Thirty-two

Ⅰ Surface integrals
for surfaces that are graphs

Ⅱ Surface integrals
for general surfaces

Ⅲ An important example

Ⓘ Surface integrals for surfaces that are graphs.

Surfaces

A surface is a thing Σ such that around every point in Σ there is a small patch that looks like a small patch ("open set") in $\mathbb{R}^2$.

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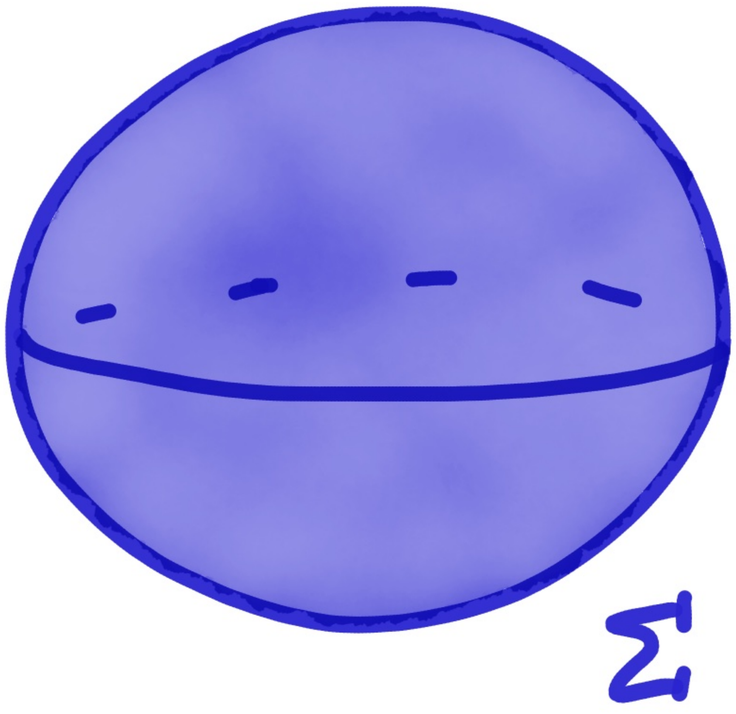

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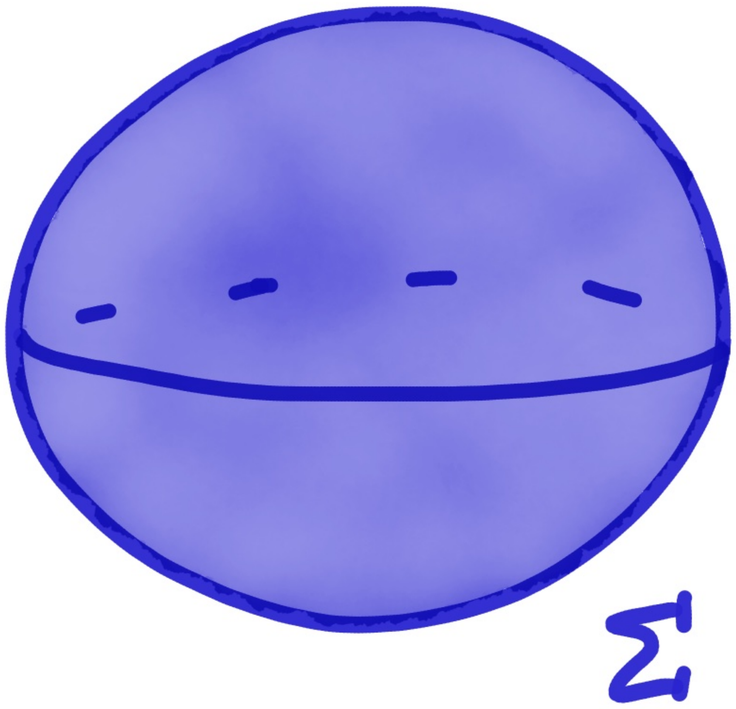

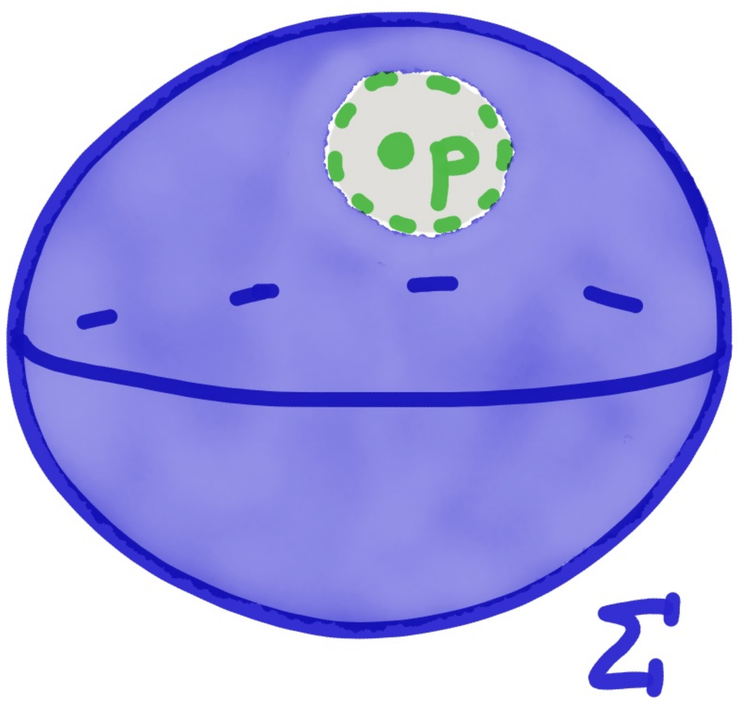

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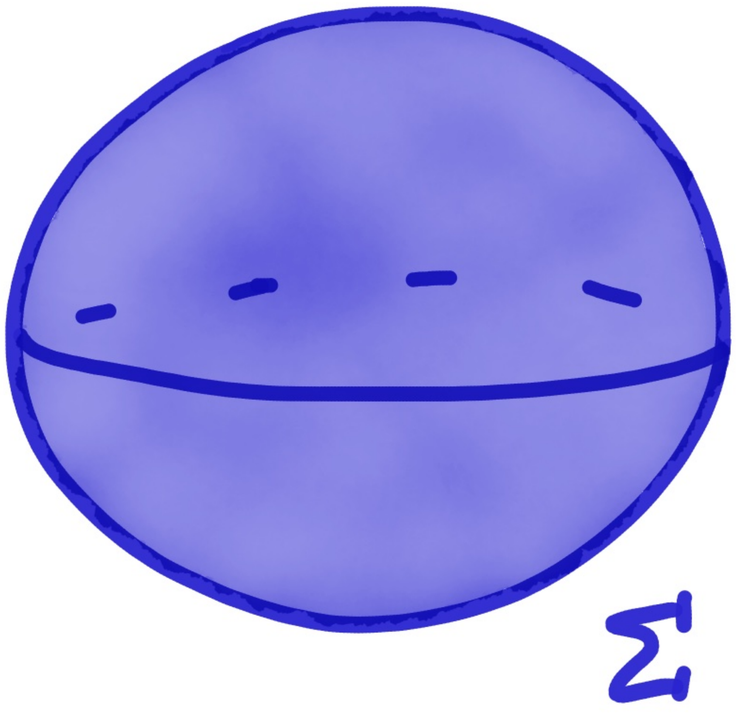

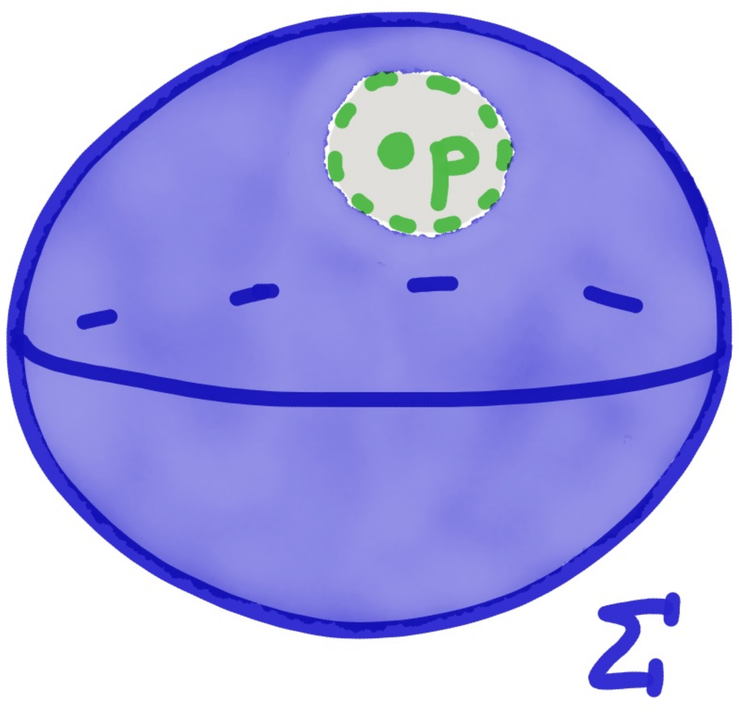

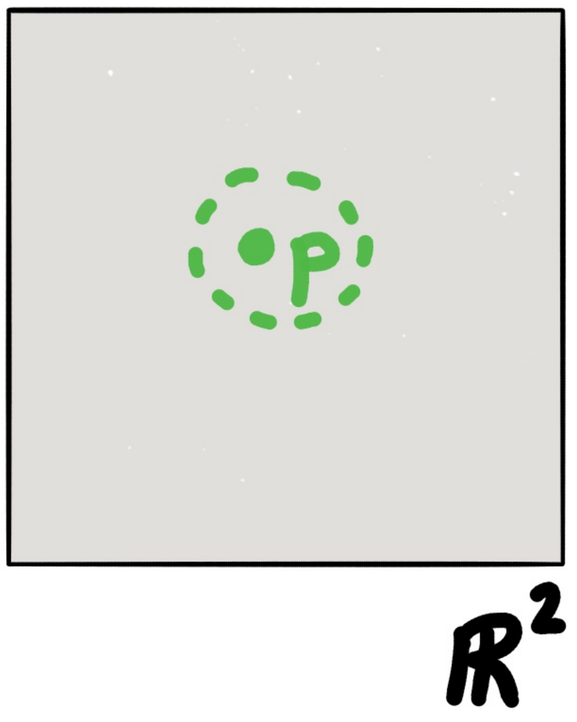

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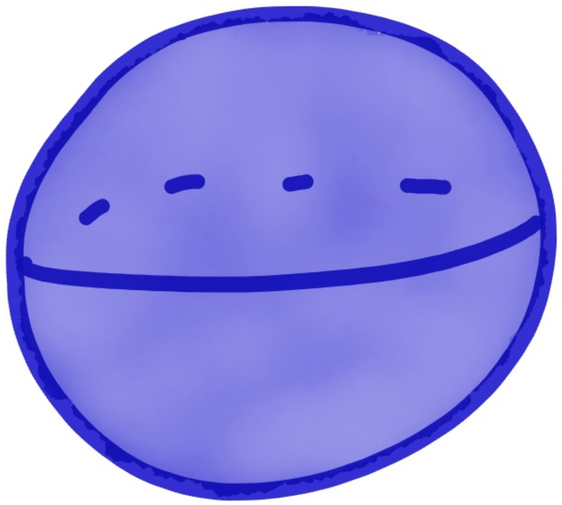
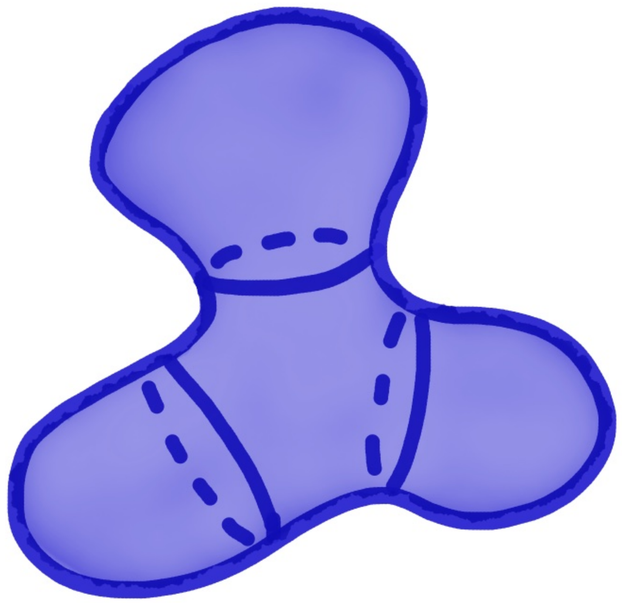
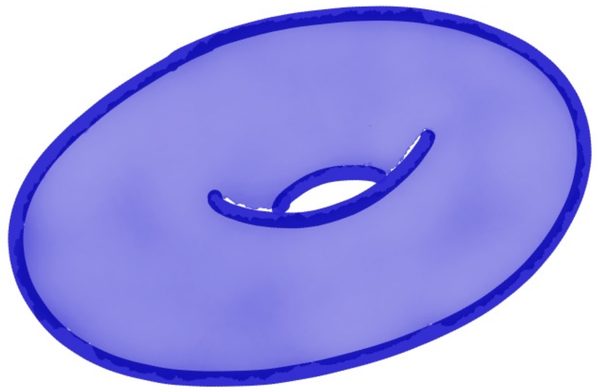
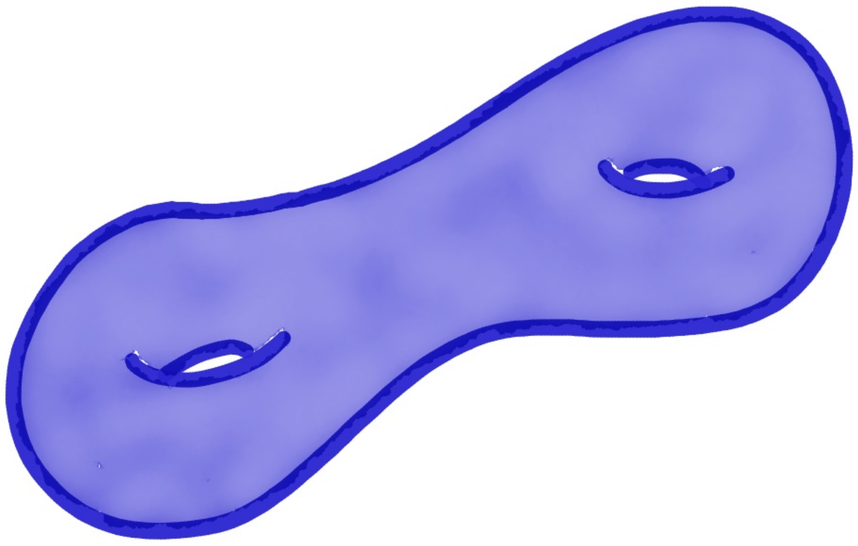
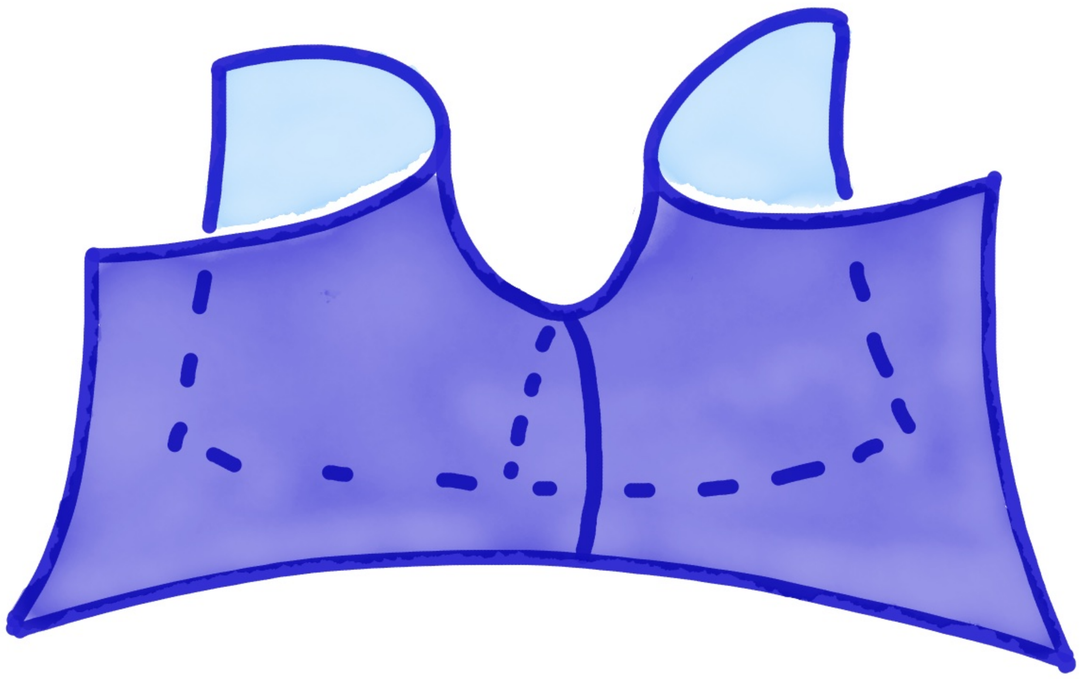
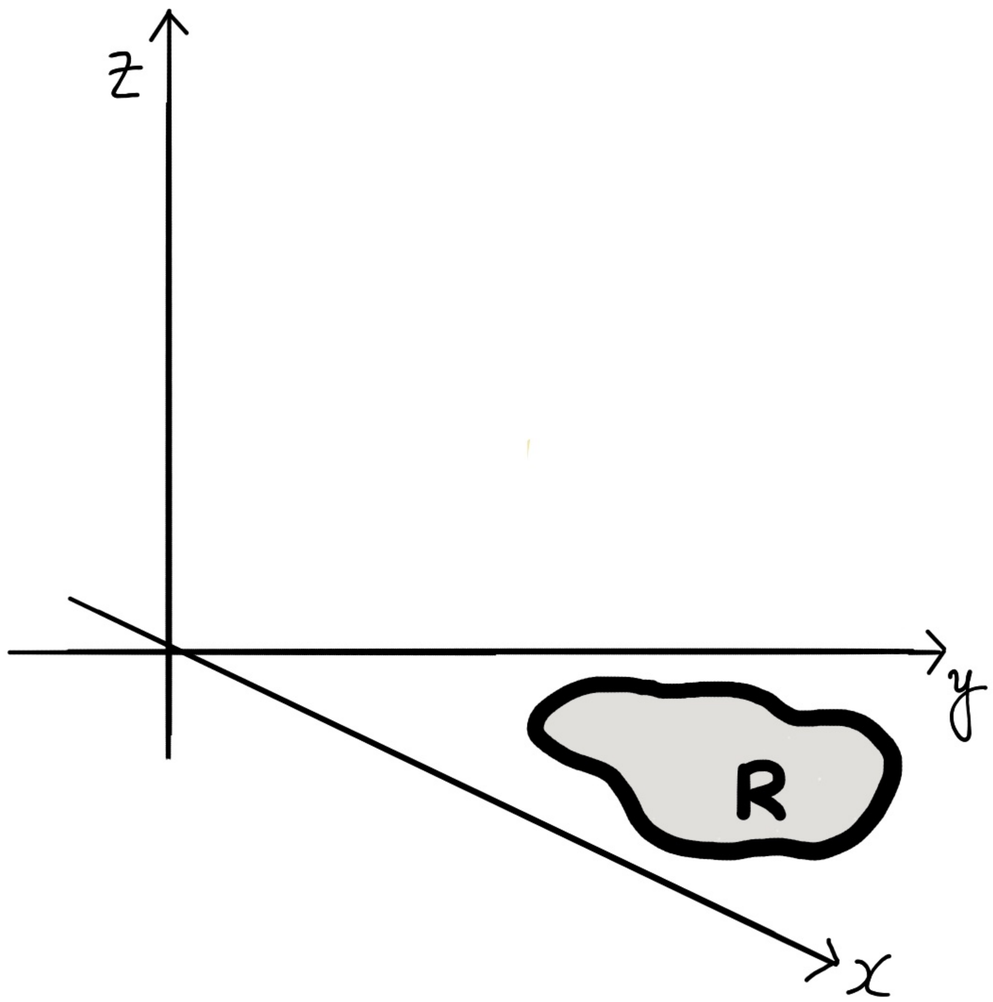






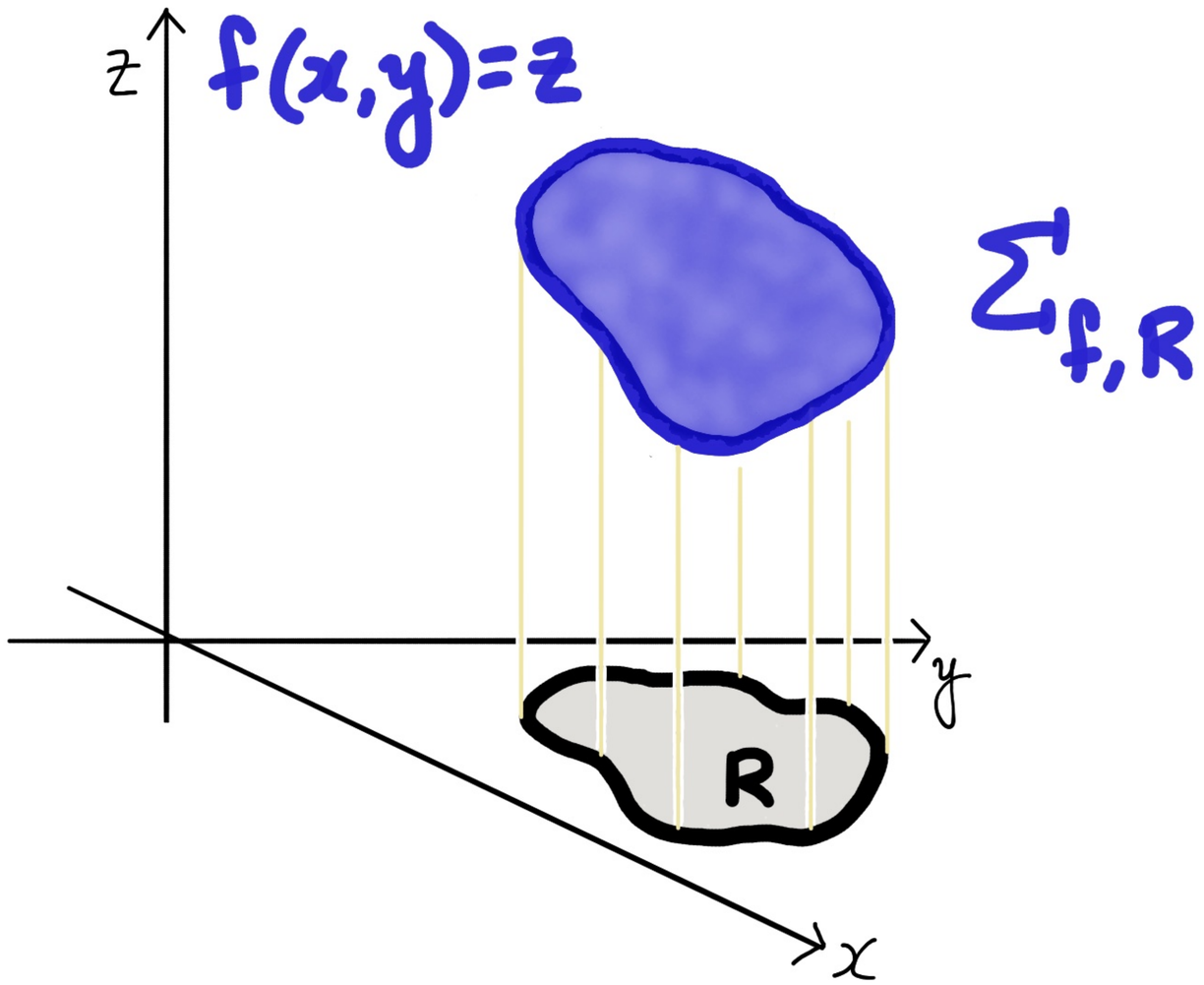






For a region R in $\mathbb{R}^2$,

and $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$,

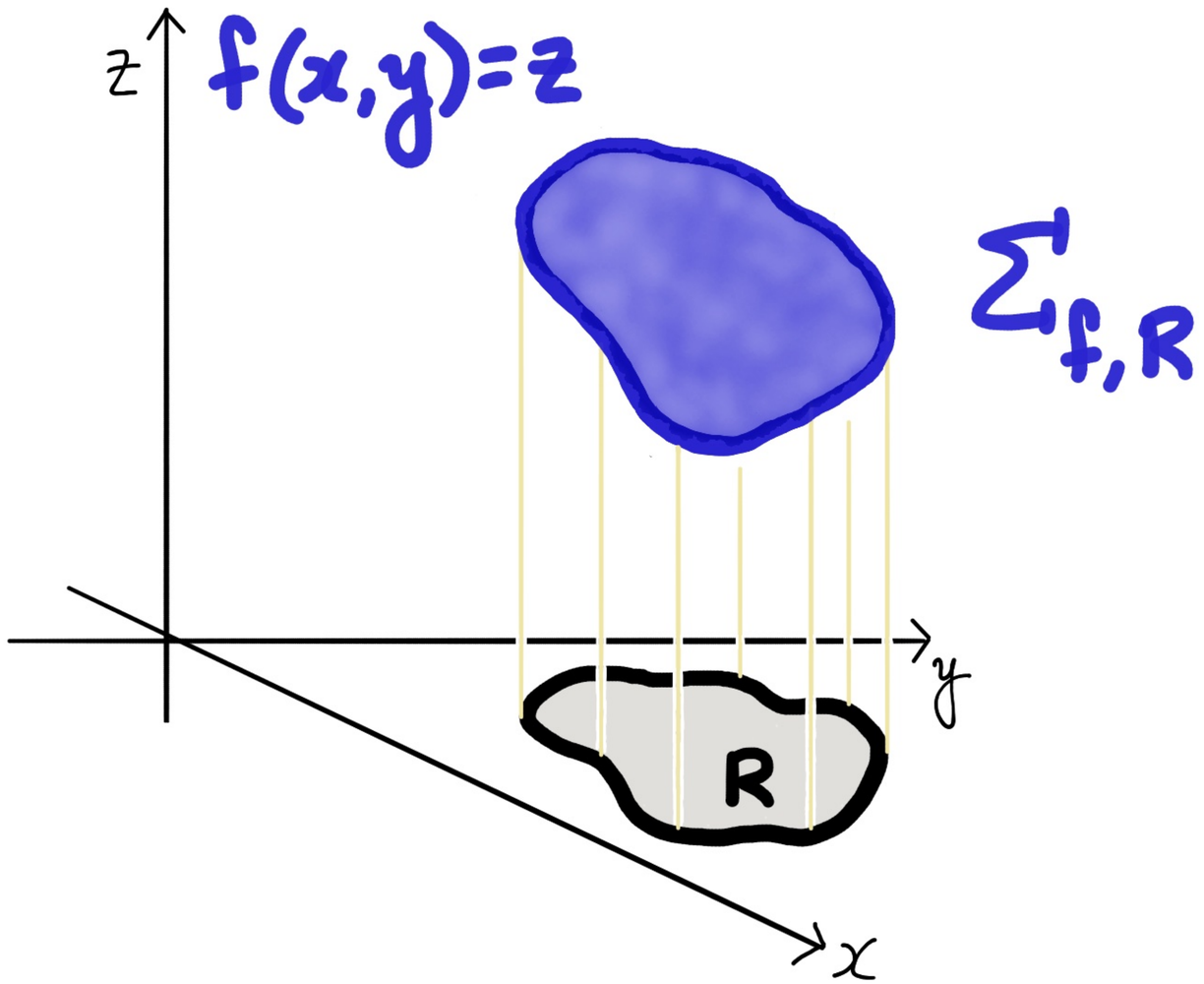


For a region R in $\mathbb{R}^2$,

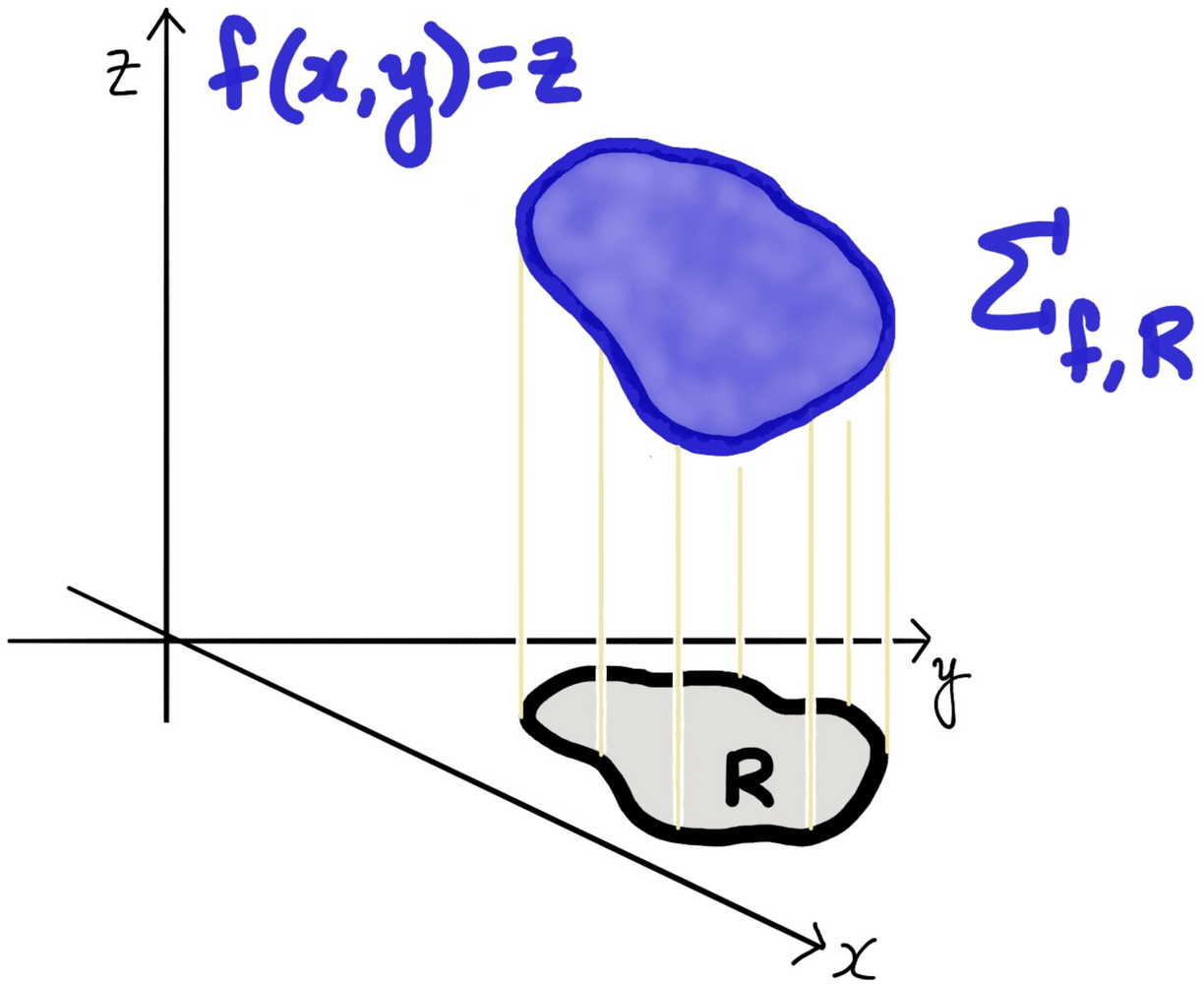
and $f: \mathbb{R}^2 \to \mathbb{R}$,

let $\Sigma_{f,R}$ be the portion of the graph

of $f(x,y)=z$ lying above (or below) R.



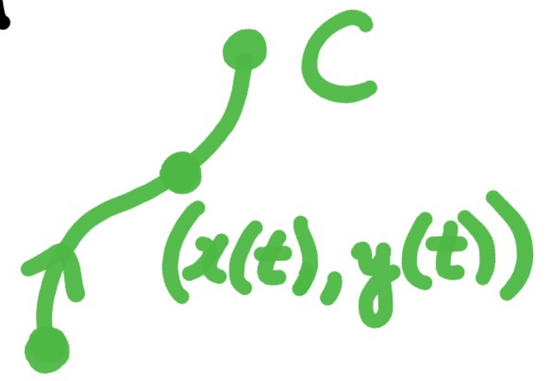
$\Sigma_{f,R}$ is a surface.



$$\text{Surface area}\left(\Sigma_{f,R}\right) = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA$$

For a curve C in $\mathbb{R}^2$ parametrized by $x(t), y(t)$, $a \le t \le b$,

$$\text{Arclength}(C) = \int_a^b \|(x'(t), y'(t))\| dt.$$



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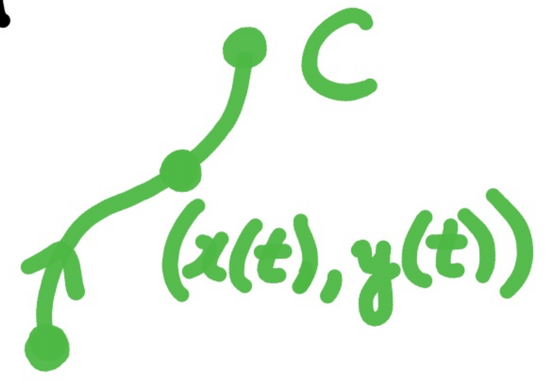

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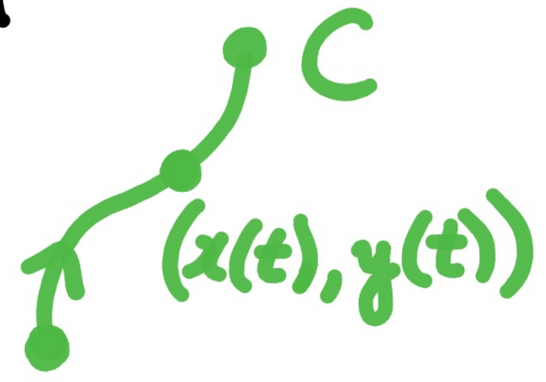

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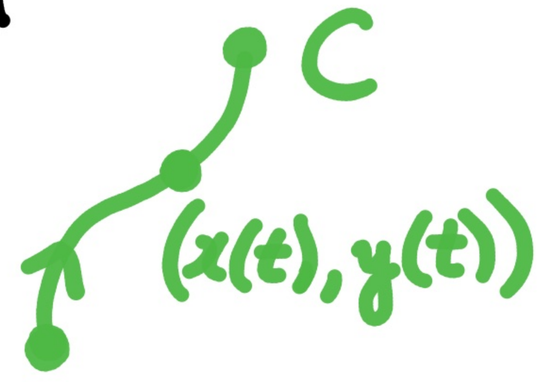

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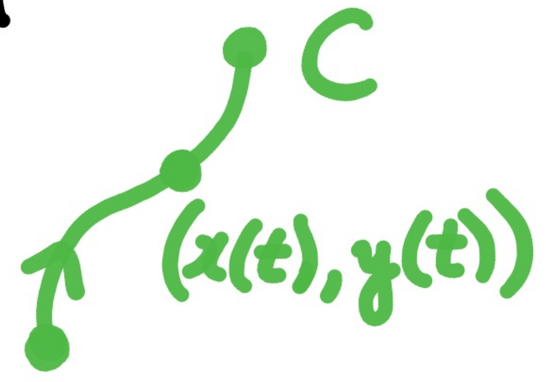

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In particular, $\text{Arclength}(C) = \int_C ds$.

$$\text{Surface area}\left(\Sigma_{f,R}\right) = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA.$$

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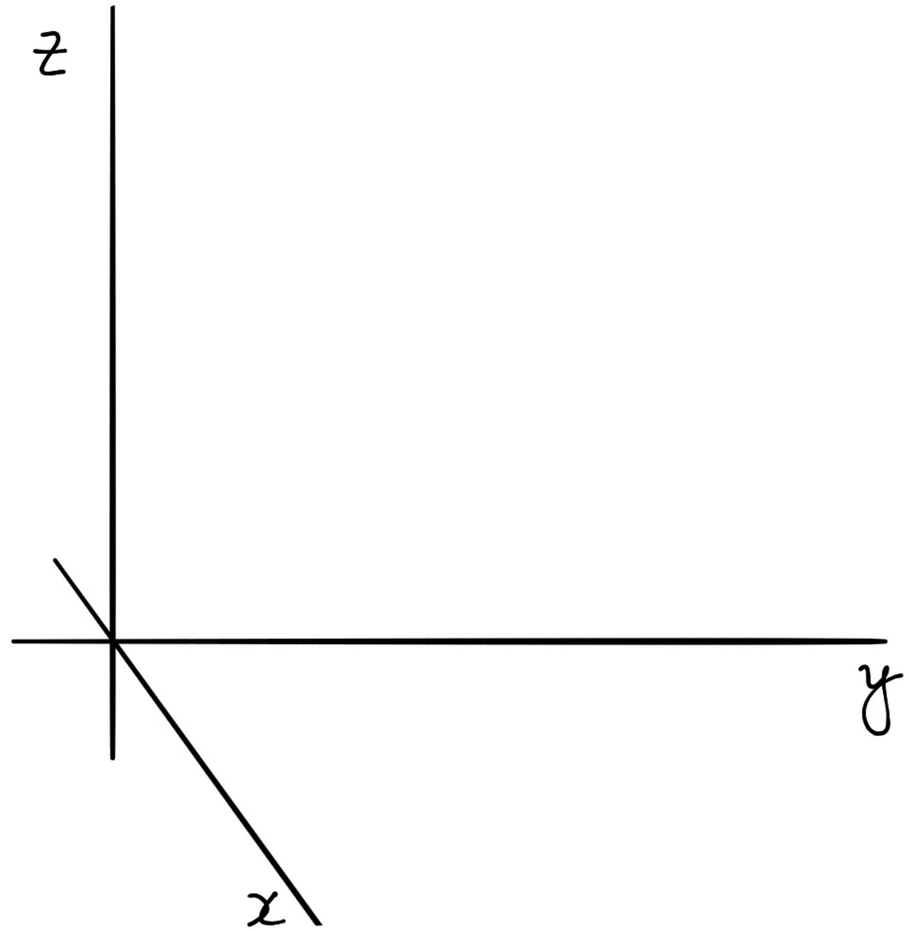
In particular, $\text{Surface area}\left(\Sigma_{f,R}\right) = \iint_{\Sigma_{f,R}} dS.$

Line integrals

Surface integrals



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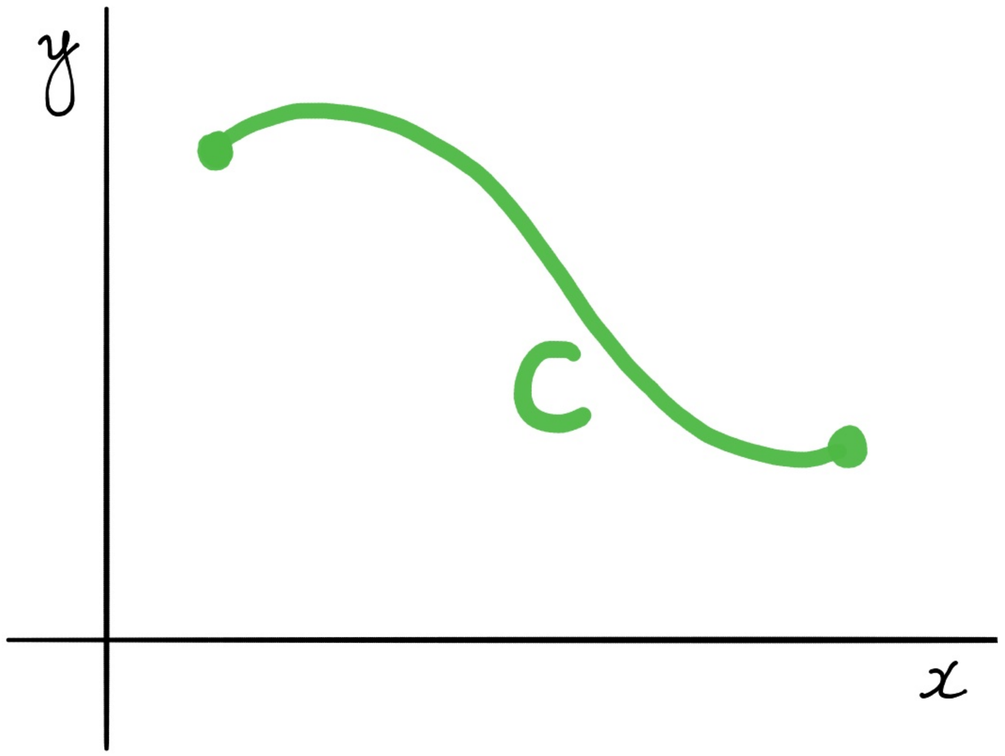

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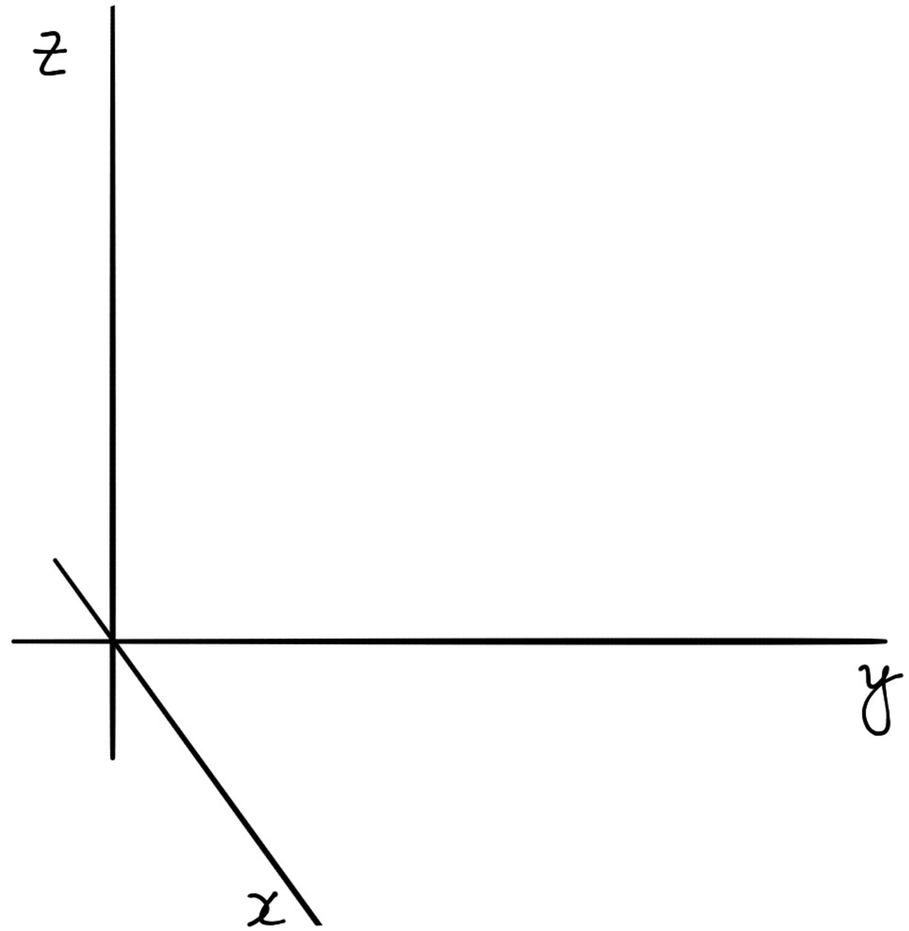

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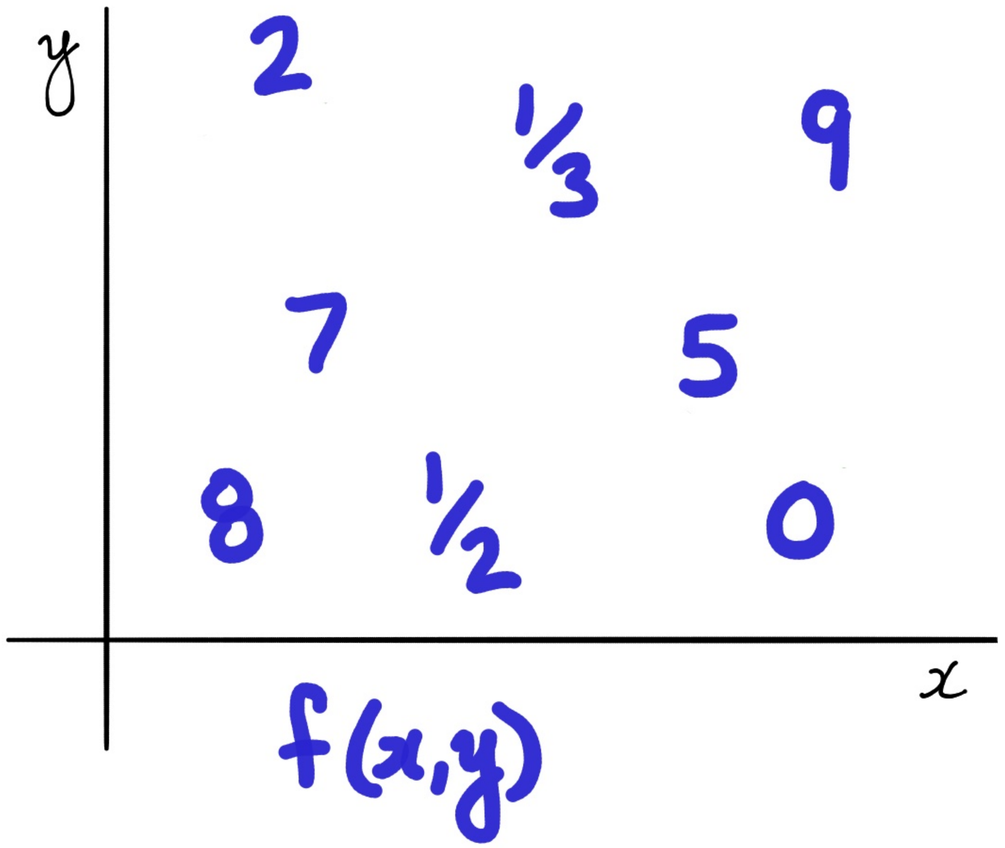

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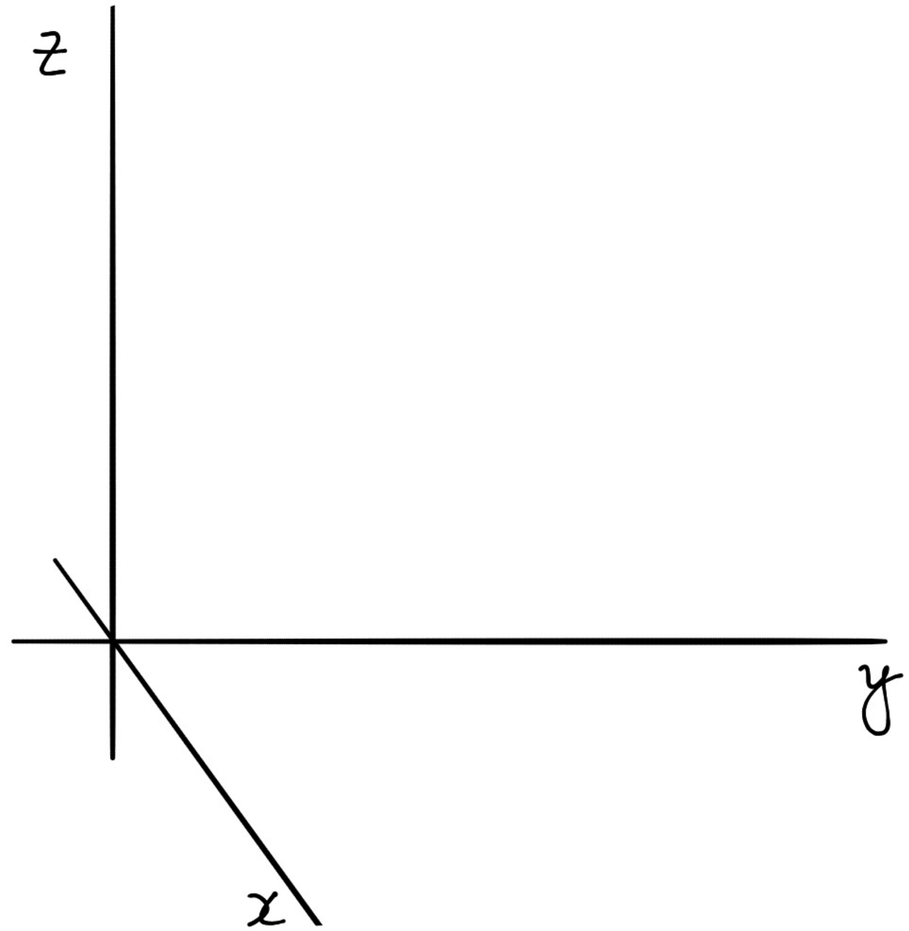

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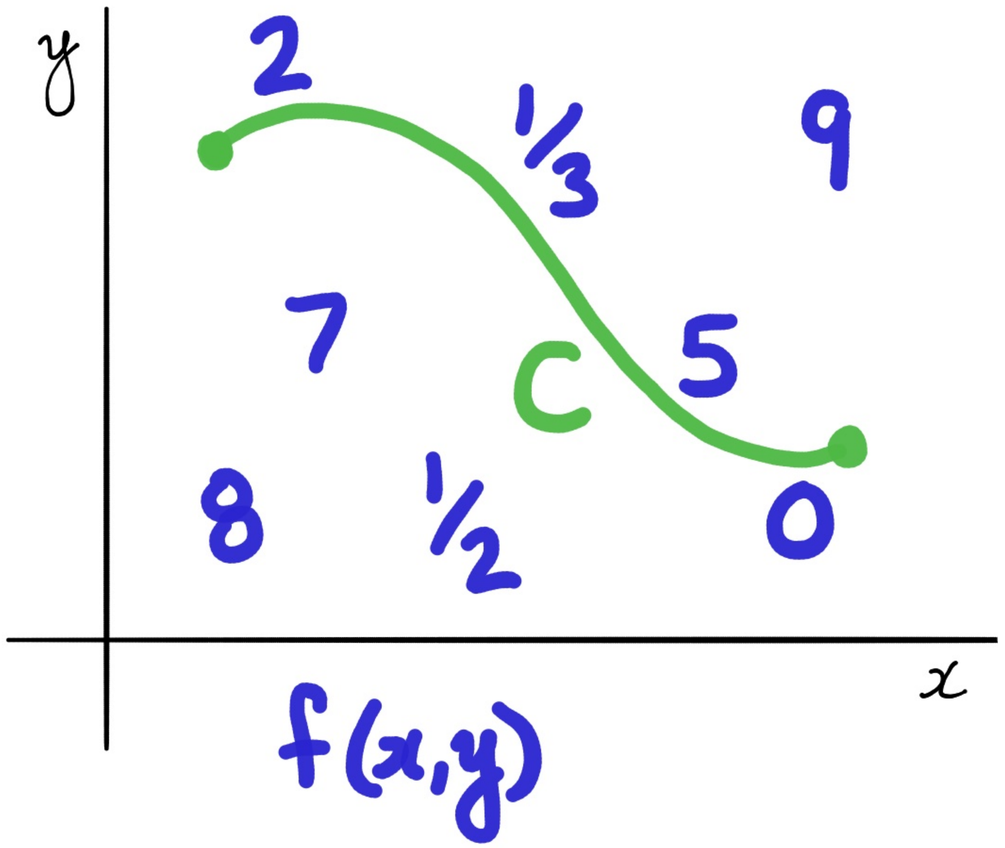

$f(x,y)$

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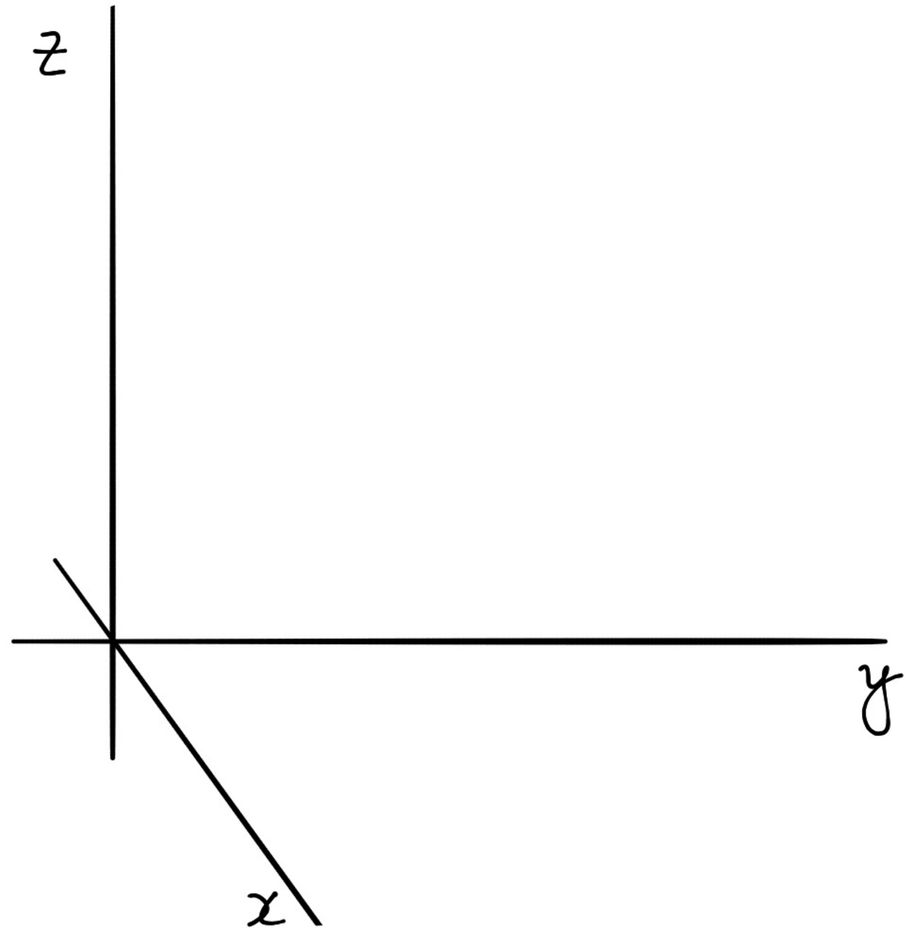

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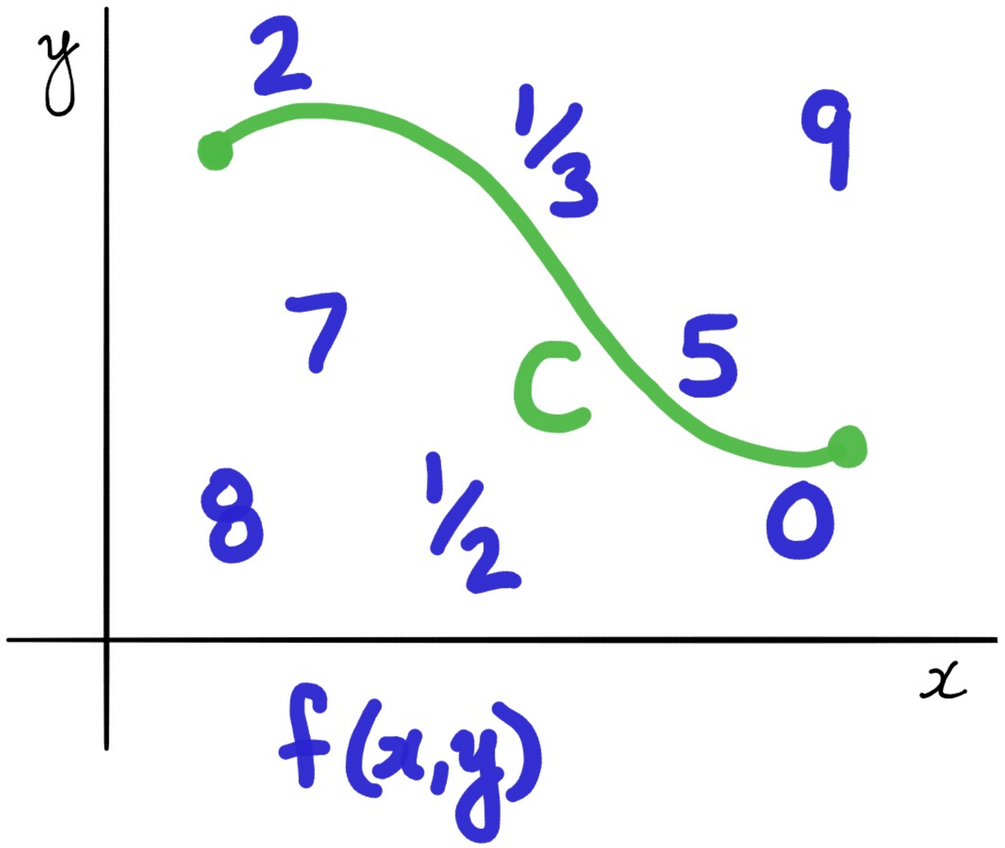

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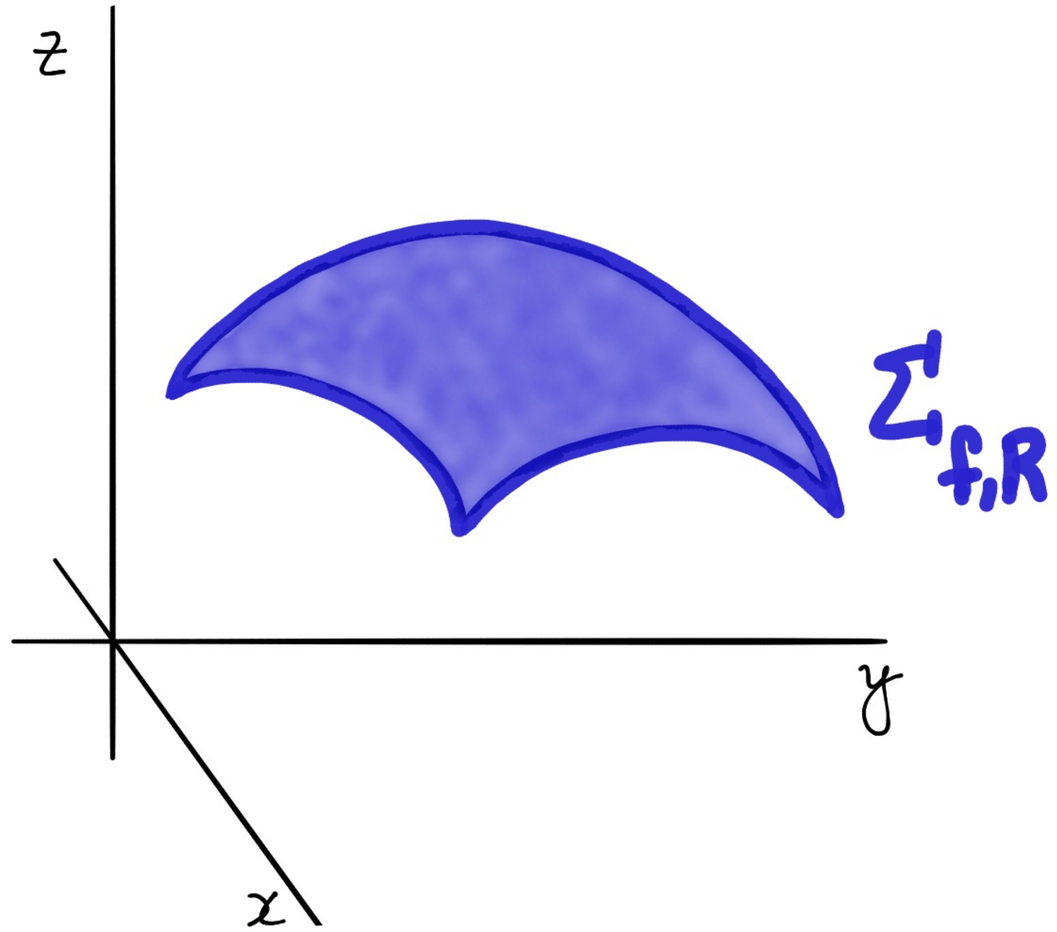

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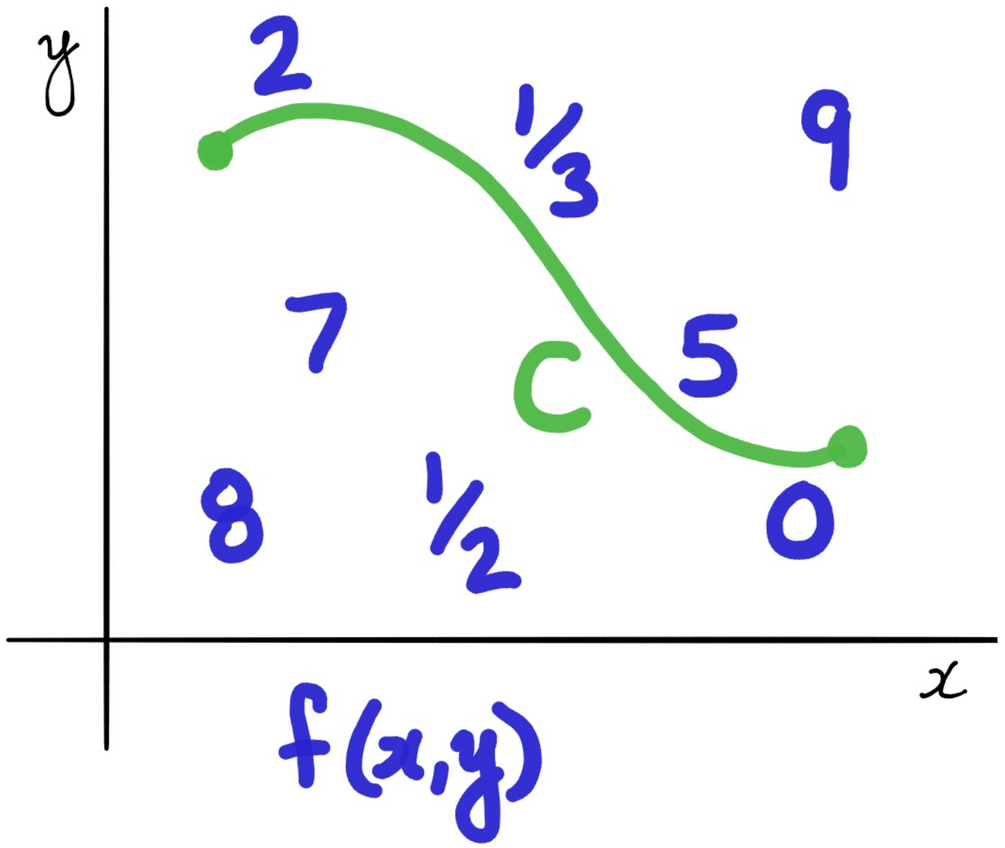

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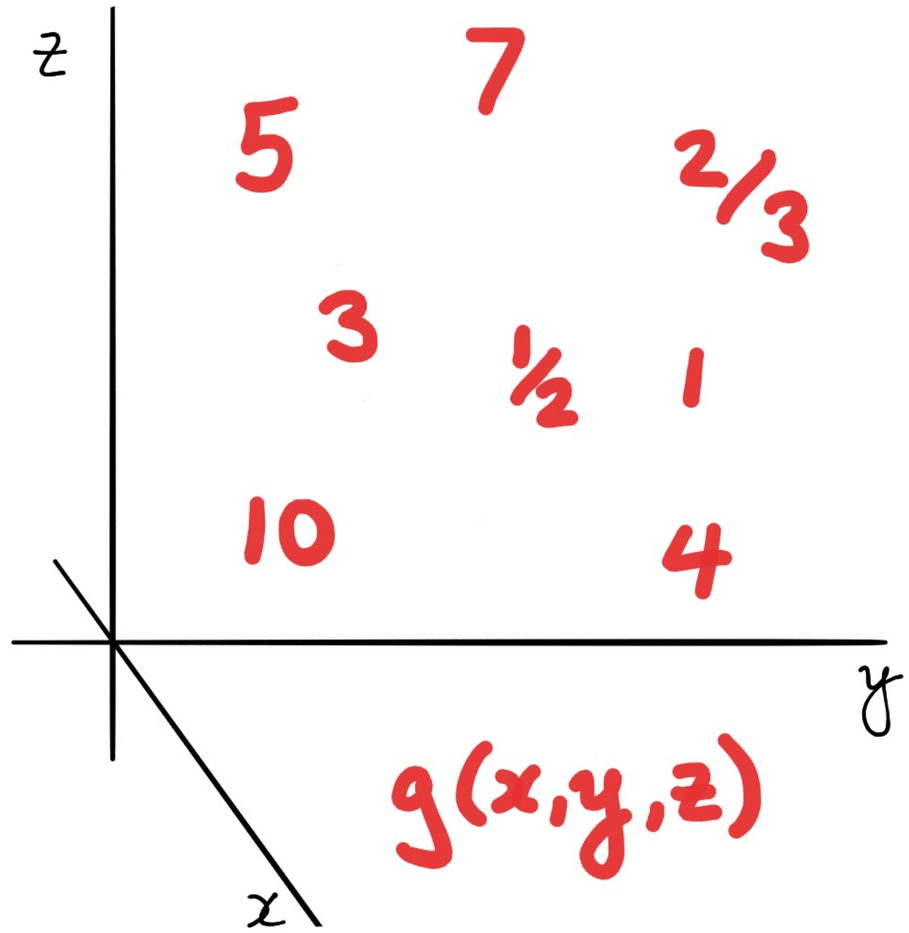

g(x,y,z)

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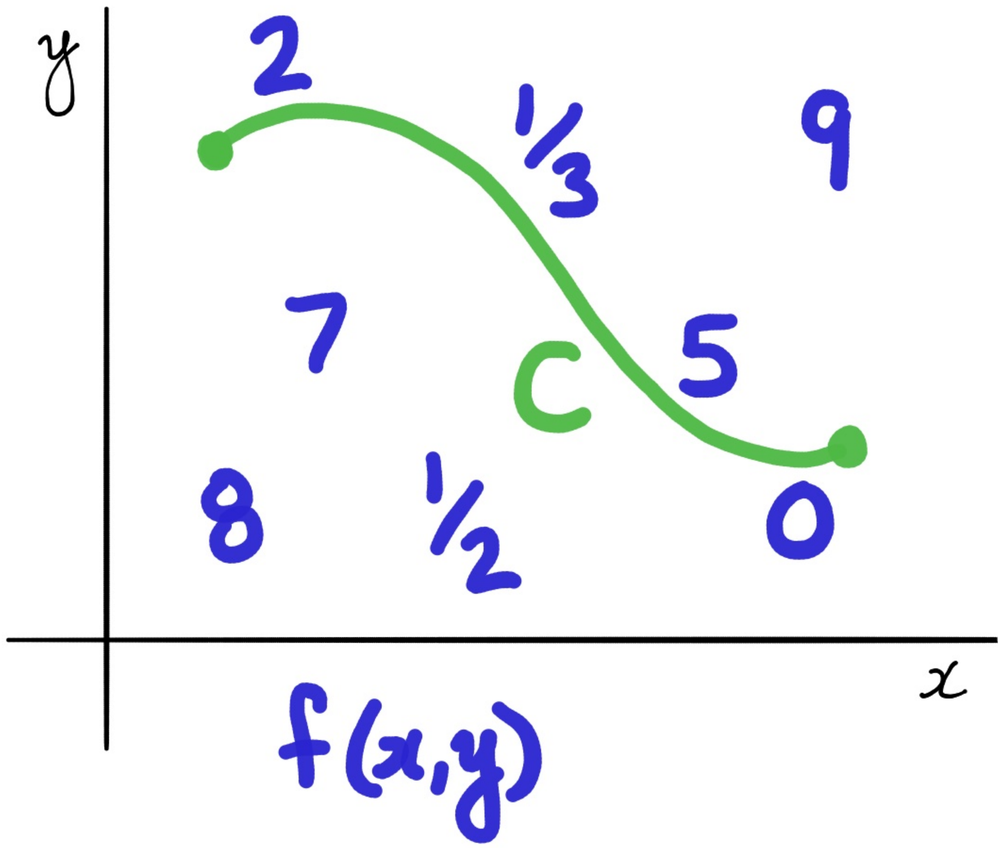

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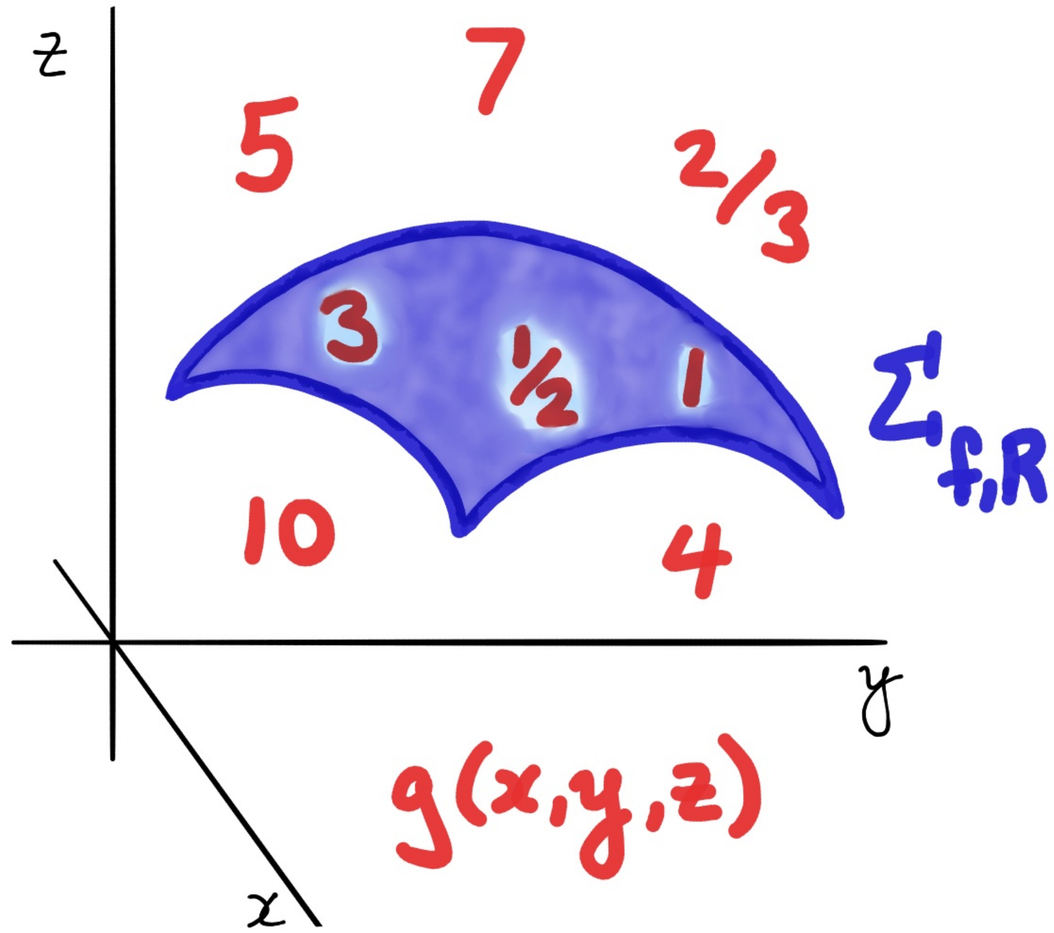


$$\iint_{\Sigma_{f,R}} g(x,y,z)\,dS$$

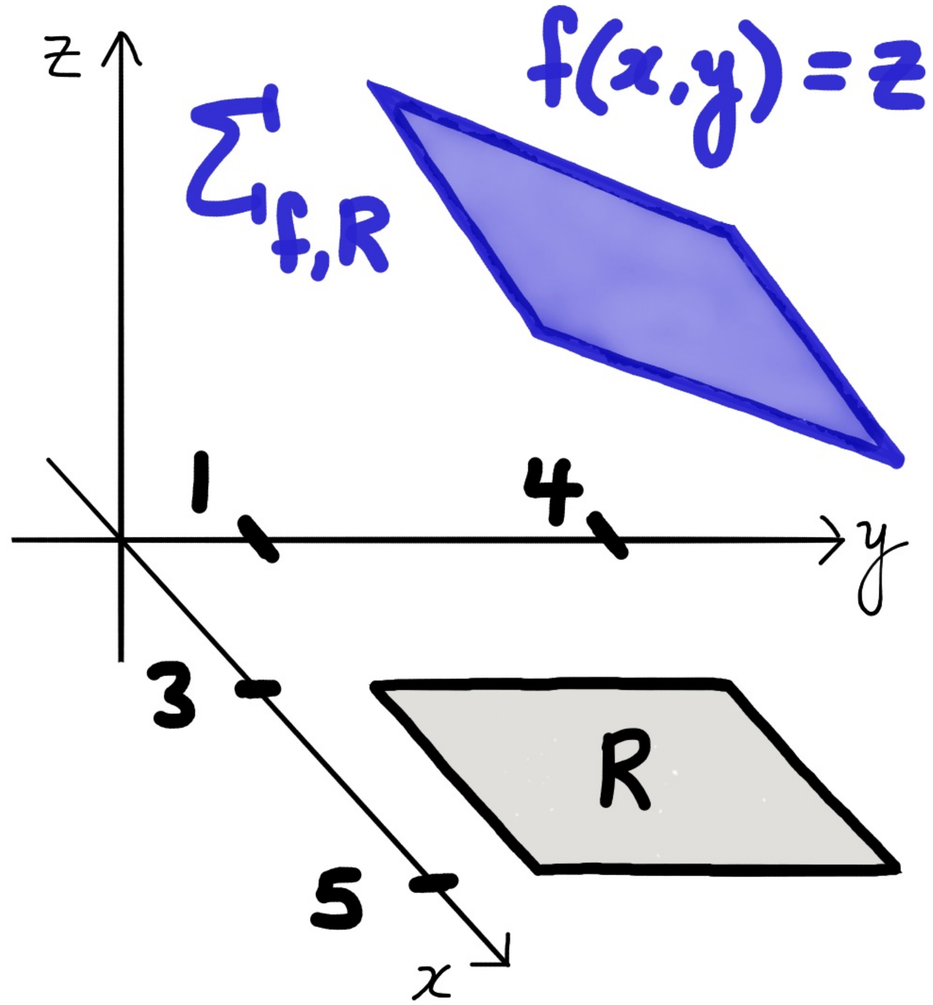
Problem:

$$f(x,y) = 20 - 2x - y$$

$$R = \{(x,y): 3 \le x \le 5,\ 1 \le y \le 4\}$$

$$g(x,y,z) = 7x + 9y + 2z + 4$$

Find $\iint_{\Sigma_{f,R}} g(x,y,z)\, dS.$



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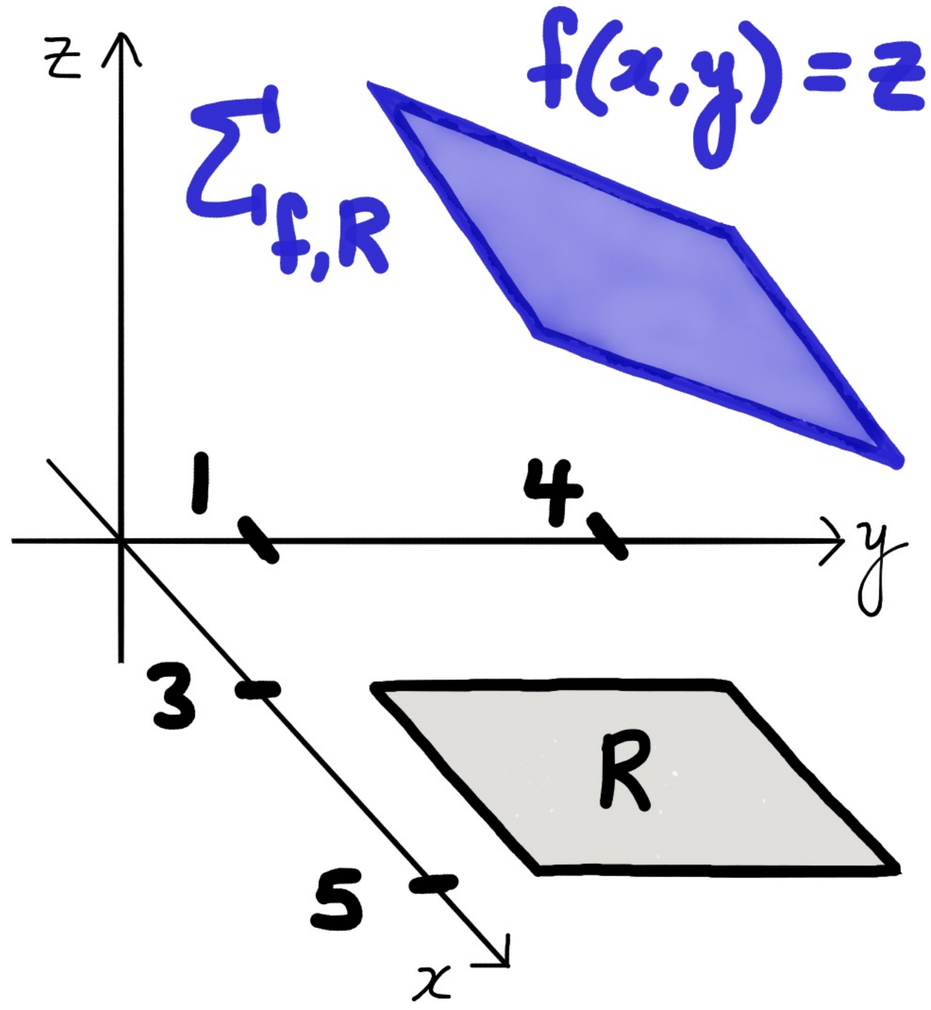

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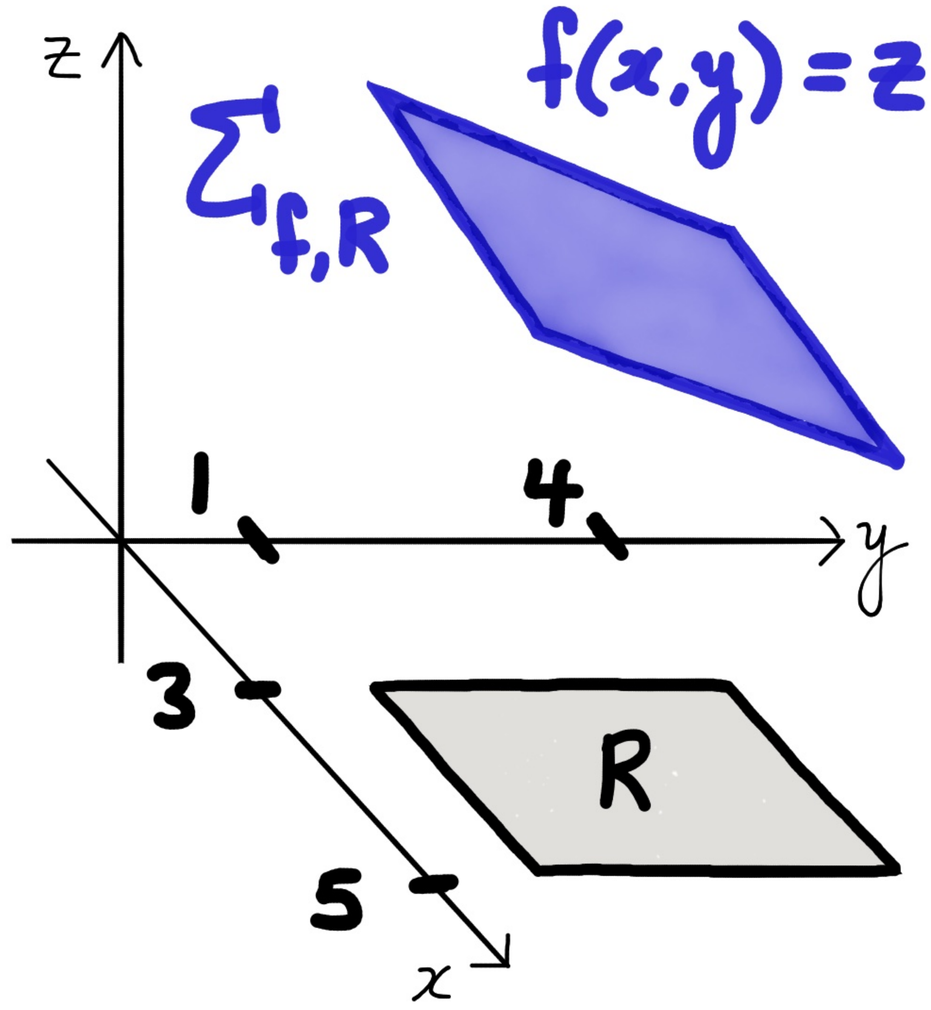

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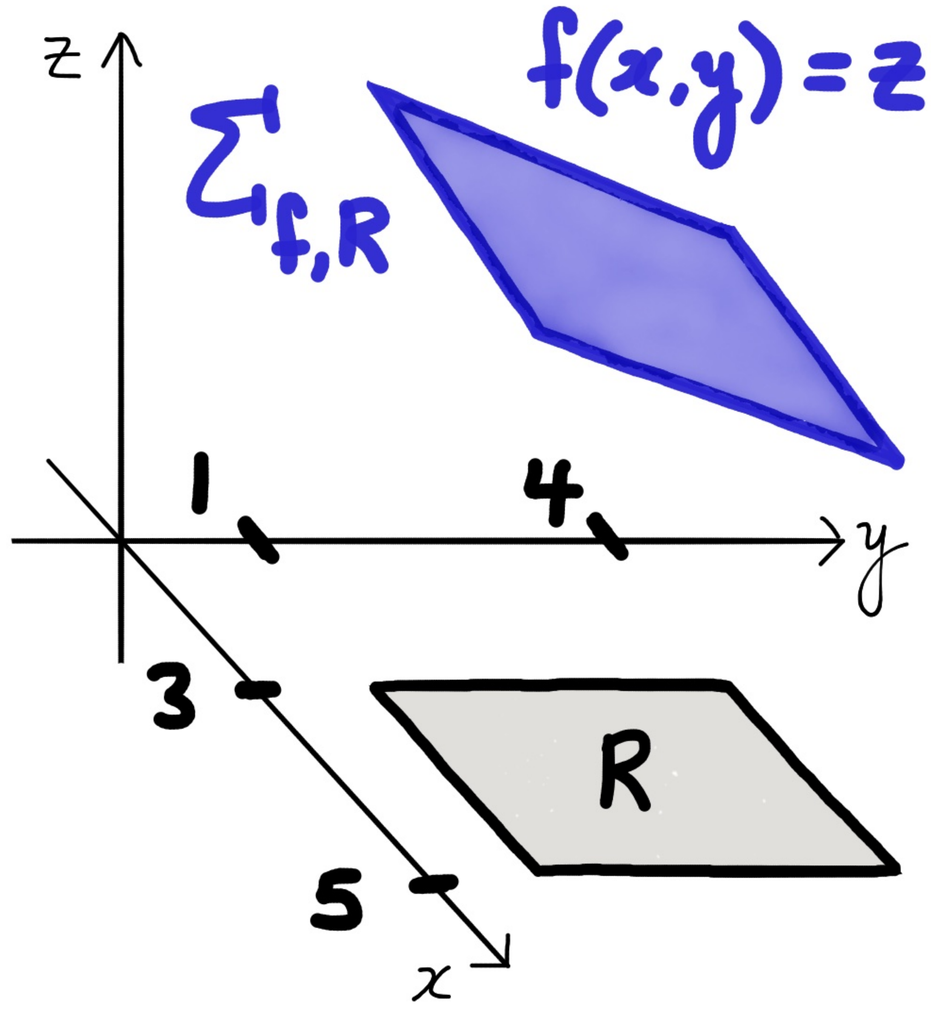

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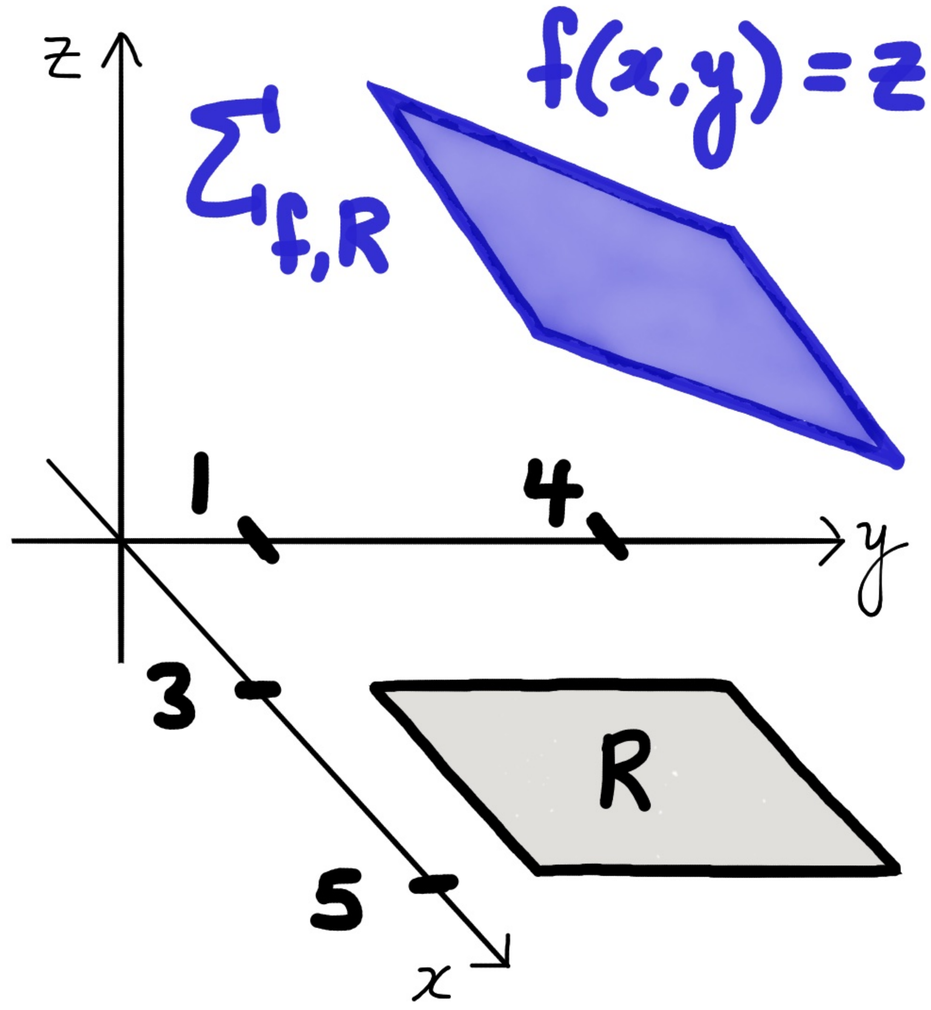

Solution:

$$\iint_{\Sigma_{f,R}} g(x,y,z)\, dS = \iint_R (7x + 9y + 2z + 4)\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA$$

$$= \int_1^4 \int_3^5 (7x + 9y + 2(20 - 2x - y) + 4)\sqrt{4 + 1 + 1}\, dx\, dy$$

Problem:

$$f(x,y) = 20 - 2x - y$$

$$R = \{(x,y): 3 \le x \le 5,\ 1 \le y \le 4\}$$

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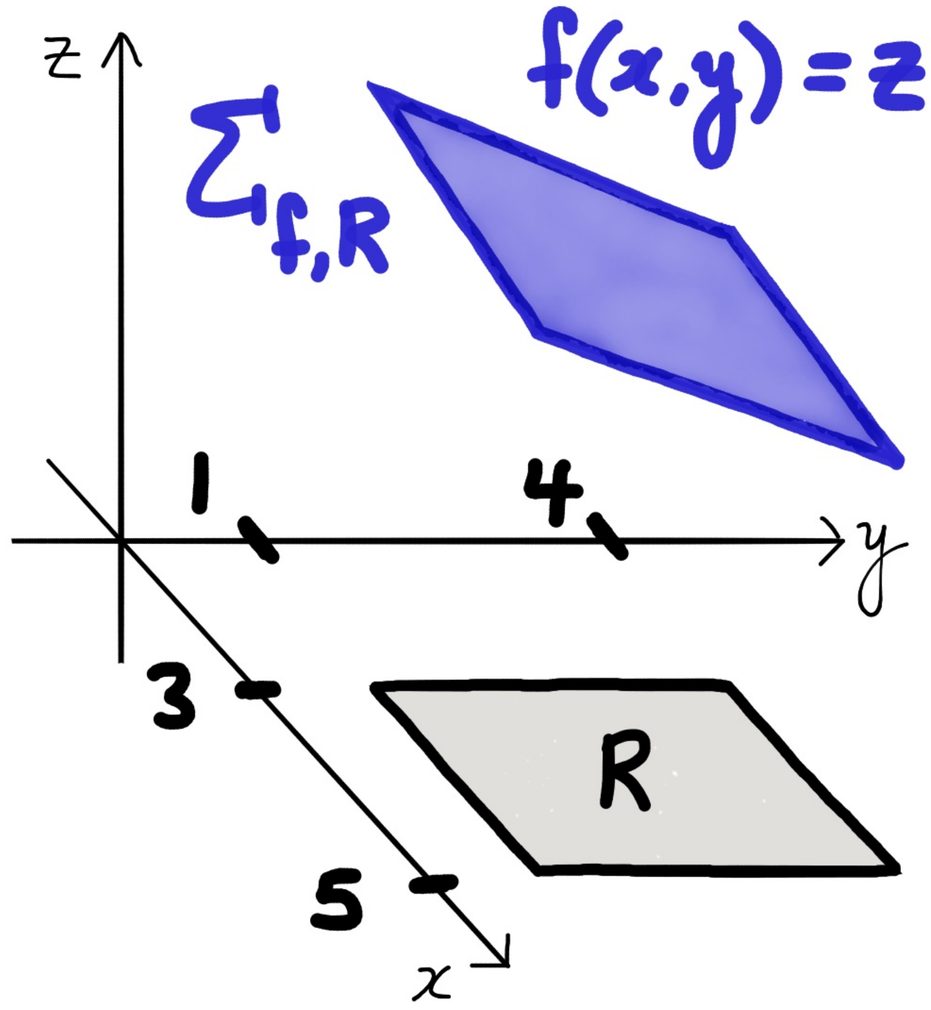

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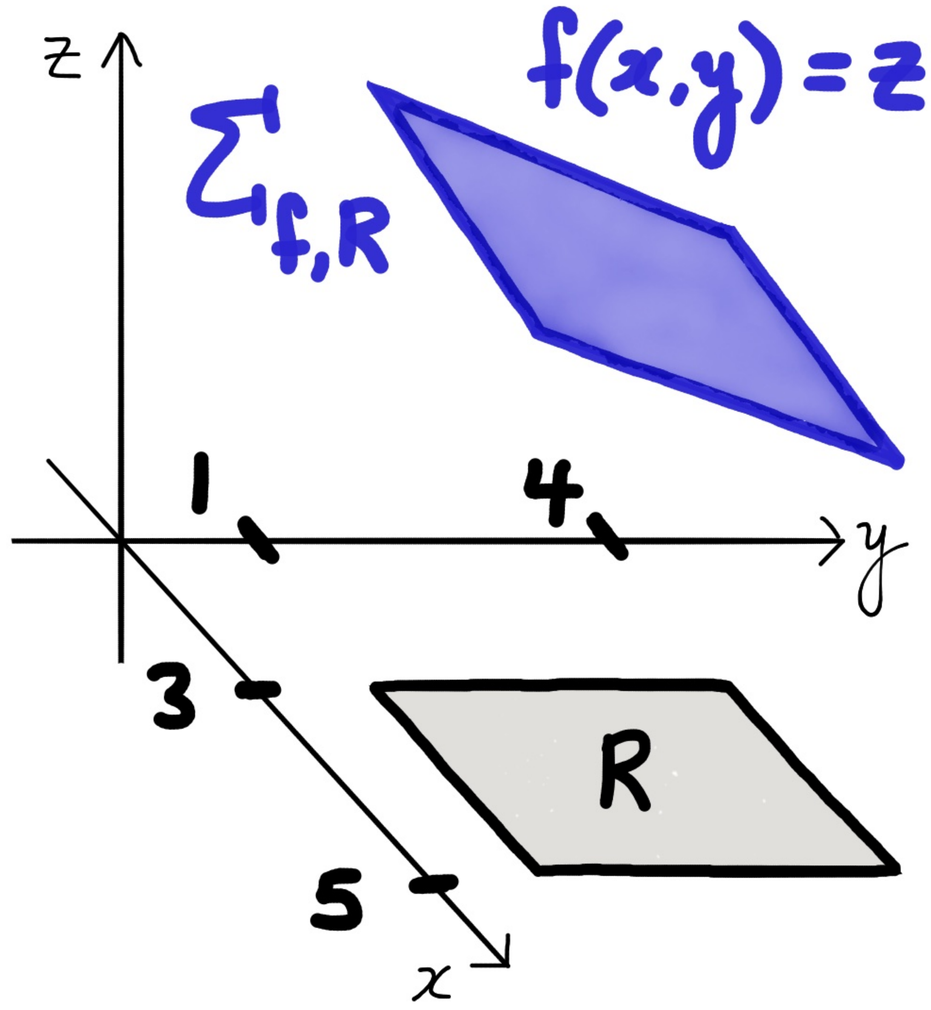

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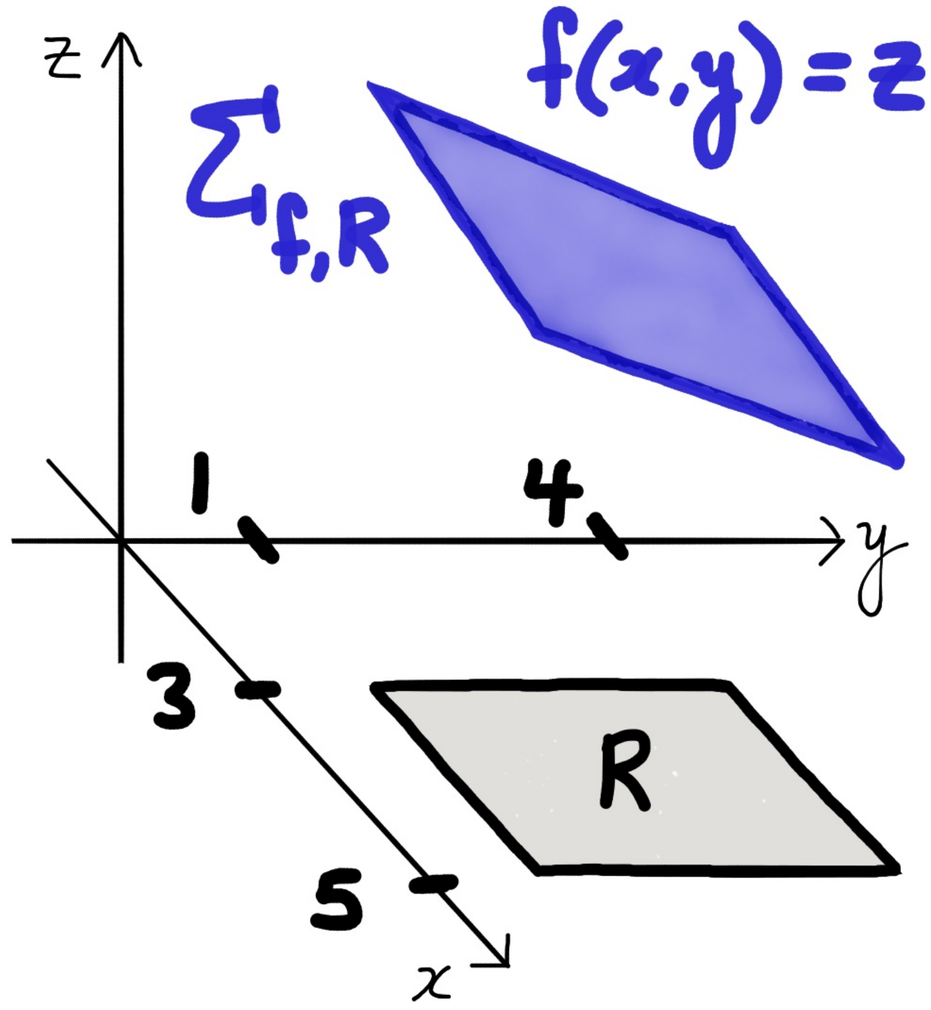

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Problem:

$$f(x,y) = 9 - x^2 - y^2$$

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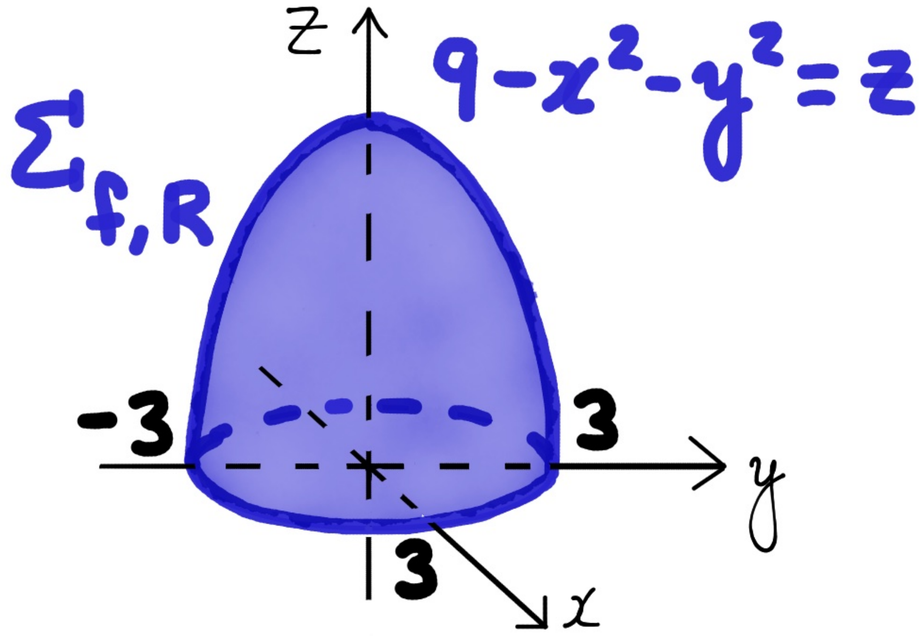

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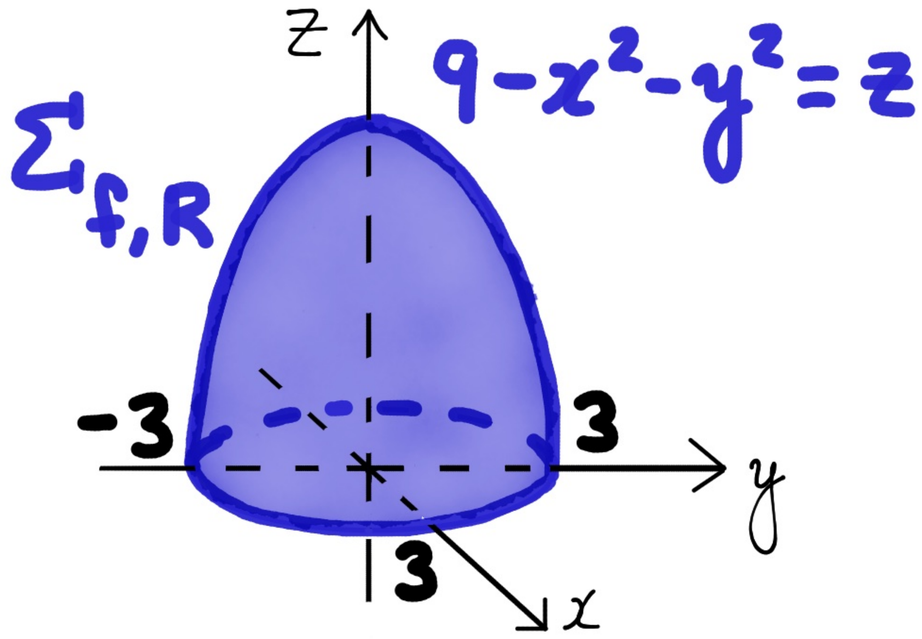

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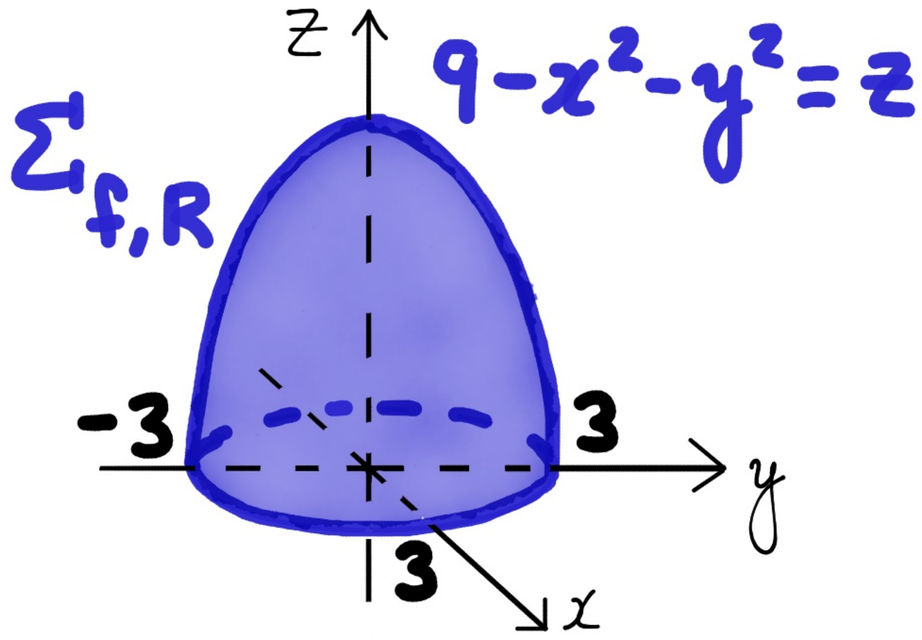

Solution:

$$\iint_{\Sigma_{f,R}} g(x,y,z)\, dS = \iint_R (x^2+y^2+z)\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dA$$

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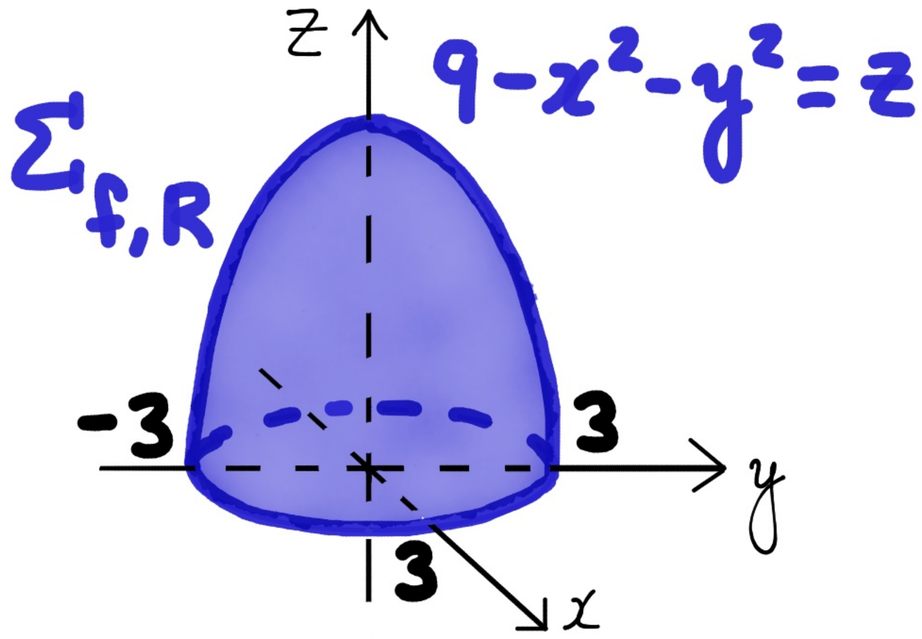

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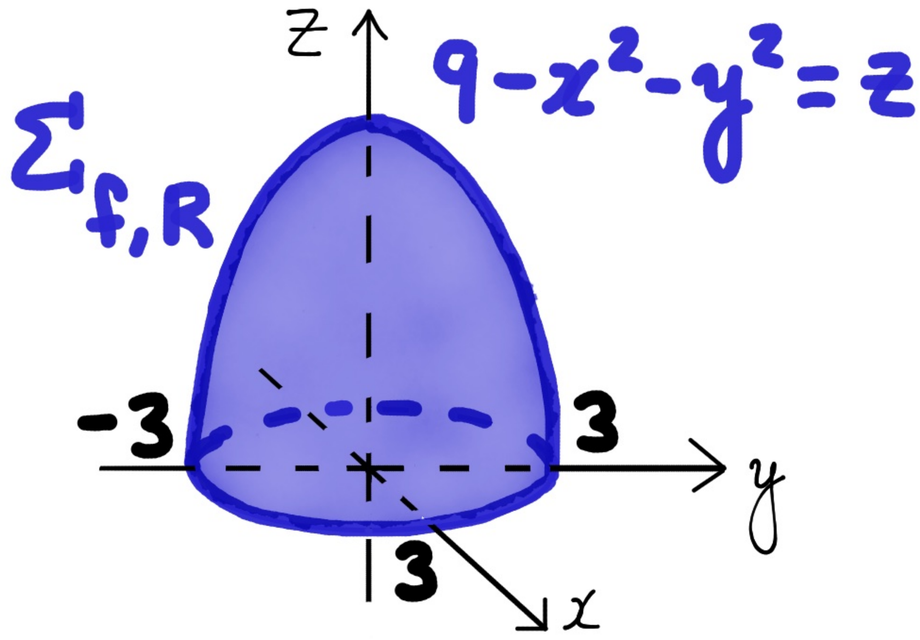

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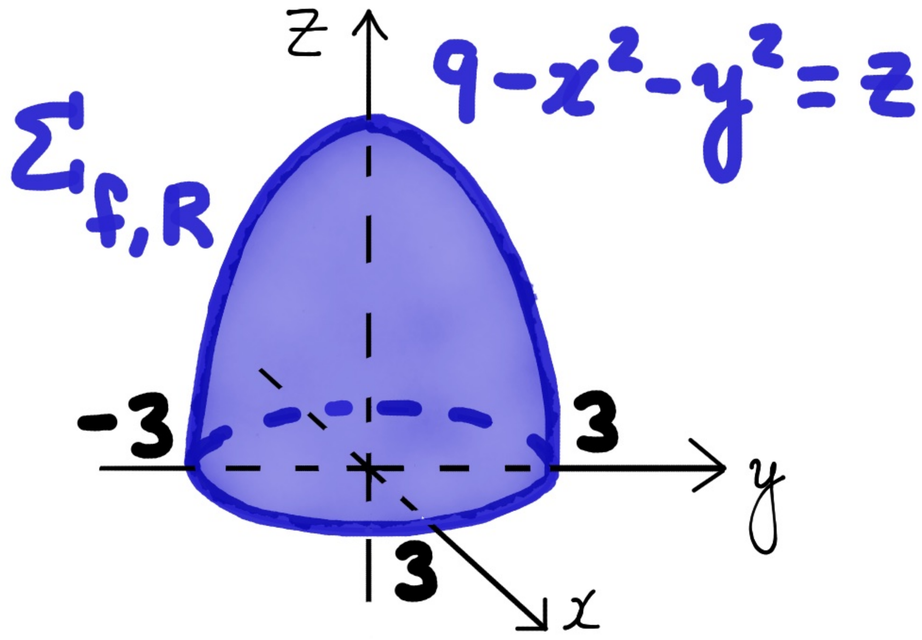

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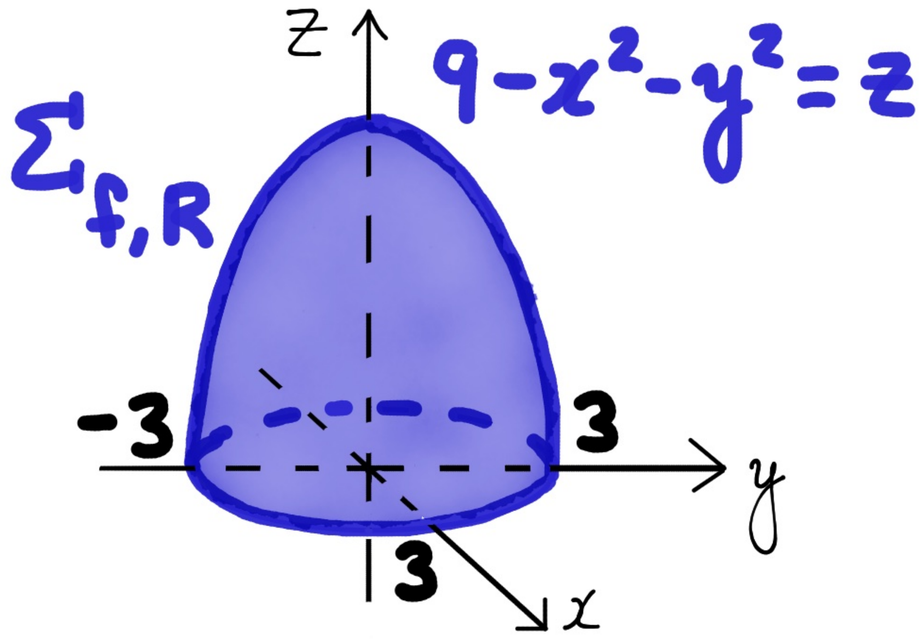

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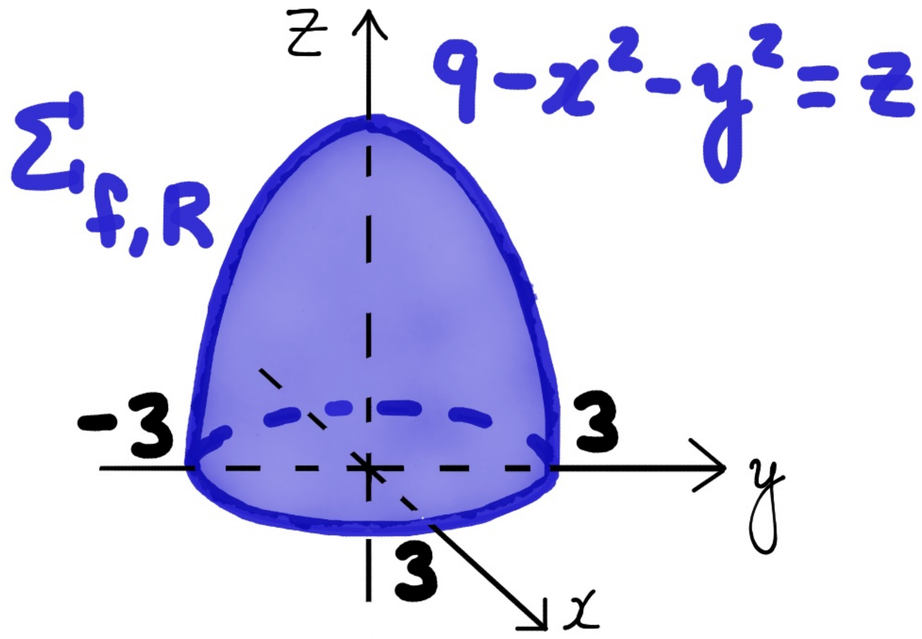

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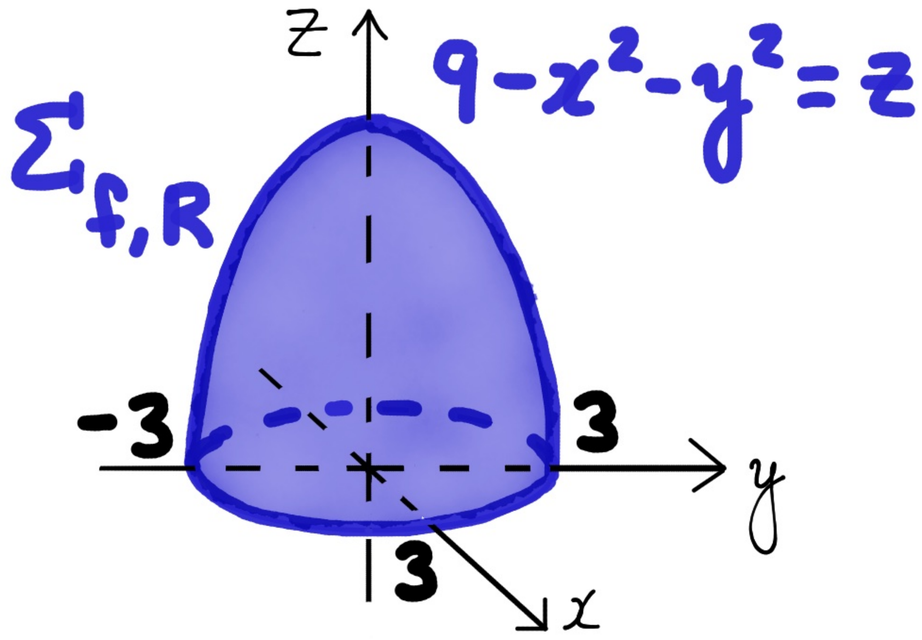

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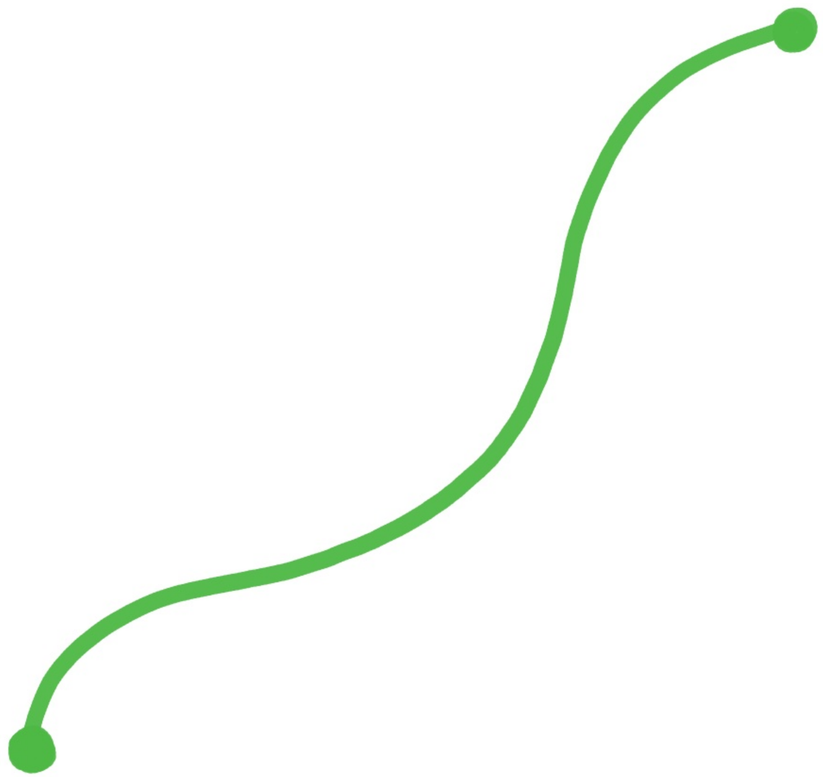
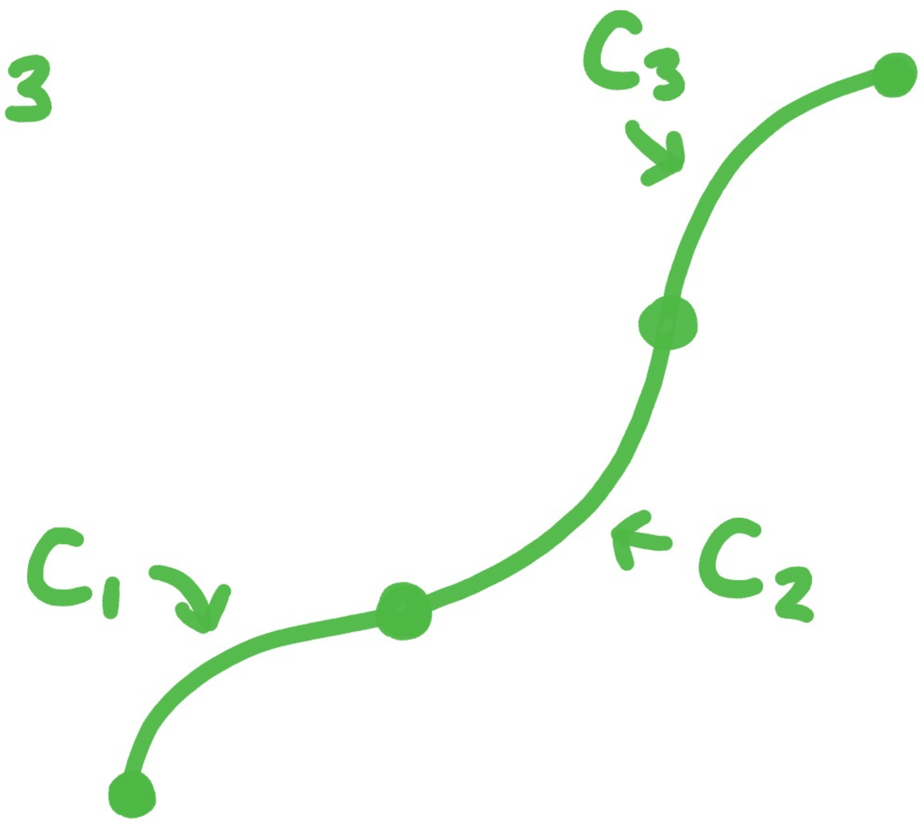
$$= \int_0^{2\pi}\int_0^3 9r\sqrt{4r^2+1}\, dr\, d\theta = \frac{3}{2}\pi\left(37^{3/2}-1\right)$$

Ⅱ Surface integrals for general surfaces

c



$C = C_1 \cup C_2 \cup C_3$



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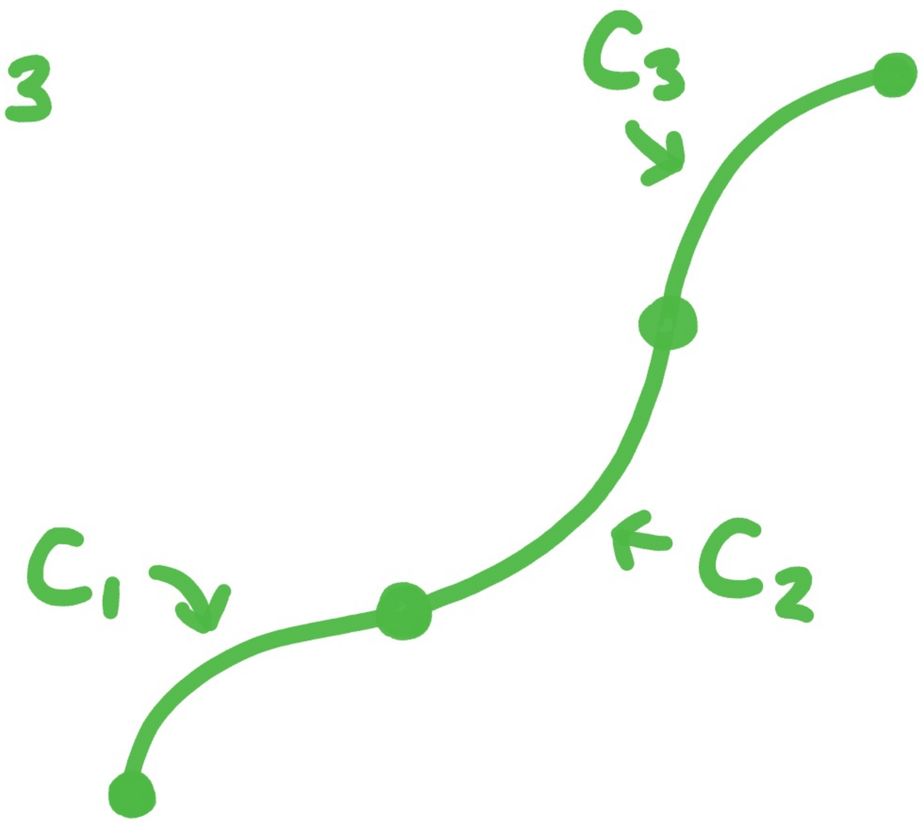

$$\int_C f(x,y)\,dA = \int_{C_1} f(x,y)\,dA + \int_{C_2} f(x,y)\,dA + \int_{C_3} f(x,y)\,dA$$

Example:

Let Σ be a sphere of radius a in $\mathbb{R}^3$.

Σ

$$z^2 = a^2 - x^2 - y^2$$

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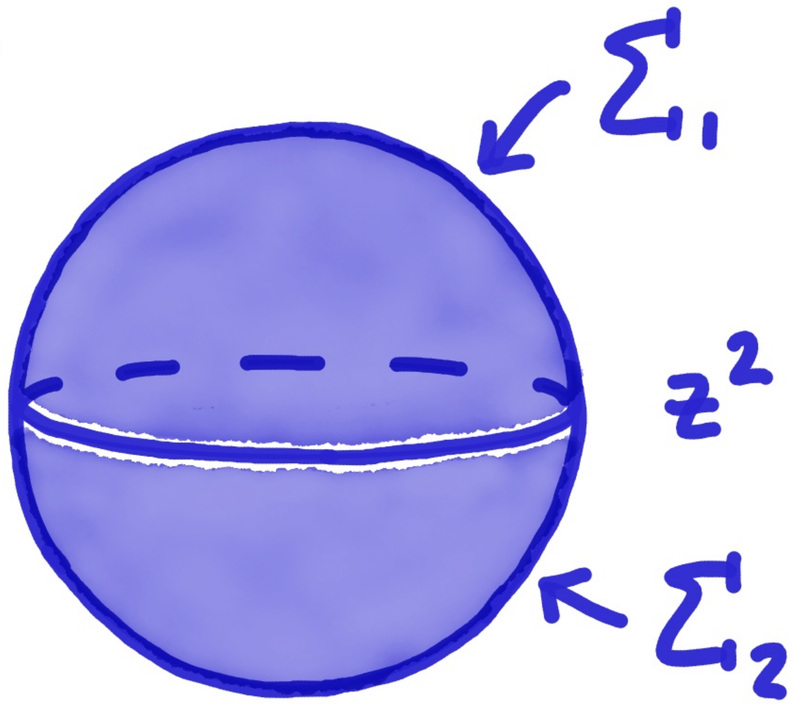
$\Sigma = \Sigma_1 \cup \Sigma_2$



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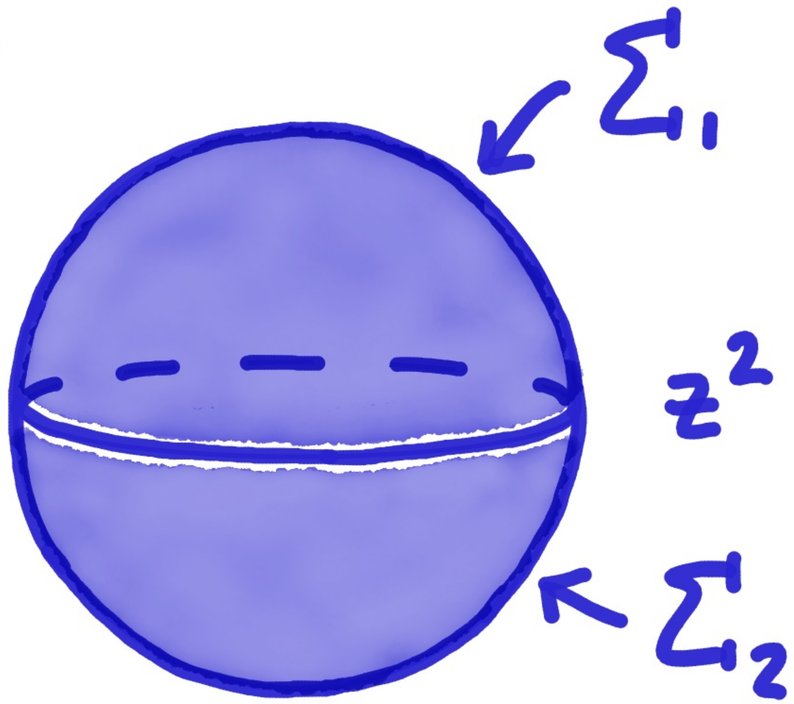

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$$z^2 = a^2 - x^2 - y^2$$

$$R = \{(x,y) : x^2 + y^2 \leq a^2\}.$$

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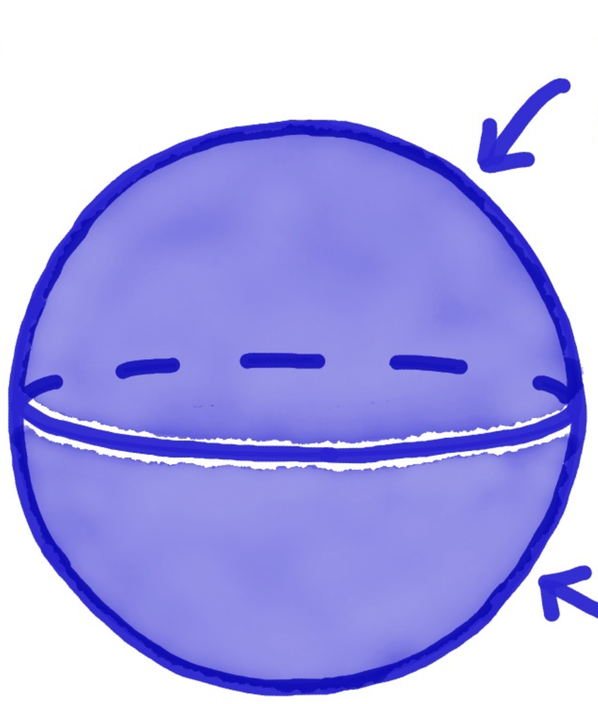
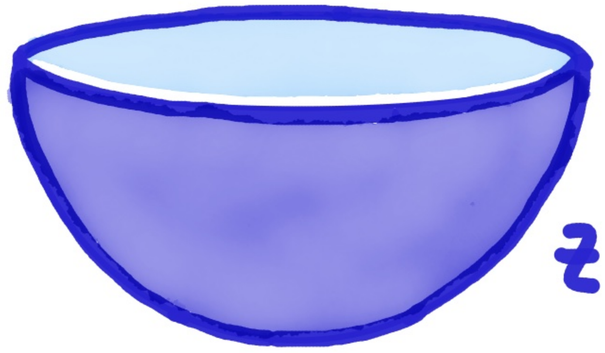


$z^2 = a^2 - x^2 - y^2$

$R = \{(x,y) : x^2 + y^2 \le a^2\}$.

$z = \sqrt{a^2 - x^2 - y^2}$

$\Sigma_1 = \Sigma_{\sqrt{a^2-x^2-y^2}, R}$



$z = -\sqrt{a^2 - x^2 - y^2}$

$\Sigma_2 = \Sigma_{-\sqrt{a^2-x^2-y^2}, R}$

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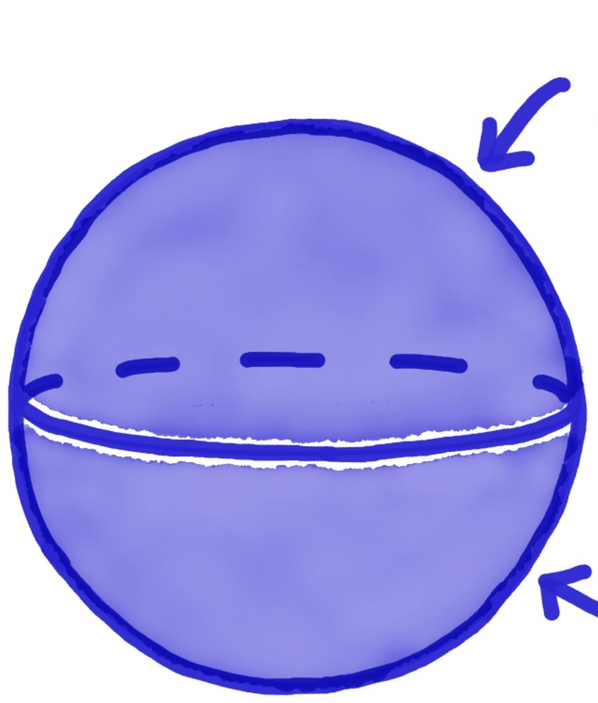

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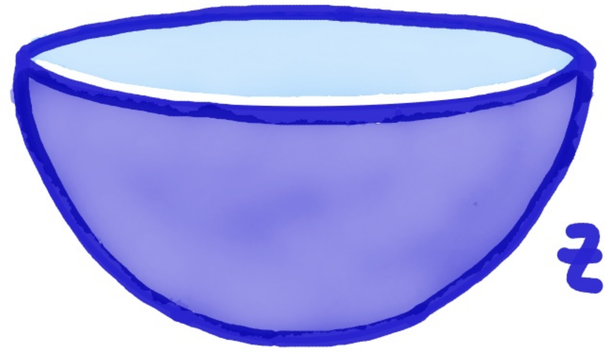

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$$\iint_{\Sigma} dS = \iint_{\Sigma_1} dS + \iint_{\Sigma_2} dS$$

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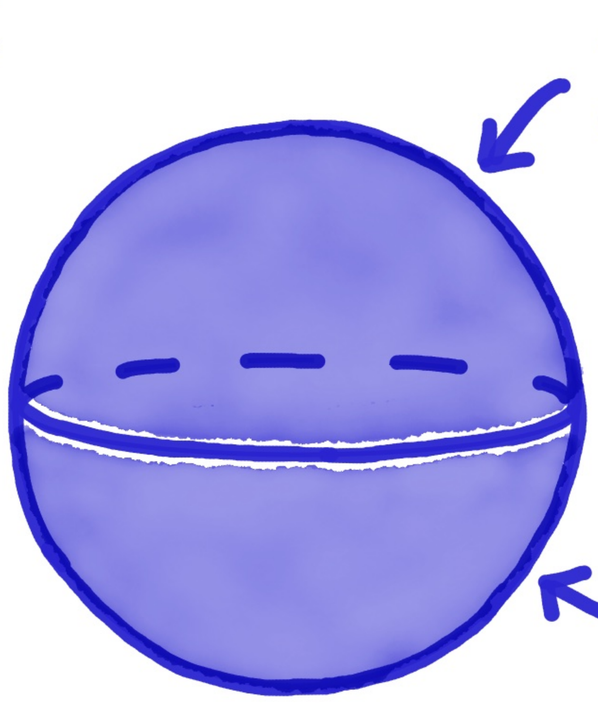

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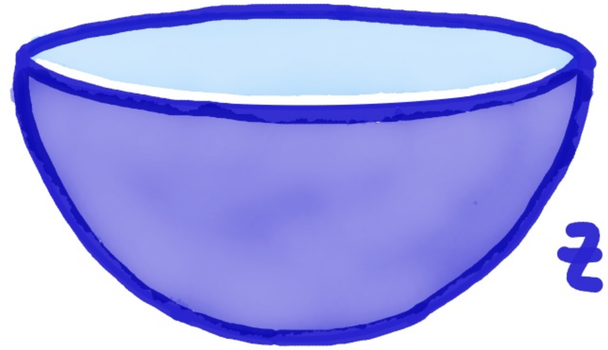

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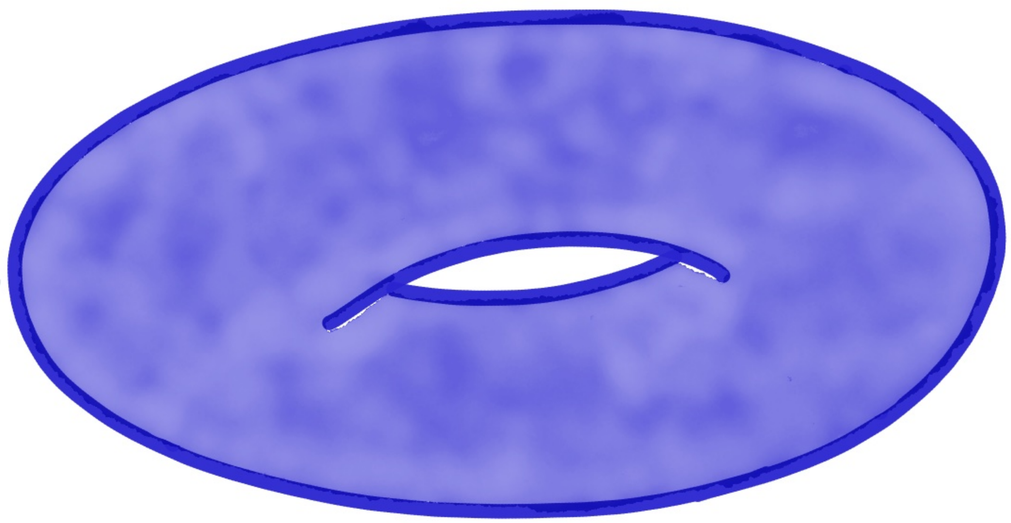
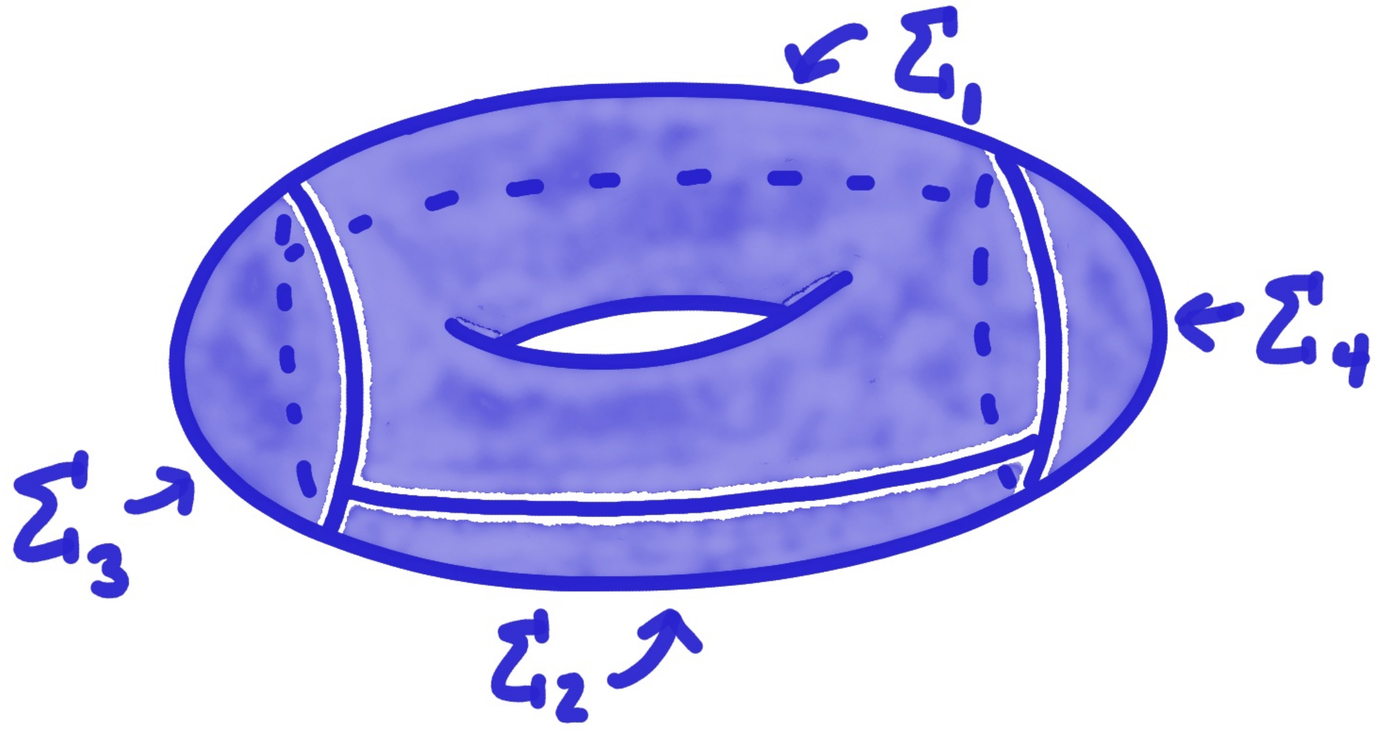
$$\iint_{\Sigma} dS = \iint_{\Sigma_1} dS + \iint_{\Sigma_2} dS = 2\pi a^2 + 2\pi a^2 = 4\pi a^2$$



3

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$$



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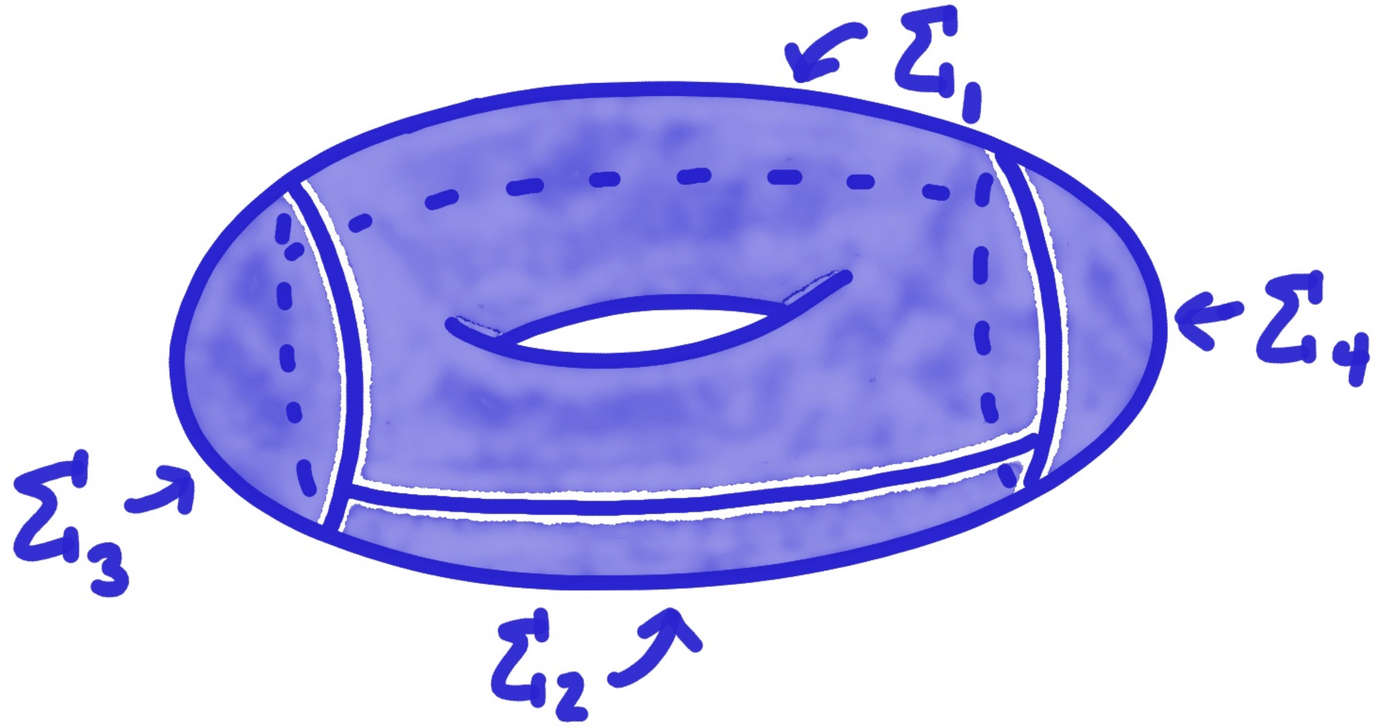

$$\iint_{\Sigma} g(x,y,z)\,dS$$

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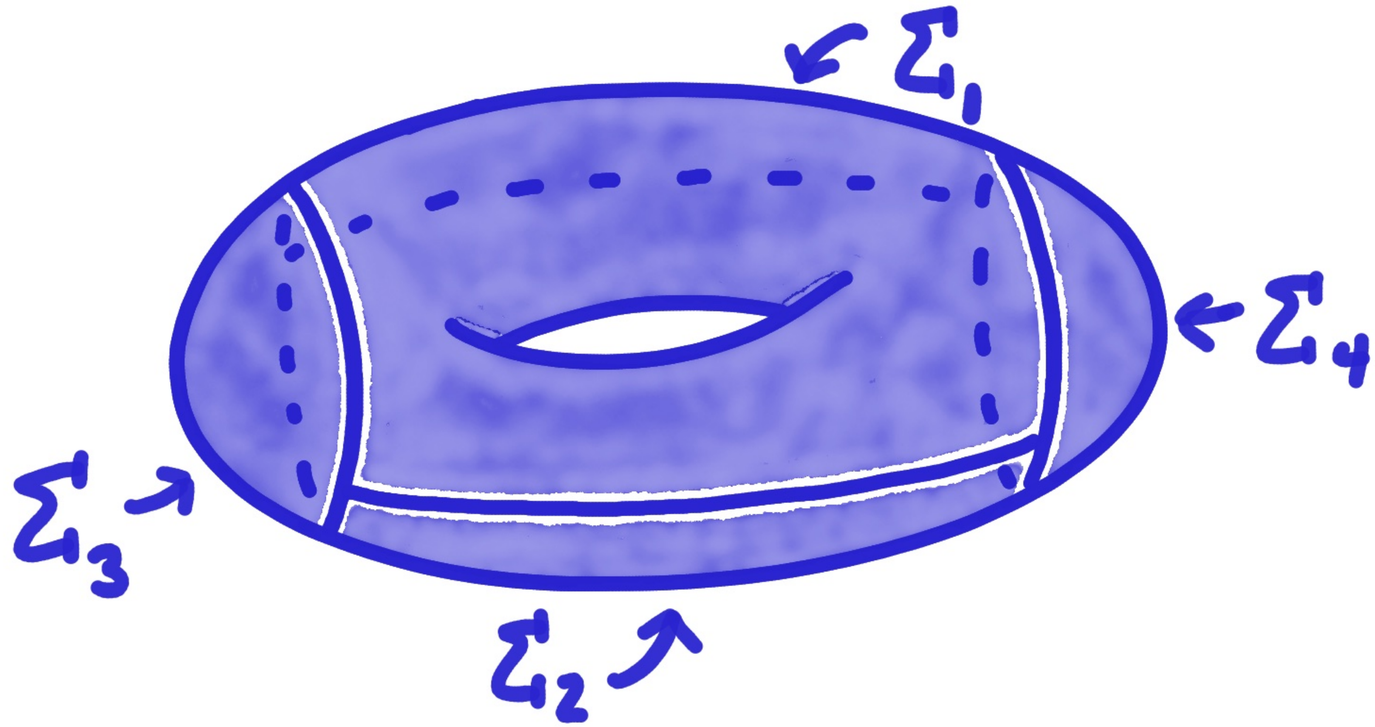

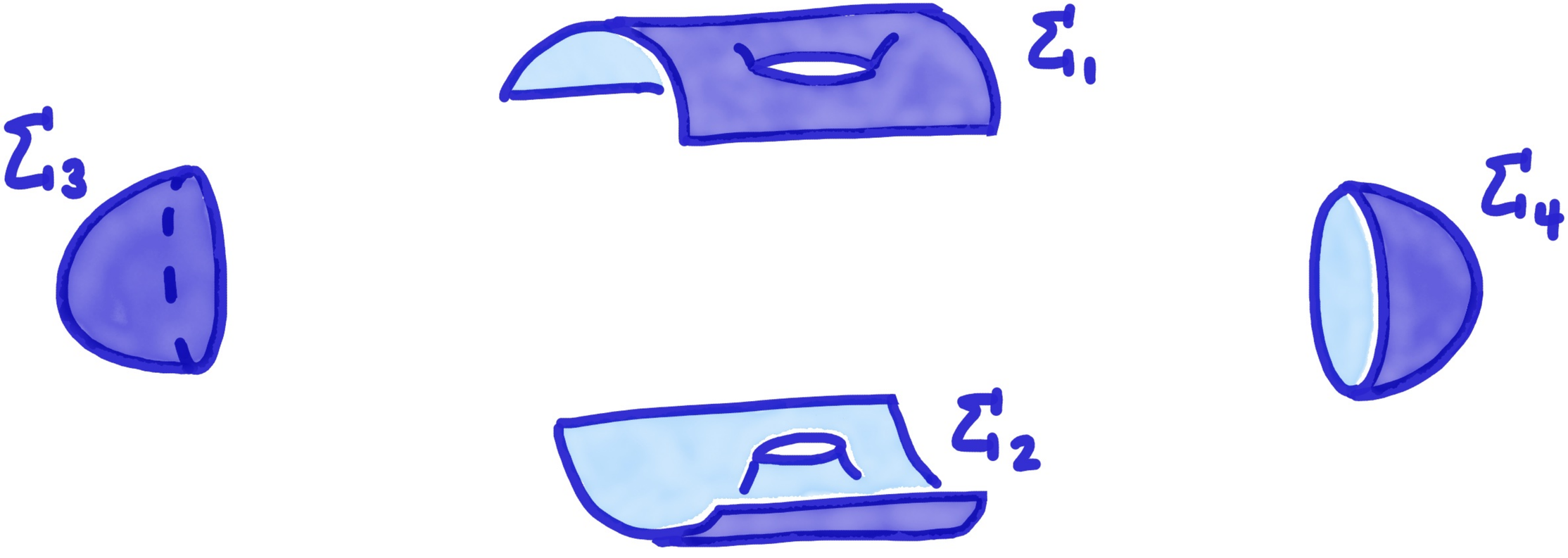

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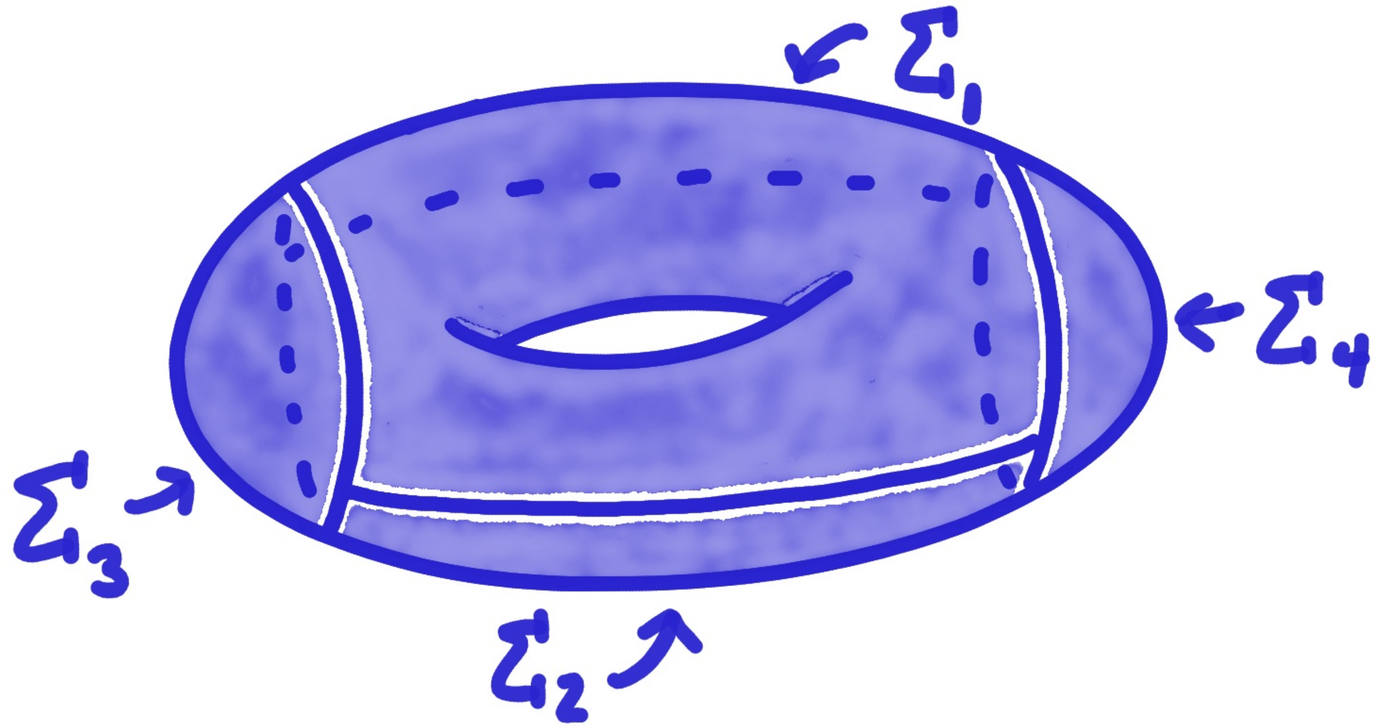
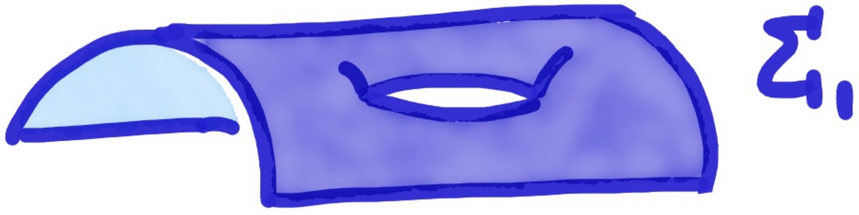
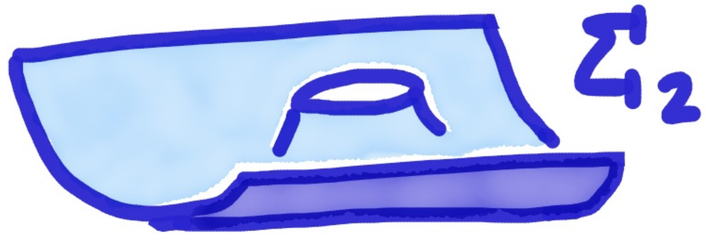


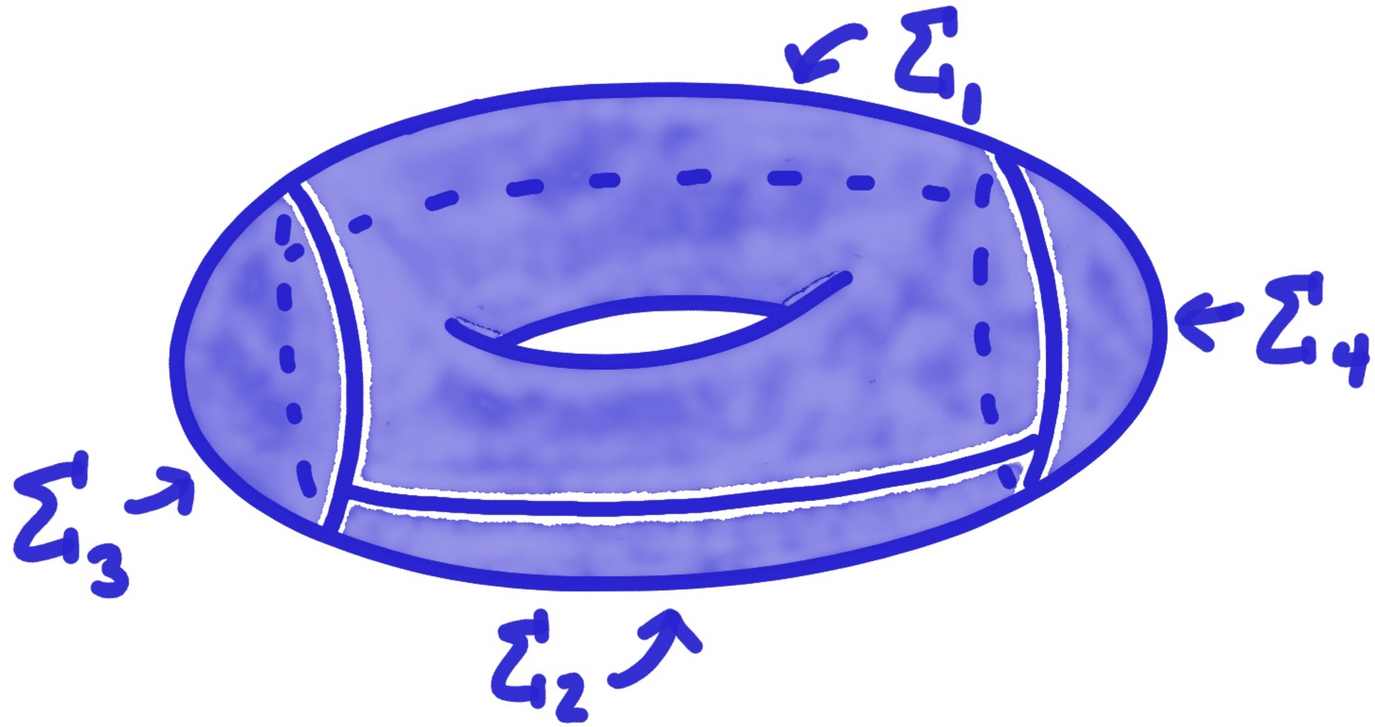


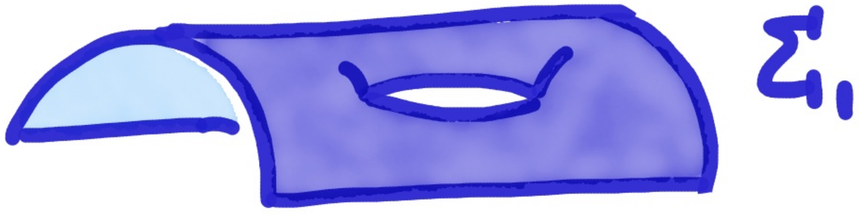
$z = f_1(x,y)$



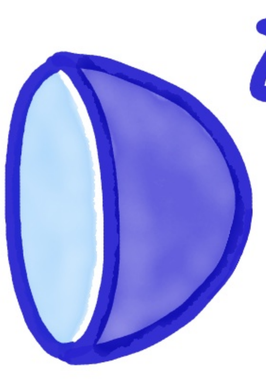
$z = f_2(x,y)$

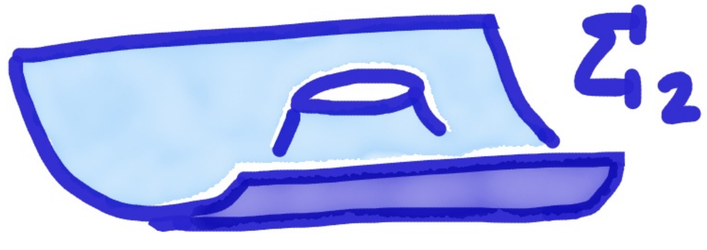
$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$$





$z = f_1(x,y)$

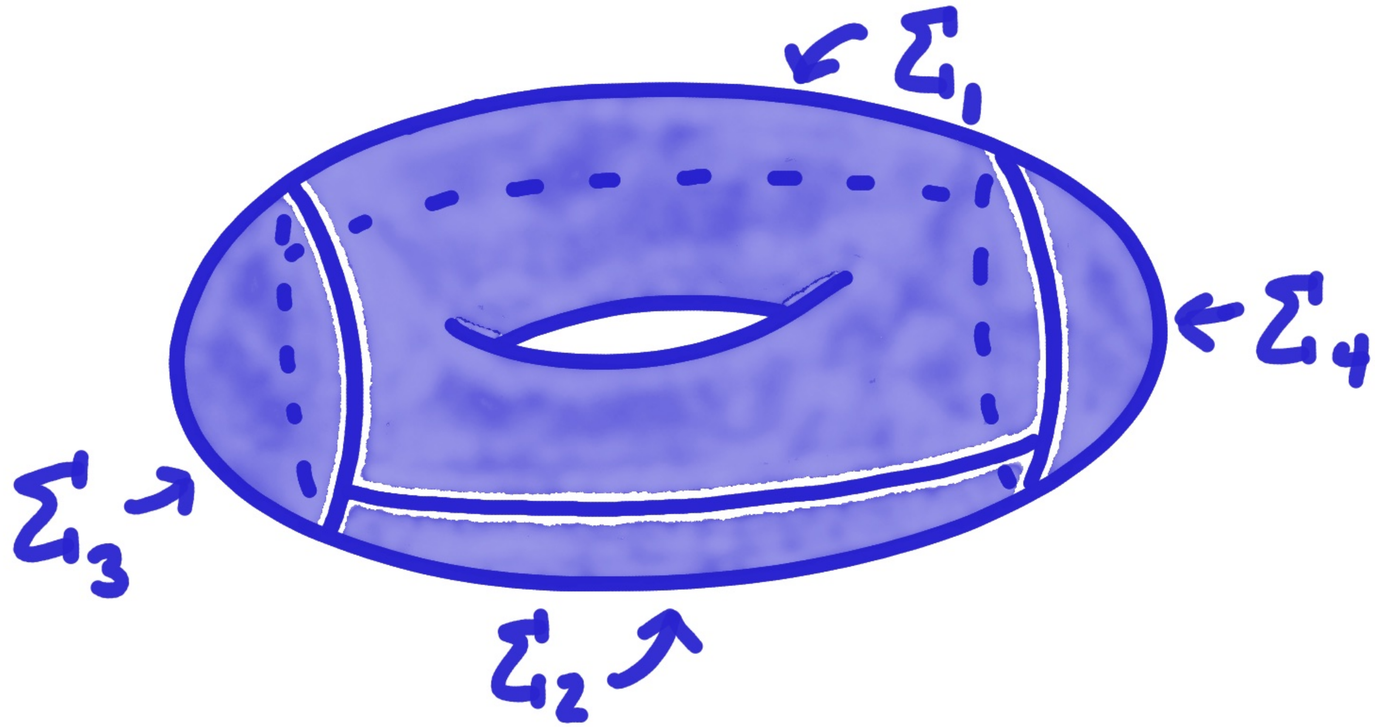


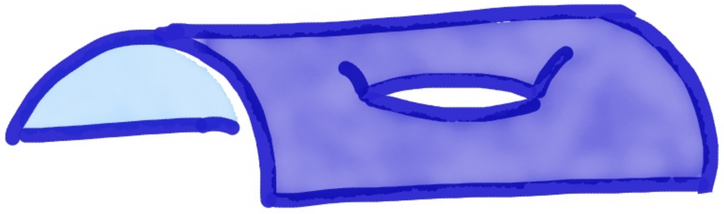


$z = f_2(x,y)$

$y = f_4(x,z)$

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$$

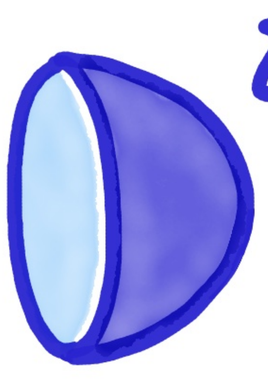




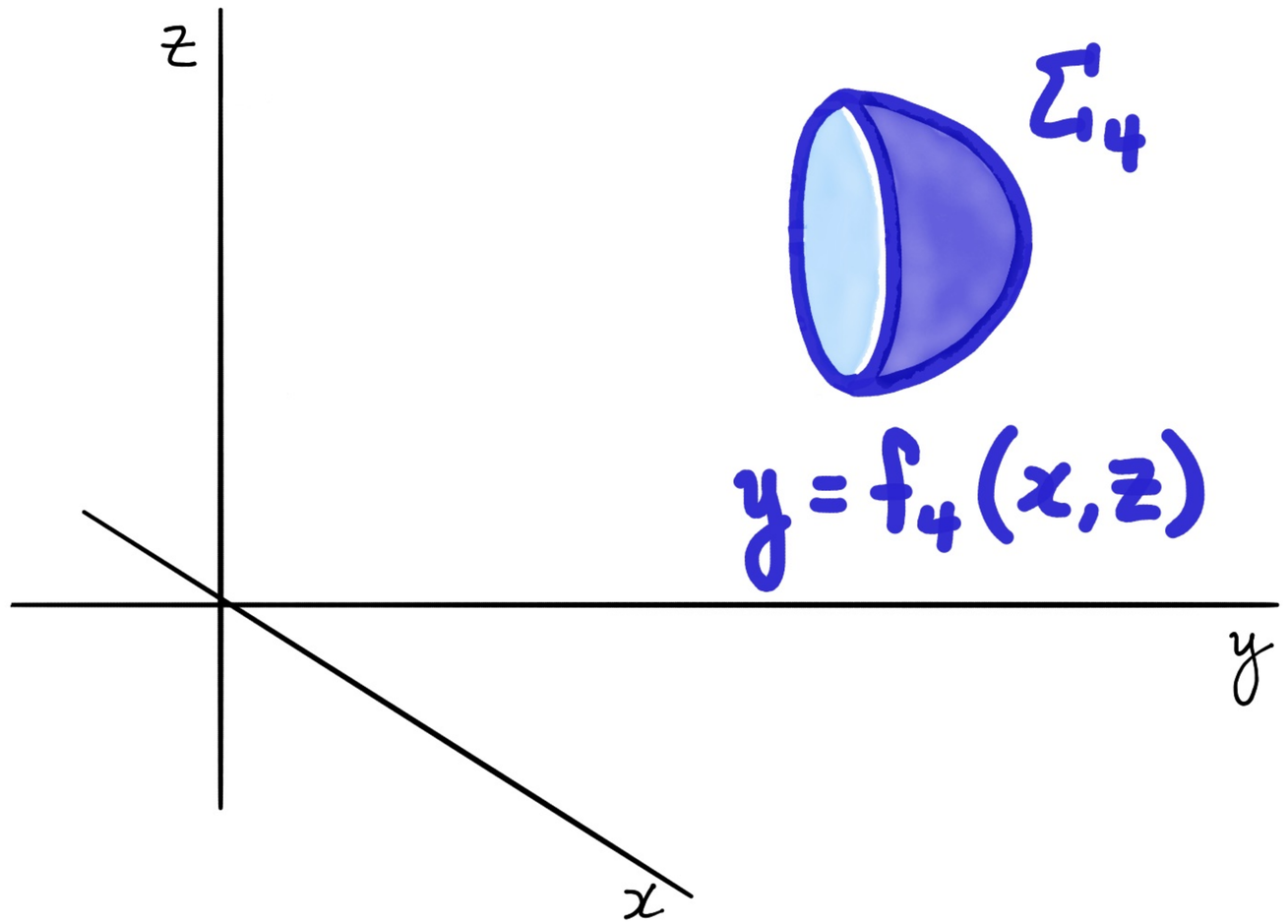
$z = f_1(x,y)$

$y = f_3(x,z)$

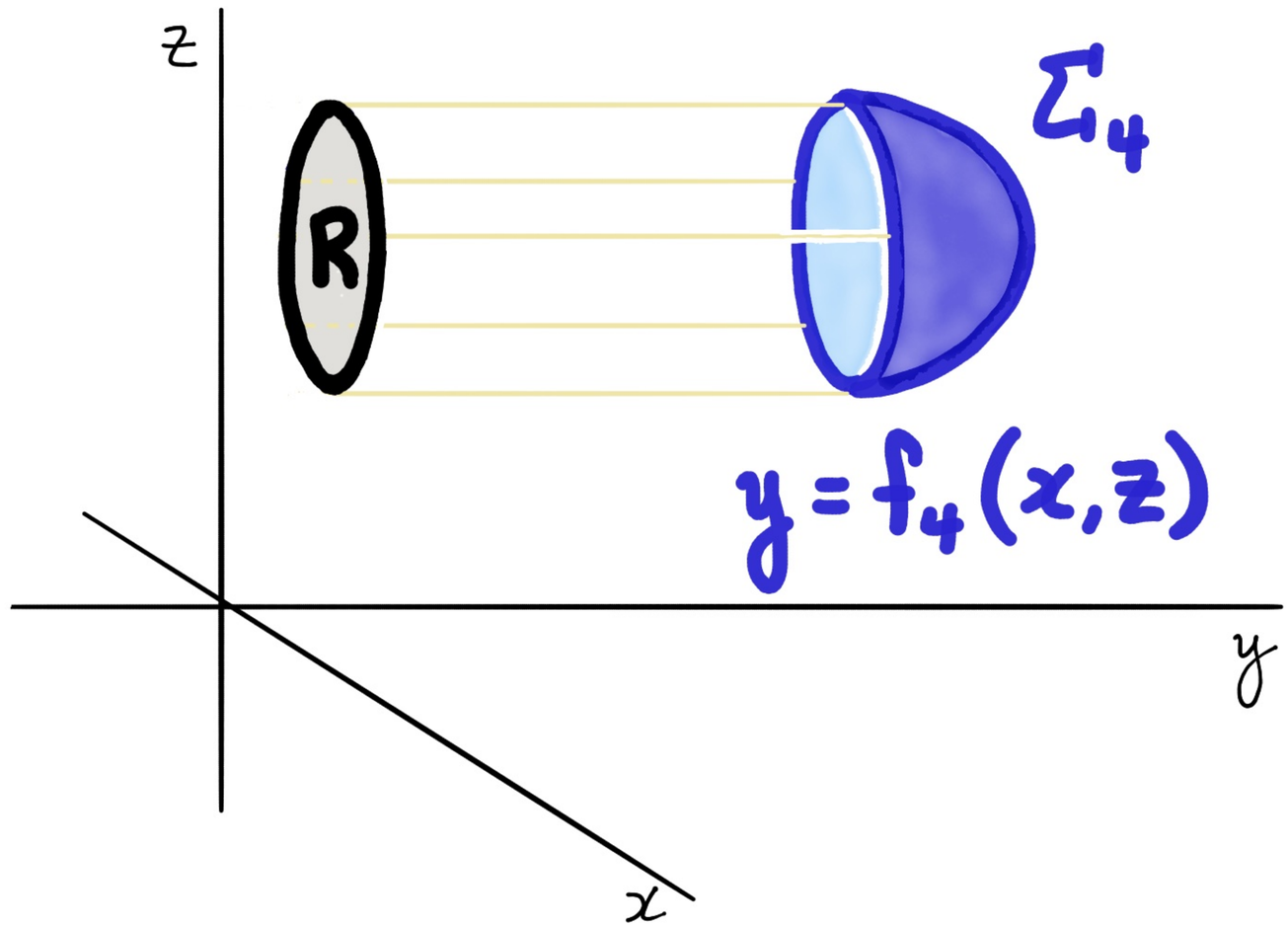
$z = f_2(x,y)$



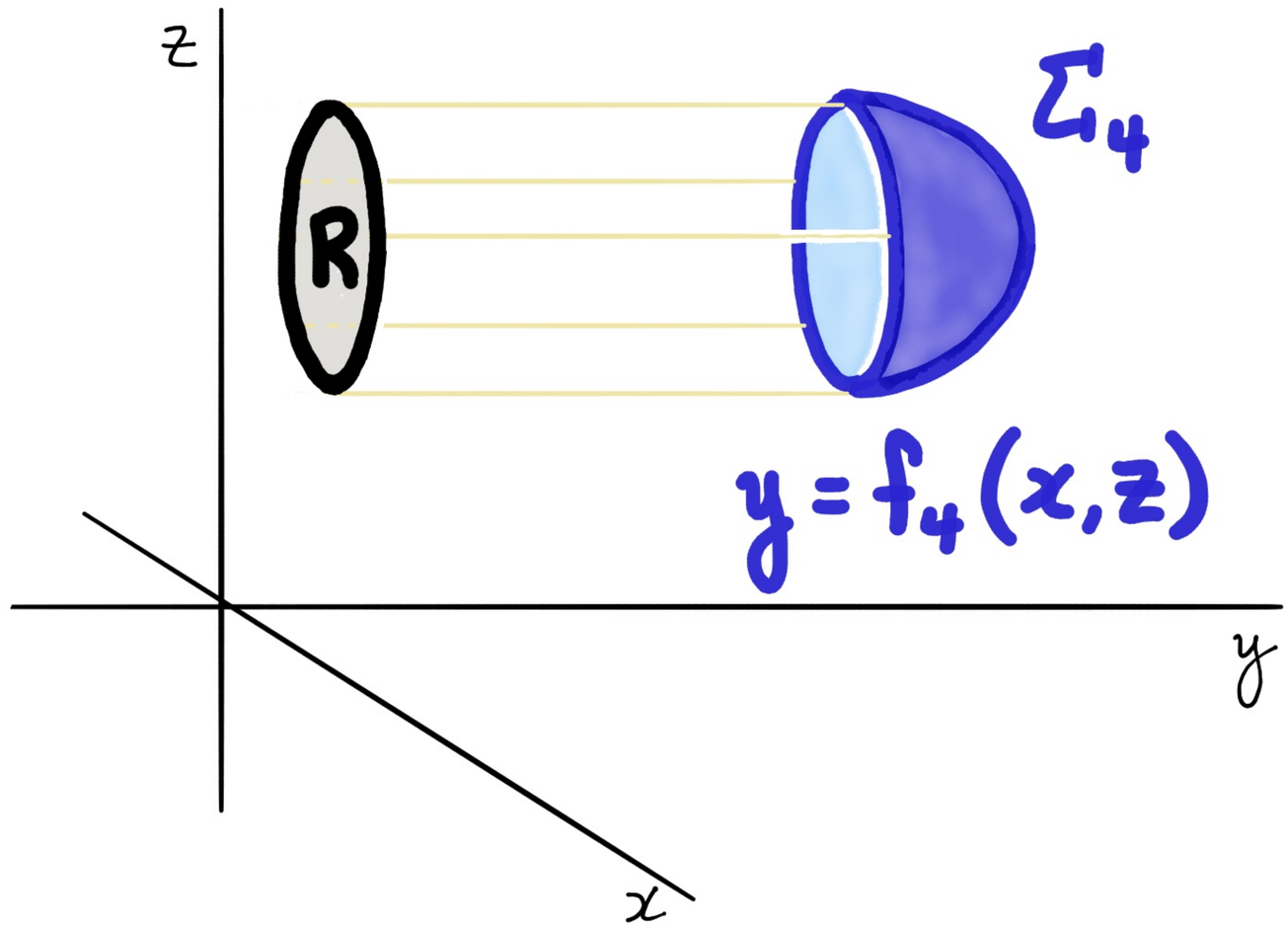
$y = f_4(x,z)$


z
y
x
Σ_4
$y = f_4(x,z)$

R is the domain of $f_4(x,z)$ in the xz-plane



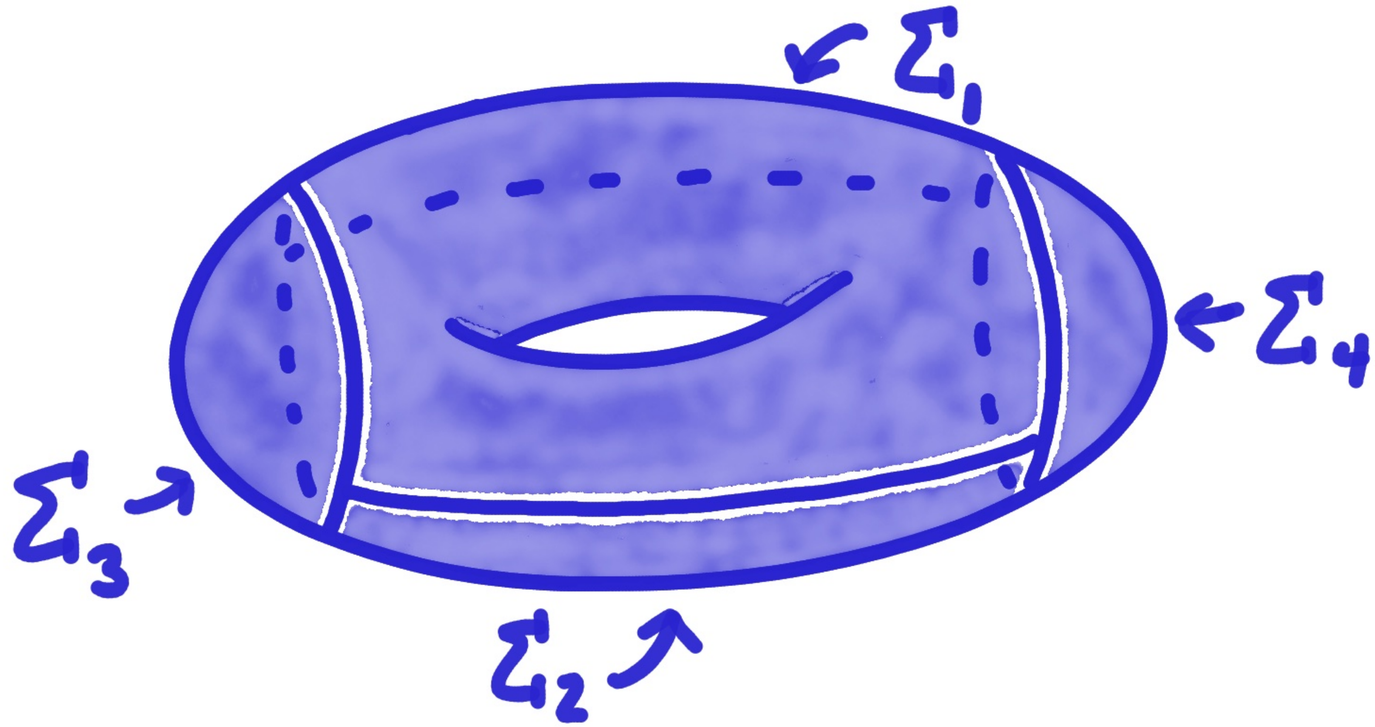
R is the domain of $f_4(x,z)$ in the xz-plane

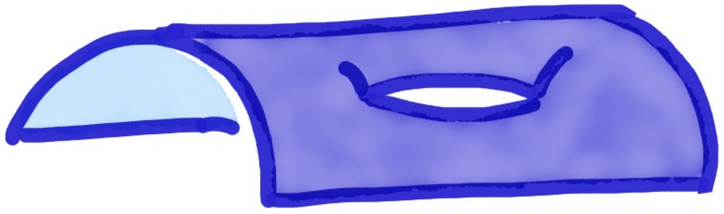


For $g: \mathbb{R}^3 \to \mathbb{R}$,

$$\iint_{\Sigma_4} g(x,y,z)\,dS = \iint_R g(x, f_4(x,z), z)\sqrt{\left(\frac{\partial f_4}{\partial x}\right)^2 + \left(\frac{\partial f_4}{\partial z}\right)^2 + 1}\,dx\,dz$$

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$$

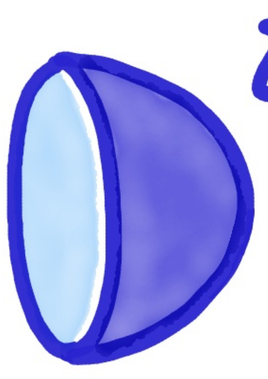




$z = f_1(x,y)$

$y = f_3(x,z)$

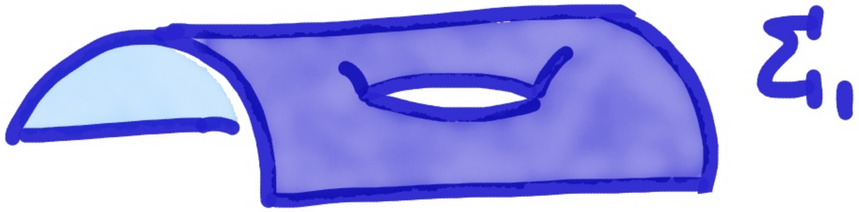
$z = f_2(x,y)$



$y = f_4(x,z)$

For $g: \mathbb{R}^3 \to \mathbb{R}$,

$$\iint_{\Sigma} g(x,y,z)\,dS = \sum_{i=1}^{4} \iint_{\Sigma_i} g(x,y,z)\,dS .$$



$z = f_1(x,y)$

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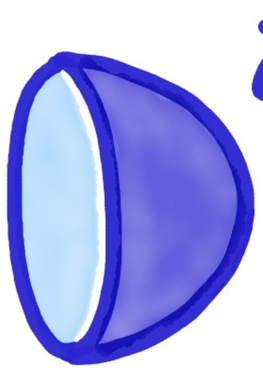
$z = f_2(x,y)$

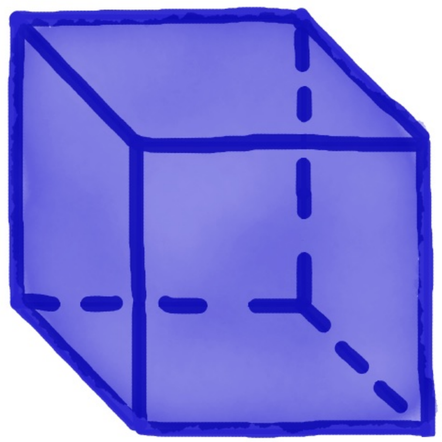


$y = f_4(x,z)$

Ⅲ An important example

Example: Let Σ be the surface of the box $R = \{(x,y,z): 0 \le x,y,z \le 1\}$.

Σ



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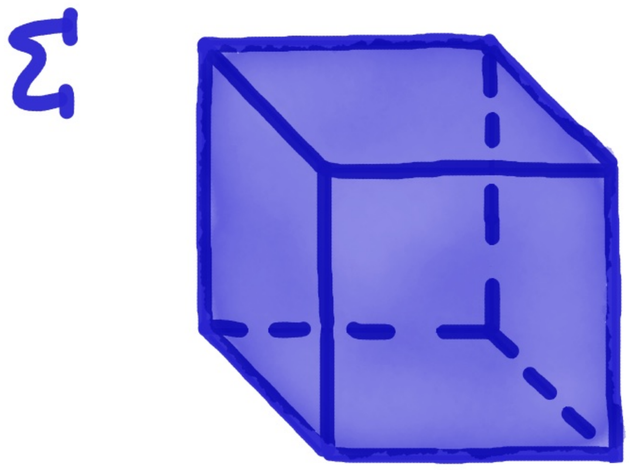
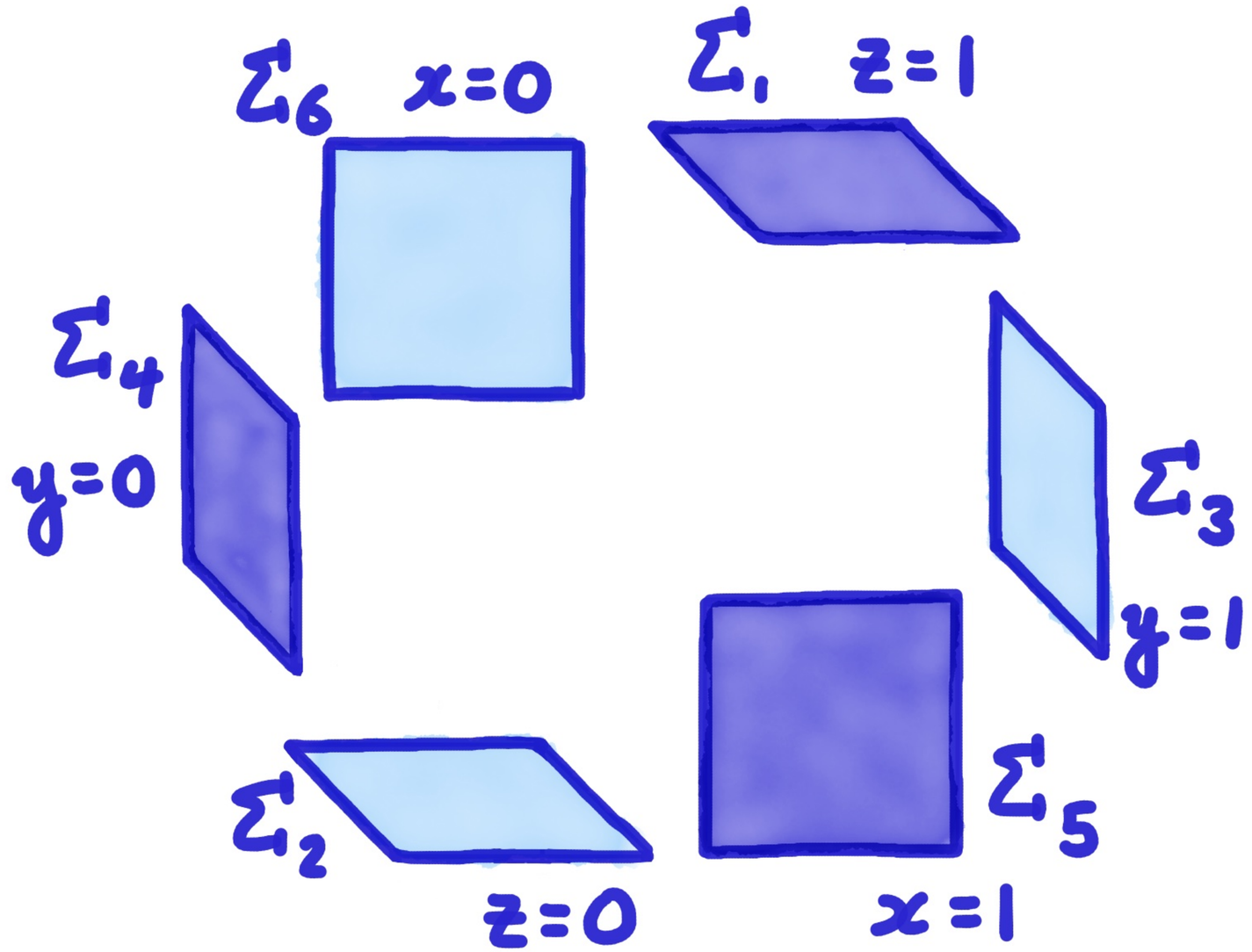
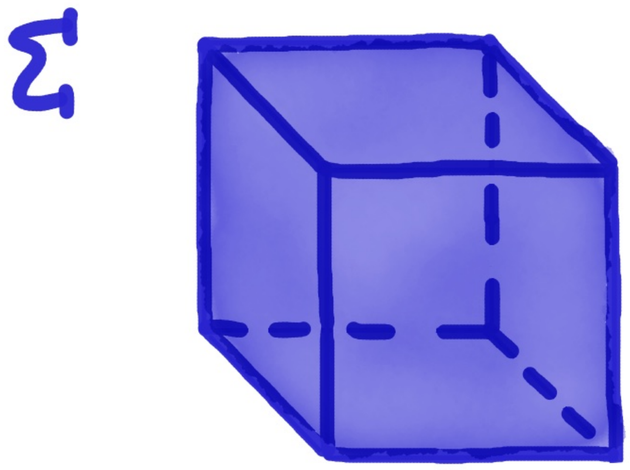
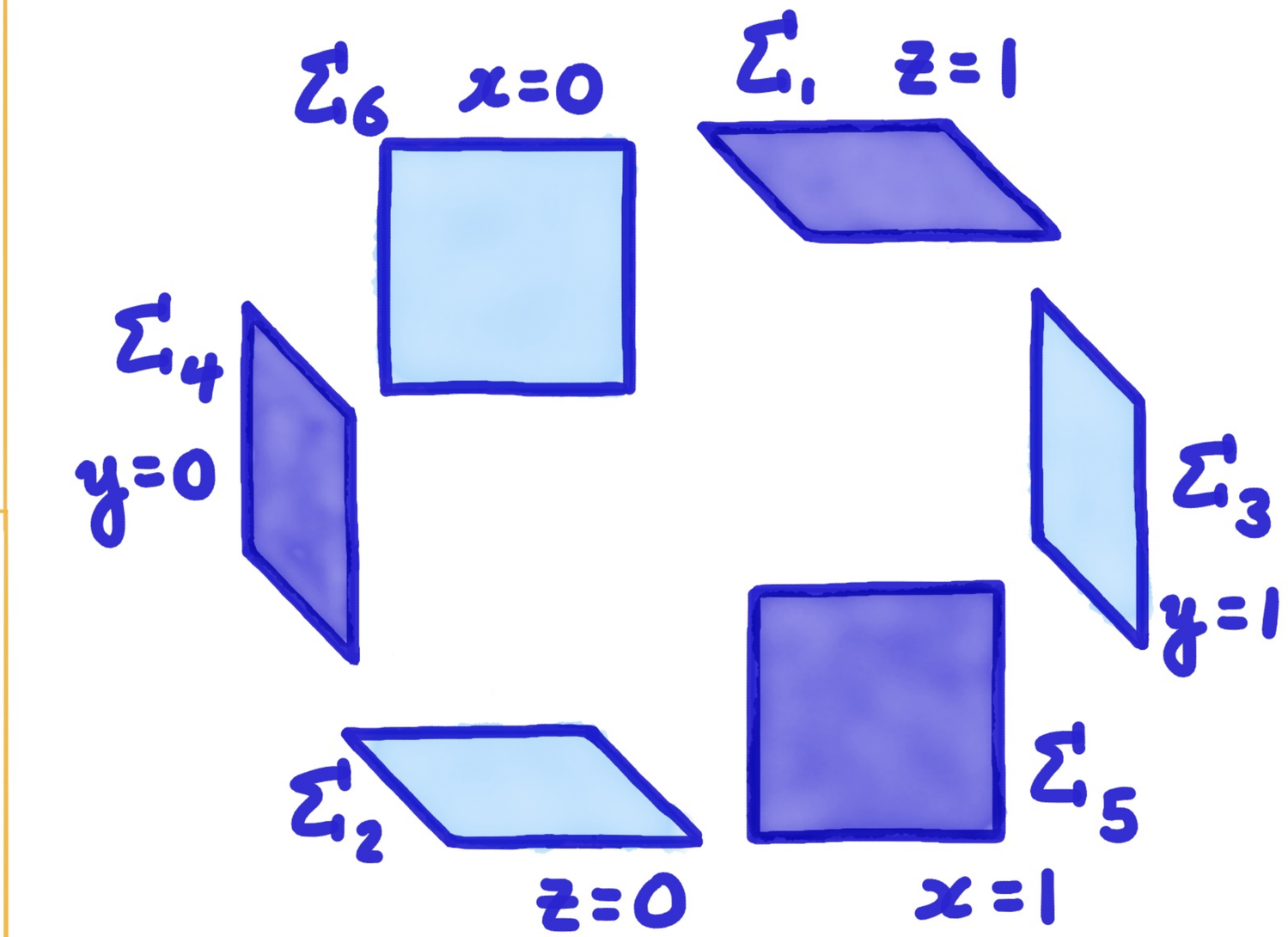




Example: Let Σ be the surface of the box $R = \{(x,y,z) : 0 \le x,y,z \le 1\}$.



Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\vec{F} = (F_1, F_2, F_3)$.



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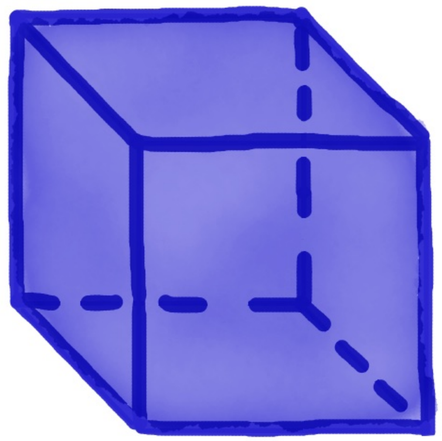
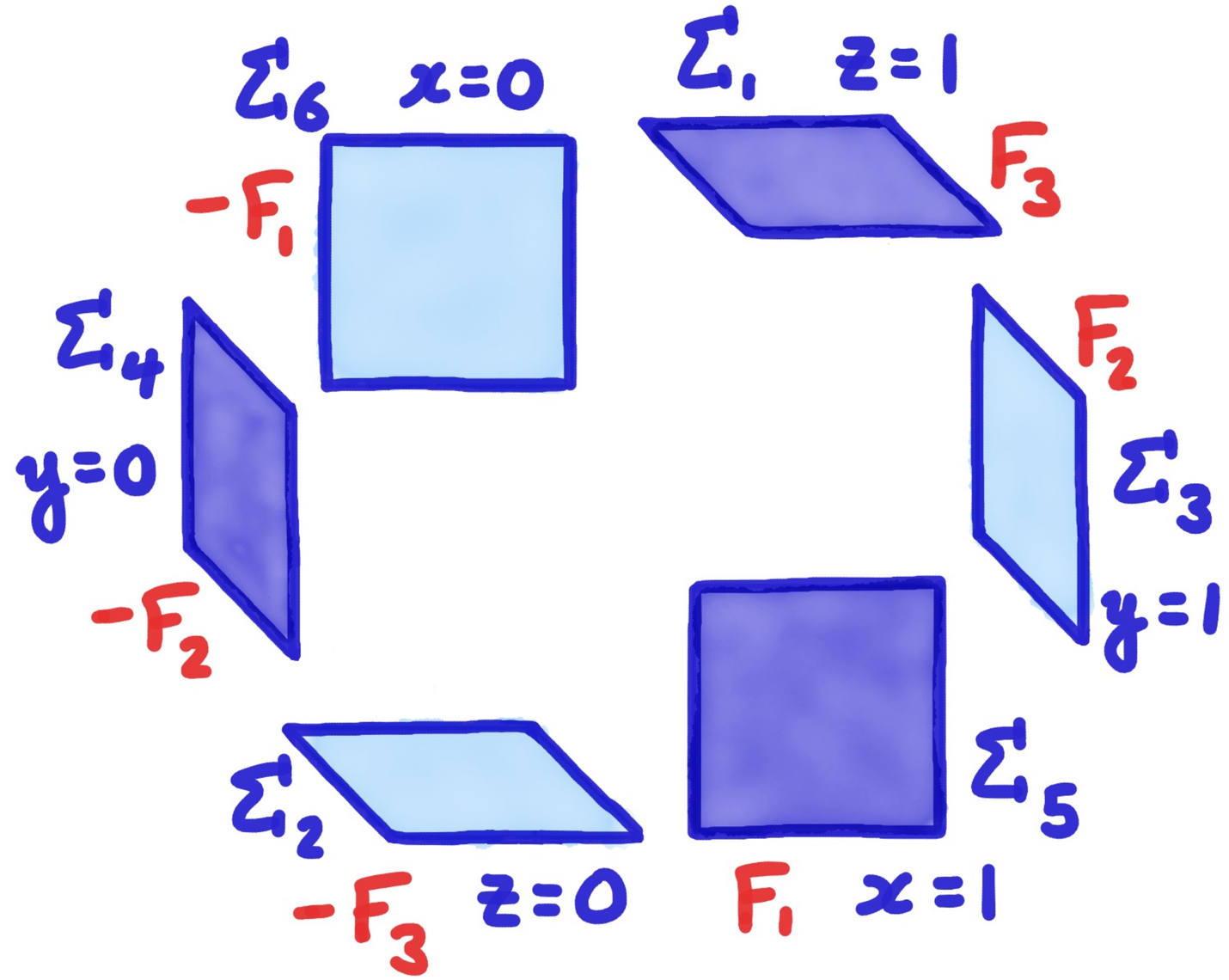


Let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\vec{F} = (F_1, F_2, F_3)$.

Define $P_{\vec{F}} : \mathbb{R}^3 \to \mathbb{R}$ as above.



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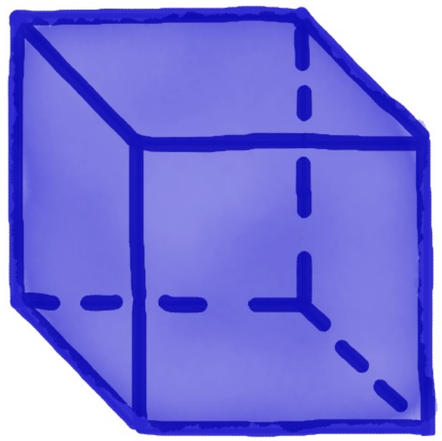

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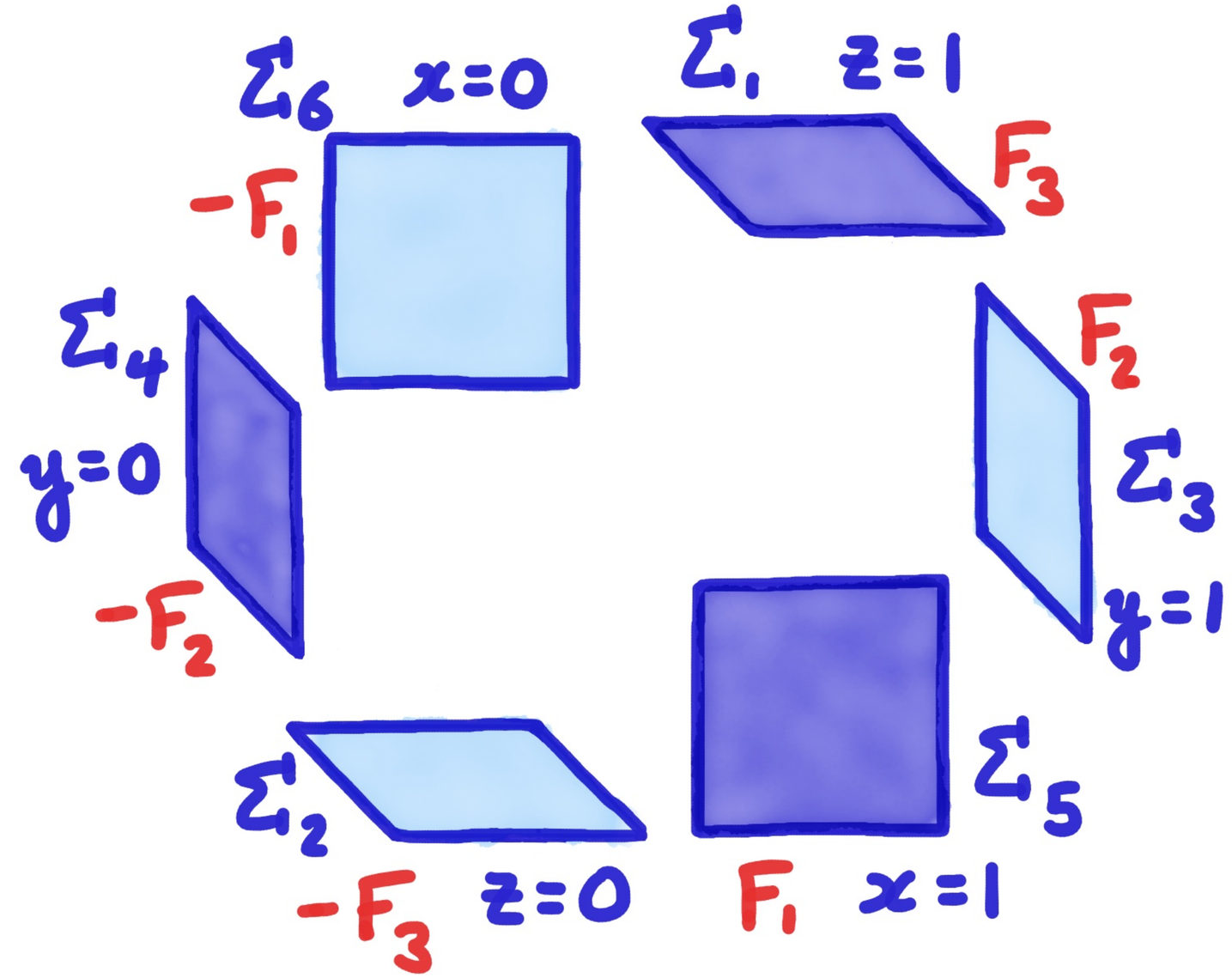
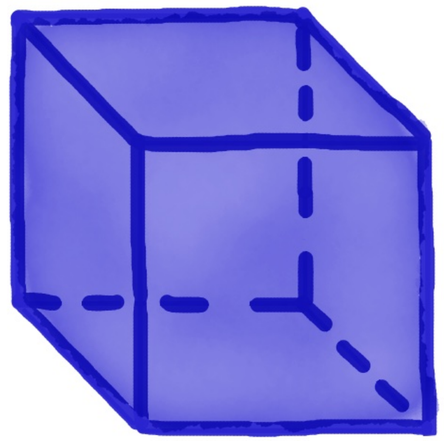
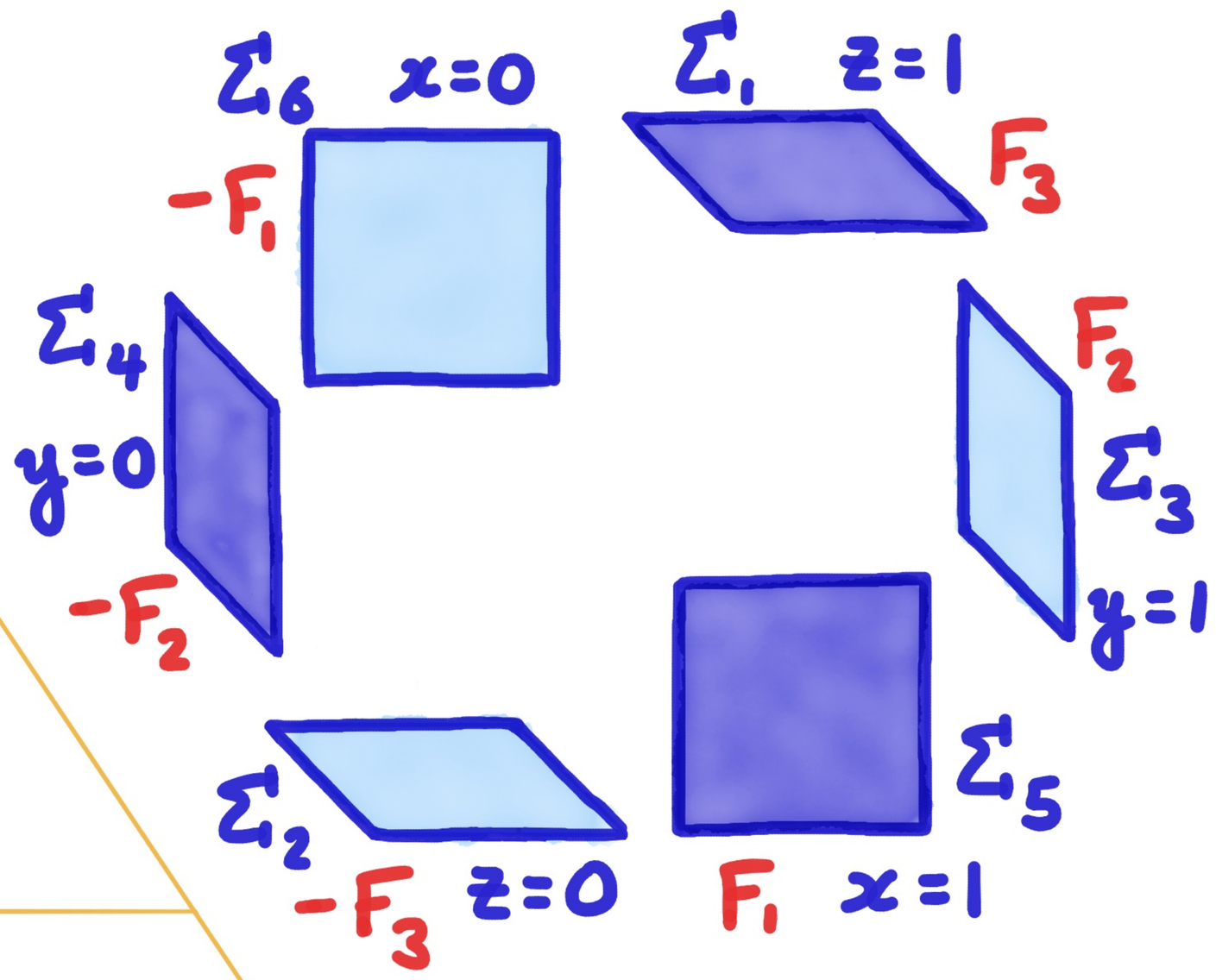


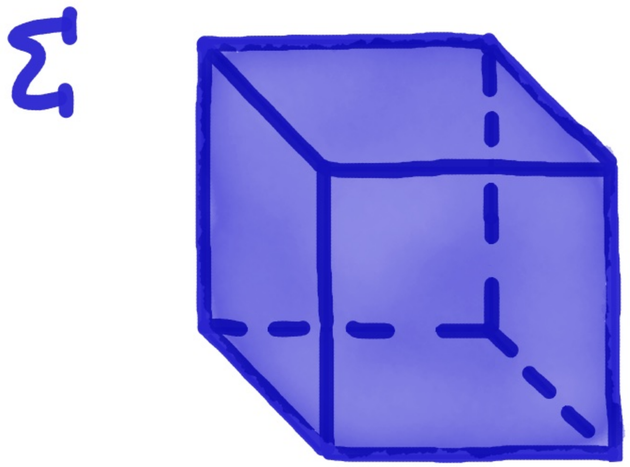
Find $\iint_{\Sigma} P_{\vec{F}} \, dS$.

Σ

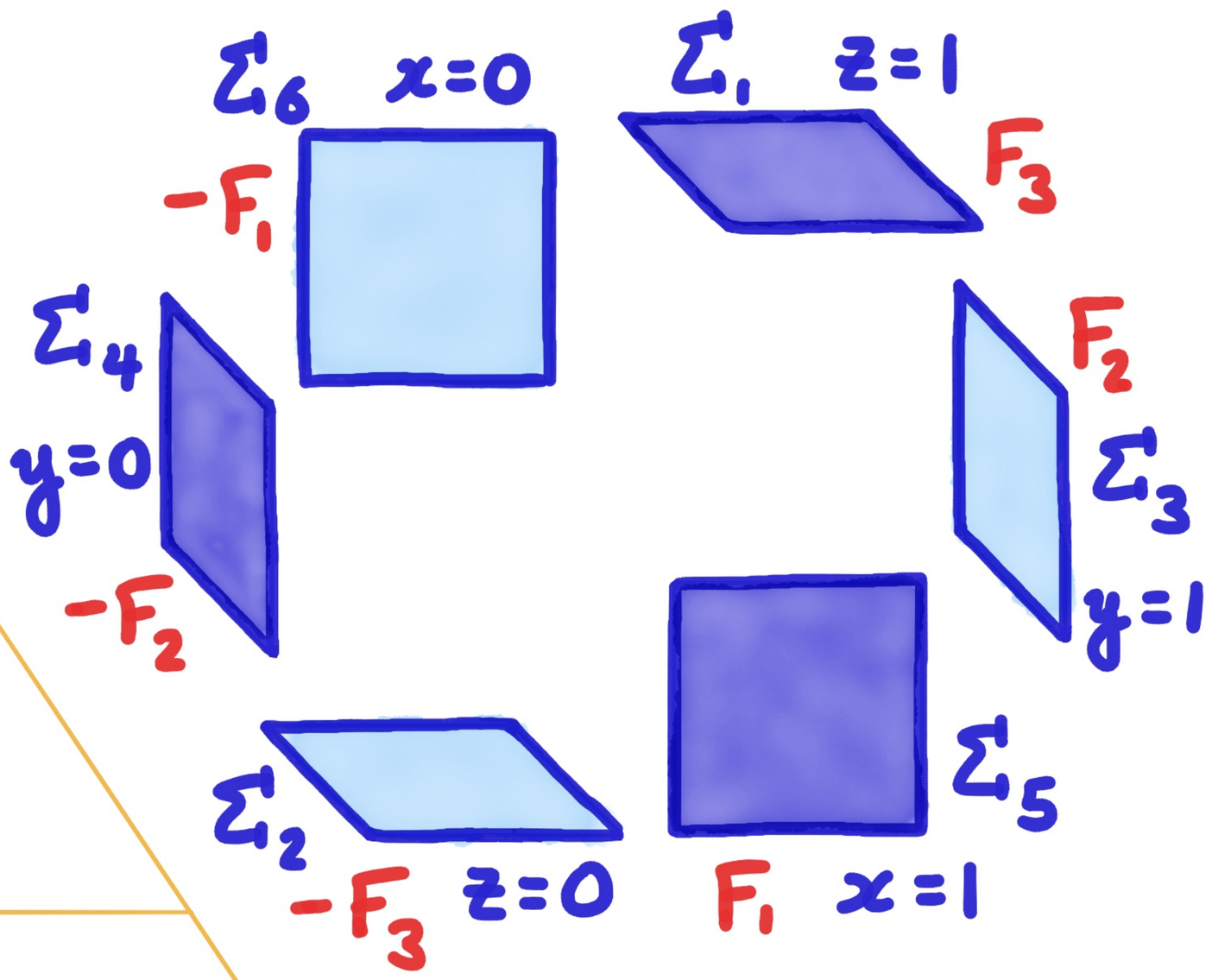


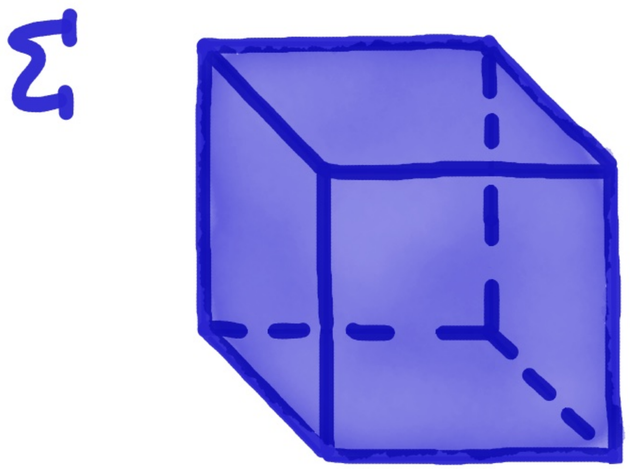
$$\iint_{\Sigma} \vec{P_F}\, dS =$$

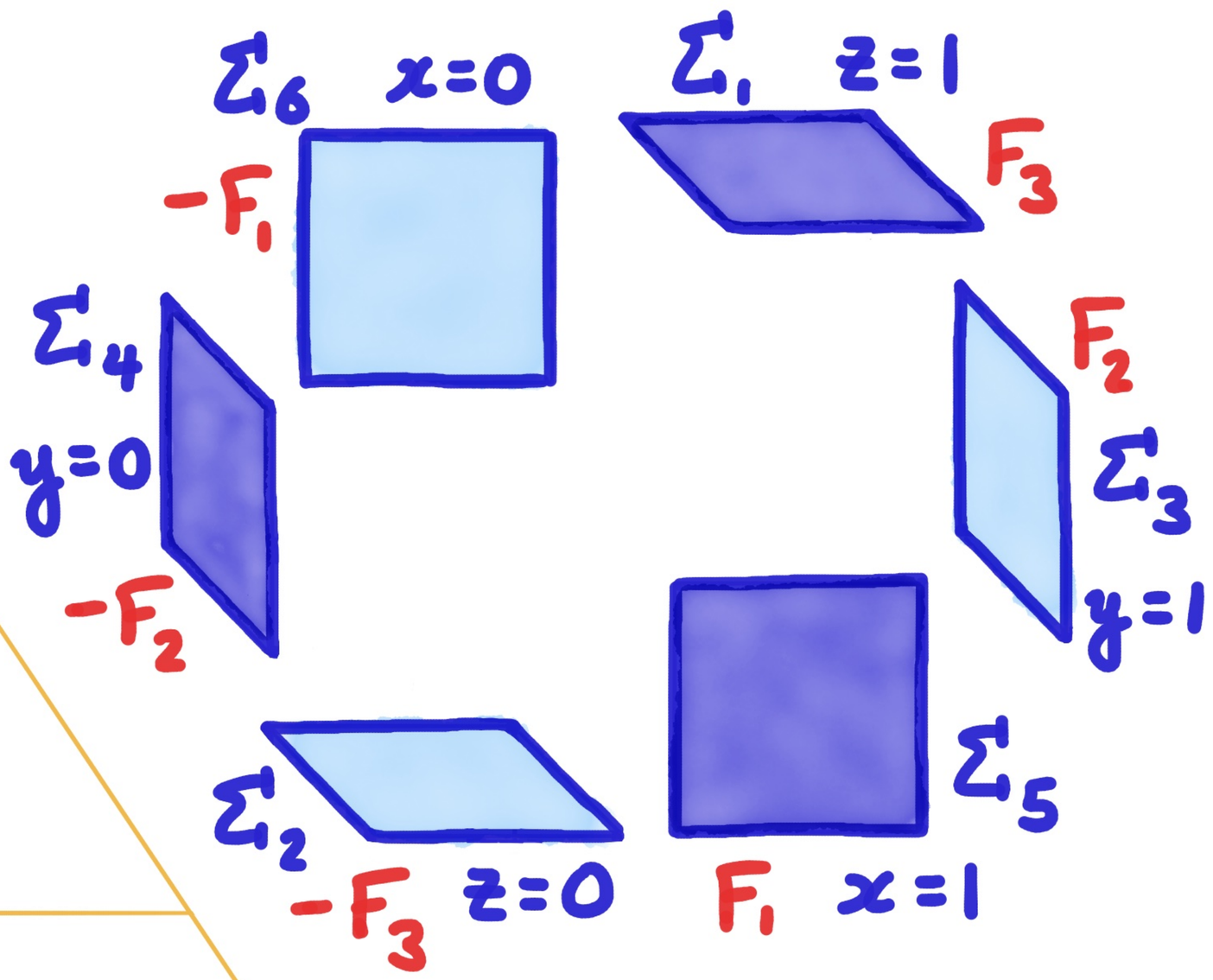




$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS.$$



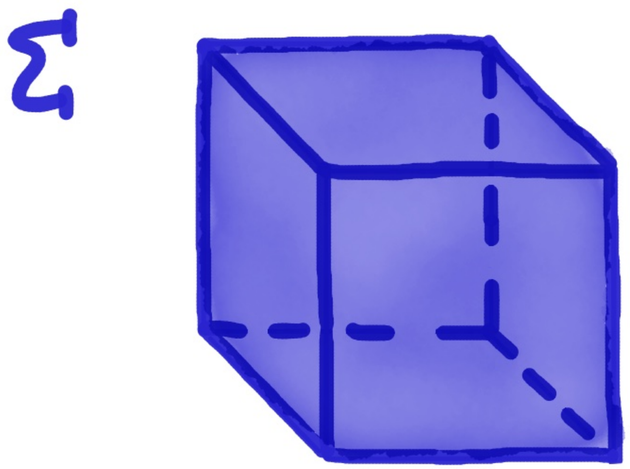


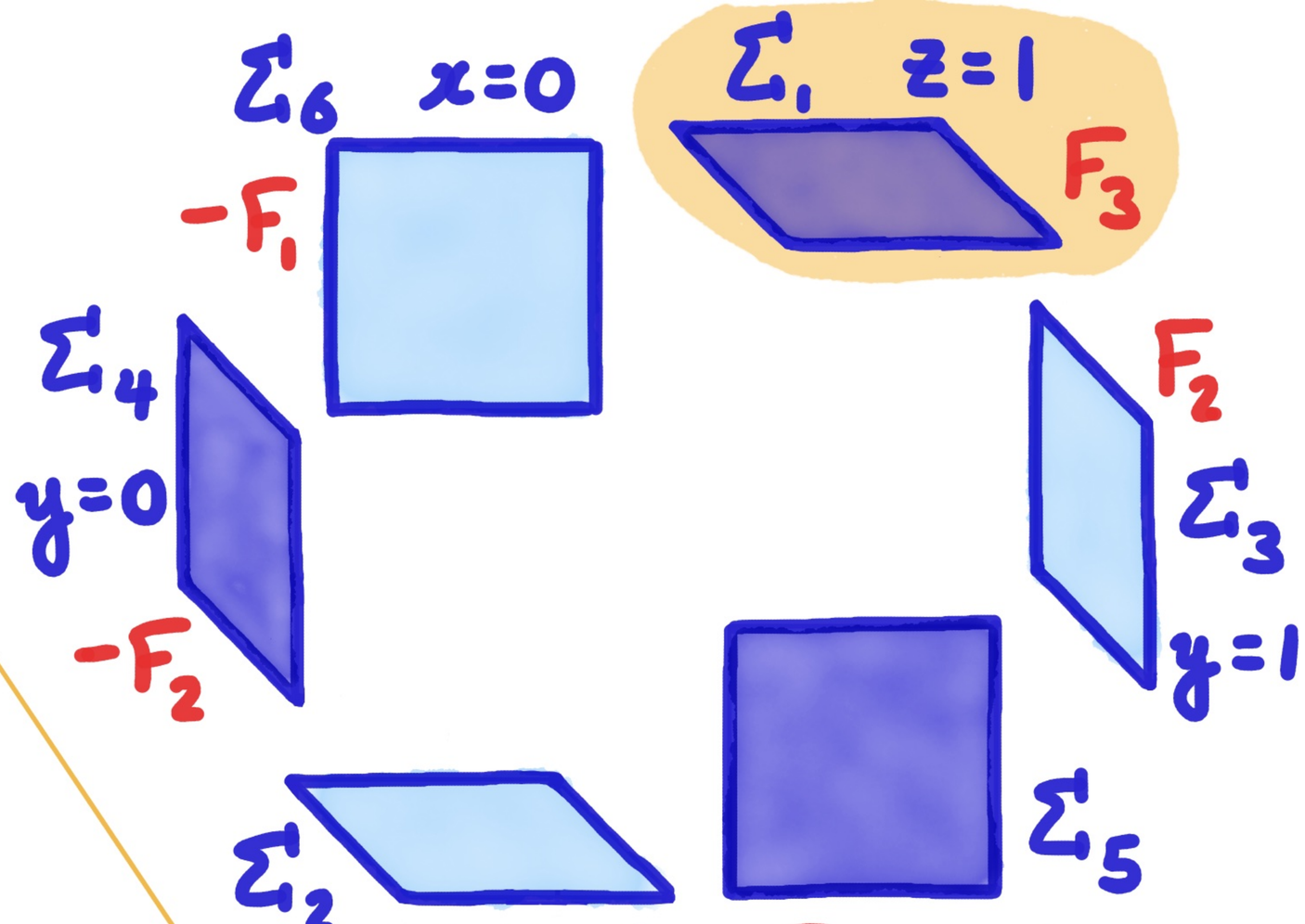


$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS.$$

Note,

$$\iint_{\Sigma_1} \vec{P_F}\, dS =$$





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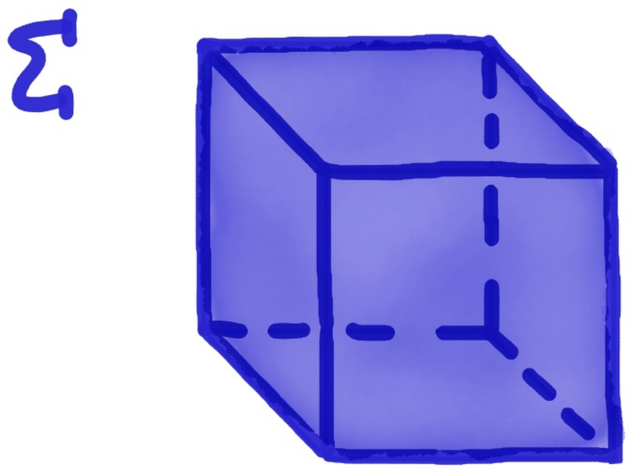

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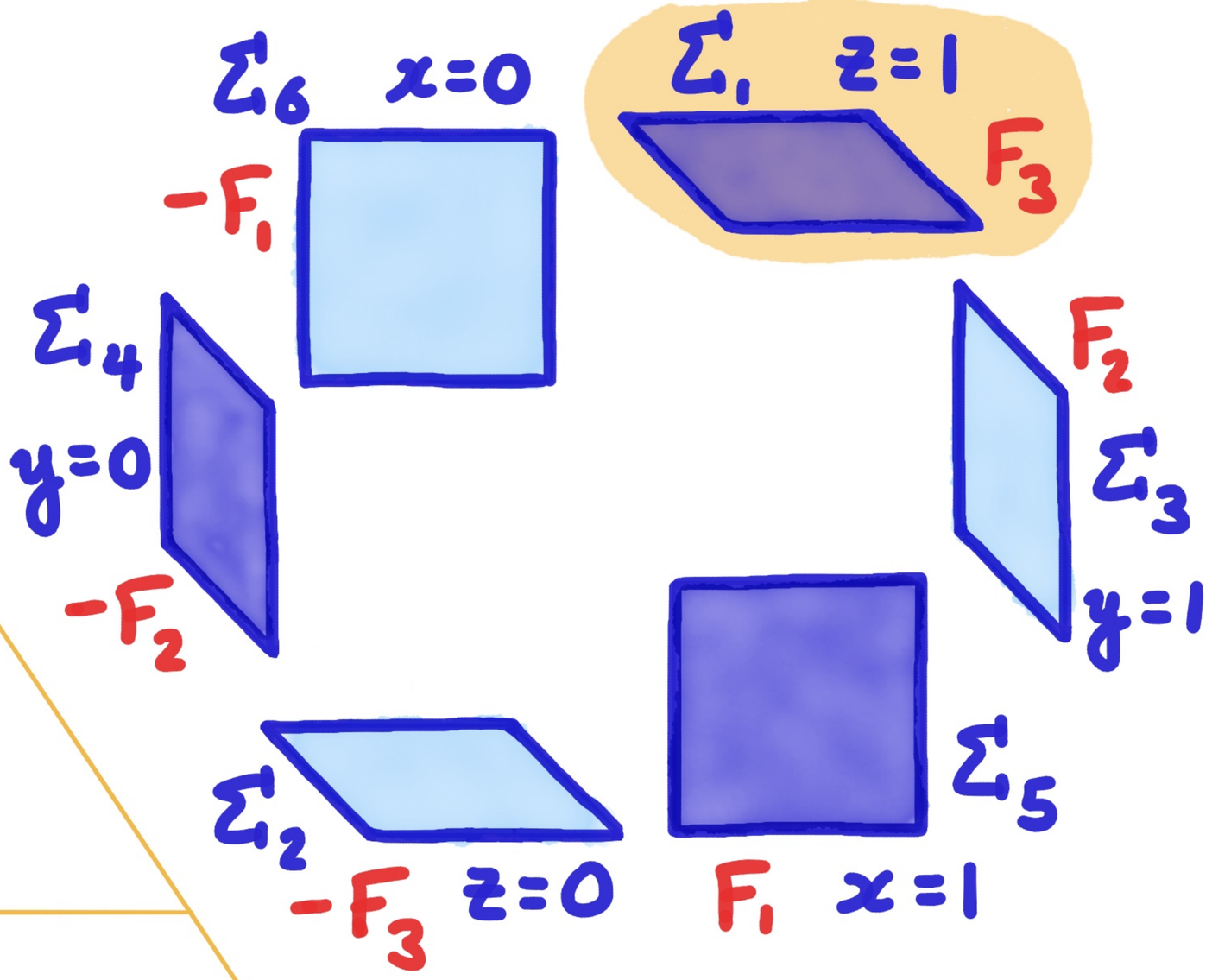
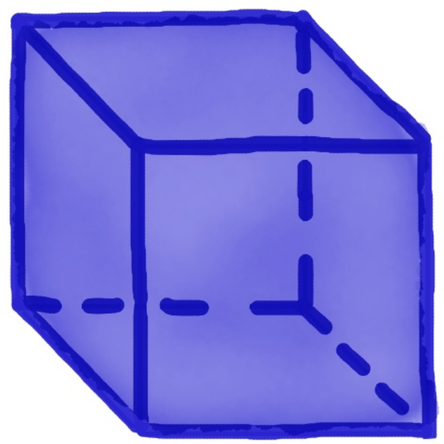


Note,

$$\iint_{\Sigma_1} \vec{P_F}\, dS = \iint_{\Sigma_1} F_3(x,y,z)\, dS$$

Σ



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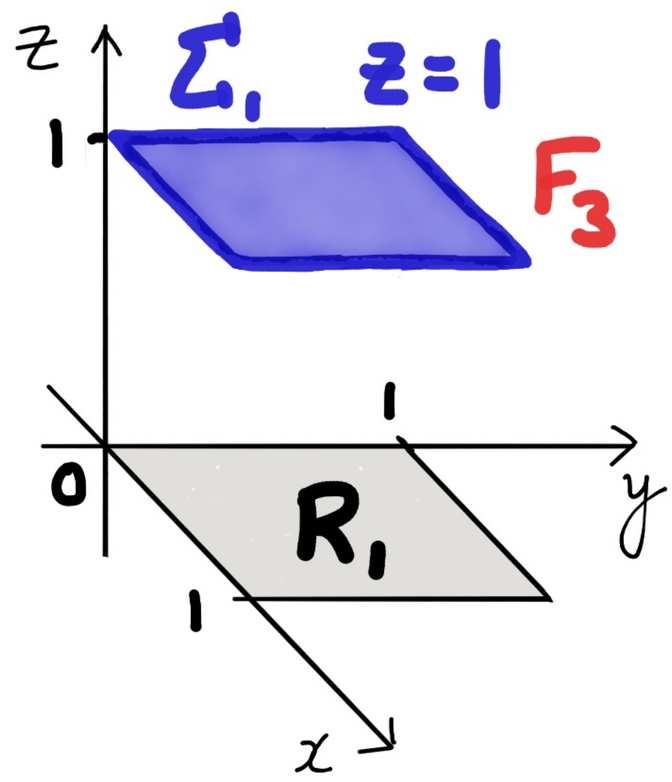

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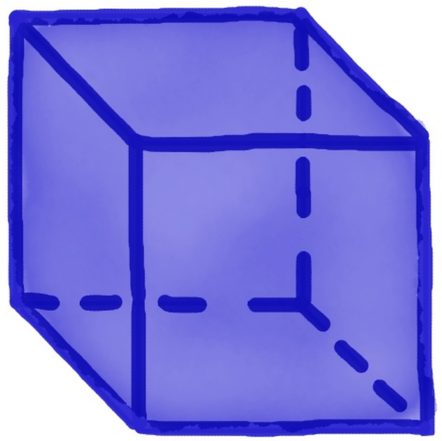
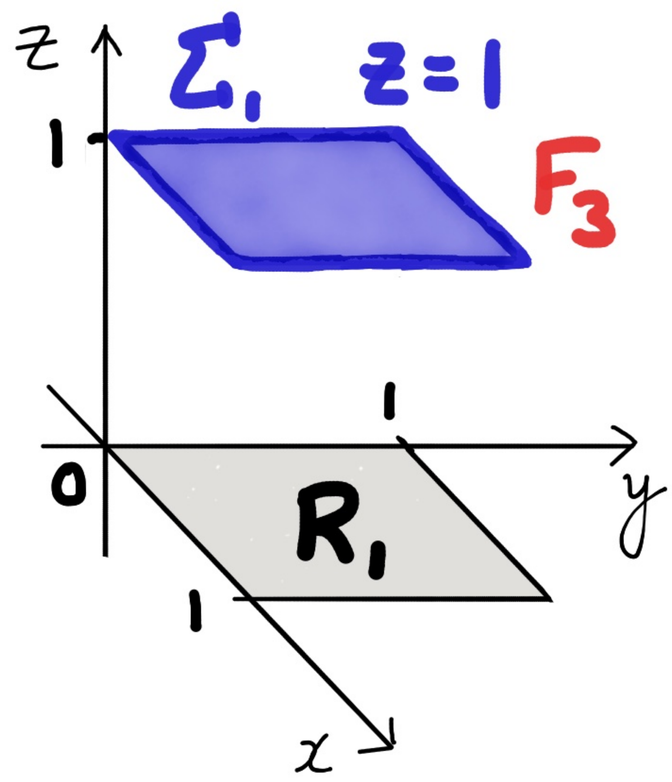


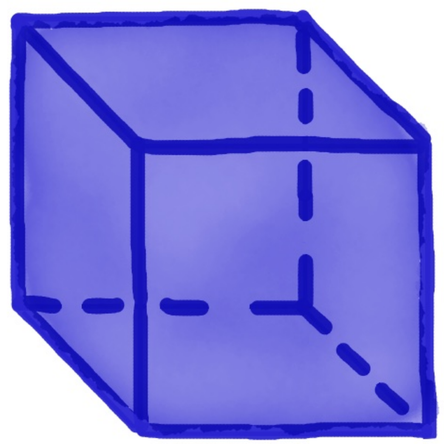


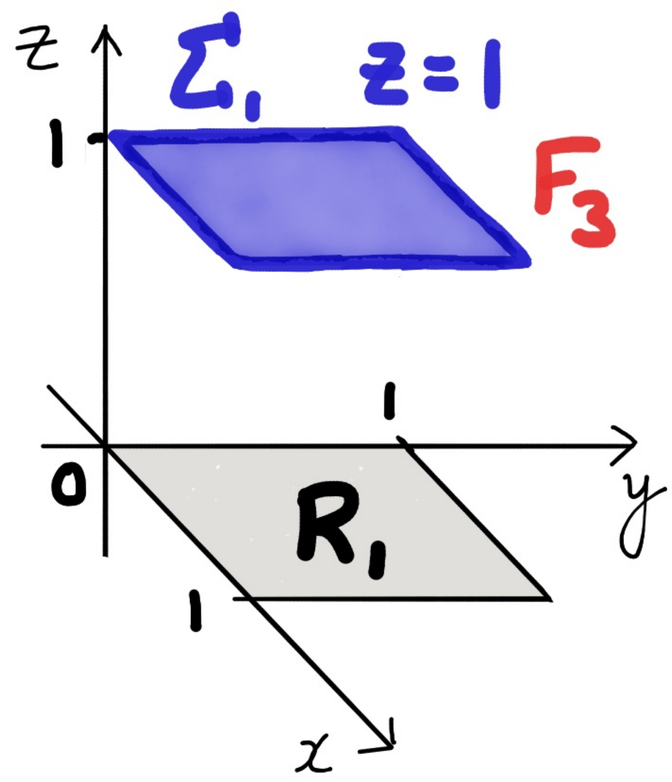
$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS.$$

Note,

$$\iint_{\Sigma_1} \vec{P_F}\, dS = \iint_{\Sigma_1} F_3(x,y,z)\, dS = \iint_{R_1} F_3(x,y,1) \sqrt{\frac{\partial 1}{\partial x}^2 + \frac{\partial 1}{\partial y}^2 + 1}\, dA$$

Σ





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Note,

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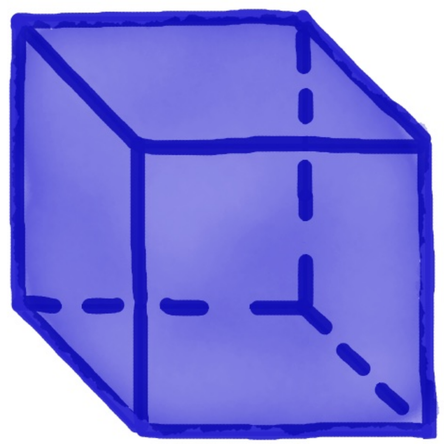
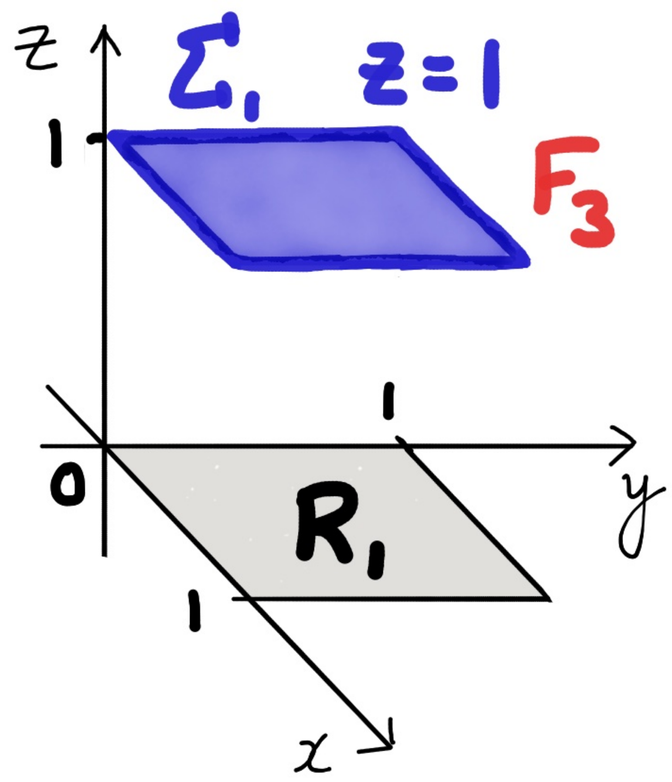
$$= \int_0^1 \int_0^1 F_3(x,y,1) \sqrt{1}\, dx\, dy$$

Σ

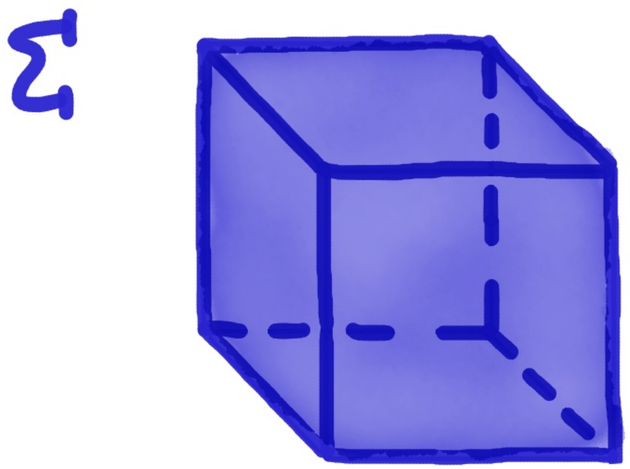




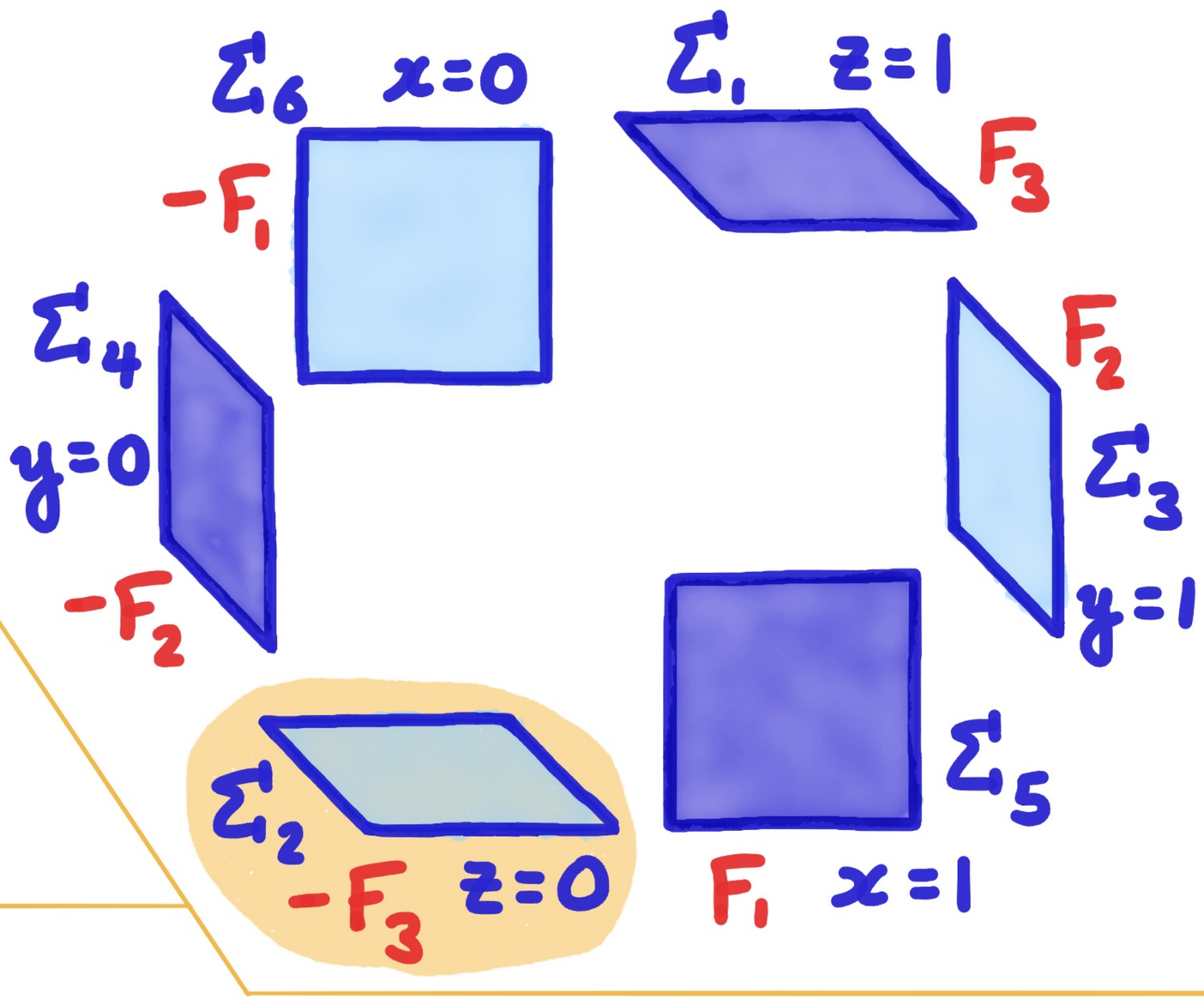
$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS.$$

Note,

$$\iint_{\Sigma_1} \vec{P_F}\, dS = \iint_{\Sigma_1} F_3(x,y,z)\, dS = \int_0^1 \int_0^1 F_3(x,y,1)\, dx\, dy \ ,$$



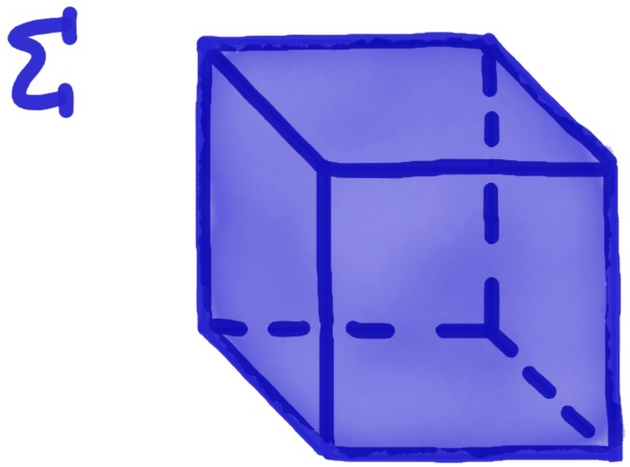
$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS.$$

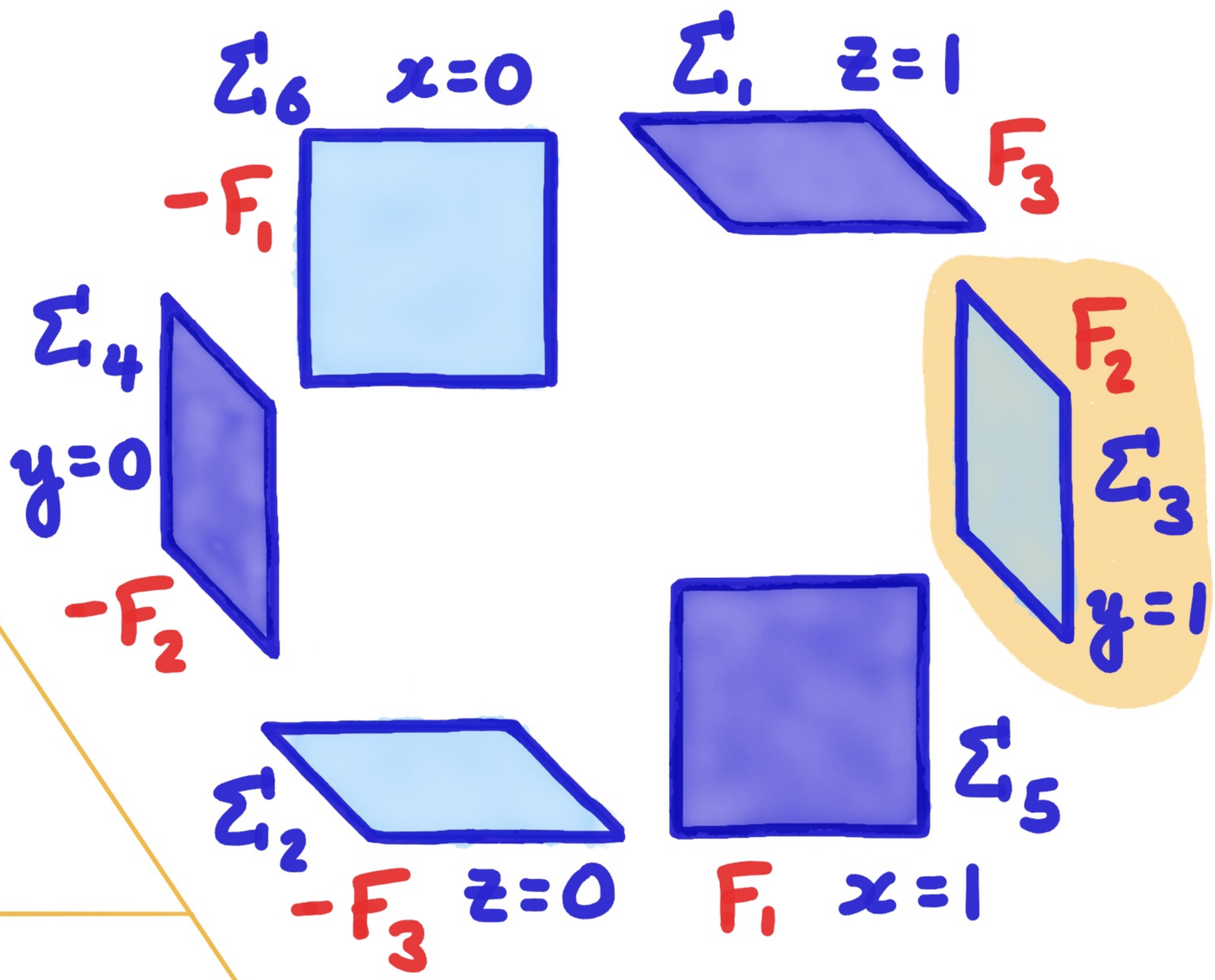


Note,

$$\iint_{\Sigma_1} \vec{P_F}\, dS = \iint_{\Sigma_1} F_3(x,y,z)\, dS = \int_0^1 \int_0^1 F_3(x,y,1)\, dx\, dy \ ,$$

$$\iint_{\Sigma_2} \vec{P_F}\, dS = \iint_{\Sigma_2} -F_3(x,y,z)\, dS = \int_0^1 \int_0^1 -F_3(x,y,0)\, dx\, dy.$$

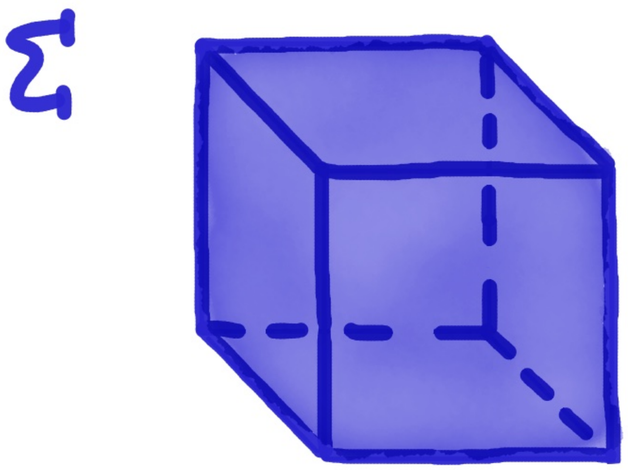




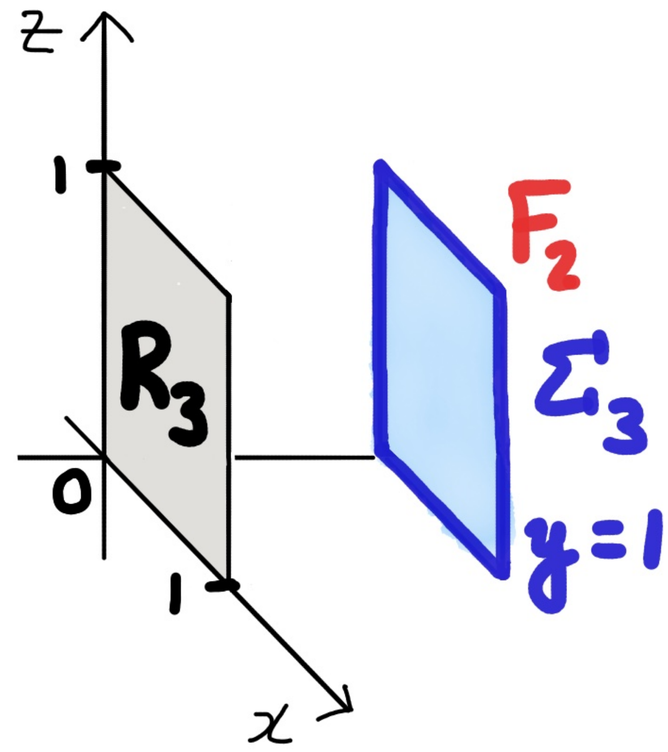
$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS.$$

Similarly,

$$\iint_{\Sigma_3} \vec{P_F}\, dS =$$



$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS.$$

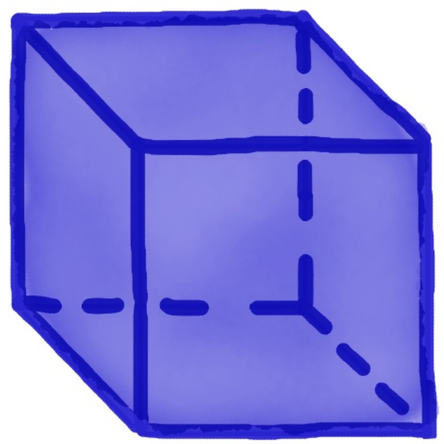


Similarly,

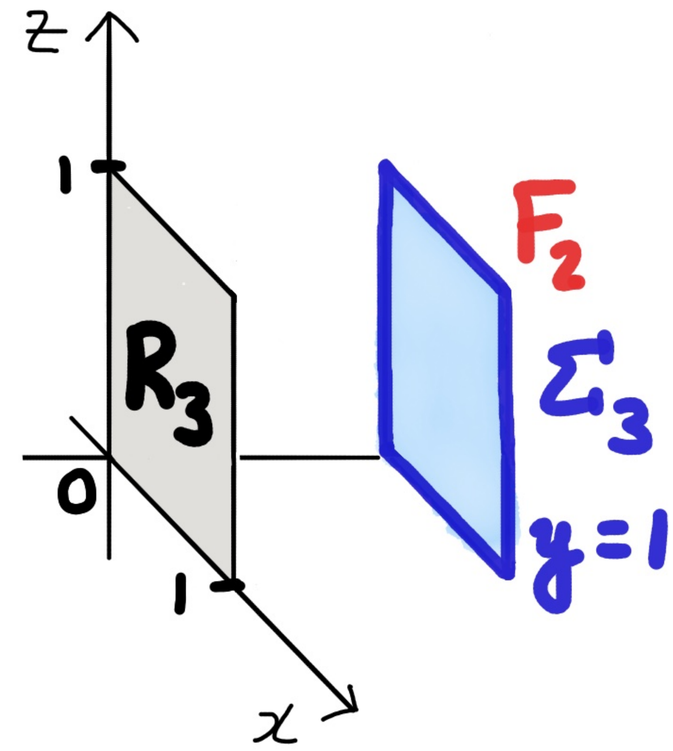
$$\iint_{\Sigma_3} \vec{P_F}\, dS = \iint_{\Sigma_3} F_2(x,y,z)\, dS = \iint_{R_3} F_2(x,1,z) \sqrt{\left(\frac{\partial 1}{\partial x}\right)^2 + \left(\frac{\partial 1}{\partial z}\right)^2 + 1}\, dA$$

$$= \int_0^1 \int_0^1 F_2(x,1,z) \sqrt{1}\, dx\, dz$$

Σ

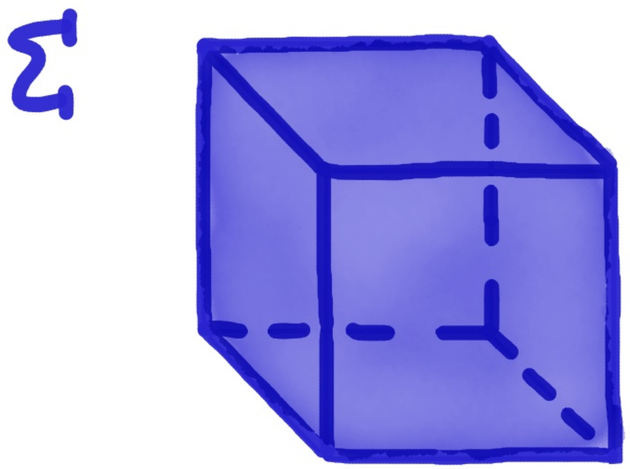


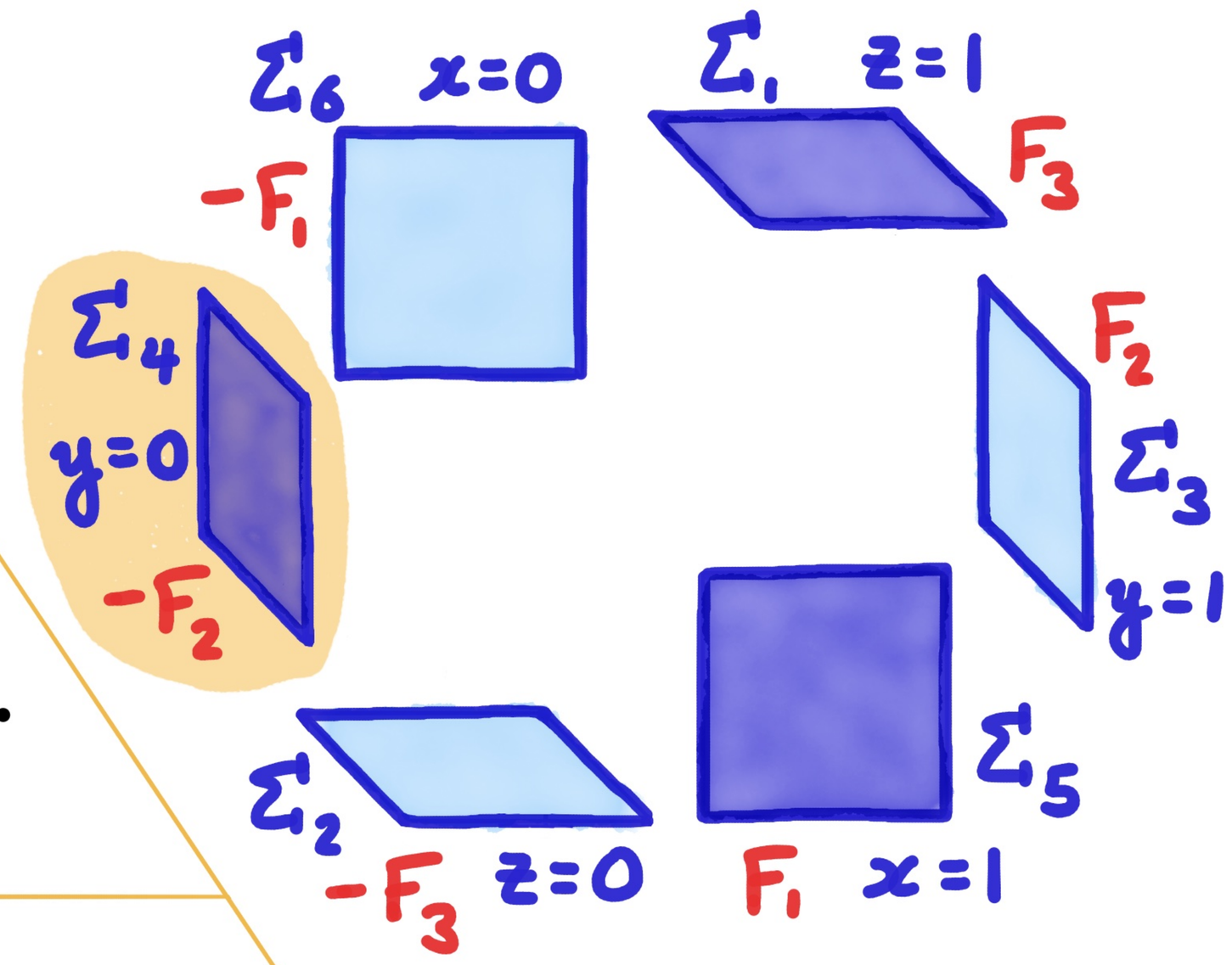
$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS.$$



Similarly,

$$\iint_{\Sigma_3} \vec{P_F}\, dS = \iint_{\Sigma_3} F_2(x,y,z)\, dS = \int_0^1 \int_0^1 F_2(x,1,z)\, dx\, dz \quad ,$$



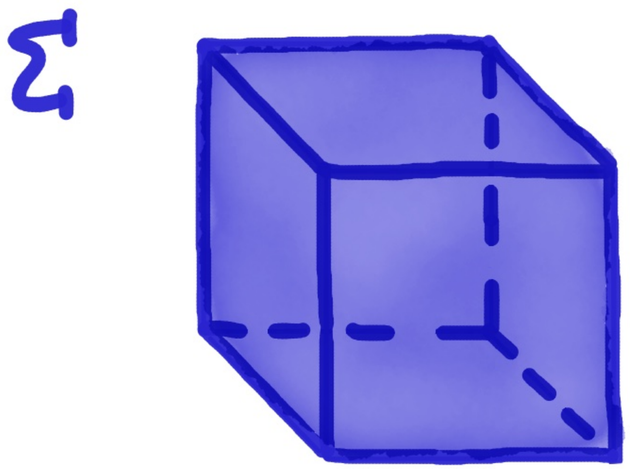


$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS.$$

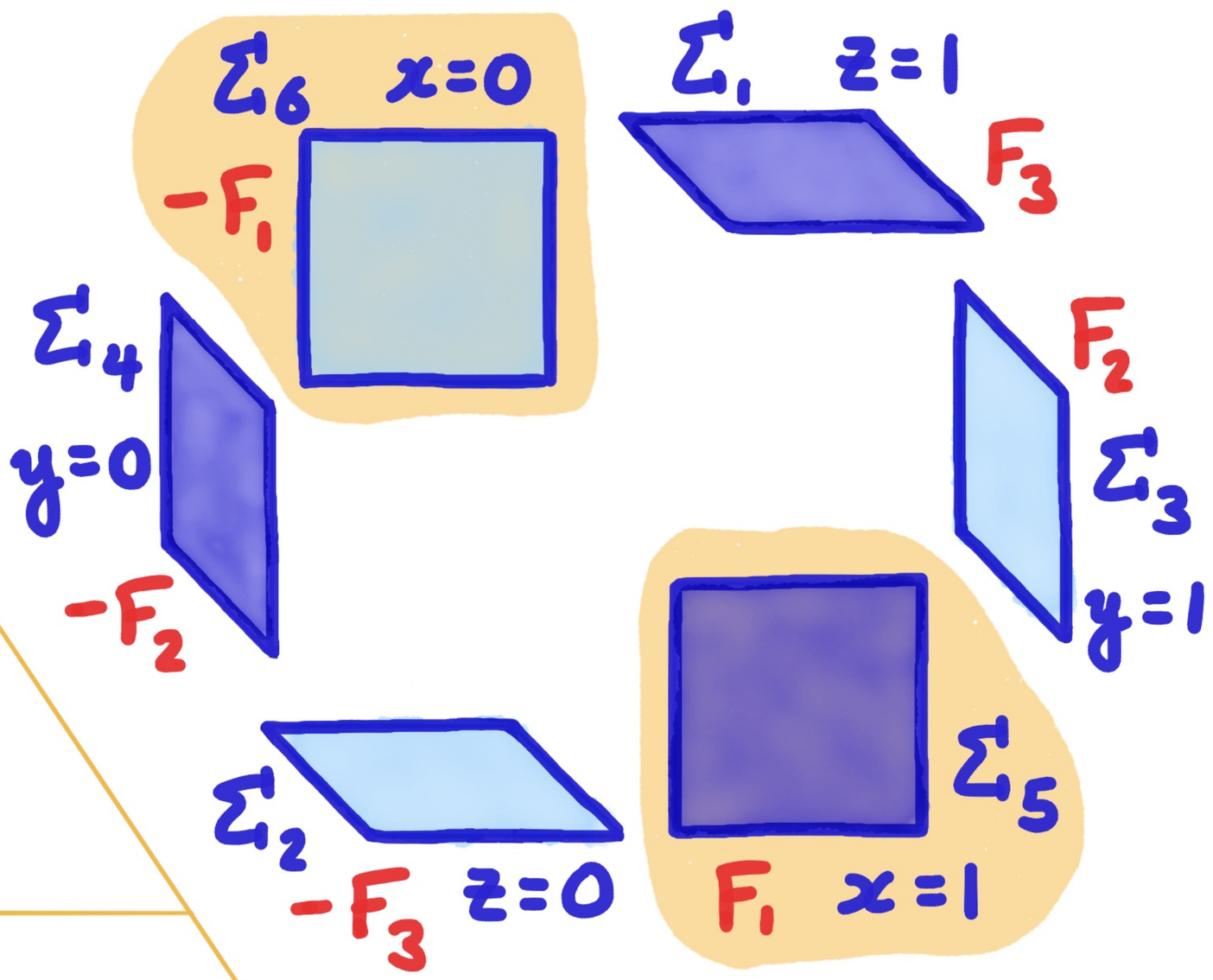
Similarly,

$$\iint_{\Sigma_3} \vec{P_F}\, dS = \iint_{\Sigma_3} F_2(x,y,z)\, dS = \int_0^1 \int_0^1 F_2(x,1,z)\, dx\, dz \quad ,$$

$$\iint_{\Sigma_4} \vec{P_F}\, dS = \iint_{\Sigma_4} -F_2(x,y,z)\, dS = \int_0^1 \int_0^1 -F_2(x,0,z)\, dx\, dz.$$



$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS.$$



Similarly,

$$\iint_{\Sigma_5} \vec{P_F}\, dS = \iint_{\Sigma_5} F_1(x,y,z)\, dS = \int_0^1 \int_0^1 F_1(1,y,z)\, dy\, dz \quad ,$$

$$\iint_{\Sigma_6} \vec{P_F}\, dS = \iint_{\Sigma_6} -F_1(x,y,z)\, dS = \int_0^1 \int_0^1 -F_1(0,y,z)\, dy\, dz.$$

$$\iint_{\Sigma} \vec{P_F}\, dS =$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS$$

$$= \iint_{\Sigma_1} \vec{P_F}\, dS + \iint_{\Sigma_2} \vec{P_F}\, dS$$

$$+ \iint_{\Sigma_3} \vec{P_F}\, dS + \iint_{\Sigma_4} \vec{P_F}\, dS$$

$$+ \iint_{\Sigma_5} \vec{P_F}\, dS + \iint_{\Sigma_6} \vec{P_F}\, dS$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS$$

$$= \int_0^1 \int_0^1 F_3(x,y,1)\, dx\, dy + \int_0^1 \int_0^1 -F_3(x,y,0)\, dx\, dy$$

$$+ \int_0^1 \int_0^1 F_2(x,1,z)\, dx\, dz + \int_0^1 \int_0^1 -F_2(x,0,z)\, dx\, dz$$

$$+ \int_0^1 \int_0^1 F_1(1,y,z)\, dy\, dz + \int_0^1 \int_0^1 -F_1(0,y,z)\, dy\, dz$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS$$

$$= \int_0^1 \int_0^1 F_3(x,y,1)\, dx\, dy + \int_0^1 \int_0^1 -F_3(x,y,0)\, dx\, dy$$

$$+ \int_0^1 \int_0^1 F_2(x,1,z)\, dx\, dz + \int_0^1 \int_0^1 -F_2(x,0,z)\, dx\, dz$$

$$+ \int_0^1 \int_0^1 F_1(1,y,z)\, dy\, dz + \int_0^1 \int_0^1 -F_1(0,y,z)\, dy\, dz$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS$$

$$= \int_0^1 \int_0^1 F_3(x,y,1) - F_3(x,y,0)\, dx\,dy$$

$$+ \int_0^1 \int_0^1 F_2(x,1,z) - F_2(x,0,z)\, dx\,dz$$

$$+ \int_0^1 \int_0^1 F_1(1,y,z) - F_1(0,y,z)\, dy\,dz$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \sum_{i=1}^{6} \iint_{\Sigma_i} \vec{P_F}\, dS$$

$$= \int_0^1 \int_0^1 F_3(x,y,1) - F_3(x,y,0)\, dx\,dy$$

$$+ \int_0^1 \int_0^1 F_2(x,1,z) - F_2(x,0,z)\, dx\,dz$$

$$+ \int_0^1 \int_0^1 F_1(1,y,z) - F_1(0,y,z)\, dy\,dz$$

$$= \int_0^1 \int_0^1 \left[F_3\right]_{z=0}^{z=1} dx\,dy$$

$$+ \int_0^1 \int_0^1 \left[F_2\right]_{y=0}^{y=1} dx\,dz$$

$$+ \int_0^1 \int_0^1 \left[F_1\right]_{x=0}^{x=1} dy\,dz$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \int_0^1\int_0^1 \left[F_3\right]_{z=0}^{z=1} dx\,dy$$

$$+ \int_0^1\int_0^1 \left[F_2\right]_{y=0}^{y=1} dx\,dz$$

$$+ \int_0^1\int_0^1 \left[F_1\right]_{x=0}^{x=1} dy\,dz$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \int_0^1\int_0^1 \Big[F_3\Big]_{z=0}^{z=1} dx\,dy$$

$$+ \int_0^1\int_0^1 \Big[F_2\Big]_{y=0}^{y=1} dx\,dz$$

$$+ \int_0^1\int_0^1 \Big[F_1\Big]_{x=0}^{x=1} dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 \frac{\partial F_3}{\partial z}\, dz\,dx\,dy$$

$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_2}{\partial y}\, dy\,dx\,dz$$

$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_1}{\partial x}\, dx\,dy\,dz$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \int_0^1\int_0^1 \left[F_3\right]_{z=0}^{z=1} dx\,dy$$

$$+ \int_0^1\int_0^1 \left[F_2\right]_{y=0}^{y=1} dx\,dz$$

$$+ \int_0^1\int_0^1 \left[F_1\right]_{x=0}^{x=1} dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 \frac{\partial F_3}{\partial z}\, dz\,dx\,dy$$

$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_2}{\partial y}\, dy\,dx\,dz$$

$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_1}{\partial x}\, dx\,dy\,dz$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \int_0^1\int_0^1 \left[F_3\right]_{z=0}^{z=1} dx\,dy$$

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$$= \int_0^1\int_0^1\int_0^1 \frac{\partial F_3}{\partial z}\, dx\,dy\,dz$$

$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_2}{\partial y}\, dx\,dy\,dz$$

$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_1}{\partial x}\, dx\,dy\,dz$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \int_0^1\int_0^1 \left[F_3\right]_{z=0}^{z=1} dx\,dy$$

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$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_2}{\partial y}\, dx\,dy\,dz$$

$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_1}{\partial x}\, dx\,dy\,dz$$

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$$+ \int_0^1\int_0^1 \left[F_1\right]_{x=0}^{x=1} dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 \frac{\partial F_3}{\partial z}\, dx\,dy\,dz$$

$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_2}{\partial y}\, dx\,dy\,dz$$

$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_1}{\partial x}\, dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 \frac{\partial F_3}{\partial z} + \frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial x}\, dx\,dy\,dz$$

$$\iint_{\Sigma} \vec{P_F}\, dS = \int_0^1\int_0^1 \left[F_3\right]_{z=0}^{z=1} dx\,dy$$

$$+ \int_0^1\int_0^1 \left[F_2\right]_{y=0}^{y=1} dx\,dz$$

$$+ \int_0^1\int_0^1 \left[F_1\right]_{x=0}^{x=1} dy\,dz$$

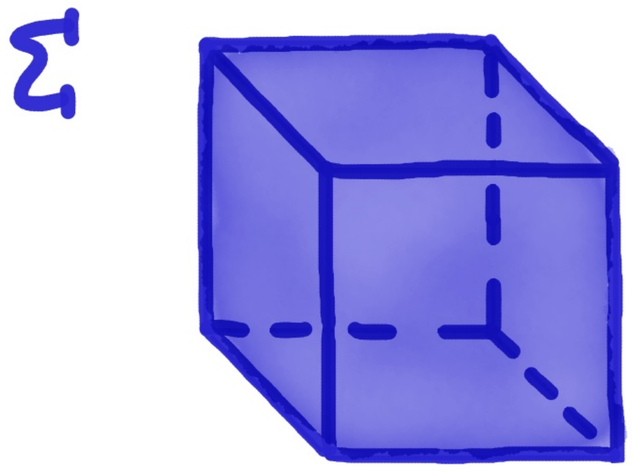
$$= \int_0^1\int_0^1\int_0^1 \frac{\partial F_3}{\partial z}\, dx\,dy\,dz$$

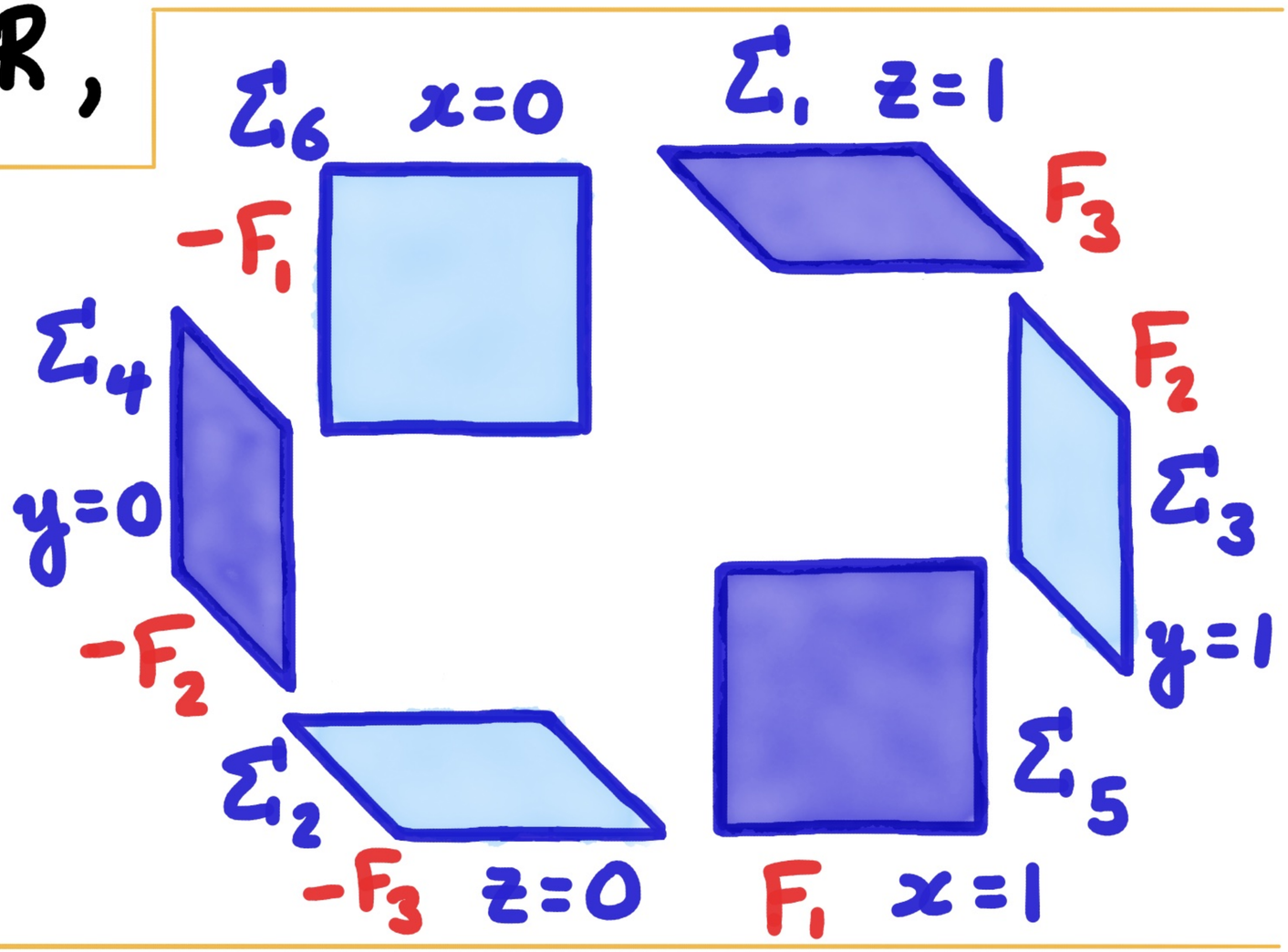
$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_2}{\partial y}\, dx\,dy\,dz$$

$$+ \int_0^1\int_0^1\int_0^1 \frac{\partial F_1}{\partial x}\, dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 \frac{\partial F_3}{\partial z} + \frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial x}\, dx\,dy\,dz$$

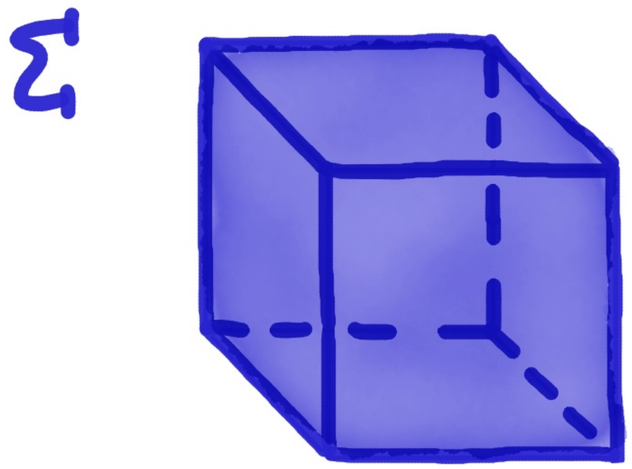
Conclusion: If Σ is the surface of the box $R = \{(x,y,z): 0 \le x,y,z \le 1\}$, and $\vec{F} = (F_1, F_2, F_3)$ then for $P_{\vec{F}} : \mathbb{R}^3 \to \mathbb{R}$,

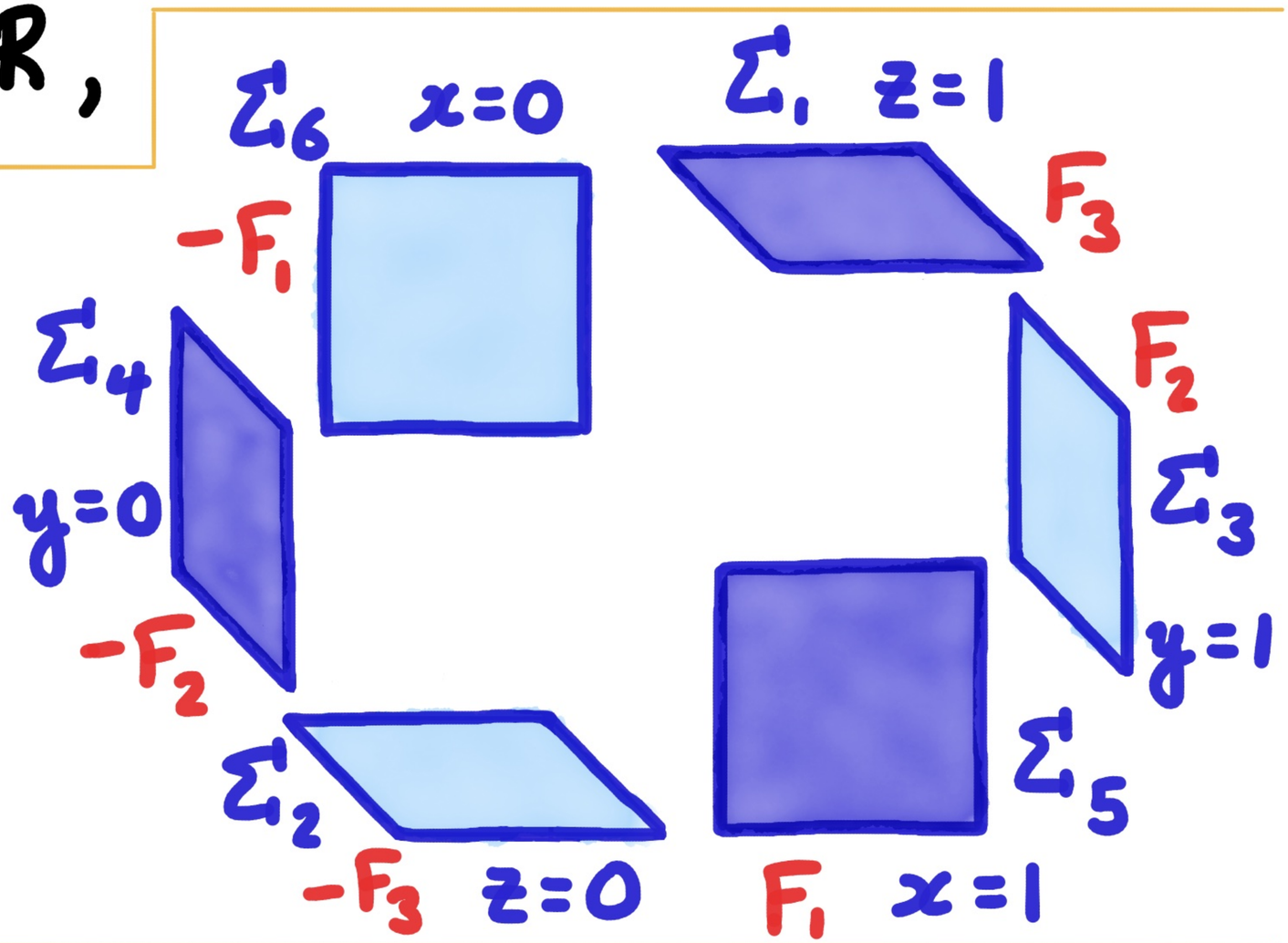




$$\iint_{\Sigma} P_{\vec{F}} \, dS = \int_0^1 \int_0^1 \int_0^1 \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \, dx\,dy\,dz$$

Conclusion: If Σ is the surface of the box $R = \{(x,y,z): 0 \le x,y,z \le 1\}$, and $\vec{F} = (F_1, F_2, F_3)$ then for $P_{\vec{F}} : \mathbb{R}^3 \to \mathbb{R}$,





$$\iint_{\Sigma} P_{\vec{F}} \, dS = \int_0^1 \int_0^1 \int_0^1 \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \, dx\,dy\,dz$$

$$= \iiint_R \operatorname{div} \vec{F} \, dV$$