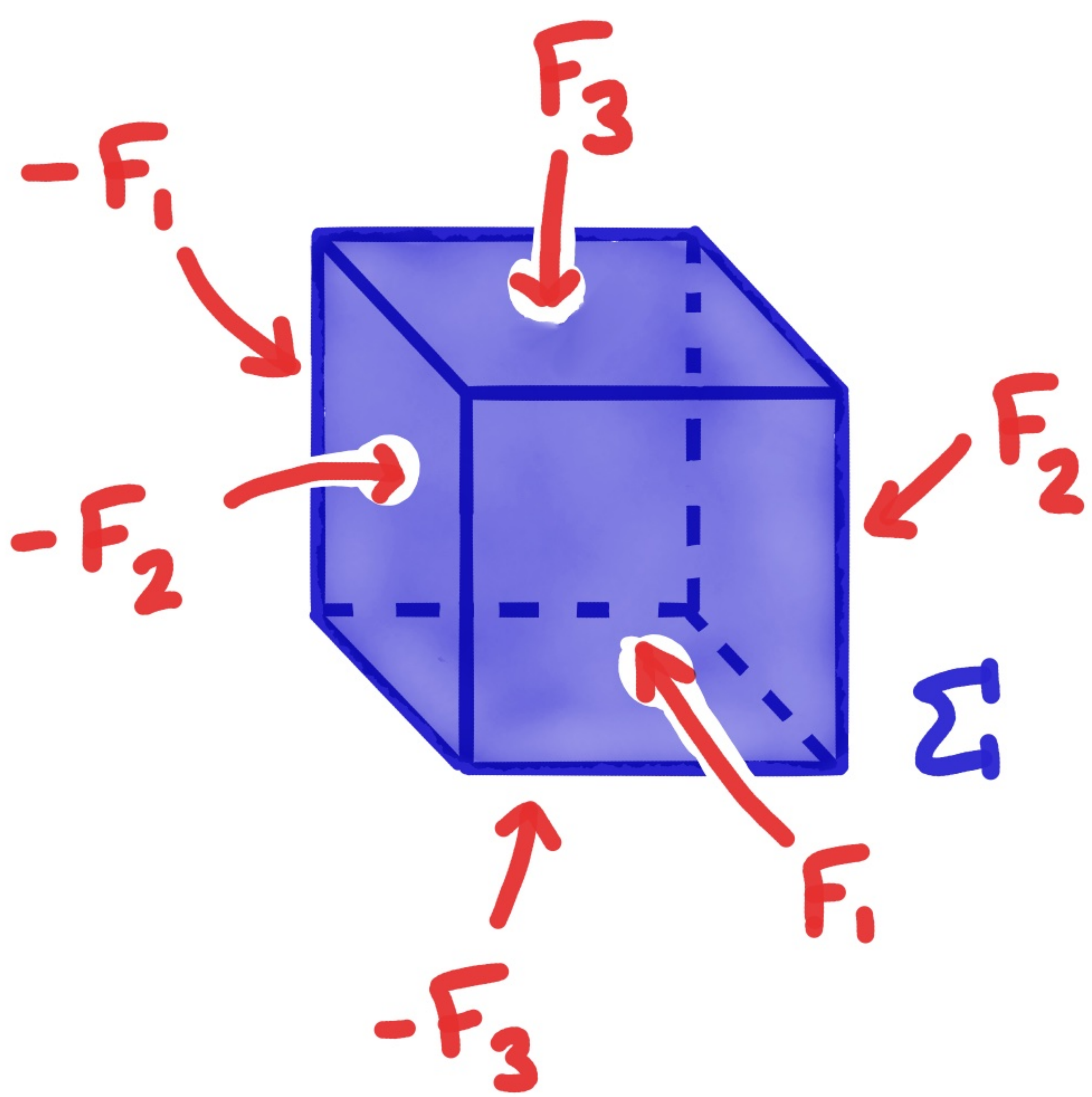
Last lecture we called $\vec{F}\cdot\vec{n}$, $P_{\vec{F}}$, and showed

$$\iint_{\Sigma} \vec{F}\cdot\vec{n}\, dS = \iint_{\Sigma} P_{\vec{F}}\, dS$$

$$= \iiint_{R} \operatorname{div}\vec{F}\, dV.$$

Proof of GDT for cube:

$\vec{F}\cdot\vec{n} = (F_1, F_2, F_3)\cdot\vec{n}$:



Last lecture we called $\vec{F}\cdot\vec{n}$, $P_{\vec{F}}$, and showed

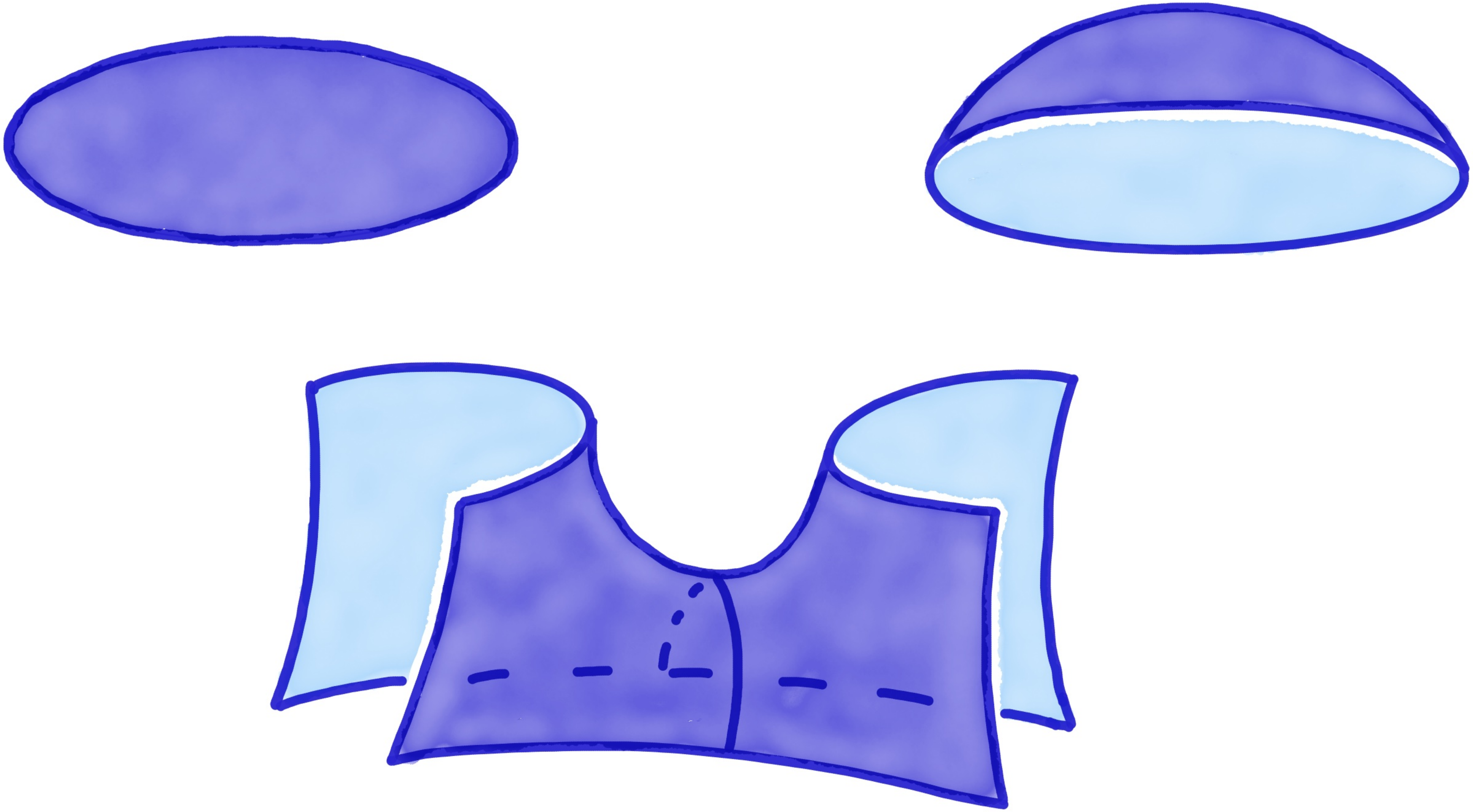
$$\iint_{\Sigma} \vec{F}\cdot\vec{n}\, dS = \iint_{\Sigma} P_{\vec{F}}\, dS$$

$$= \iiint_{R} \operatorname{div}\vec{F}\, dV.$$

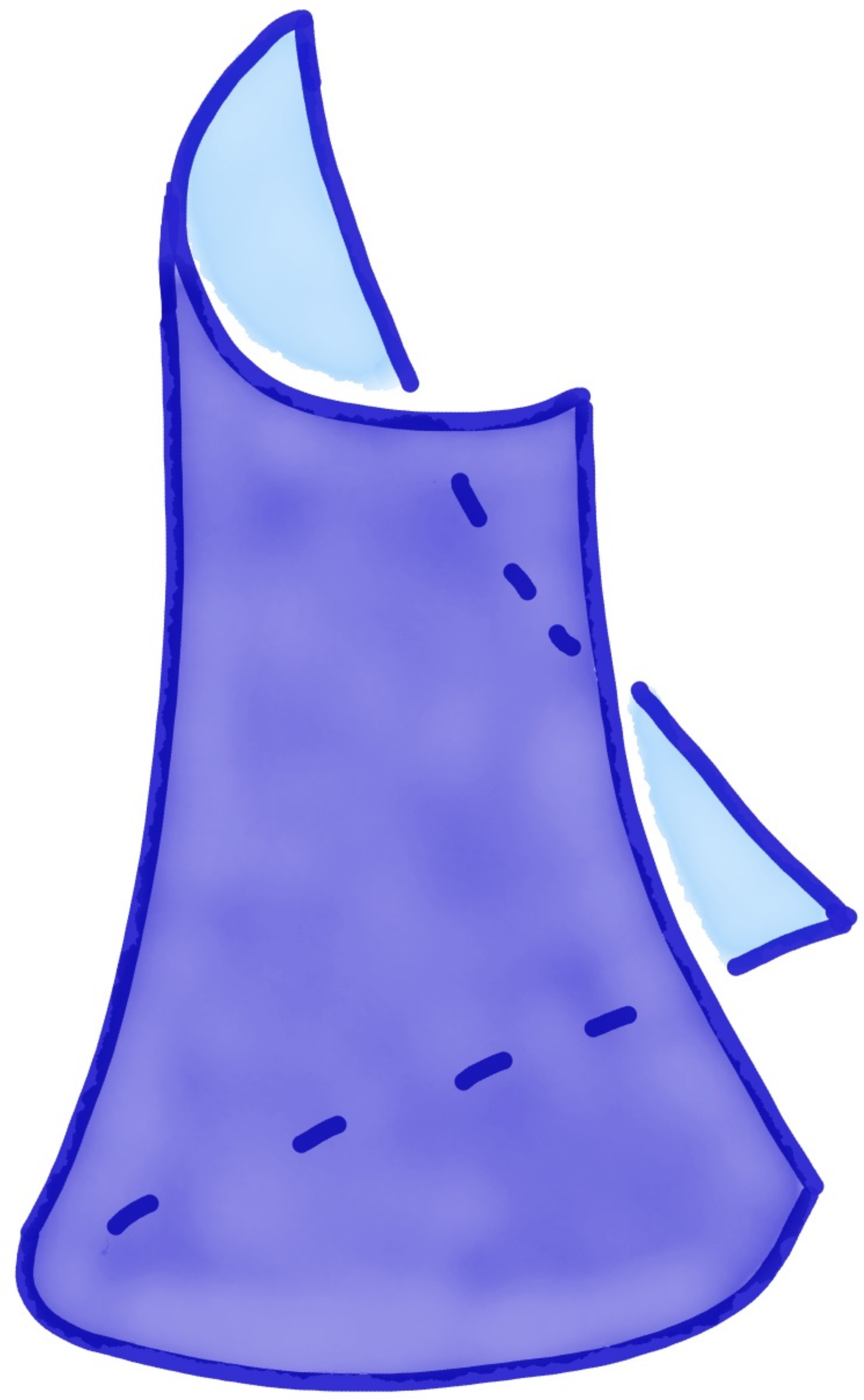
$\boxed{\text{END}}$

III Two-sided surfaces

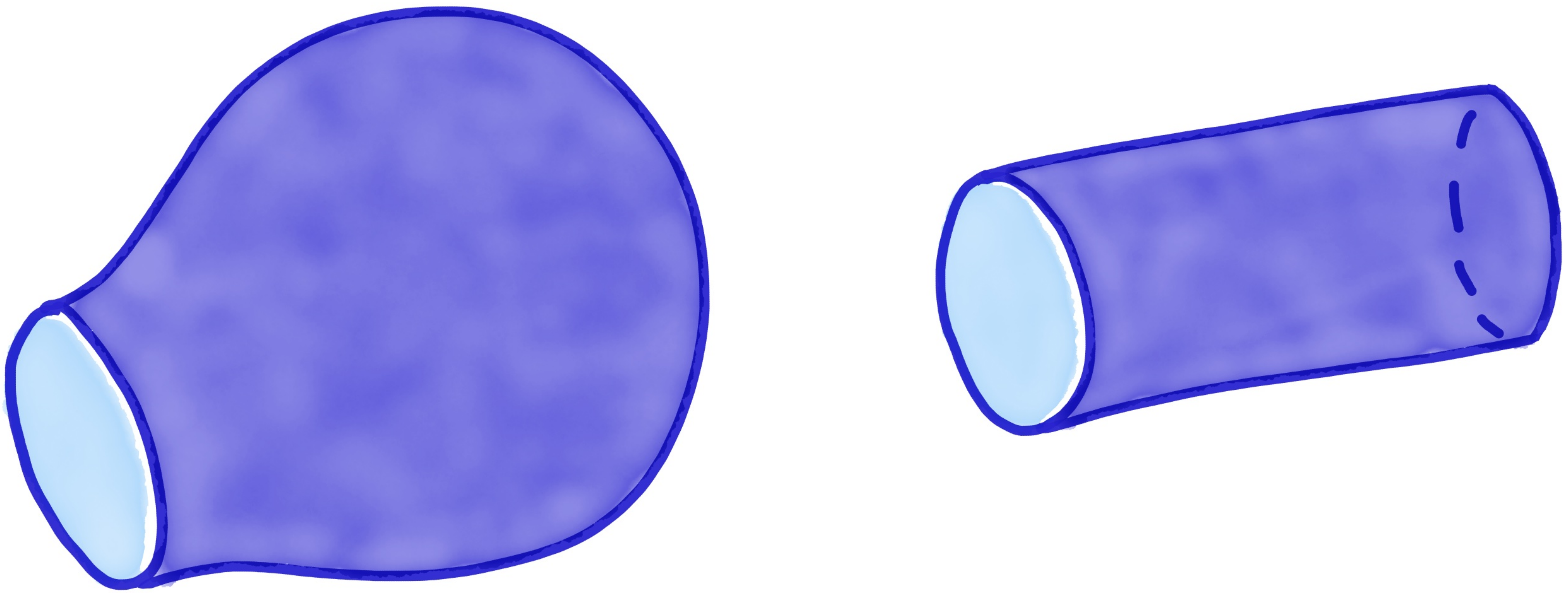
Topside / Underside



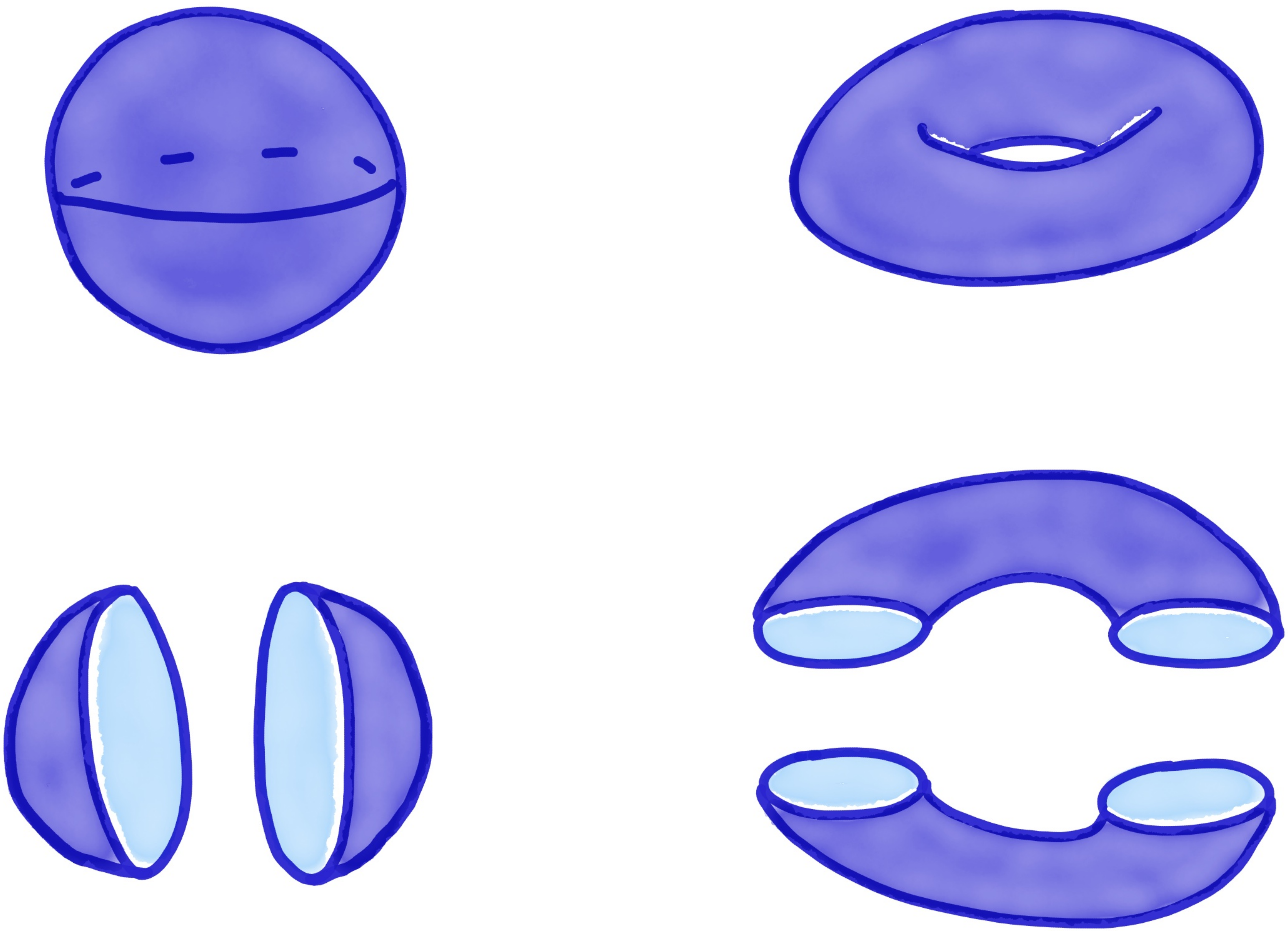
Leftside / Rightside



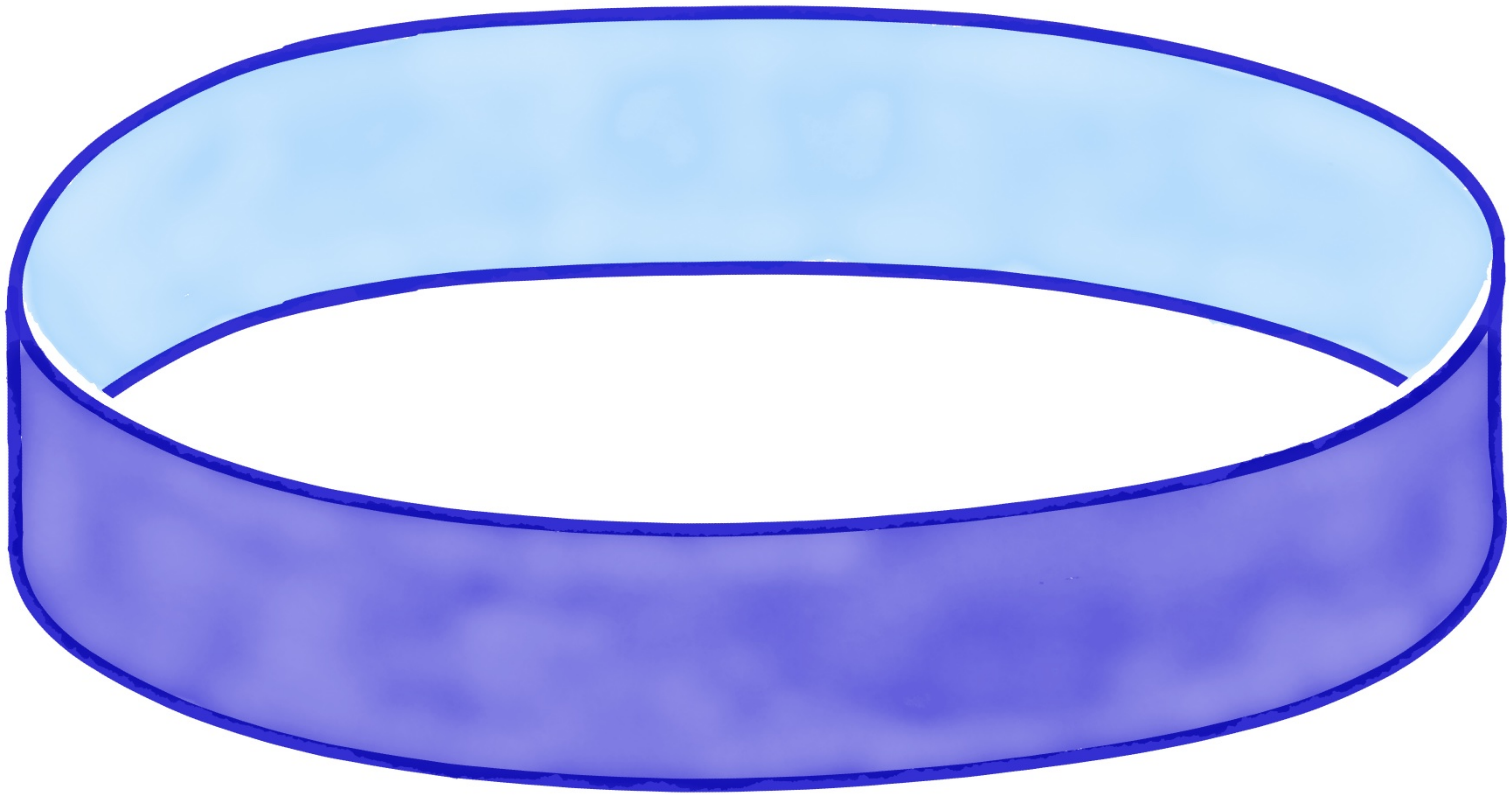
Inside/Outside



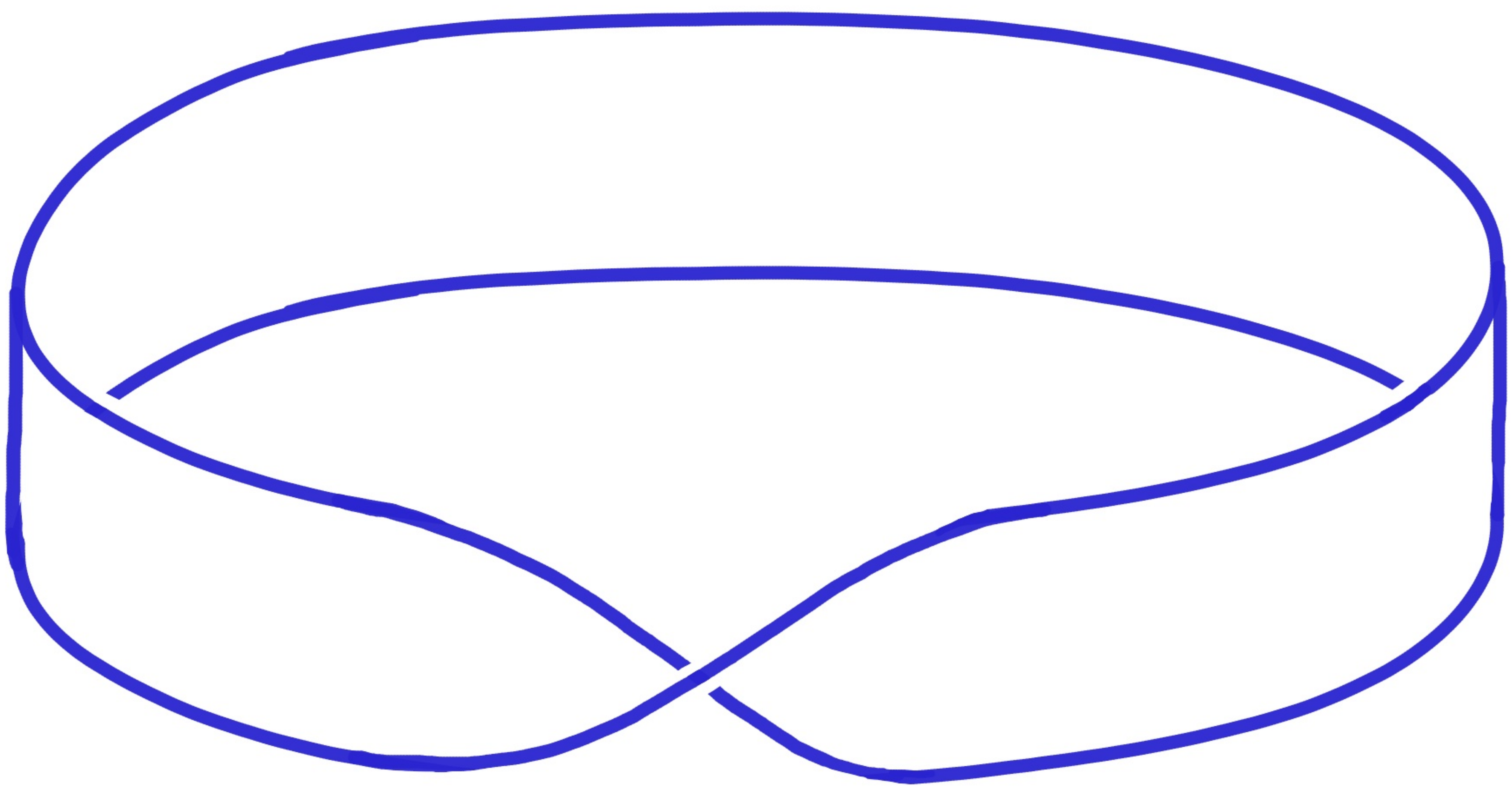
Inside / Outside



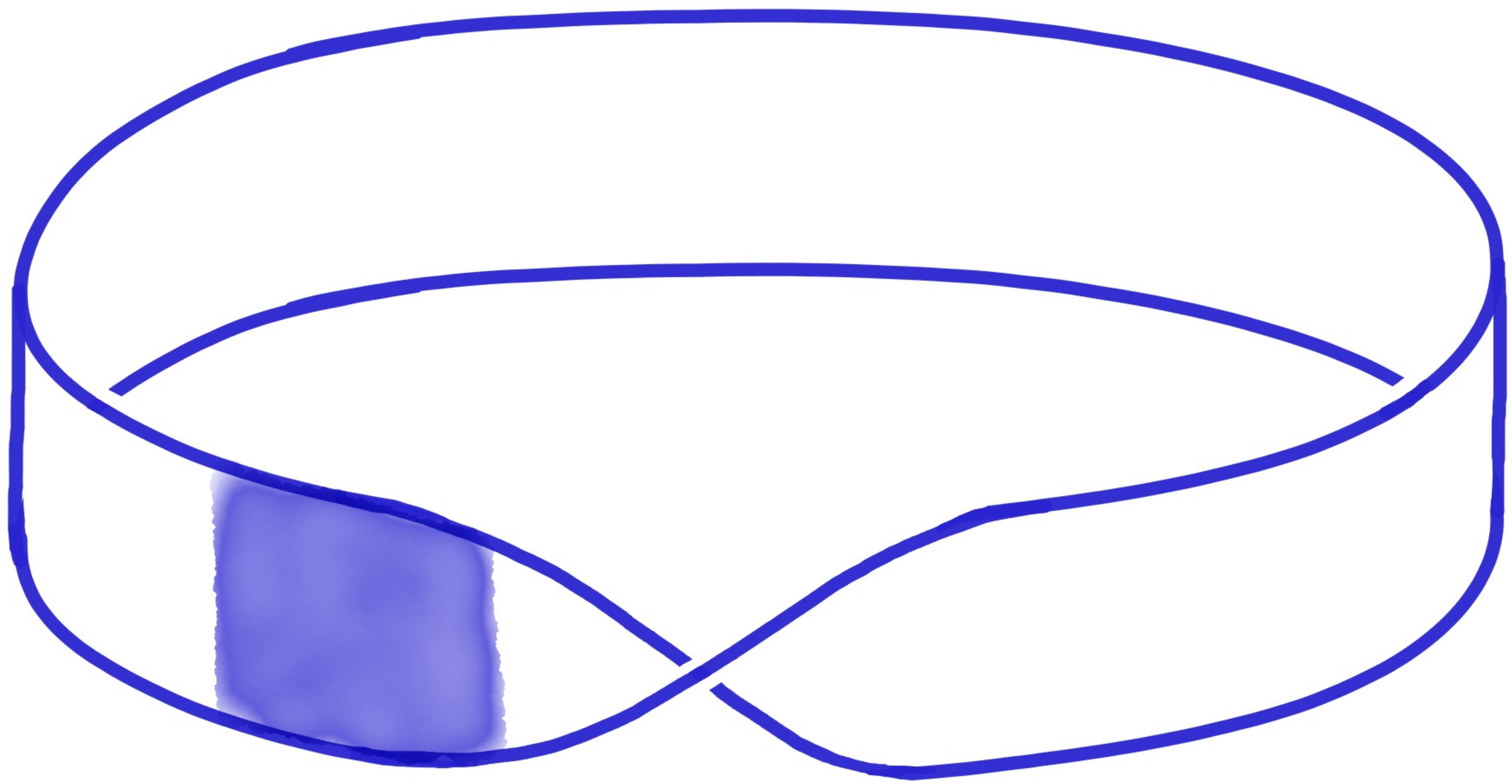
Inside / Outside



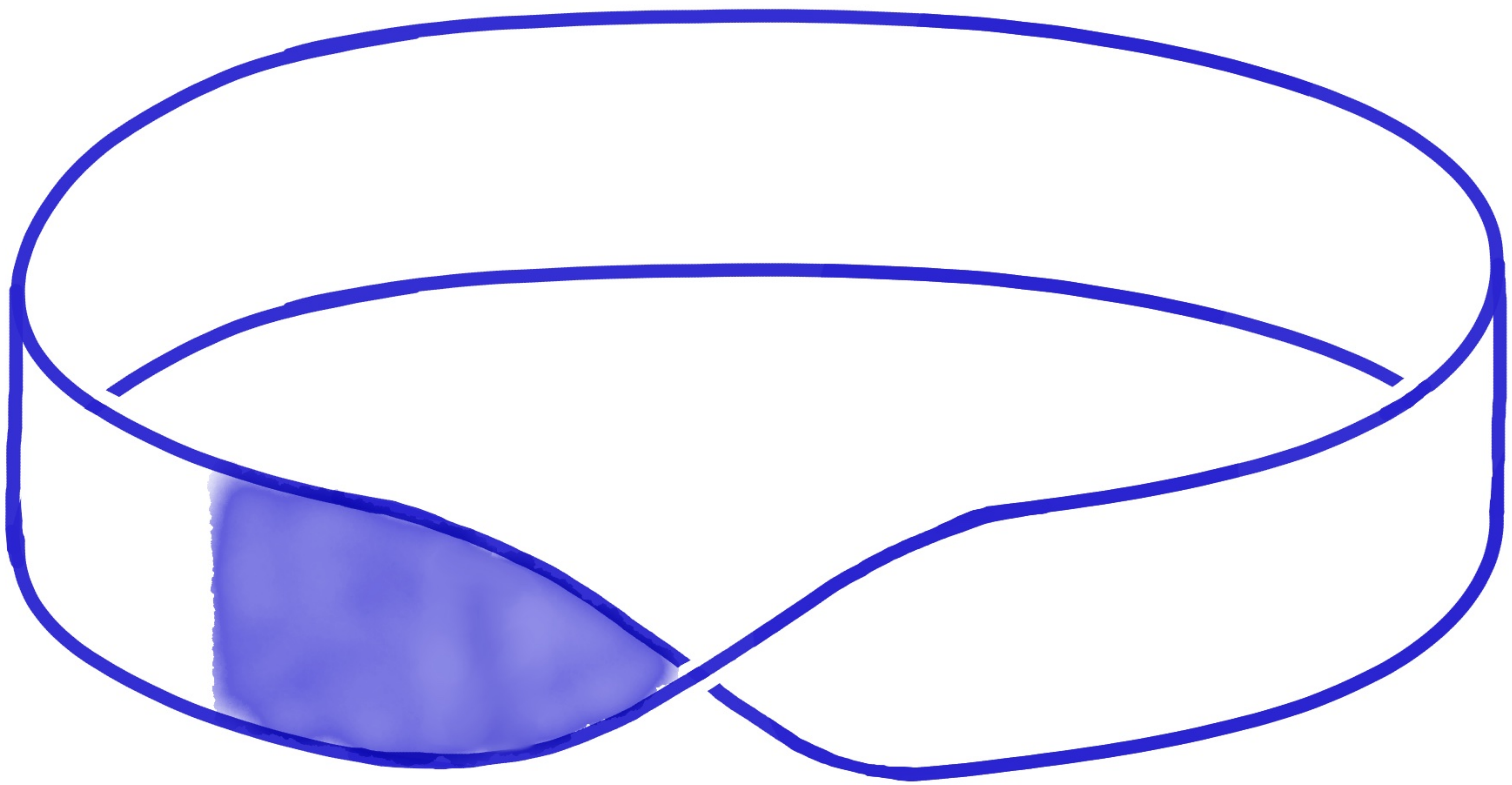
One-sided



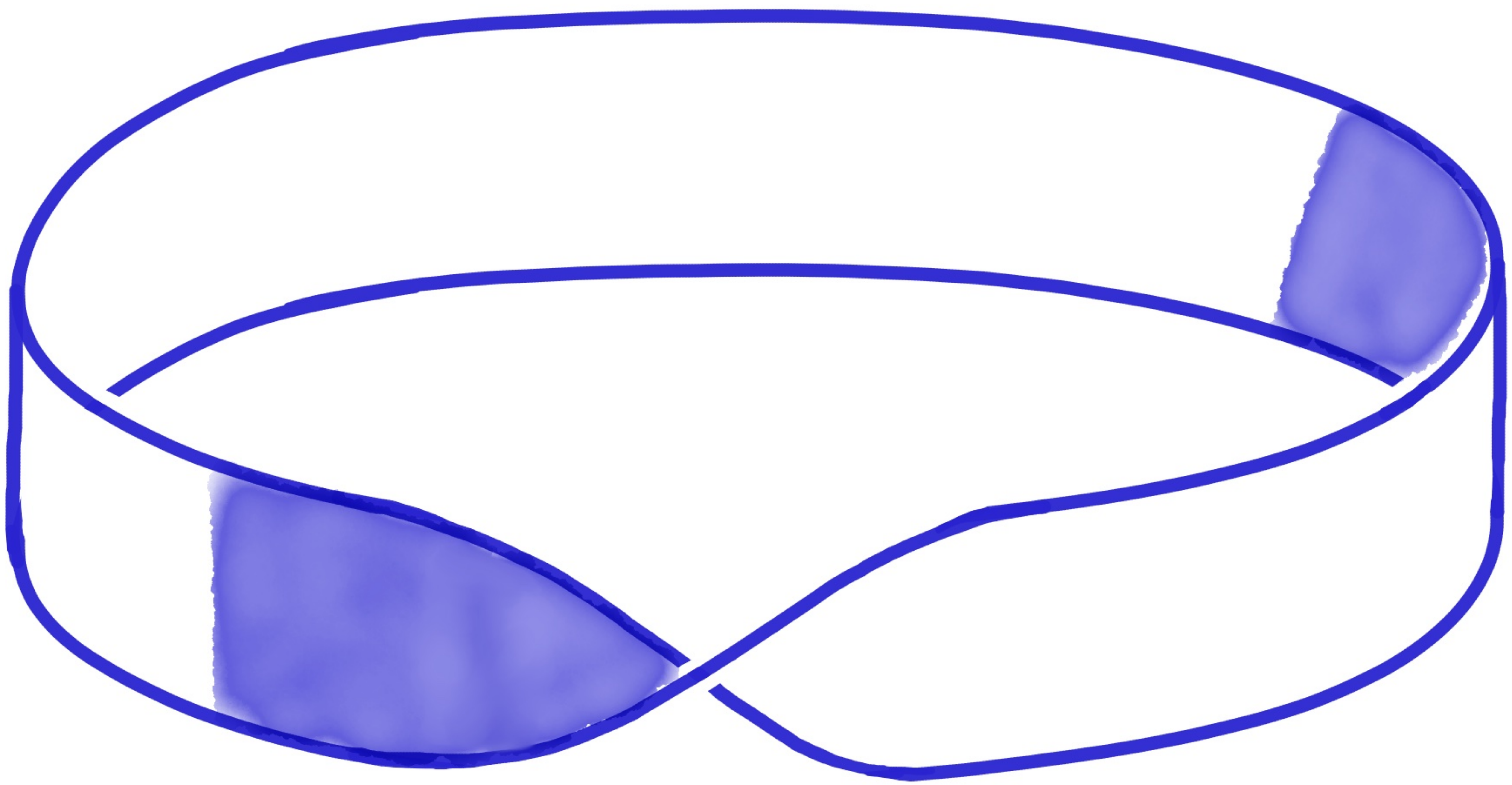
One-sided



One-sided



One - sided



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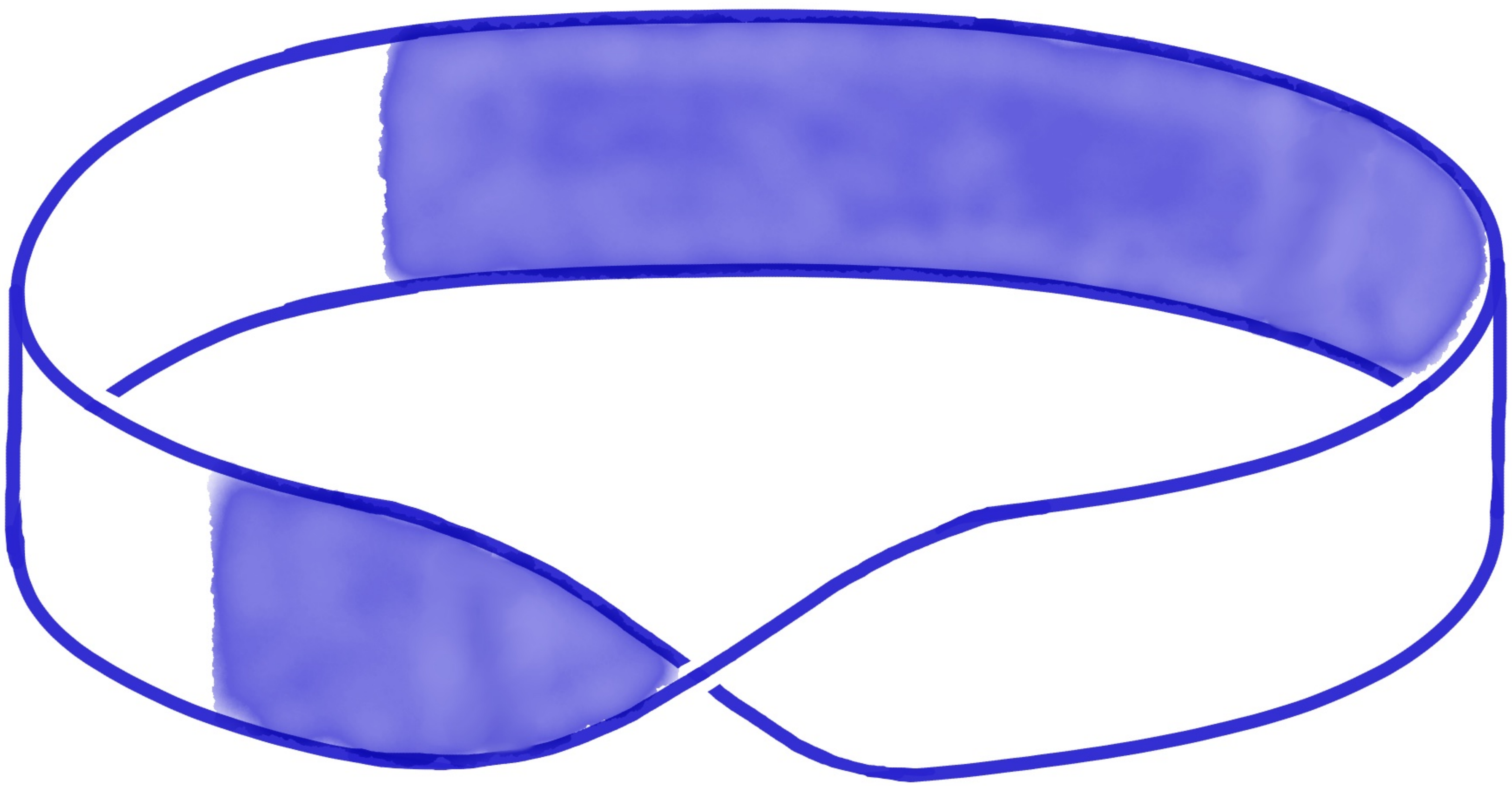
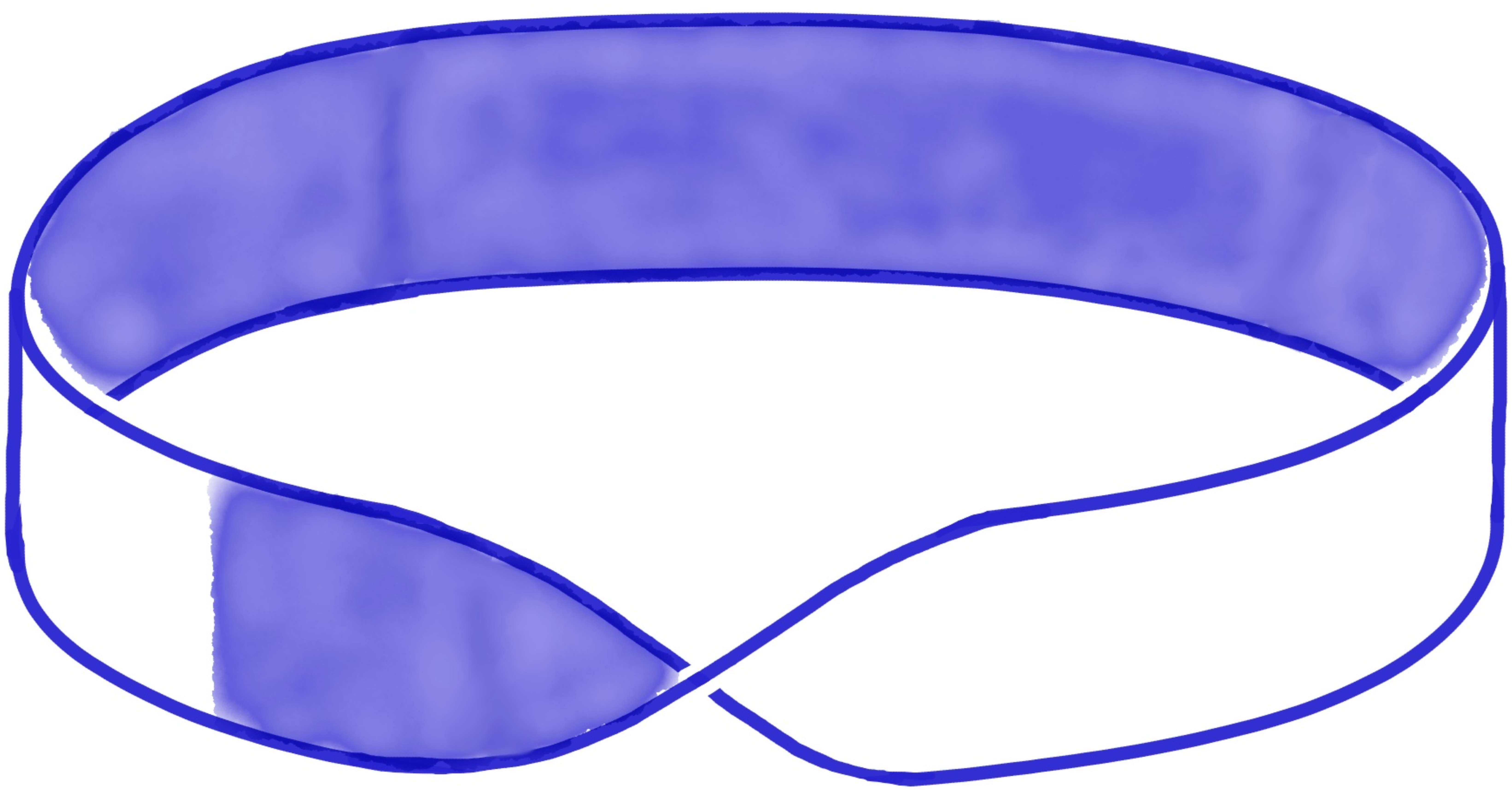
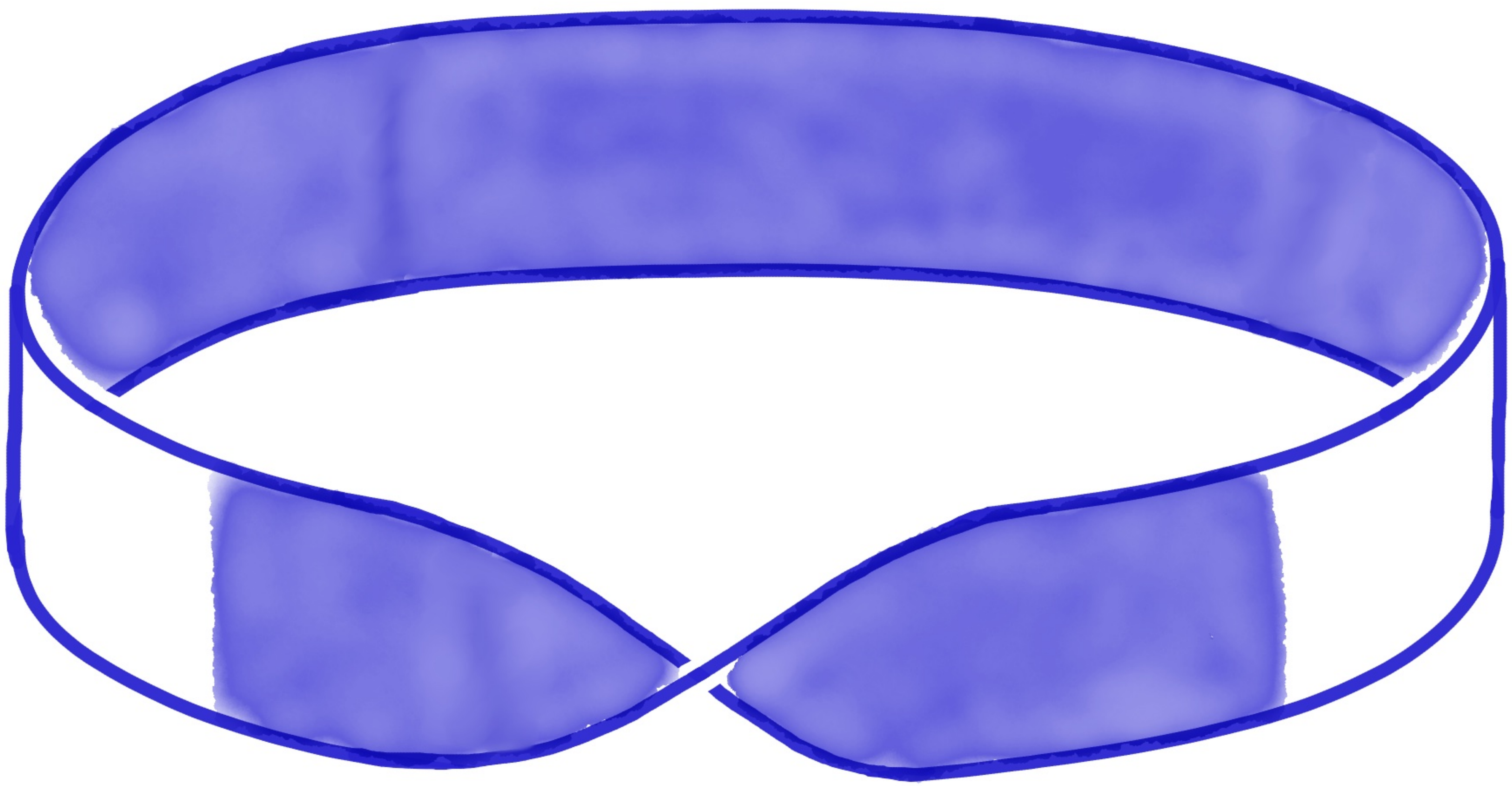


One - sided



One-sided



One-sided