Thirty-one

Ⅰ Green's Theorem

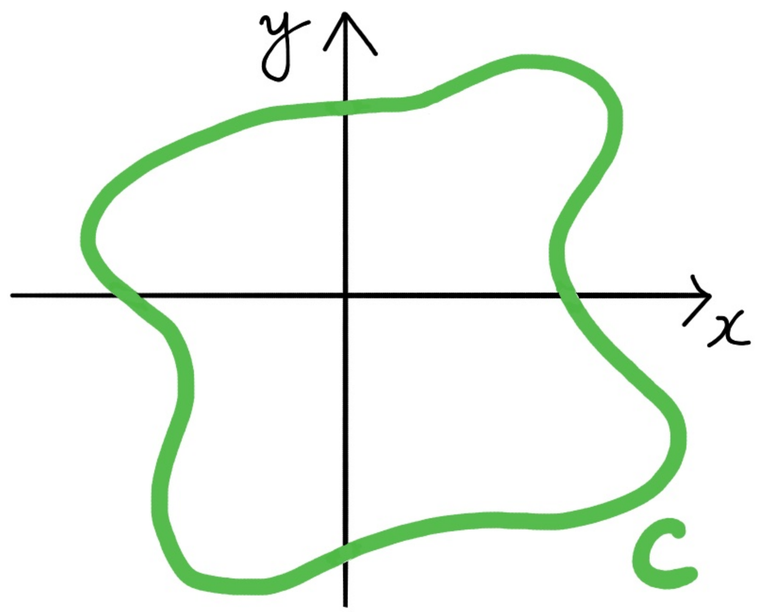
Ⅱ Return to Central Theorem

Ⅲ Final example

Ⓘ Green's Theorem

Convention

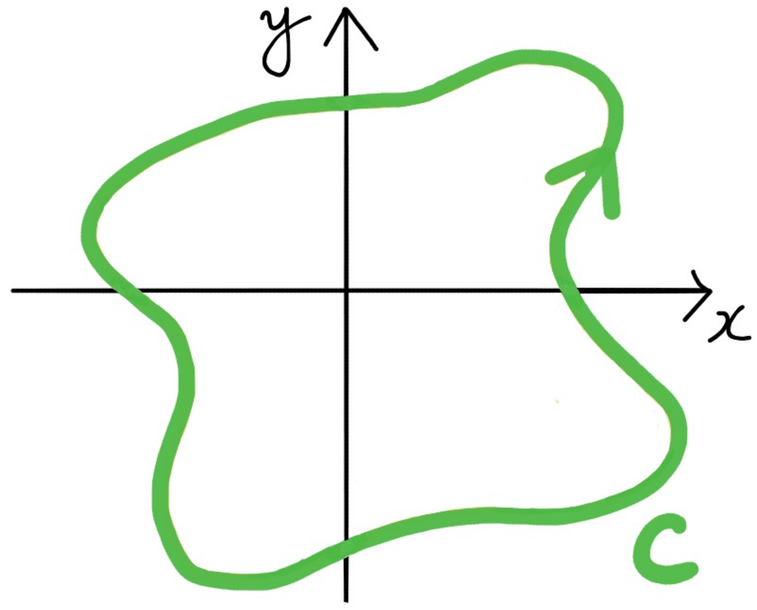
For a simple closed curve C in $\mathbb{R}^2$



$$\oint_C F_1\,dx + F_2\,dy = \int_C F_1\,dx + F_2\,dy$$

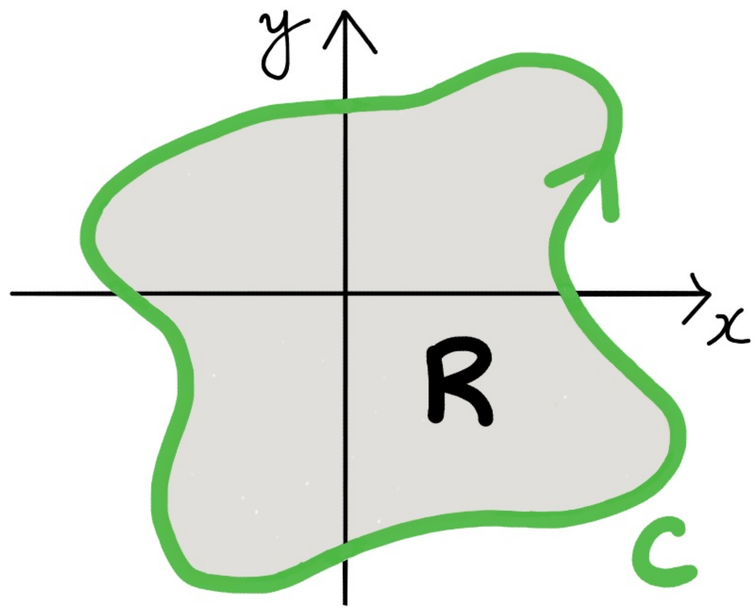
Green's Theorem:

If C is a counter clockwise simple closed curve in $\mathbb{R}^2$



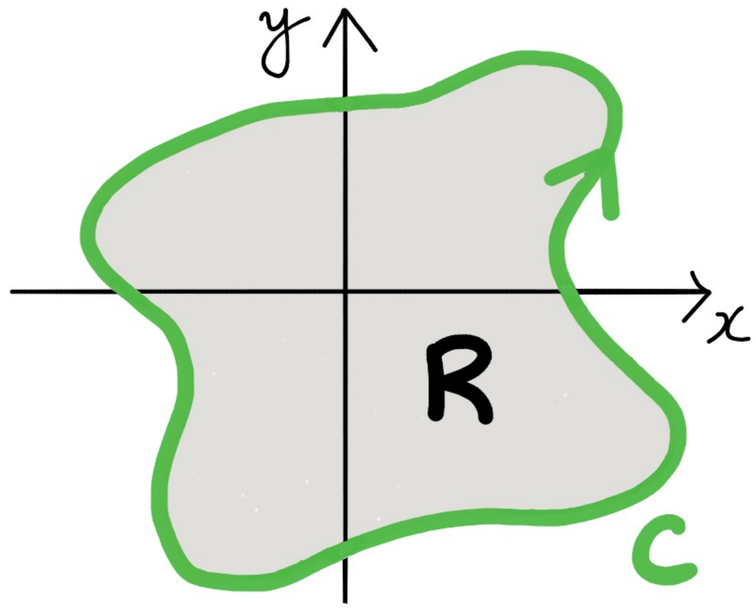
Green's Theorem:

If C is a counter clockwise simple closed curve in $\mathbb{R}^2$ that is the boundary of a region R in $\mathbb{R}^2$,



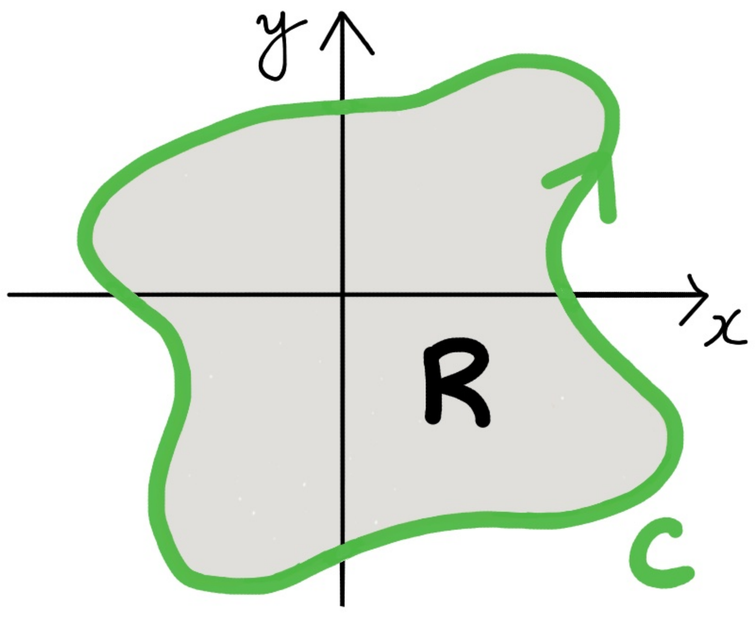
Green's Theorem:

If C is a counter clockwise simple closed curve in $\mathbb{R}^2$ that is the boundary of a region R in $\mathbb{R}^2$, and if $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field with $\vec{F} = (F_1, F_2)$



Green's Theorem:

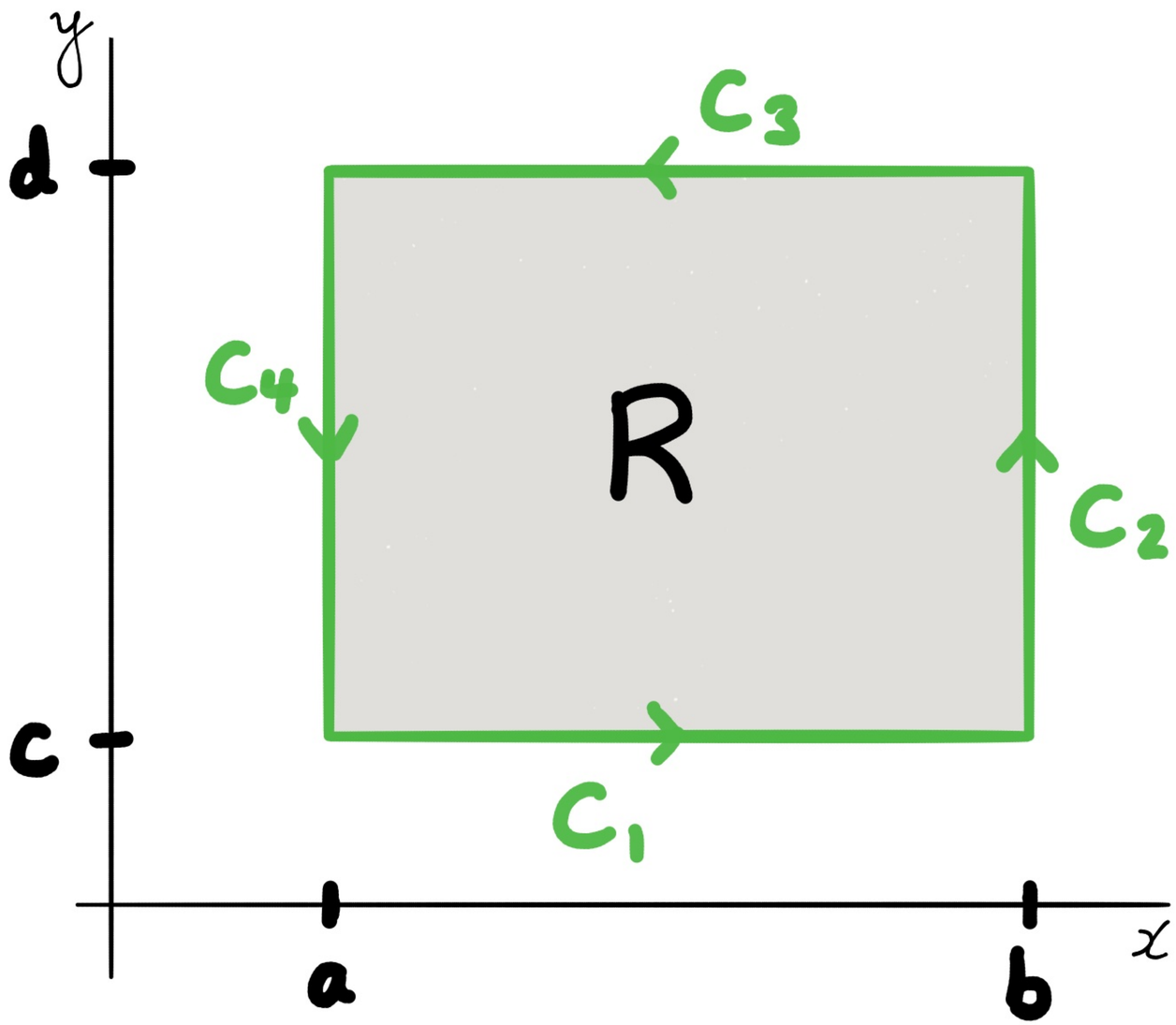
If C is a counter clockwise simple closed curve in $\mathbb{R}^2$ that is the boundary of a region R in $\mathbb{R}^2$, and if $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field with $\vec{F} = (F_1, F_2)$, then



$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

proof: (for rectangle)

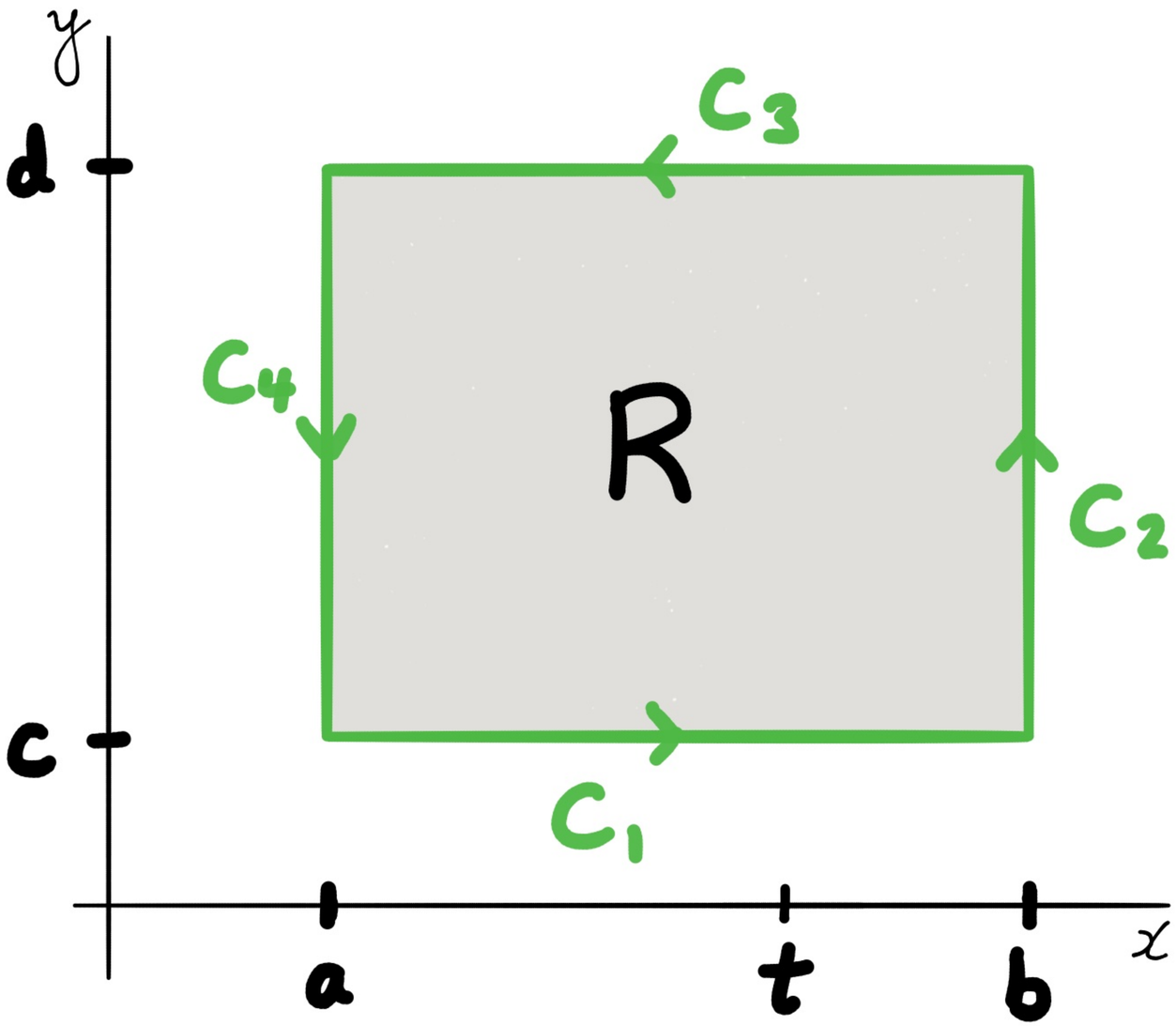
proof: (for rectangle)



$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

proof: (for rectangle)

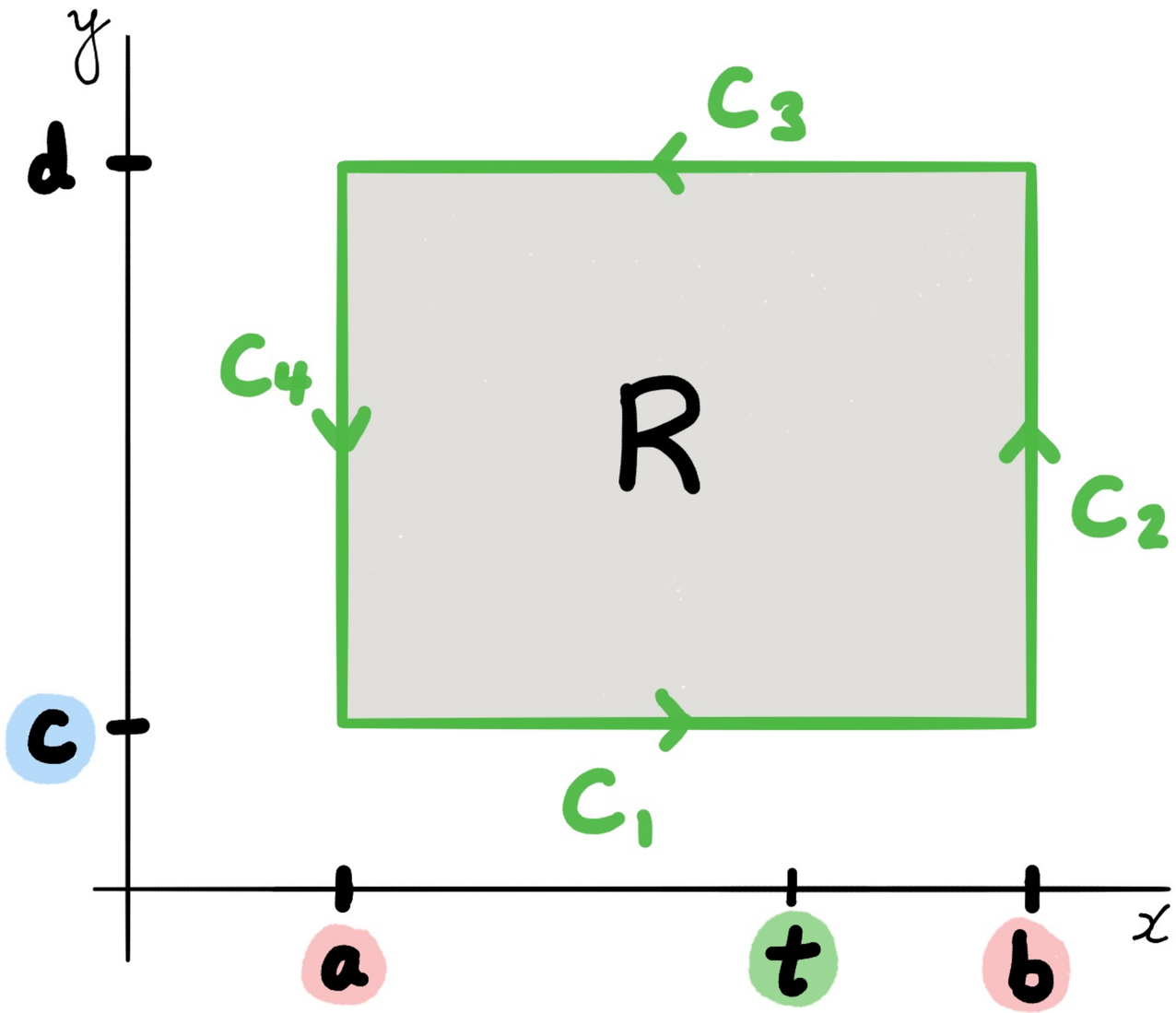


$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

C_1: $x = t,\ y = c,\ a \le t \le b$

proof: (for rectangle)

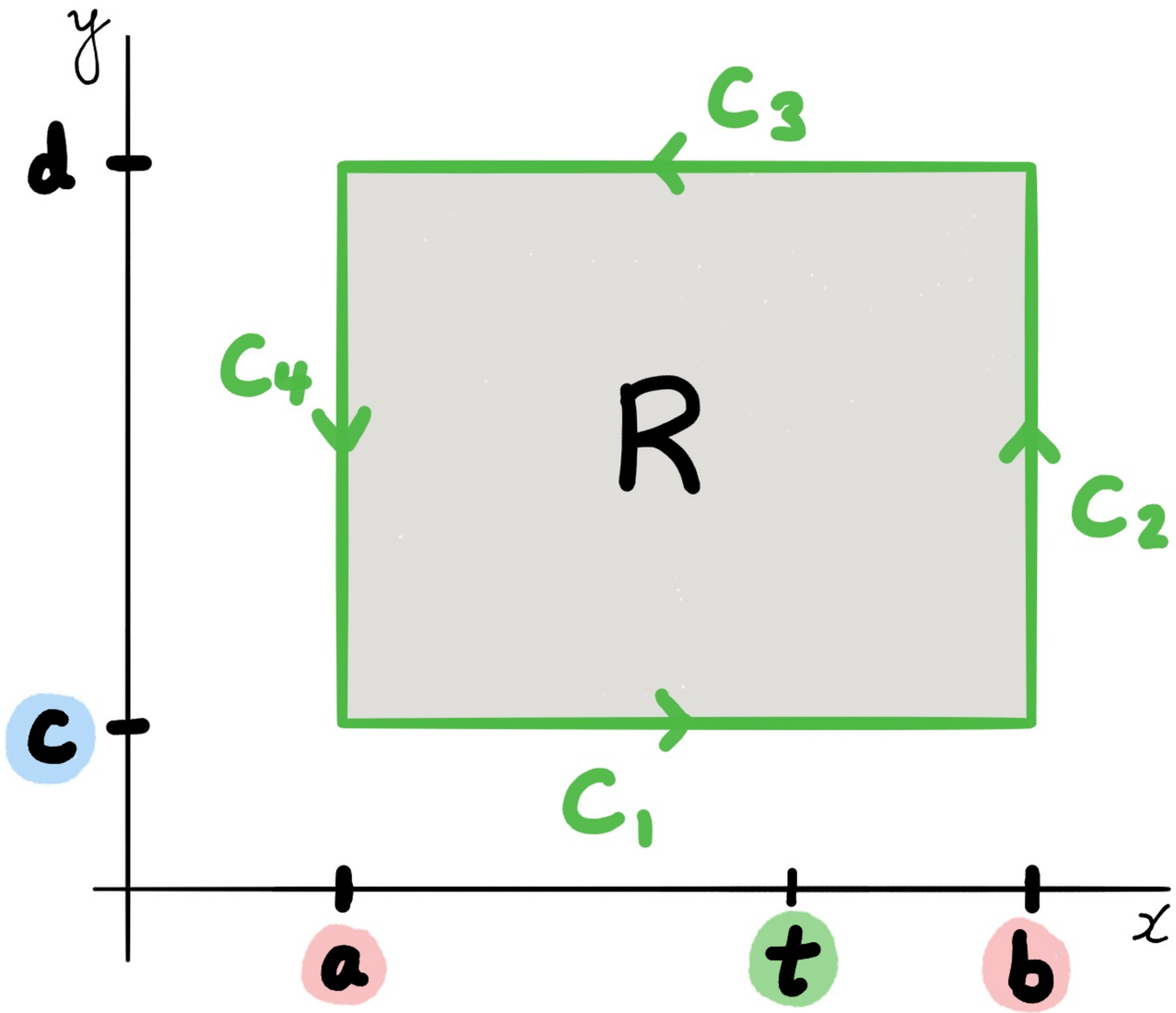


$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

C_1: $x = t$, $y = c$, $a \le t \le b$

proof: (for rectangle)



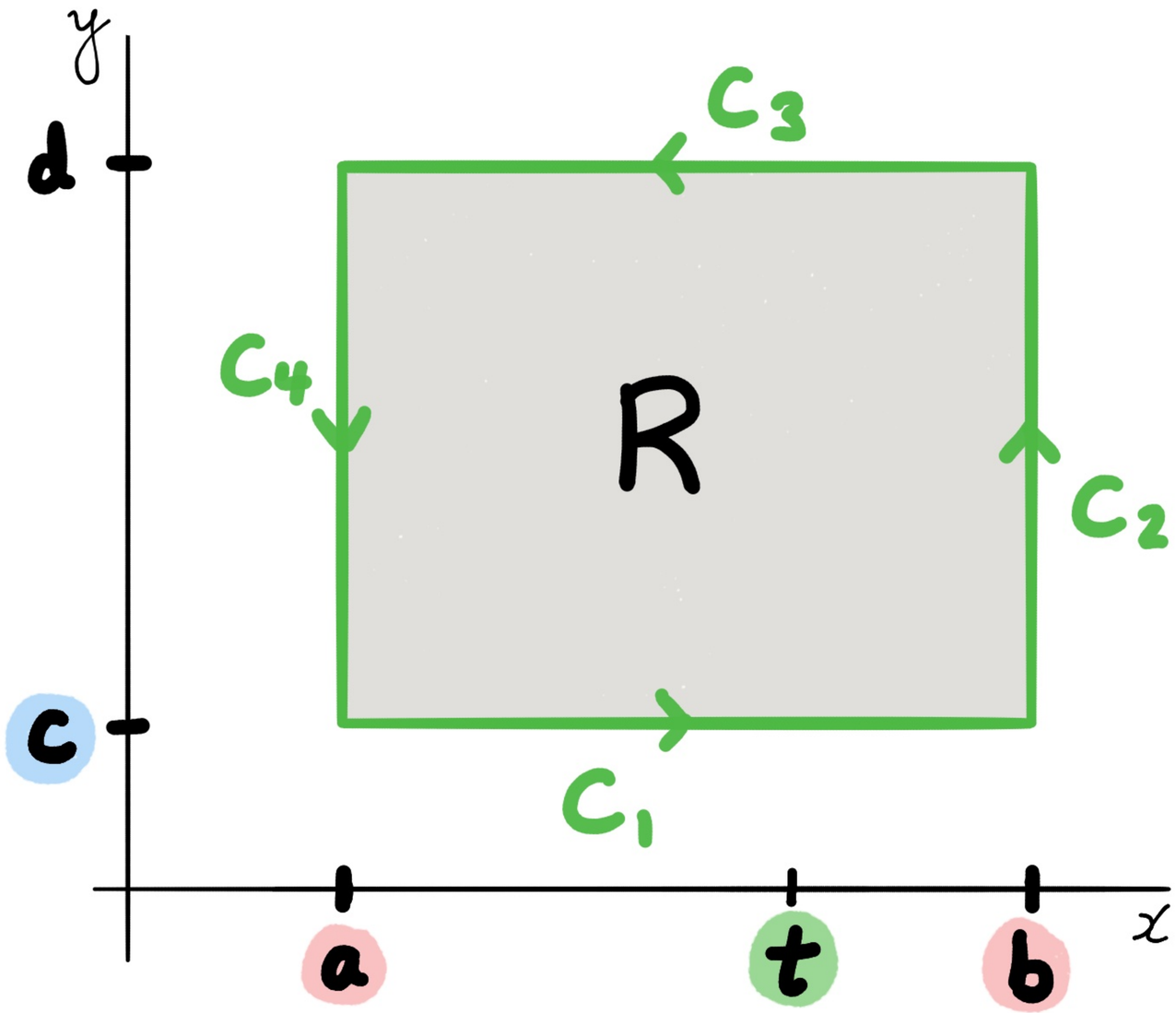
$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

C_1: $x = t$, $y = c$, $a \le t \le b$

$$\int_{C_1} F_1\,dx + F_2\,dy =$$

proof: (for rectangle)



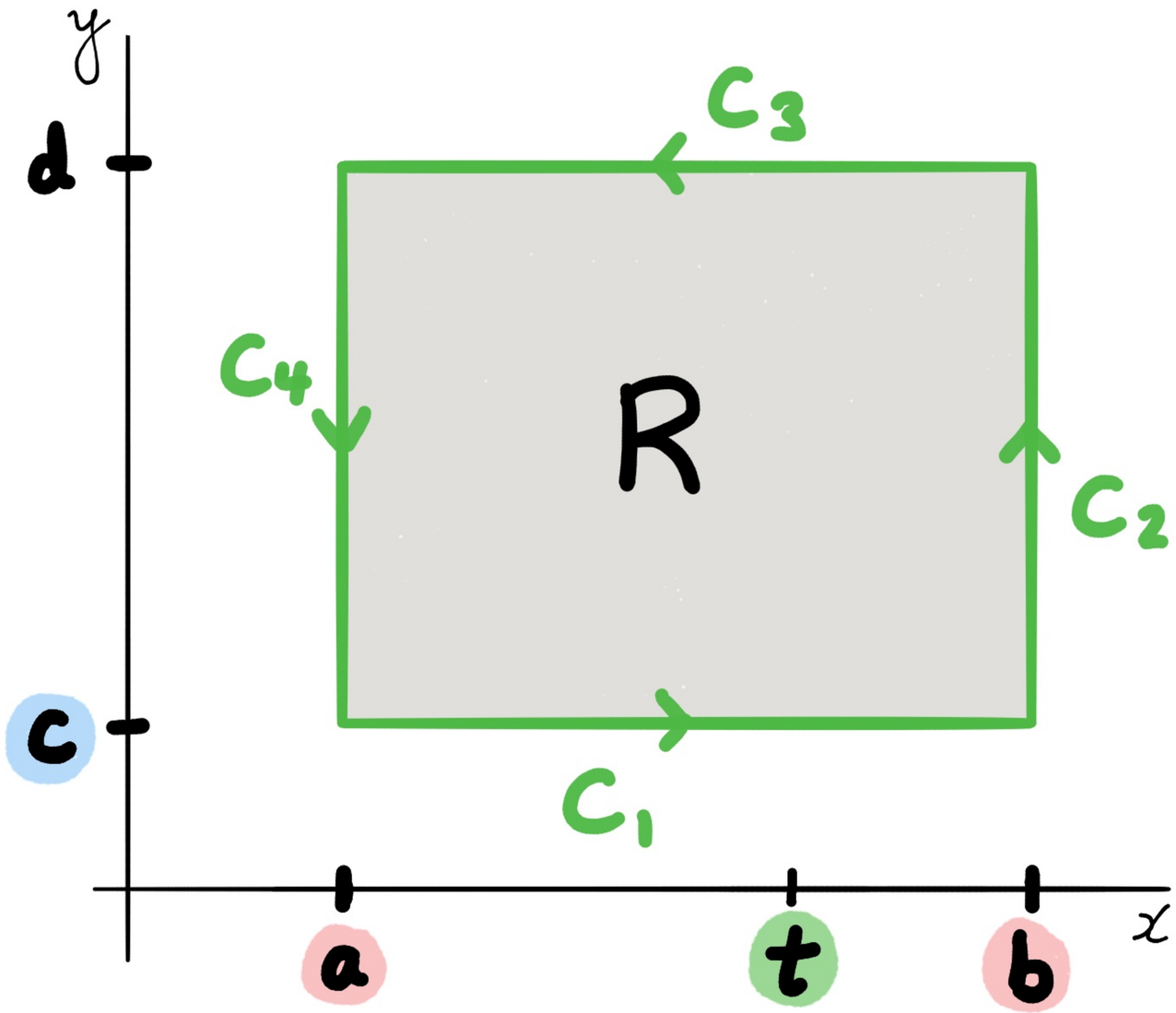
$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

C_1: $x = t$, $y = c$, $a \le t \le b$

$$\int_{C_1} F_1\,dx + F_2\,dy = \int_{C_1} F_1\,dx$$

proof: (for rectangle)



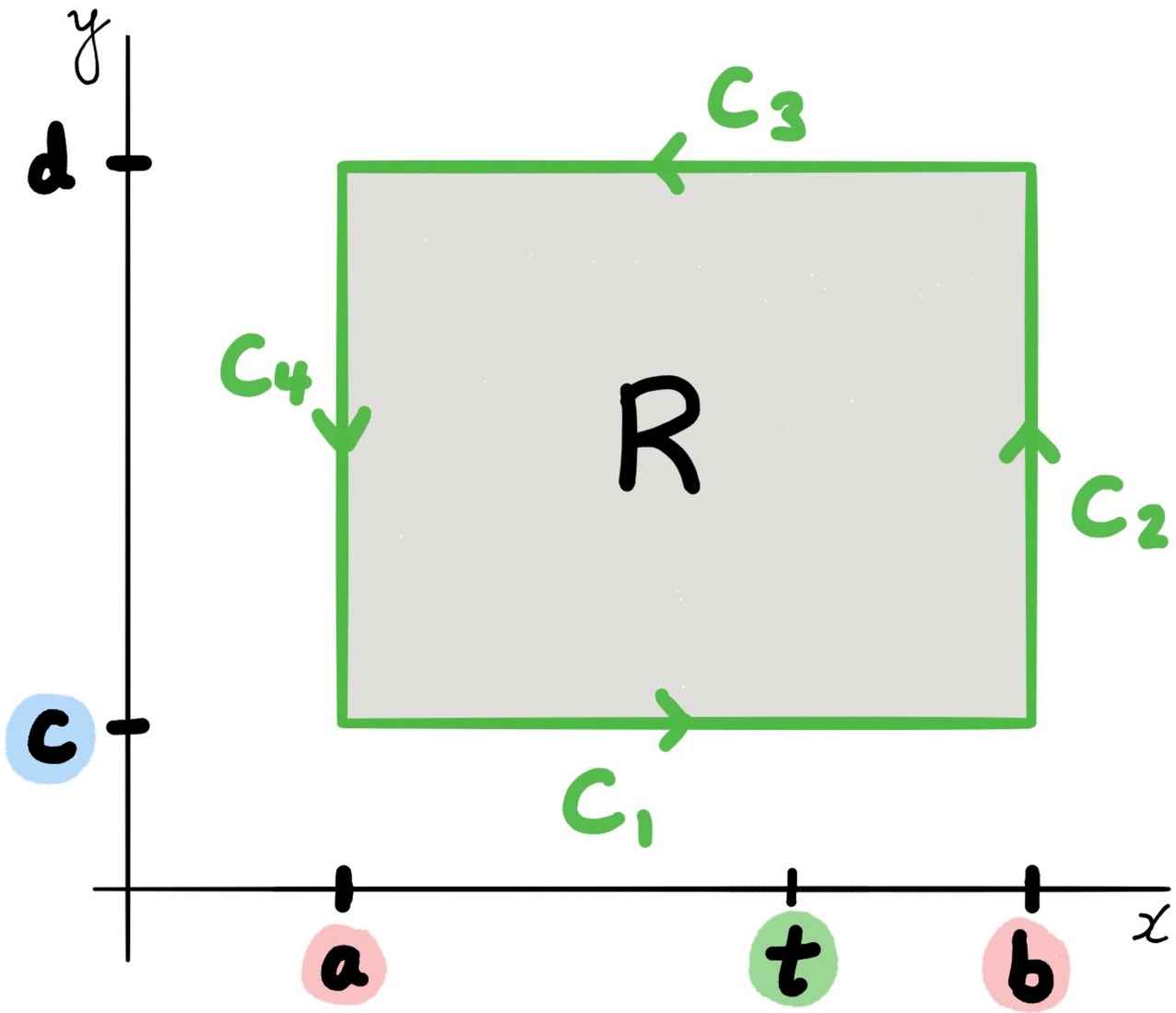
$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

C_1: $x = t$, $y = c$, $a \le t \le b$

$$\int_{C_1} F_1\,dx + F_2\,dy = \int_{C_1} F_1\,dx = \int_{C_1} F_1(x,y)\,dx$$

proof: (for rectangle)



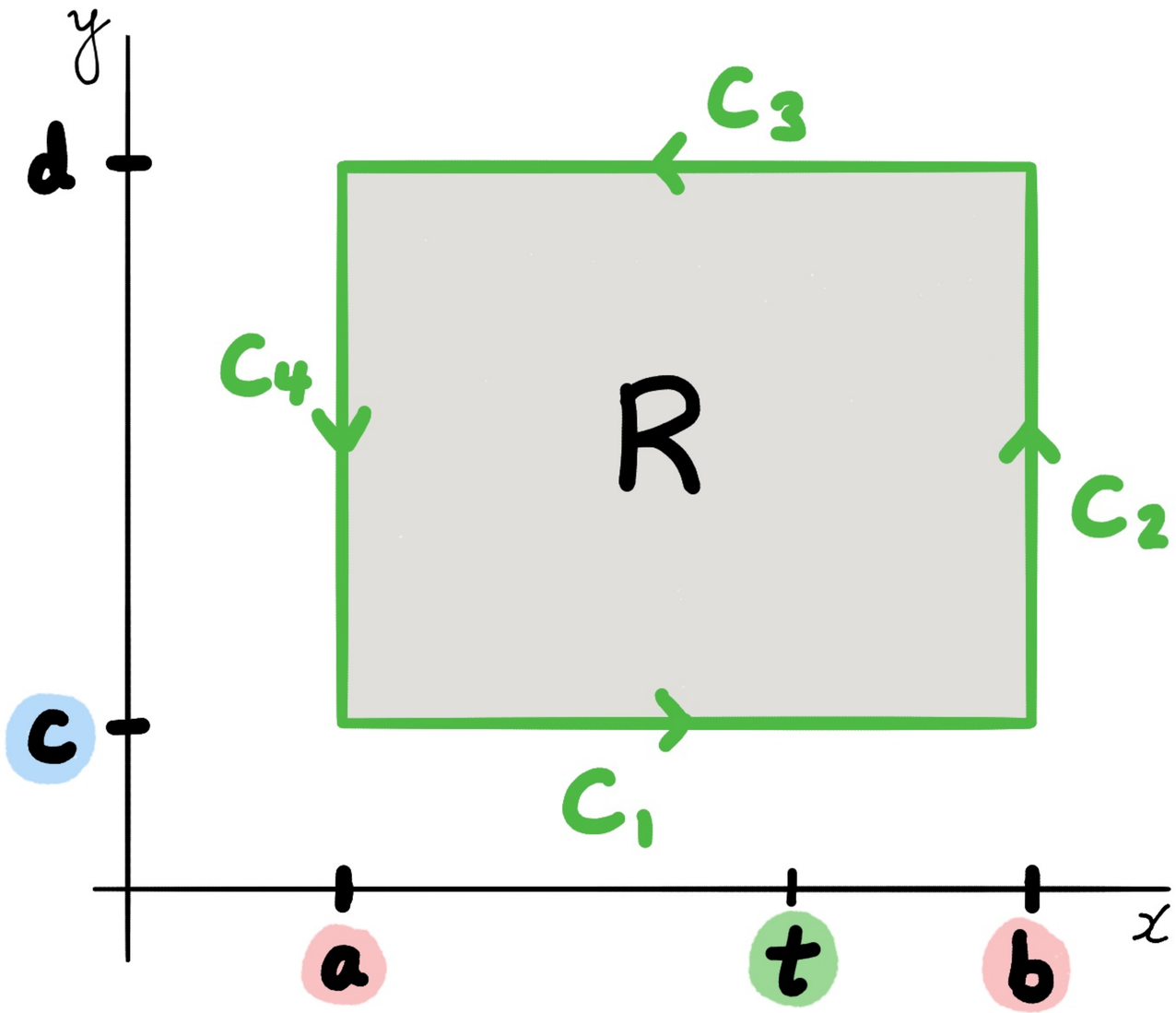
$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

C_1: $x = t$, $y = c$, $a \le t \le b$

$$\int_{C_1} F_1\,dx + F_2\,dy = \int_{C_1} F_1\,dx = \int_{C_1} F_1(x,y)\,dx = \int_a^b F_1(t,c)\,dt$$

proof: (for rectangle)



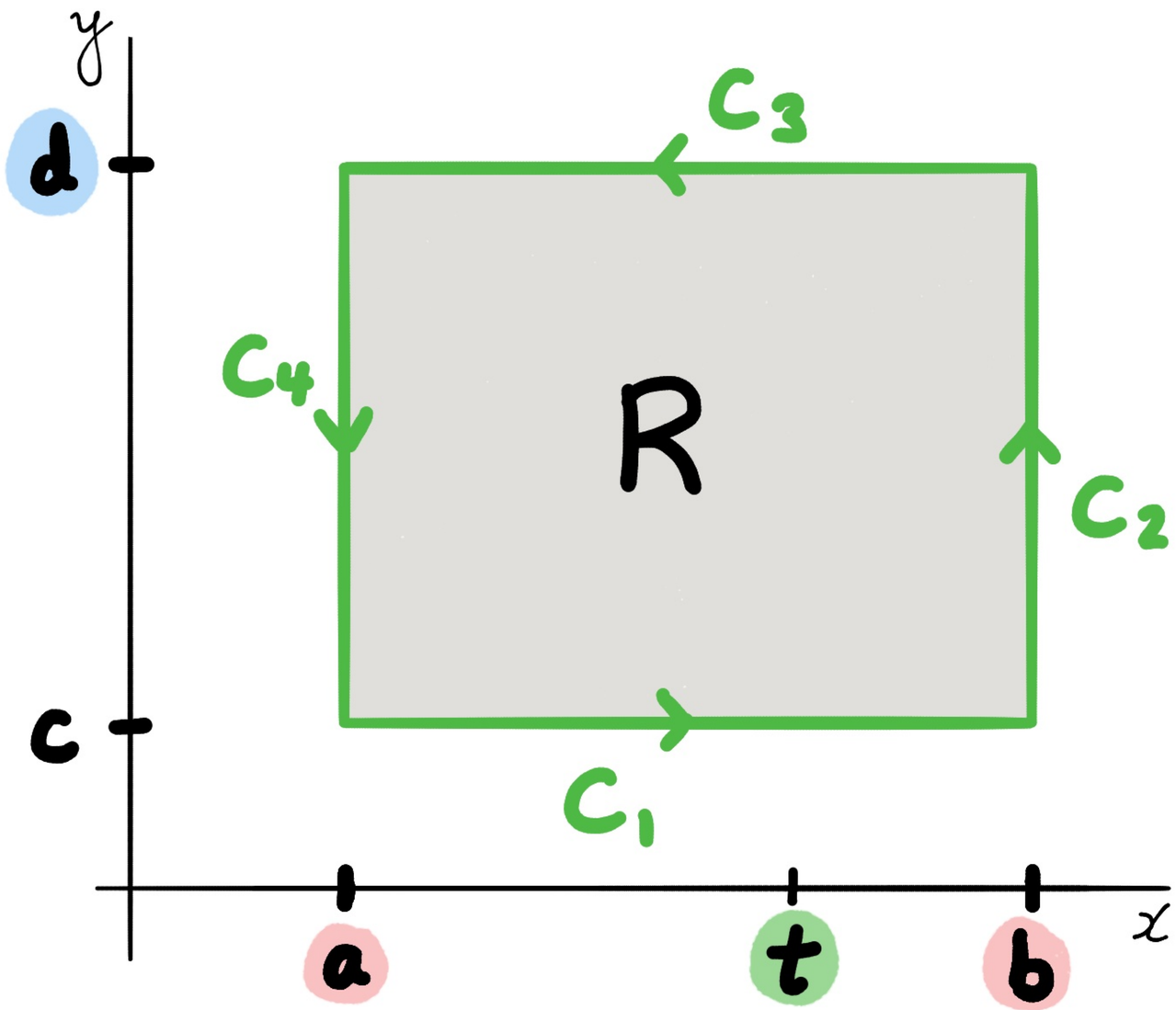
$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

C_1: $x = t,\ y = c,\ a \le t \le b$

$$\int_{C_1} F_1\,dx + F_2\,dy = \int_a^b F_1(t,c)\,dt$$

proof: (for rectangle)



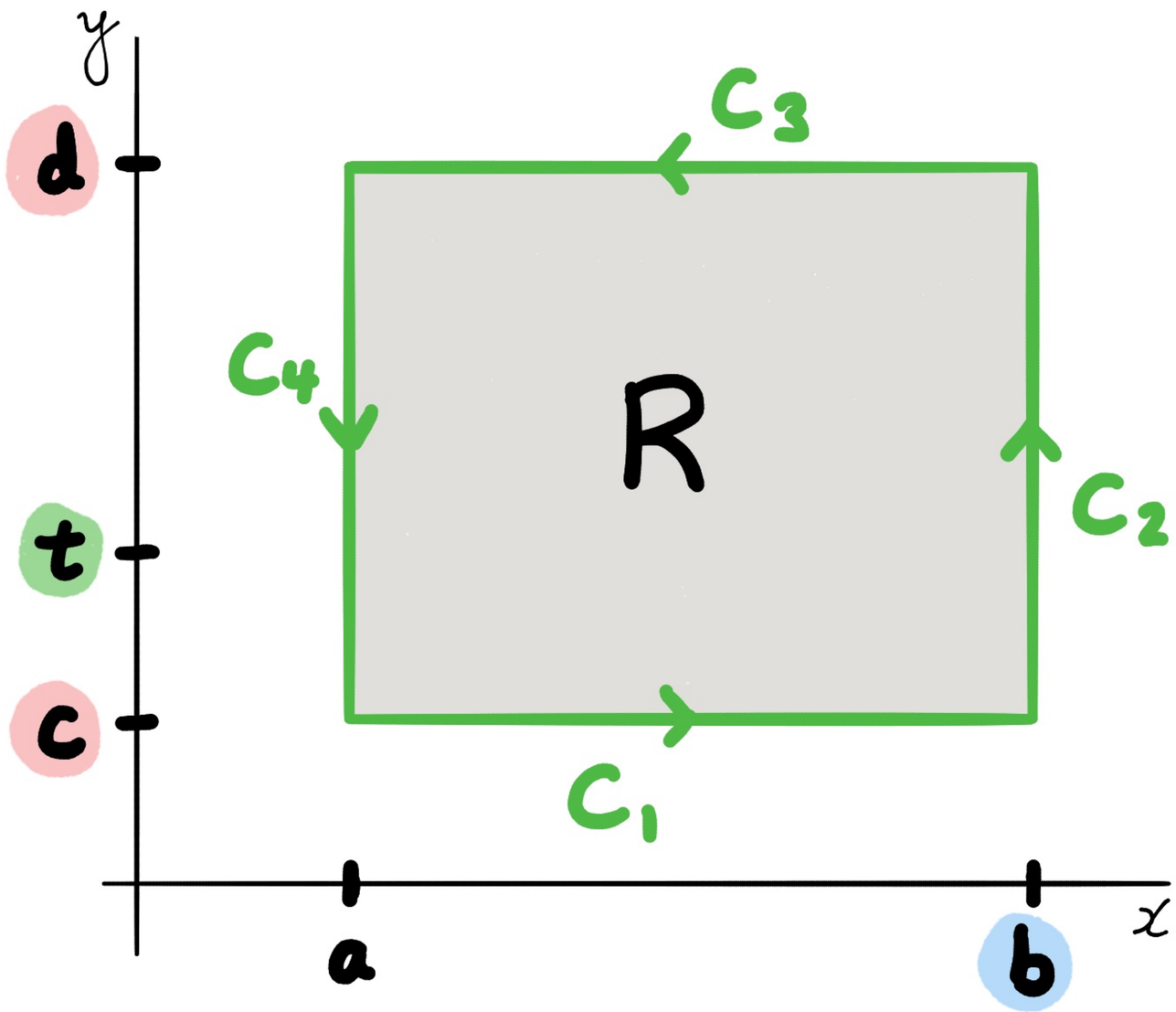
$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

$$\int_{C_1} F_1\,dx + F_2\,dy = \int_a^b F_1(t,c)\,dt$$

$$\int_{C_3} F_1\,dx + F_2\,dy = -\int_a^b F_1(t,d)\,dt$$

proof: (for rectangle)



$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

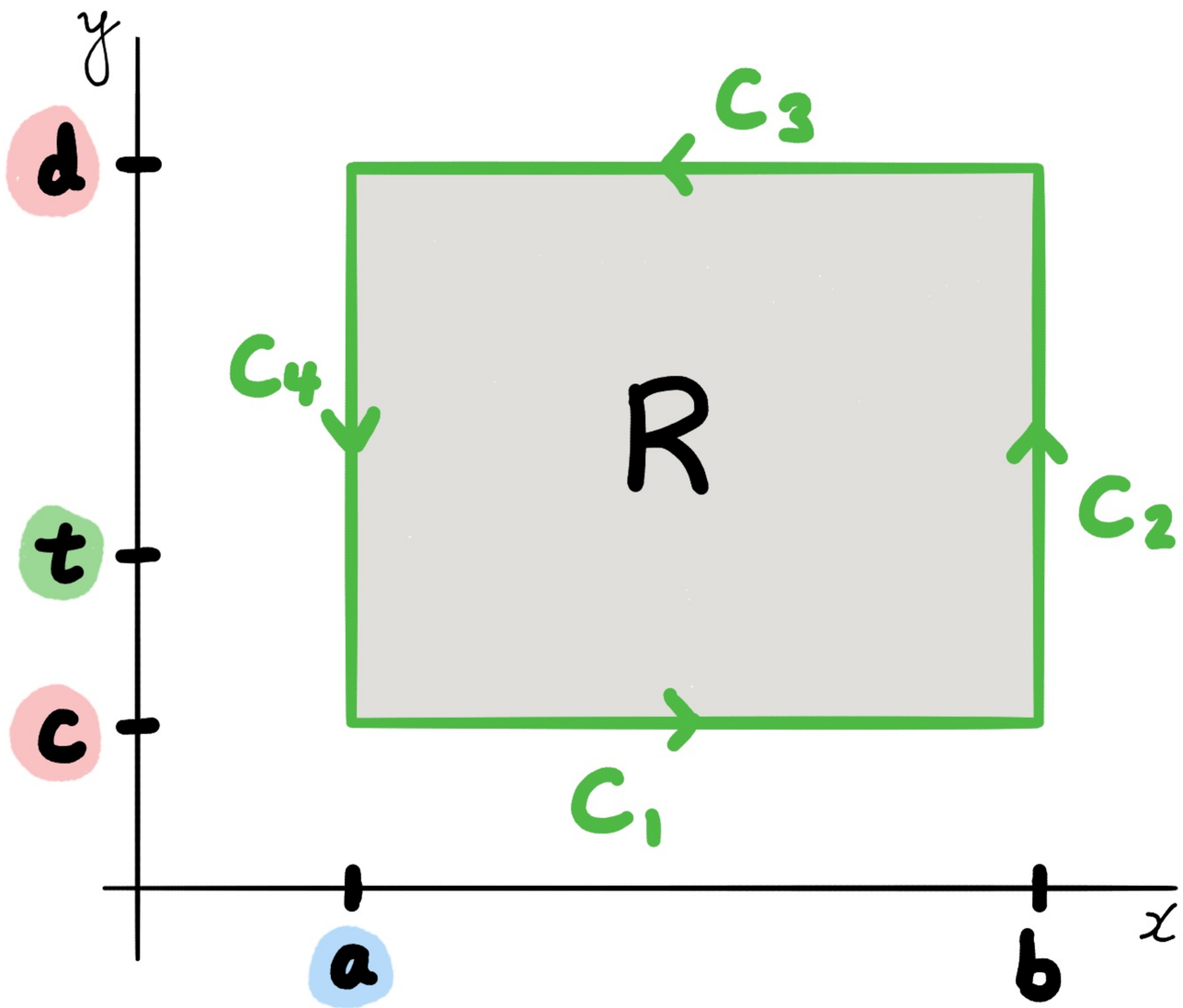
$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

$$\int_{C_1} F_1\,dx + F_2\,dy = \int_a^b F_1(t,c)\,dt$$

$$\int_{C_2} F_1\,dx + F_2\,dy = \int_c^d F_2(b,t)\,dt$$

$$\int_{C_3} F_1\,dx + F_2\,dy = -\int_a^b F_1(t,d)\,dt$$

proof: (for rectangle)



$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

$$\int_{C_1} F_1\,dx + F_2\,dy = \int_a^b F_1(t,c)\,dt$$

$$\int_{C_2} F_1\,dx + F_2\,dy = \int_c^d F_2(b,t)\,dt$$

$$\int_{C_3} F_1\,dx + F_2\,dy = -\int_a^b F_1(t,d)\,dt$$

$$\int_{C_4} F_1\,dx + F_2\,dy = -\int_c^d F_2(a,t)\,dt$$

$$\int_{C_1} F_1\,dx + F_2\,dy = \int_a^b F_1(t,c)\,dt$$

$$\int_{C_2} F_1\,dx + F_2\,dy = \int_c^d F_2(b,t)\,dt$$

$$\int_{C_3} F_1\,dx + F_2\,dy = -\int_a^b F_1(t,d)\,dt$$

$$\int_{C_4} F_1\,dx + F_2\,dy = -\int_c^d F_2(a,t)\,dt$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

$$\int_{C_1} F_1\,dx + F_2\,dy = \int_a^b F_1(t,c)\,dt$$

$$\int_{C_2} F_1\,dx + F_2\,dy = \int_c^d F_2(b,t)\,dt$$

$$\int_{C_3} F_1\,dx + F_2\,dy = -\int_a^b F_1(t,d)\,dt$$

$$\int_{C_4} F_1\,dx + F_2\,dy = -\int_c^d F_2(a,t)\,dt$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

Therefore,

$$\int_C F_1\,dx + F_2\,dy = \sum_{i=1}^{4} \int_{C_i} F_1\,dx + F_2\,dy$$

$$\int_{C_1} F_1\,dx + F_2\,dy = \int_a^b F_1(t,c)\,dt$$

$$\int_{C_2} F_1\,dx + F_2\,dy = \int_c^d F_2(b,t)\,dt$$

$$\int_{C_3} F_1\,dx + F_2\,dy = -\int_a^b F_1(t,d)\,dt$$

$$\int_{C_4} F_1\,dx + F_2\,dy = -\int_c^d F_2(a,t)\,dt$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

Therefore,

$$\int_C F_1\,dx + F_2\,dy$$

$$= \sum_{i=1}^{4} \int_{C_i} F_1\,dx + F_2\,dy$$

$$= \int_c^d F_2(b,t)\,dt - \int_c^d F_2(a,t)\,dt - \int_a^b F_1(t,d)\,dt + \int_a^b F_1(t,c)\,dt$$

$$\int_{C_1} F_1\,dx + F_2\,dy = \int_a^b F_1(t,c)\,dt$$

$$\int_{C_2} F_1\,dx + F_2\,dy = \int_c^d F_2(b,t)\,dt$$

$$\int_{C_3} F_1\,dx + F_2\,dy = -\int_a^b F_1(t,d)\,dt$$

$$\int_{C_4} F_1\,dx + F_2\,dy = -\int_c^d F_2(a,t)\,dt$$

$$C = C_1 \cup C_2 \cup C_3 \cup C_4$$

Therefore,

$$\int_C F_1\,dx + F_2\,dy$$

$$= \sum_{i=1}^{4} \int_{C_i} F_1\,dx + F_2\,dy$$

$$= \int_c^d F_2(b,t)\,dt - \int_c^d F_2(a,t)\,dt - \int_a^b F_1(t,d)\,dt + \int_a^b F_1(t,c)\,dt.$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA =$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R \frac{\partial F_2}{\partial x}\, dA - \iint_R \frac{\partial F_1}{\partial y}\, dA$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R \frac{\partial F_2}{\partial x}\, dA - \iint_R \frac{\partial F_1}{\partial y}\, dA$$

$$= \int_c^d \int_a^b \frac{\partial F_2}{\partial x}\, dx\, dy - \int_a^b \int_c^d \frac{\partial F_1}{\partial y}\, dy\, dx$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R \frac{\partial F_2}{\partial x}\, dA - \iint_R \frac{\partial F_1}{\partial y}\, dA$$

$$= \int_c^d \int_a^b \frac{\partial F_2}{\partial x}\, dx\, dy - \int_a^b \int_c^d \frac{\partial F_1}{\partial y}\, dy\, dx$$

$$= \int_c^d \left[F_2\right]_{x=a}^{x=b} dy - \int_a^b \left[F_1\right]_{y=c}^{y=d} dx$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R \frac{\partial F_2}{\partial x}\, dA - \iint_R \frac{\partial F_1}{\partial y}\, dA$$

$$= \int_c^d \int_a^b \frac{\partial F_2}{\partial x}\, dx\, dy - \int_a^b \int_c^d \frac{\partial F_1}{\partial y}\, dy\, dx$$

$$= \int_c^d \left[F_2\right]_{x=a}^{x=b} dy - \int_a^b \left[F_1\right]_{y=c}^{y=d} dx$$

$$= \int_c^d \Big(F_2(b,y) - F_2(a,y)\Big)\, dy - \int_a^b \Big(F_1(x,d) - F_1(x,c)\Big)\, dx$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R \frac{\partial F_2}{\partial x}\, dA - \iint_R \frac{\partial F_1}{\partial y}\, dA$$

$$= \int_c^d \int_a^b \frac{\partial F_2}{\partial x}\, dx\, dy - \int_a^b \int_c^d \frac{\partial F_1}{\partial y}\, dy\, dx$$

$$= \int_c^d \left[F_2\right]_{x=a}^{x=b} dy - \int_a^b \left[F_1\right]_{y=c}^{y=d} dx$$

$$= \int_c^d \Big(F_2(b,y) - F_2(a,y)\Big)\, dy - \int_a^b \Big(F_1(x,d) - F_1(x,c)\Big)\, dx$$

$$= \int_c^d F_2(b,y)\, dy - \int_c^d F_2(a,y)\, dy - \int_a^b F_1(x,d)\, dx + \int_a^b F_1(x,c)\, dx$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R \frac{\partial F_2}{\partial x}\, dA - \iint_R \frac{\partial F_1}{\partial y}\, dA$$

$$= \int_c^d \int_a^b \frac{\partial F_2}{\partial x}\, dx\, dy - \int_a^b \int_c^d \frac{\partial F_1}{\partial y}\, dy\, dx$$

$$= \int_c^d \left[F_2\right]_{x=a}^{x=b} dy - \int_a^b \left[F_1\right]_{y=c}^{y=d} dx$$

$$= \int_c^d \Big(F_2(b,y) - F_2(a,y)\Big)\, dy - \int_a^b \Big(F_1(x,d) - F_1(x,c)\Big)\, dx$$

$$= \int_c^d F_2(b,y)\, dy - \int_c^d F_2(a,y)\, dy - \int_a^b F_1(x,d)\, dx + \int_a^b F_1(x,c)\, dx$$

$$= \int_c^d F_2(b,t)\, dt - \int_c^d F_2(a,t)\, dt - \int_a^b F_1(t,d)\, dt + \int_a^b F_1(t,c)\, dt$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R \frac{\partial F_2}{\partial x}\, dA - \iint_R \frac{\partial F_1}{\partial y}\, dA$$

$$= \int_c^d \int_a^b \frac{\partial F_2}{\partial x}\, dx\, dy - \int_a^b \int_c^d \frac{\partial F_1}{\partial y}\, dy\, dx$$

$$= \int_c^d \left[F_2\right]_{x=a}^{x=b} dy - \int_a^b \left[F_1\right]_{y=c}^{y=d} dx$$

$$= \int_c^d \Big(F_2(b,y) - F_2(a,y)\Big)\, dy - \int_a^b \Big(F_1(x,d) - F_1(x,c)\Big)\, dx$$

$$= \int_c^d F_2(b,y)\, dy - \int_c^d F_2(a,y)\, dy - \int_a^b F_1(x,d)\, dx + \int_a^b F_1(x,c)\, dx$$

$$= \int_c^d F_2(b,t)\, dt - \int_c^d F_2(a,t)\, dt - \int_a^b F_1(t,d)\, dt + \int_a^b F_1(t,c)\, dt$$

Combining the previous two pages:

$$\int_C F_1\,dx + F_2\,dy$$

$$=$$

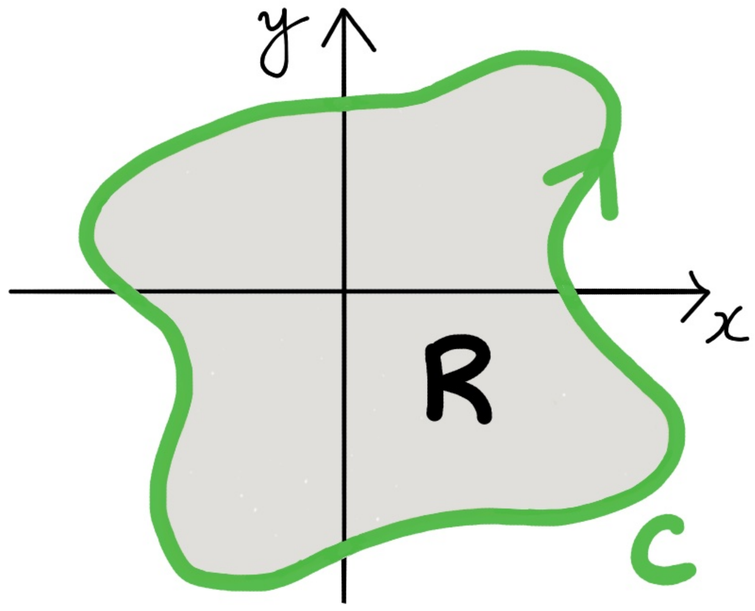
$$\int_c^d F_2(b,t)\,dt - \int_c^d F_2(a,t)\,dt - \int_a^b F_1(t,d)\,dt + \int_a^b F_1(t,c)\,dt$$

$$=$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

END

Green's Theorem:

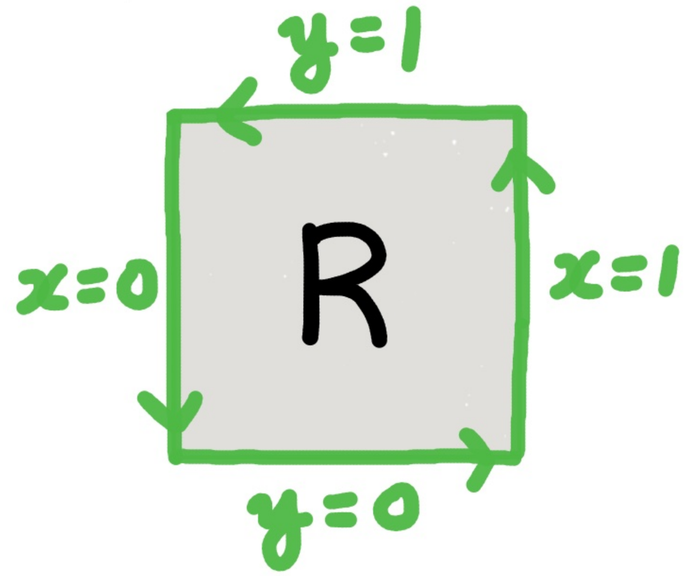


$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

Example:

Let C be the boundary of R.

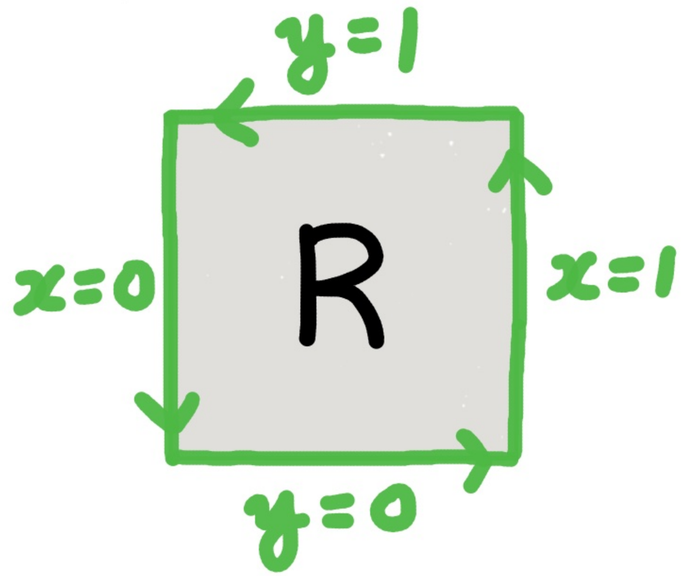
Find $\oint_C xy\,dx + 7x\,dy$.



Example:

Let C be the boundary of R.

Find $\oint_C xy\,dx + 7x\,dy$.



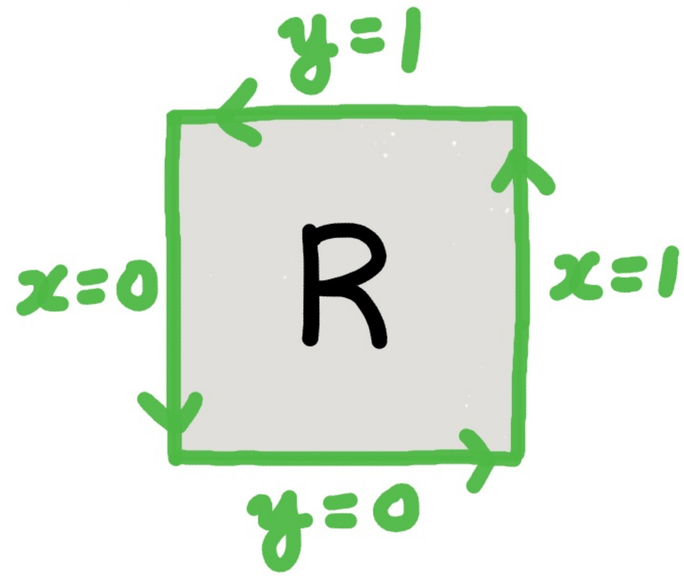
Solution:

$$\oint_C xy\,dx + 7x\,dy = \iint_R \left(\frac{\partial}{\partial x}7x - \frac{\partial}{\partial y}xy\right) dA$$

Example:

Let C be the boundary of R.

Find $\oint_C xy\,dx + 7x\,dy$.



Solution:

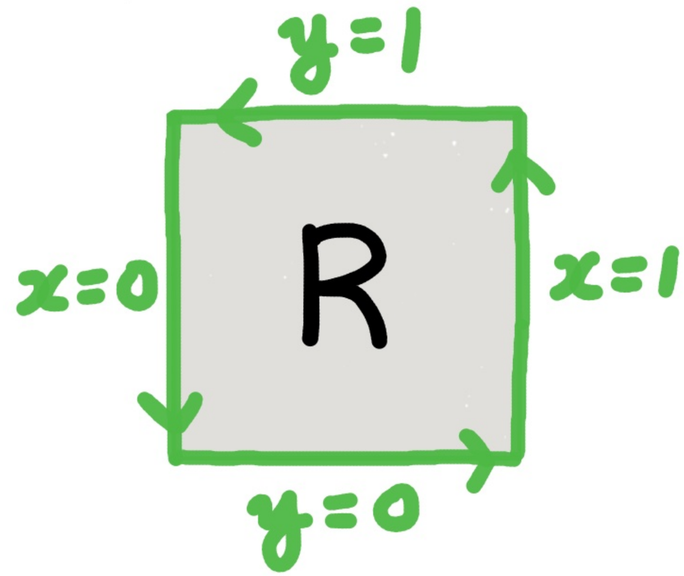
$$\oint_C xy\,dx + 7x\,dy = \iint_R \left(\frac{\partial}{\partial x}7x - \frac{\partial}{\partial y}xy\right) dA$$

$$= \int_0^1 \int_0^1 (7 - x)\,dx\,dy$$

Example:

Let C be the boundary of R.

Find $\oint_C xy\,dx + 7x\,dy$.



Solution:

$$\oint_C xy\,dx + 7x\,dy = \iint_R \left(\frac{\partial}{\partial x}7x - \frac{\partial}{\partial y}xy\right) dA$$

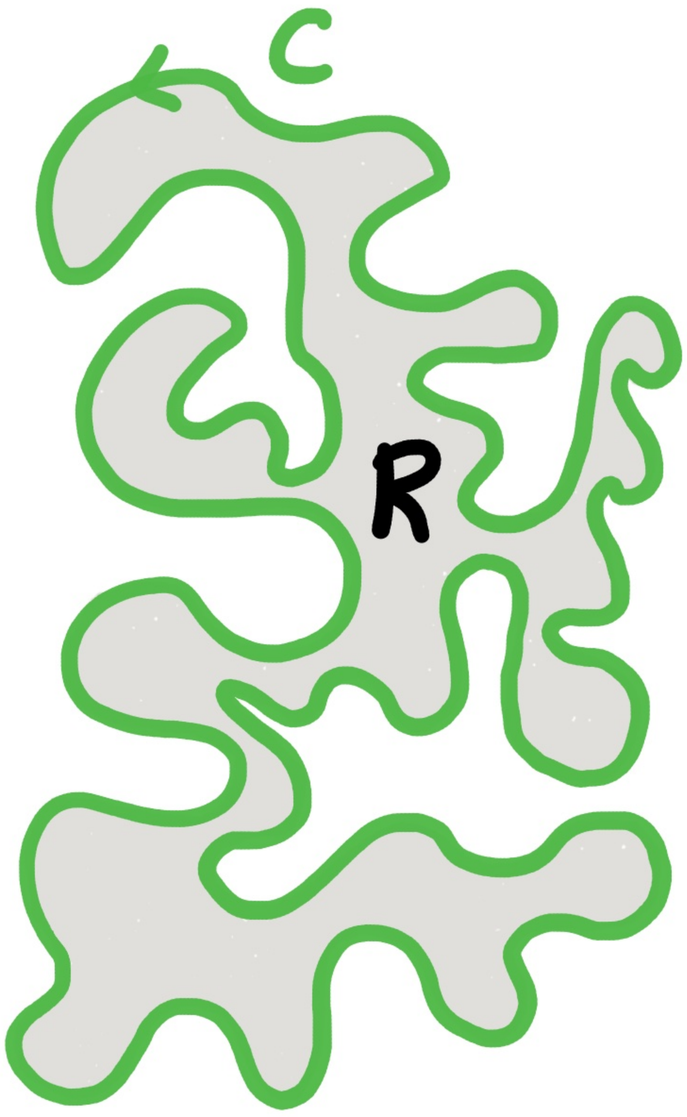
$$= \int_0^1 \int_0^1 (7 - x)\,dx\,dy$$

$$= \frac{13}{2}$$

Example:

Let R be a region in $\mathbb{R}^2$ enclosed by a simple closed counter clockwise curve C. Show

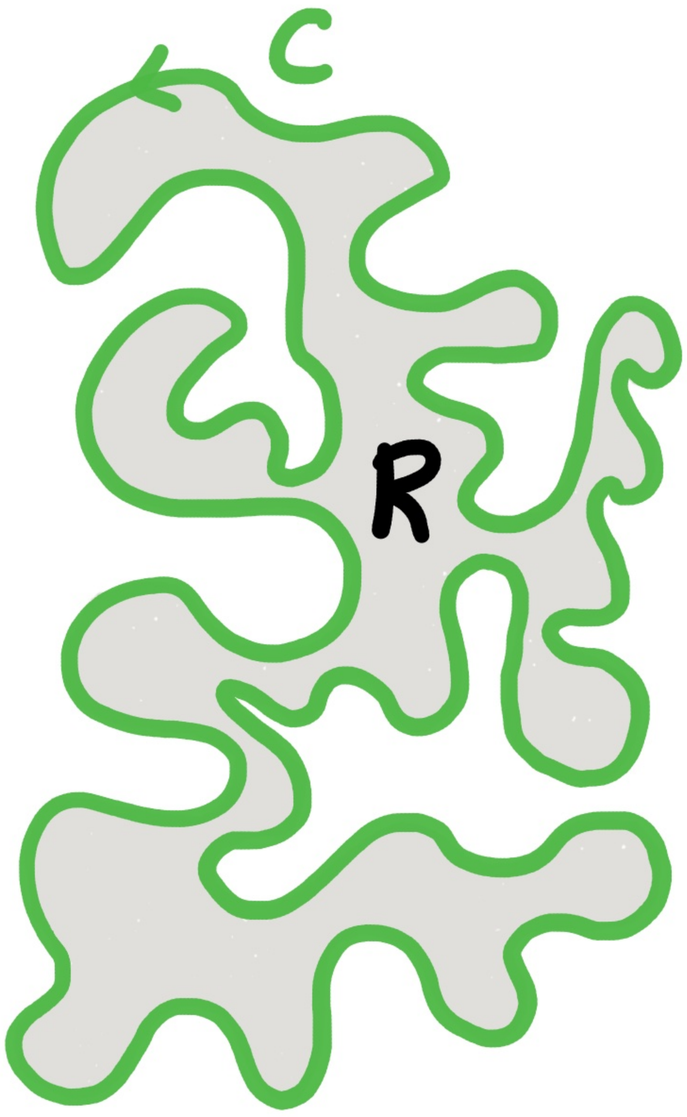
$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \text{Area}(R).$$



Example:

Let R be a region in $\mathbb{R}^2$ enclosed by a simple closed counter clockwise curve C. Show

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \text{Area}(R).$$



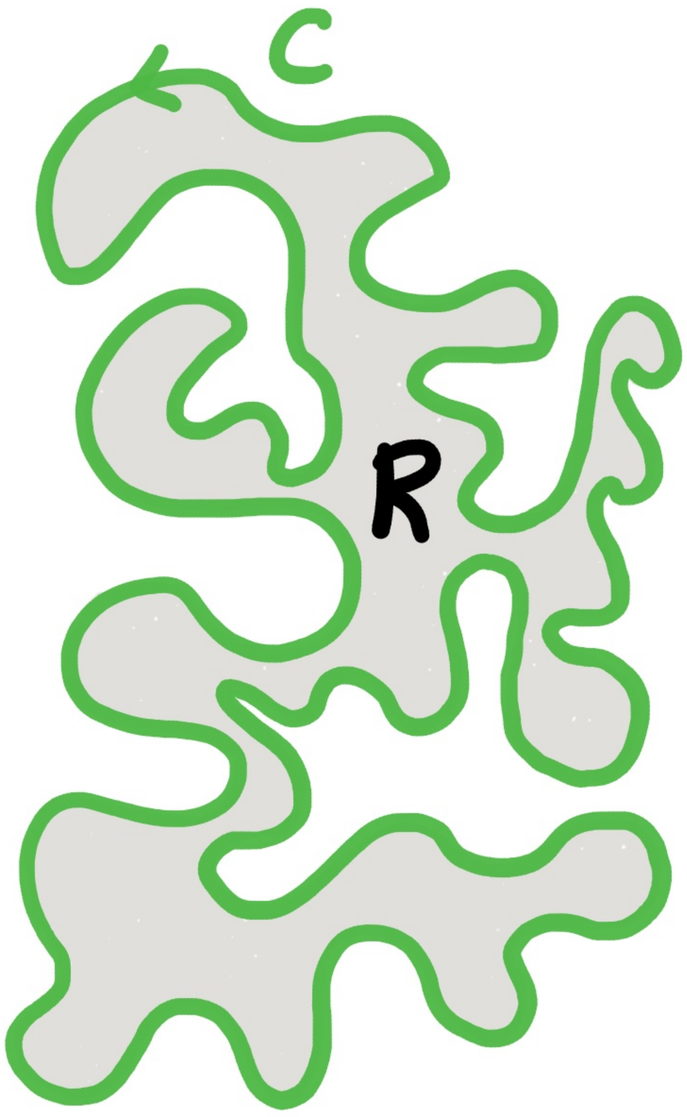
Solution:

$$\frac{1}{2}\oint_C -y\,dx + x\,dy =$$

Example:

Let R be a region in $\mathbb{R}^2$ enclosed by a simple closed counter clockwise curve C. Show

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \text{Area}(R).$$



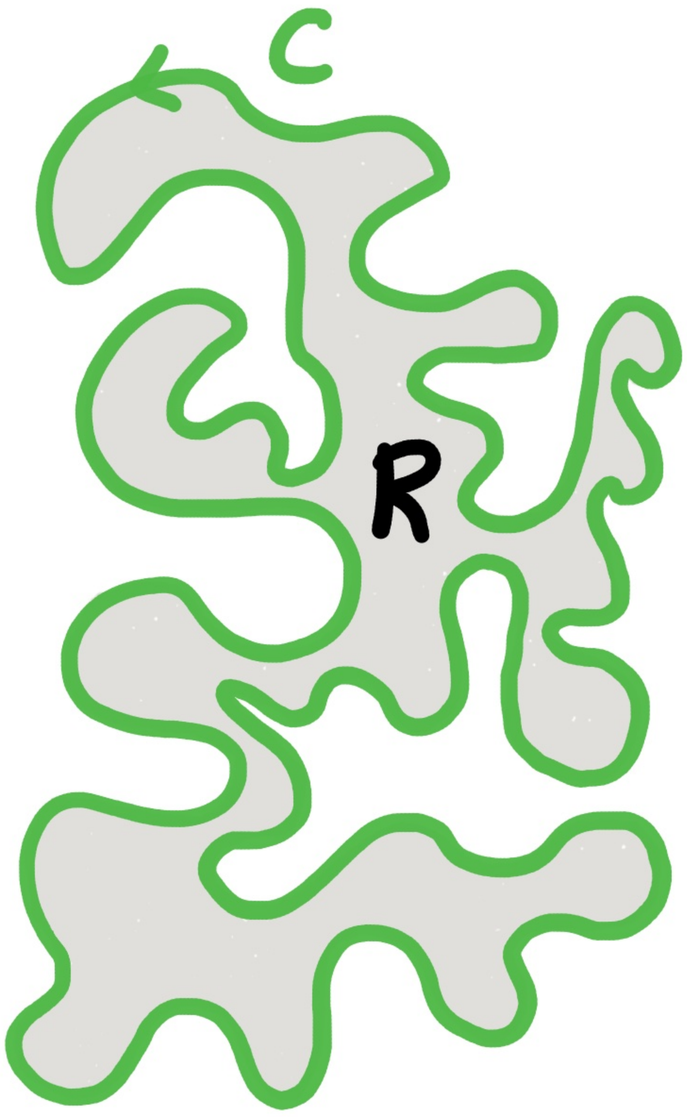
Solution:

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dA$$

Example:

Let R be a region in $\mathbb{R}^2$ enclosed by a simple closed counter clockwise curve C. Show

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \text{Area}(R).$$



Solution:

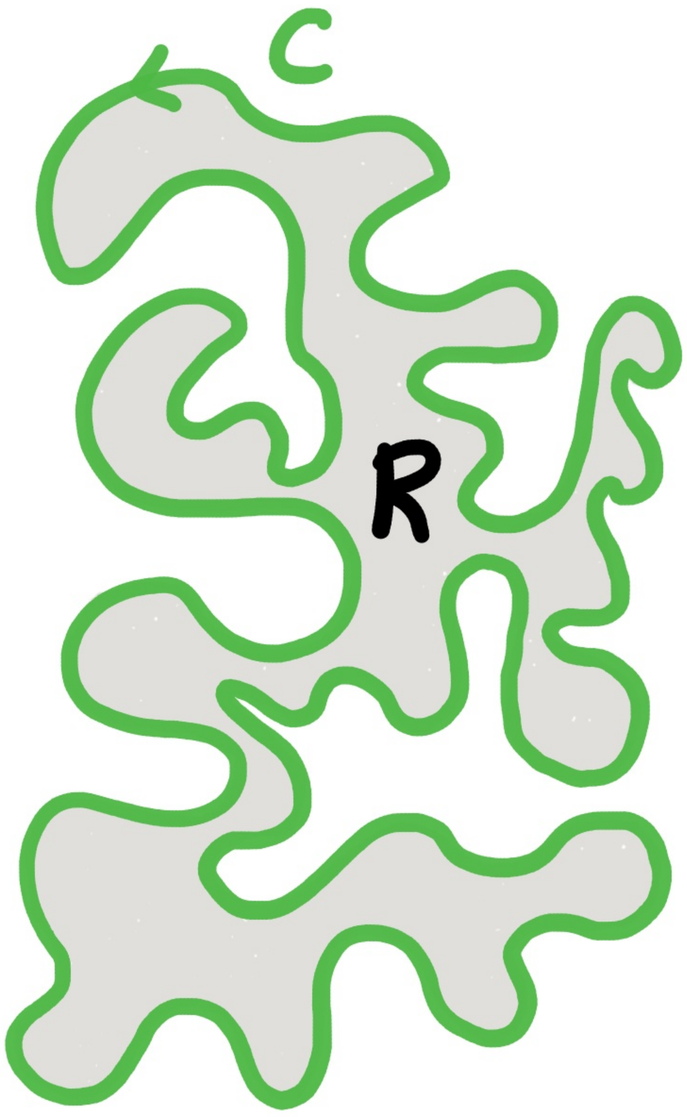
$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dA$$

$$= \frac{1}{2}\iint_R (1+1)\,dA$$

Example:

Let R be a region in $\mathbb{R}^2$ enclosed by a simple closed counter clockwise curve C. Show

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \text{Area}(R).$$



Solution:

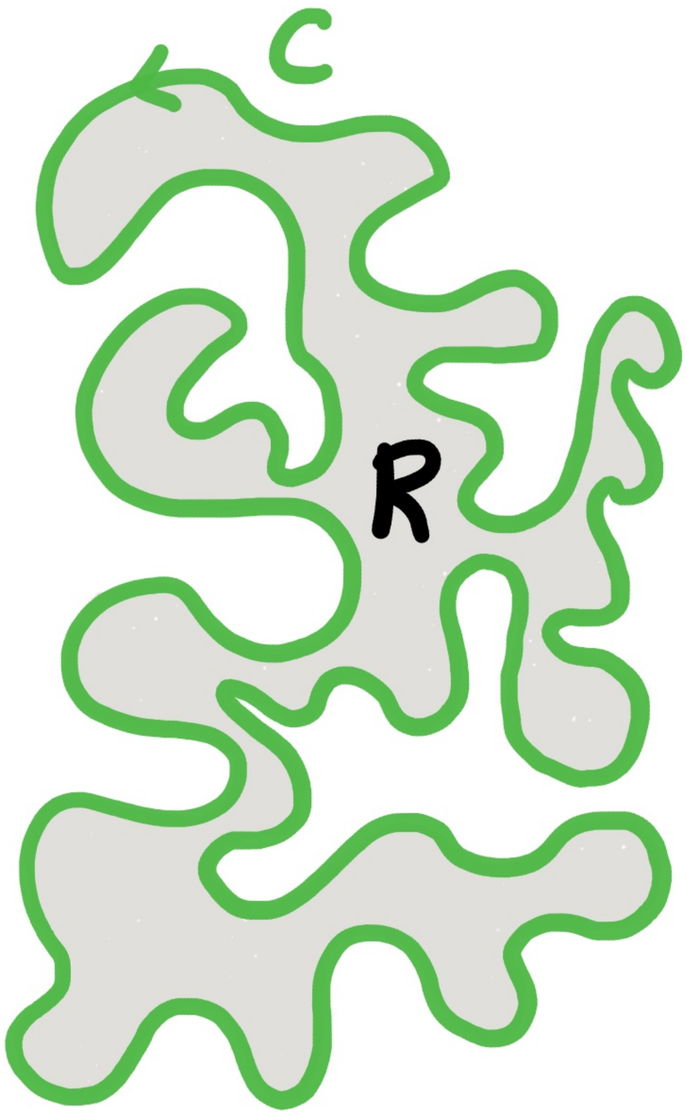
$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dA$$

$$= \frac{1}{2}\iint_R (1+1)\,dA = \frac{1}{2}\iint_R 2\,dA$$

Example:

Let R be a region in $\mathbb{R}^2$ enclosed by a simple closed counter clockwise curve C. Show

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \text{Area}(R).$$



Solution:

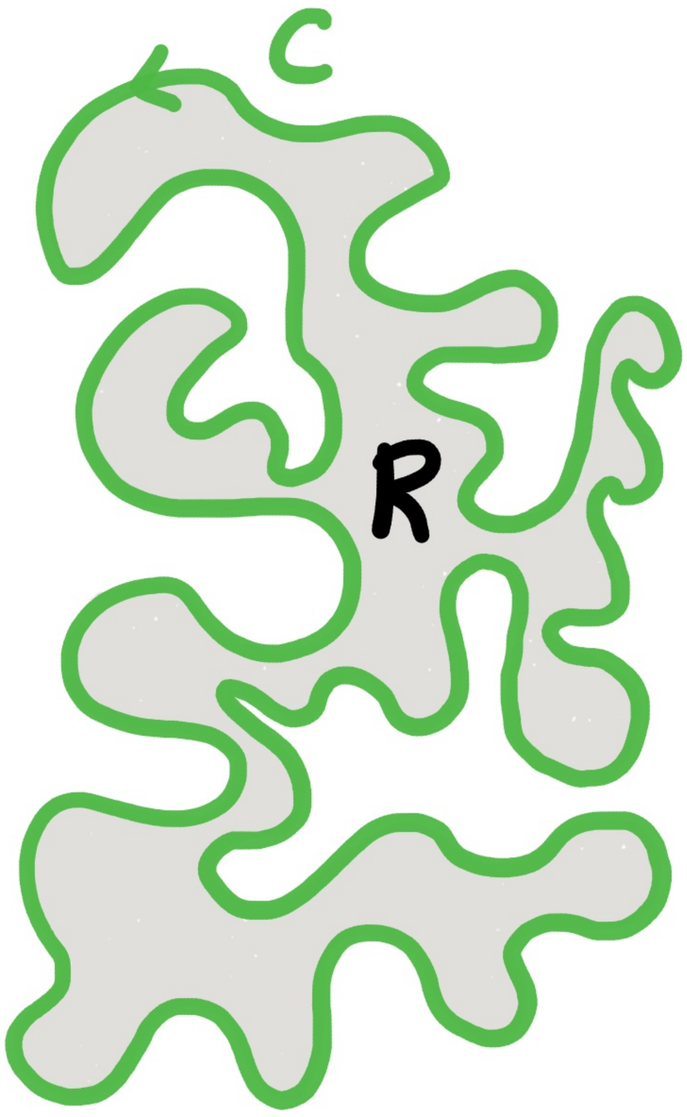
$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dA$$

$$= \frac{1}{2}\iint_R (1+1)\,dA = \frac{1}{2}\iint_R 2\,dA = \iint_R dA$$

Example:

Let R be a region in $\mathbb{R}^2$ enclosed by a simple closed counter clockwise curve C. Show

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \text{Area}(R).$$



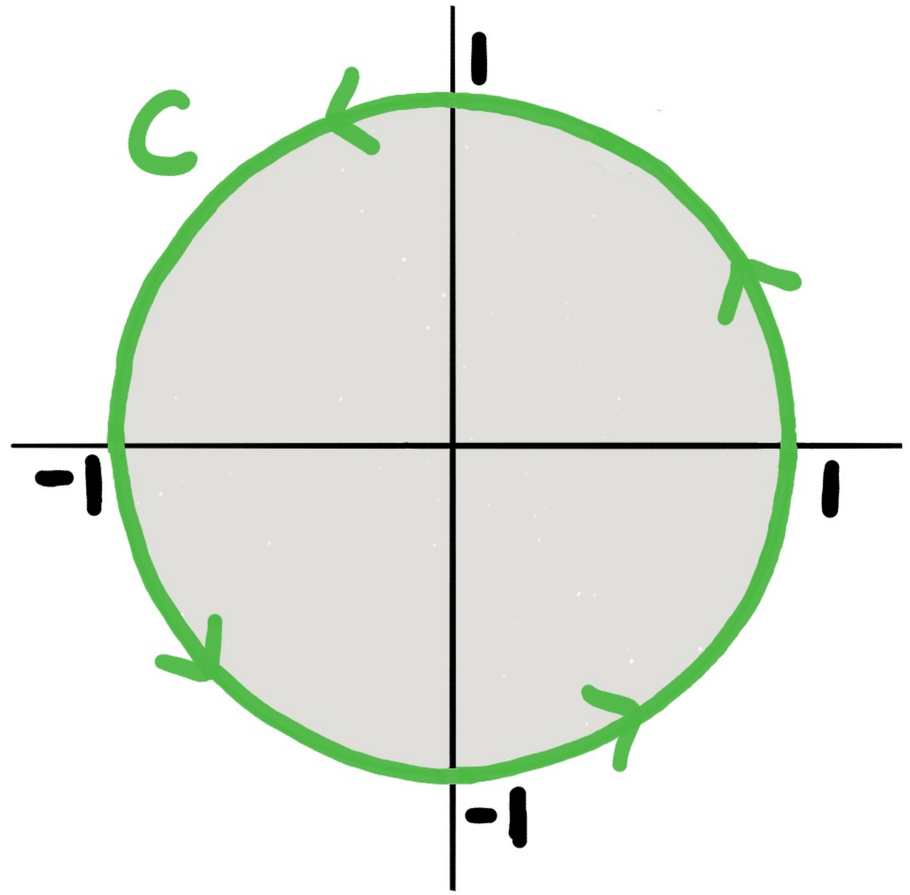
Solution:

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\iint_R \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right) dA$$

$$= \frac{1}{2}\iint_R (1+1)\,dA = \frac{1}{2}\iint_R 2\,dA = \iint_R dA = \text{Area}(R)$$

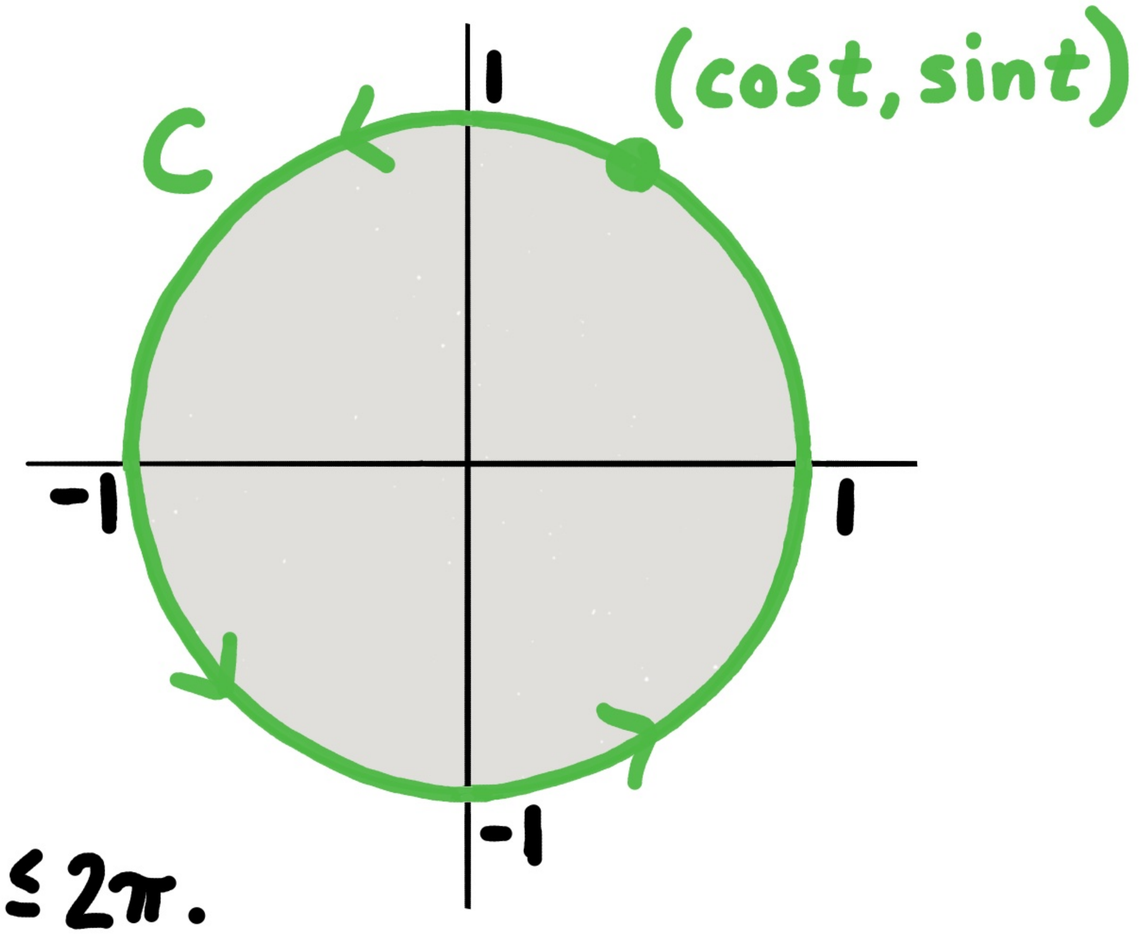
Example:

Find the area of the region enclosed by a circle of radius 1.



Example:

Find the area of the region enclosed by a circle of radius 1.

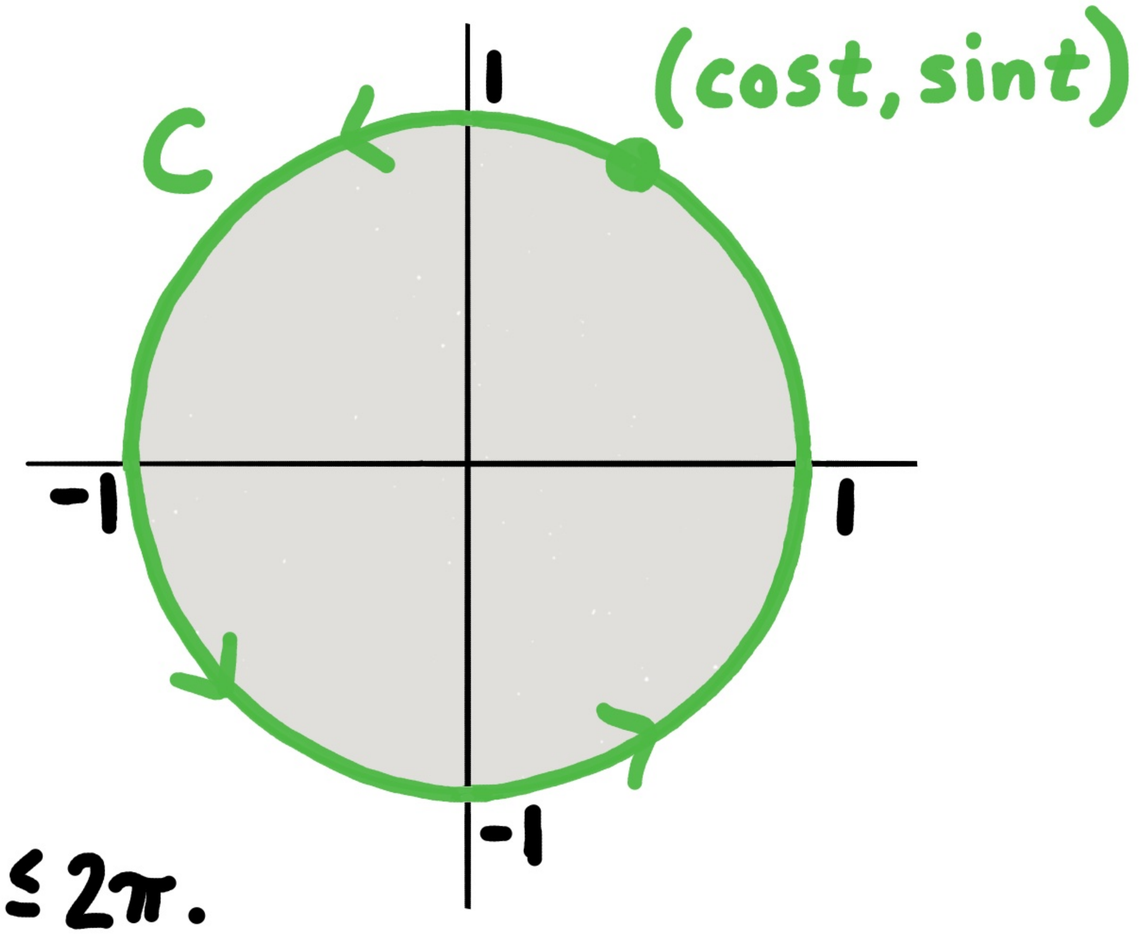


Solution:

$C : x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi.$

Example:

Find the area of the region enclosed by a circle of radius 1.



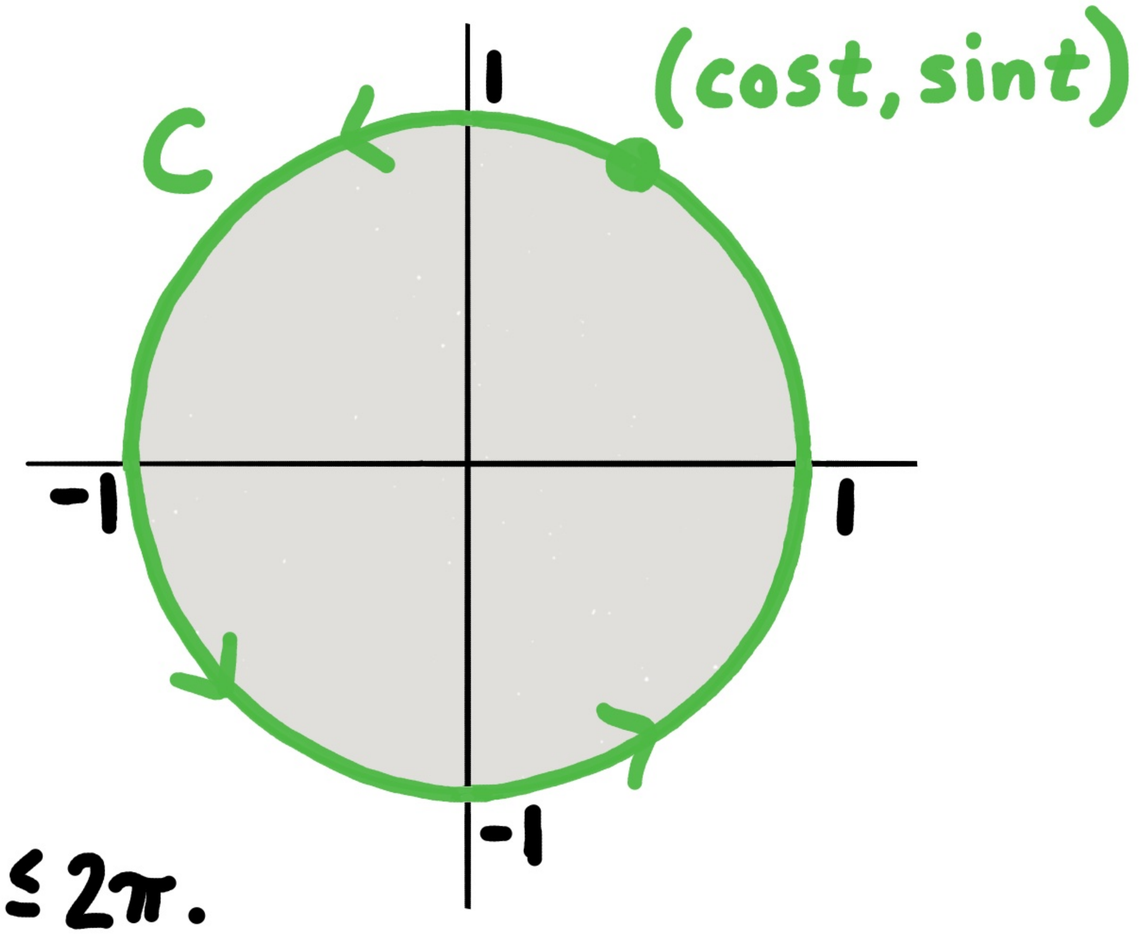
Solution:

$C : x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi.$

Area: $\frac{1}{2}\oint_C -y\,dx + x\,dy$

Example:

Find the area of the region enclosed by a circle of radius 1.



Solution:

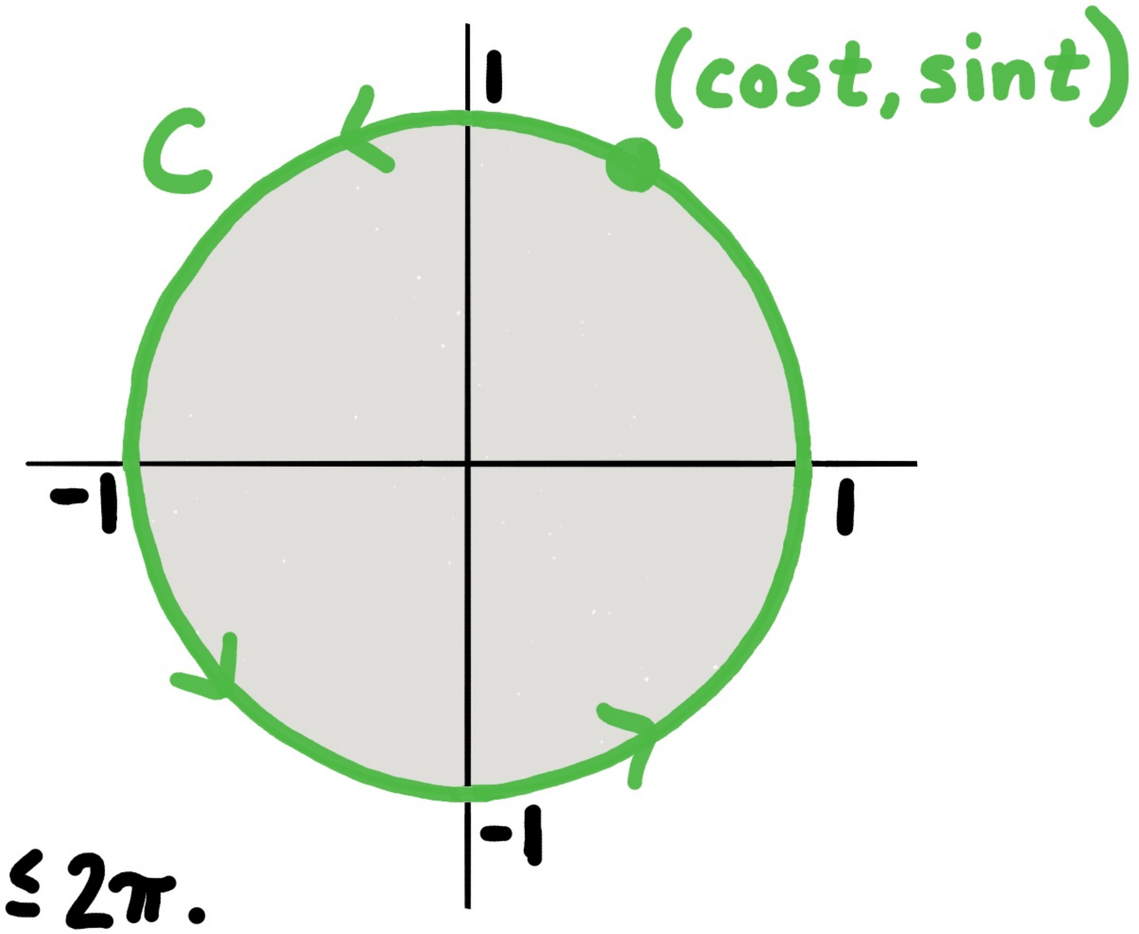
$C : x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi.$

Area: $\frac{1}{2}\oint_C -y\,dx + x\,dy$

$$= \frac{1}{2}\int_0^{2\pi} \left(-\sin t(-\sin t) + \cos t \cdot \cos t\right) dt$$

Example:

Find the area of the region enclosed by a circle of radius 1.



Solution:

$C : x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi.$

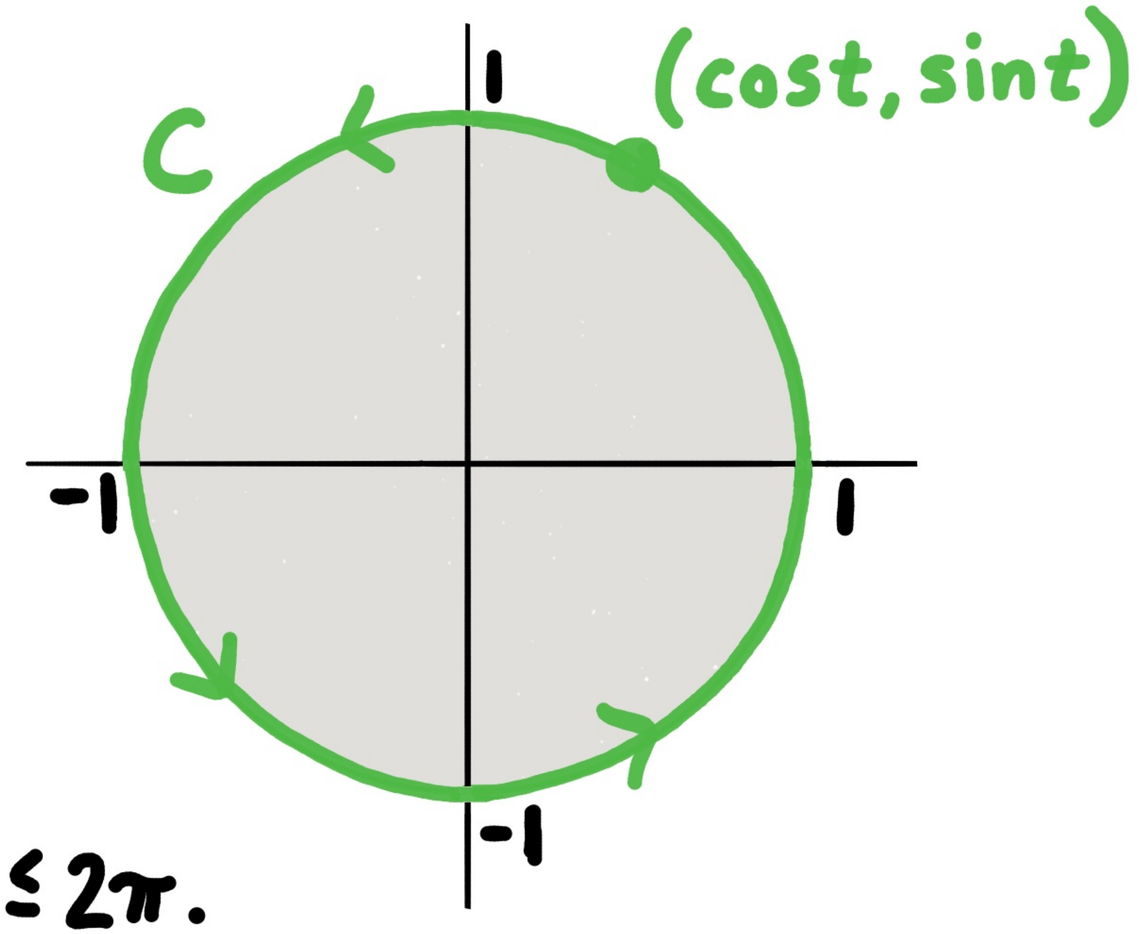
Area: $\frac{1}{2}\oint_C -y\,dx + x\,dy$

$$= \frac{1}{2}\int_0^{2\pi} \left(-\sin t(-\sin t) + \cos t \cdot \cos t\right) dt$$

$$= \frac{1}{2}\int_0^{2\pi} dt$$

Example:

Find the area of the region enclosed by a circle of radius 1.



Solution:

$C : x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi.$

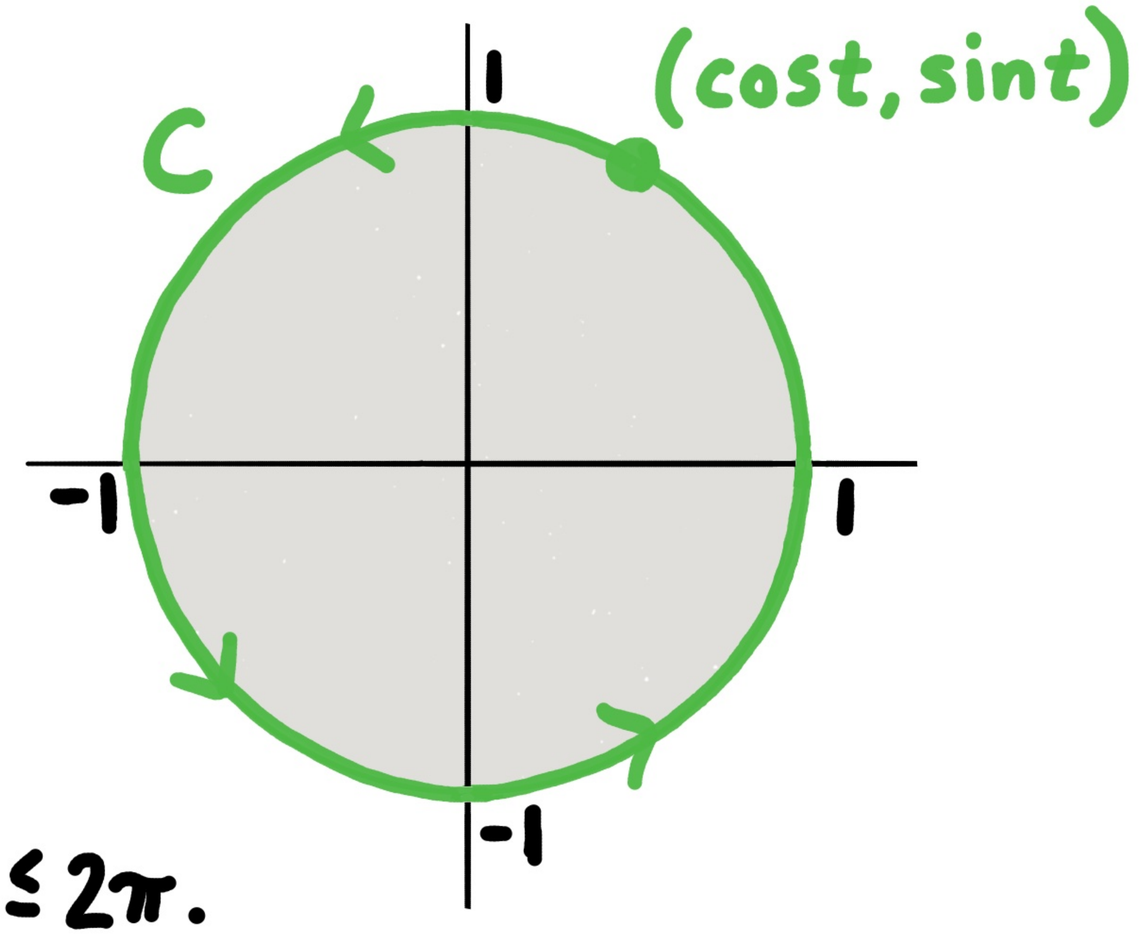
Area: $\frac{1}{2}\oint_C -y\,dx + x\,dy$

$$= \frac{1}{2}\int_0^{2\pi} \left(-\sin t(-\sin t) + \cos t \cdot \cos t\right) dt$$

$$= \frac{1}{2}\int_0^{2\pi} dt = \frac{1}{2}\,2\pi$$

Example:

Find the area of the region enclosed by a circle of radius 1.



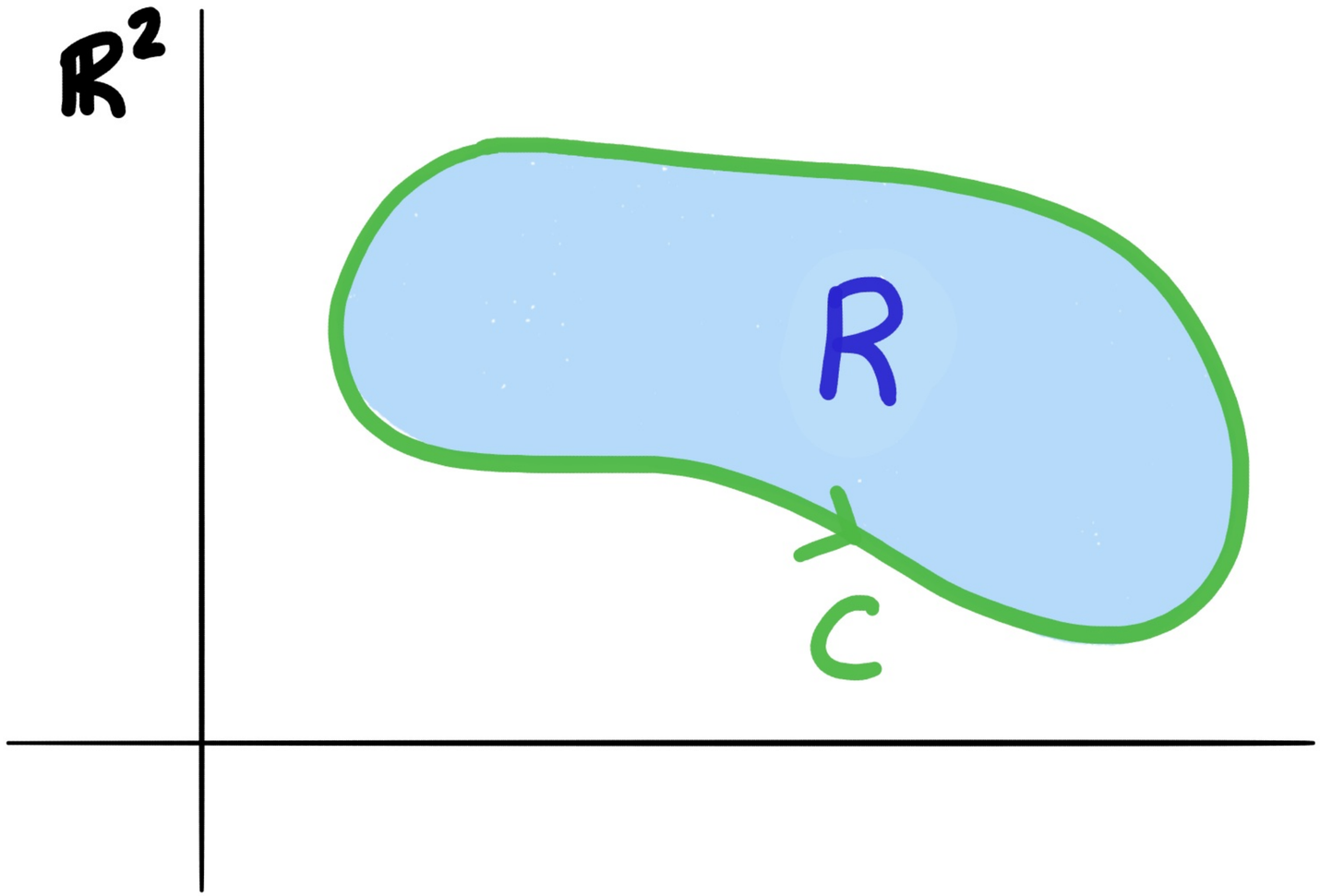
Solution:

$C : x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi.$

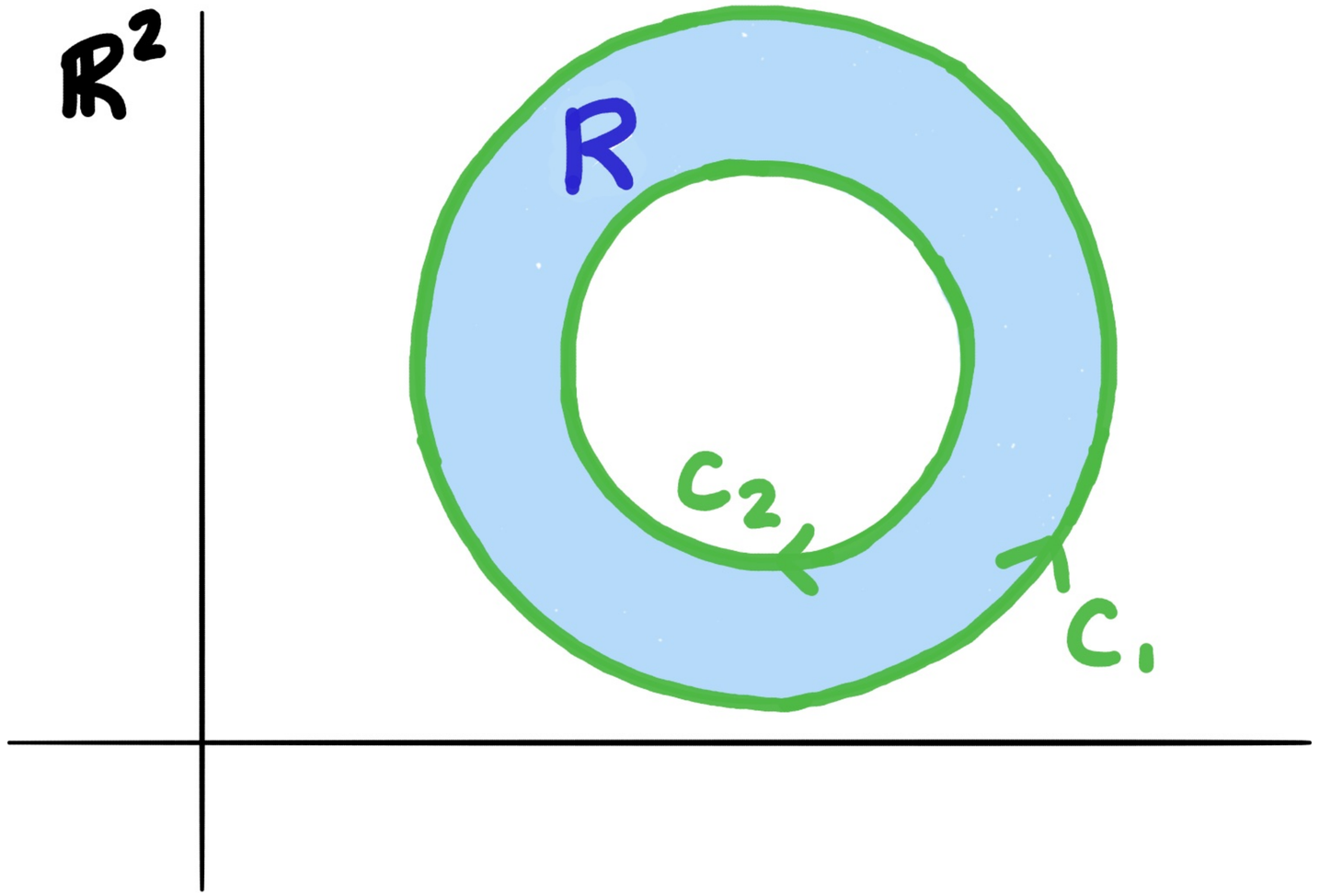
Area: $\frac{1}{2}\oint_C -y\,dx + x\,dy$

$$= \frac{1}{2}\int_0^{2\pi} \left(-\sin t(-\sin t) + \cos t \cdot \cos t\right) dt$$

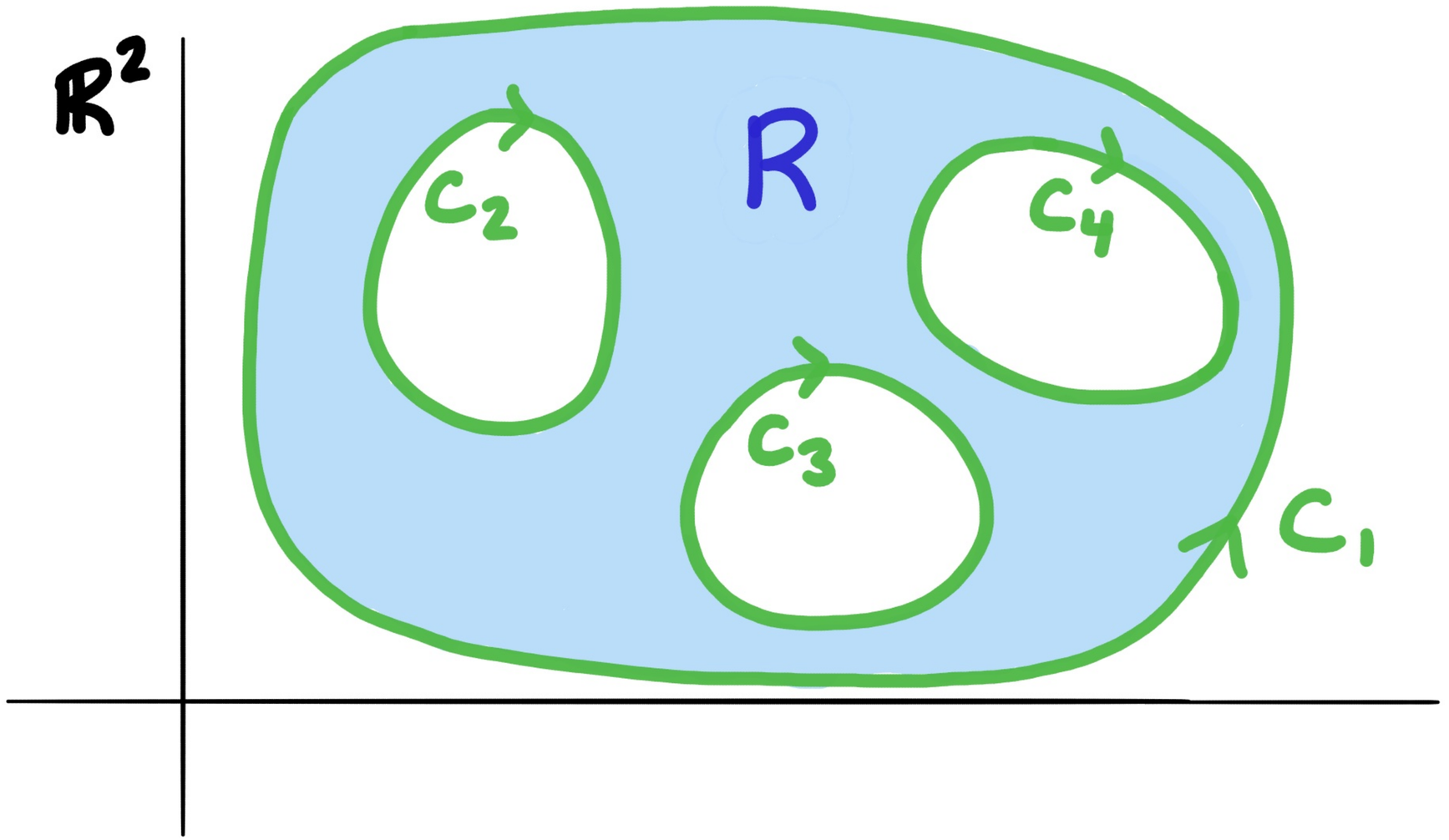
$$= \frac{1}{2}\int_0^{2\pi} dt = \frac{1}{2}\,2\pi = \pi$$



$$\int_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$



$$\int_{C_1 \cup C_2} F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$



$$\int_{C_1 \cup C_2 \cup C_3 \cup C_4} F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

Ⅱ Return to Central Theorem

Theorem:

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$, $\vec{F} = (F_1, \dots, F_n)$.

The following are equivalent:

① $\vec{F} = \nabla f$.

② $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ for $1 \le i, j \le n$.

③ $\oint_C \sum_{i=1}^{n} F_i \, dx_i = 0$ for simple closed curves C.

④ $\int_C \sum_{i=1}^{n} F_i \, dx_i$ is independent of path.

$$\vec{F} = \nabla f$$

1 ✓

2 $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ ✓

3 $\oint_C \sum_{i=1}^{n} F_i \, dx_i = 0$ ✓

4 $\int_C \sum_{i=1}^{n} F_i \, dx_i$ ind. of path

proof: (② ⇒ ③) (n=2)

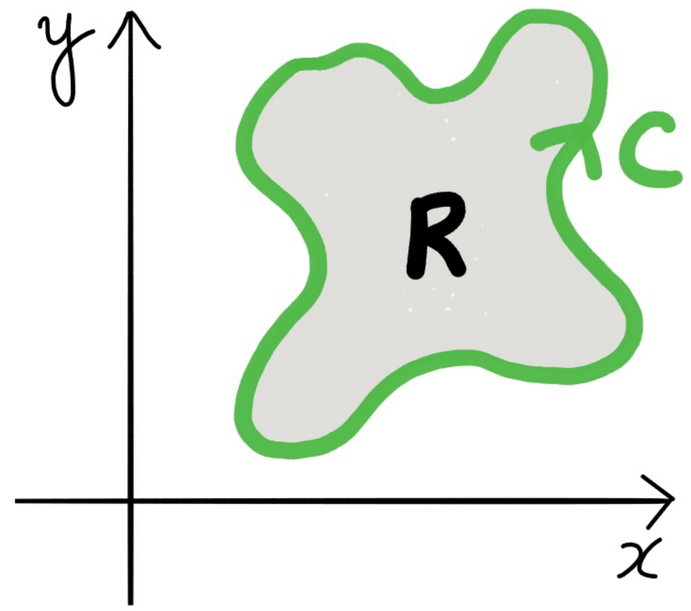
proof: (② ⇒ ③) $(n=2)$

Suppose $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$.

proof: $(② \Rightarrow ③)$ $(n=2)$

Suppose $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$.

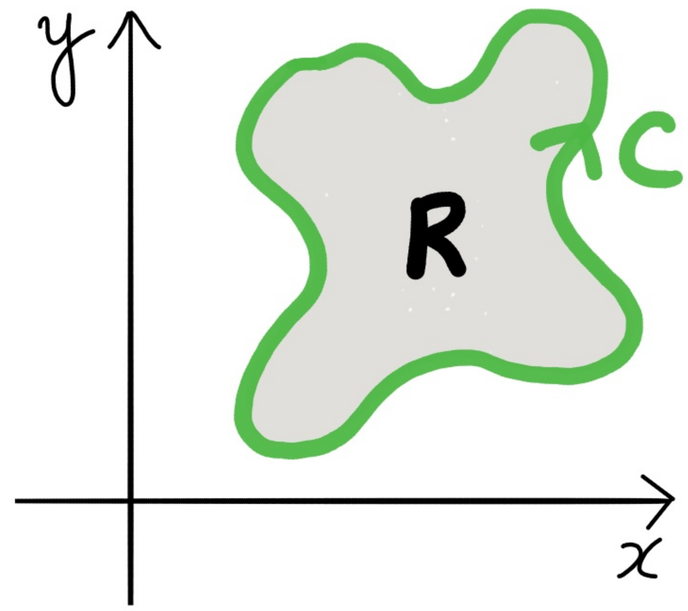
By Green's Theorem, for any simple closed curve C in $\mathbb{R}^2$,



proof: (② ⇒ ③) (n=2)

Suppose $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$.

By Green's Theorem, for any simple closed curve C in $\mathbb{R}^2$, we have

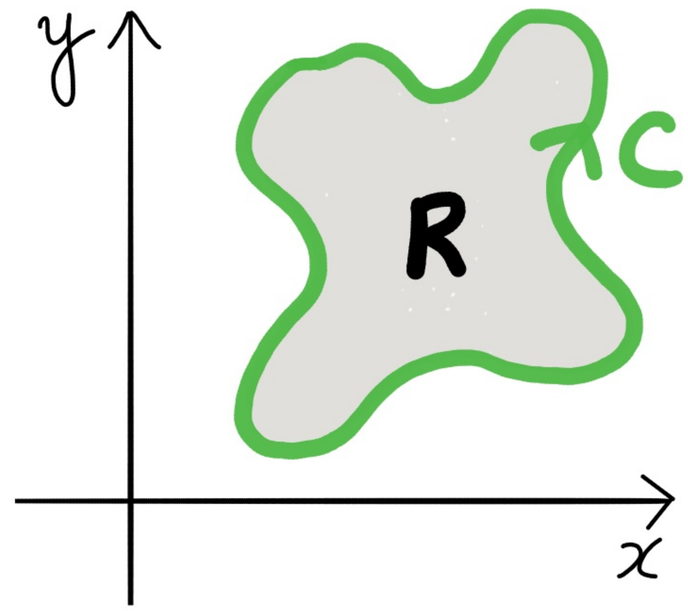


$$\oint_C F_1\,dx + F_2\,dy =$$

proof: (② ⇒ ③) $(n=2)$

Suppose $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$.

By Green's Theorem, for any simple closed curve C in $\mathbb{R}^2$, we have

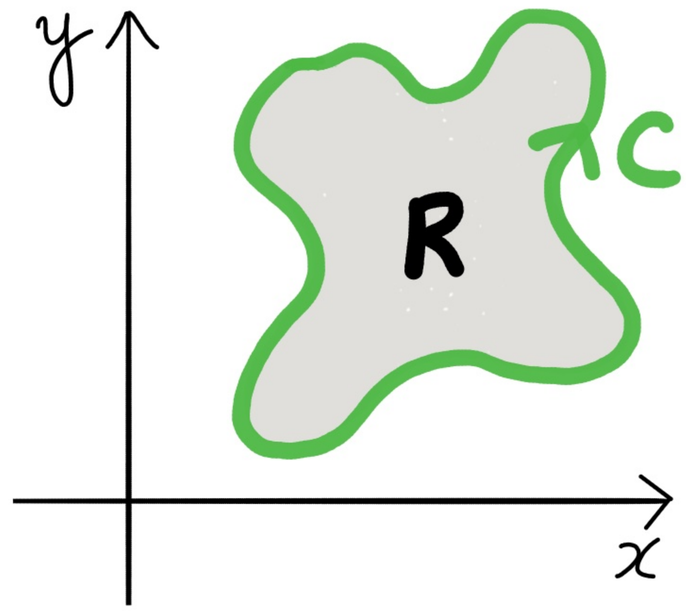


$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

proof: (② ⇒ ③) $(n=2)$

Suppose $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$.

By Green's Theorem, for any simple closed curve C in $\mathbb{R}^2$, we have

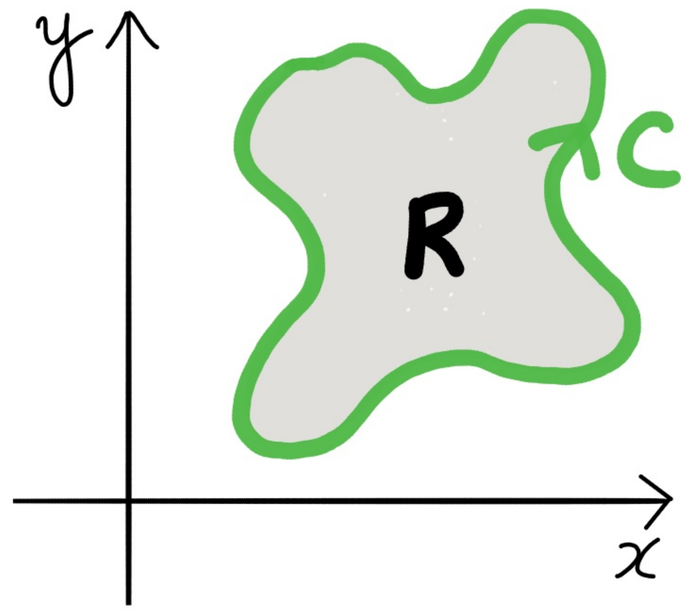


$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

proof: (② ⇒ ③) (n=2)

Suppose $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$.

By Green's Theorem, for any simple closed curve C in $\mathbb{R}^2$, we have

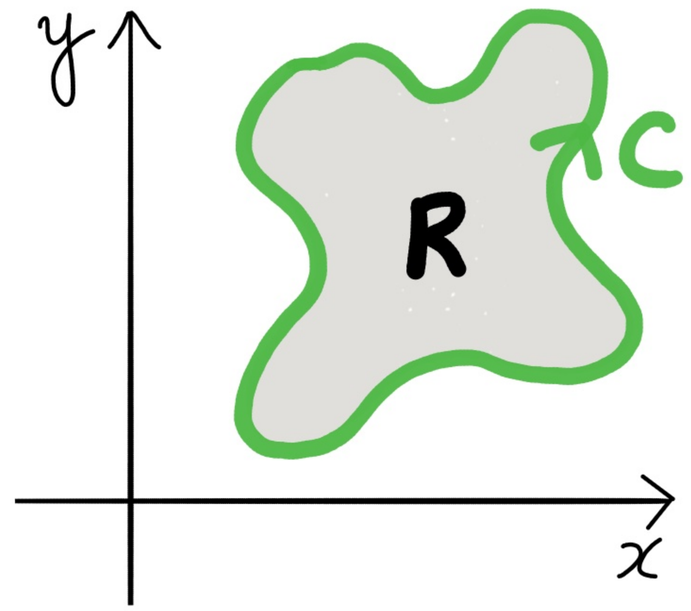


$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R 0\,dA$$

proof: (② ⇒ ③) (n=2)

Suppose $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$.

By Green's Theorem, for any simple closed curve C in $\mathbb{R}^2$, we have

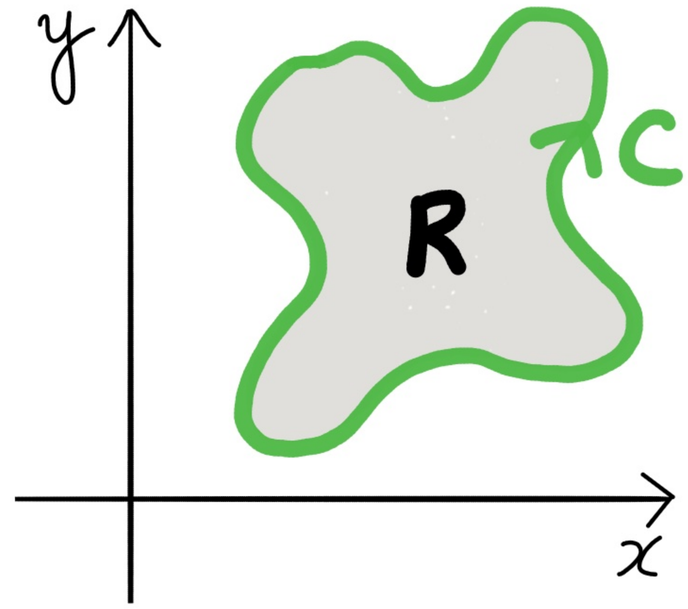


$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R 0\,dA = 0.$$

proof: (② ⇒ ③) (n=2)

Suppose $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$.

By Green's Theorem, for any simple closed curve C in $\mathbb{R}^2$, we have



$$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA = \iint_R 0\,dA = 0.$$

END

$$\vec{F} = \nabla f$$

✓ ① ✓

$$\int_C \sum_{i=1}^{n} F_i \, dx_i$$ ④ ind. of path

② $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$

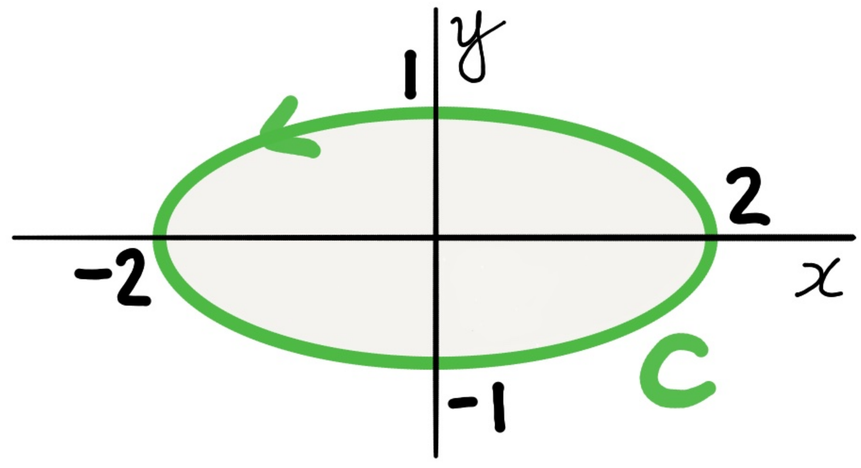
✓ ③ ✓ (for n=2)

$$\oint_C \sum_{i=1}^{n} F_i \, dx_i = 0$$

Ⅲ Final Example

Problem:

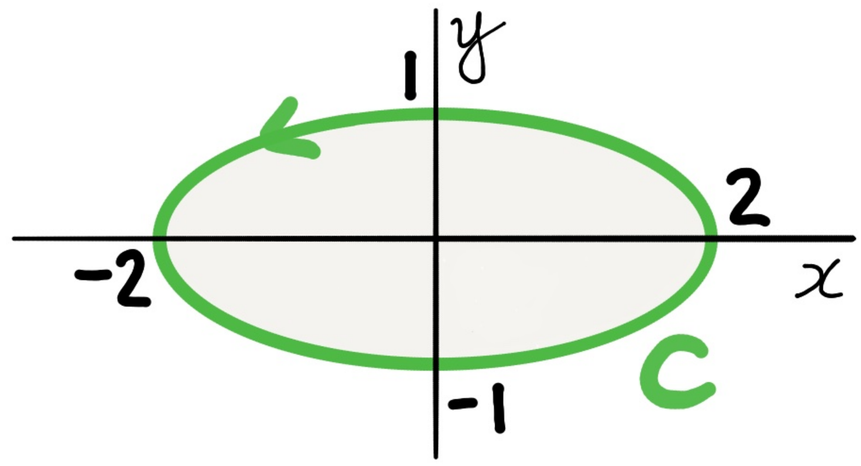
Find $\oint_C -4y^3\,dx + x^3\,dy$



where C is the counter-clockwise oriented ellipse $\frac{x^2}{4} + y^2 = 1$.

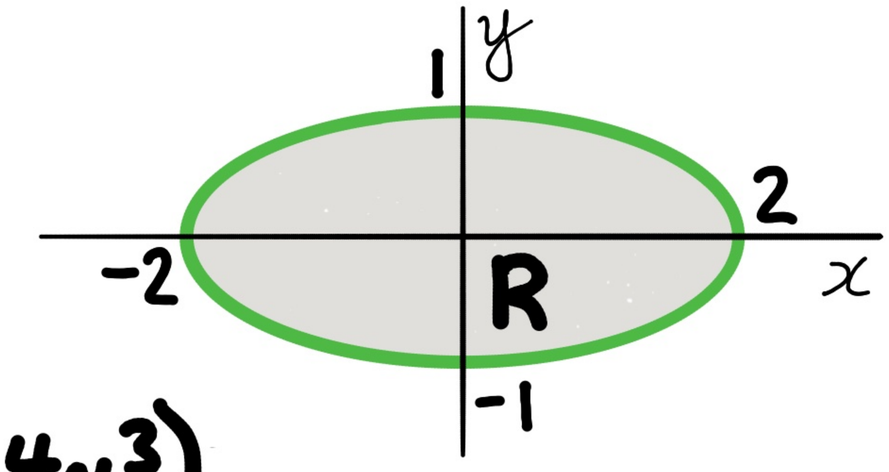
Problem:

Find $\oint_C -4y^3\,dx + x^3\,dy$



where C is the counter-clockwise oriented ellipse $\frac{x^2}{4} + y^2 = 1$.

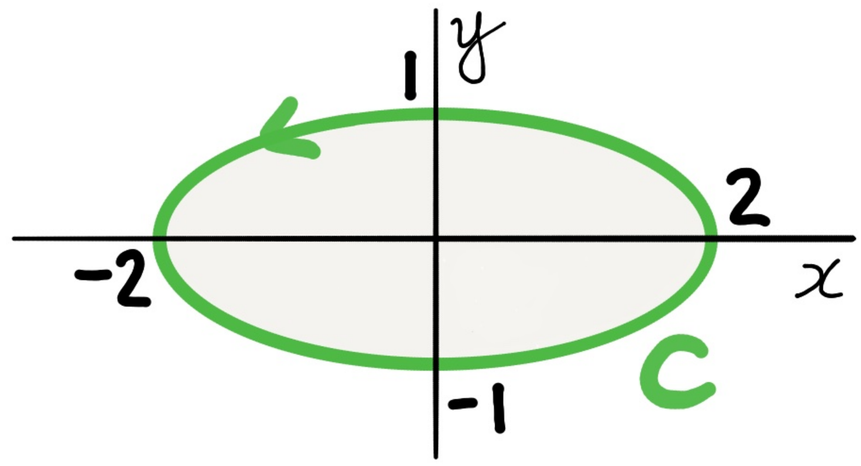
Solution:



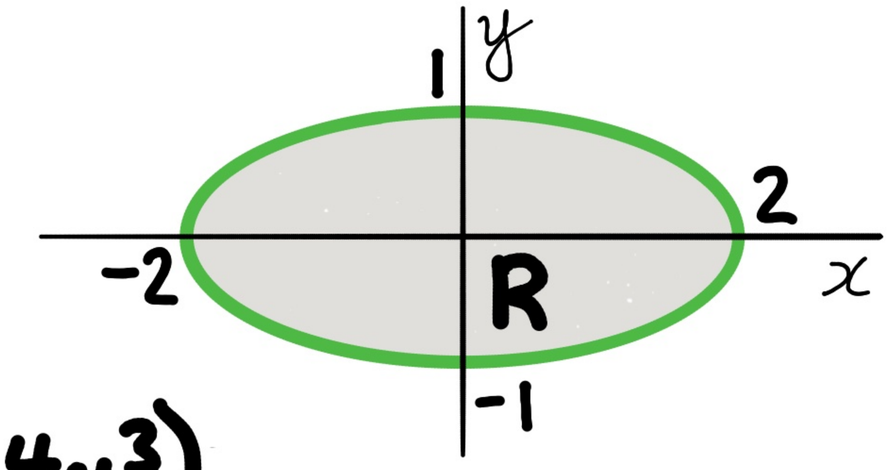
$$\oint_C -4y^3\,dx + x^3\,dy = \iint_R \frac{\partial x^3}{\partial x} - \frac{\partial(-4y^3)}{\partial y}\,dA$$

Problem:

Find $\oint_C -4y^3\,dx + x^3\,dy$ where C is the counter-clockwise oriented ellipse $\frac{x^2}{4} + y^2 = 1$.



Solution:

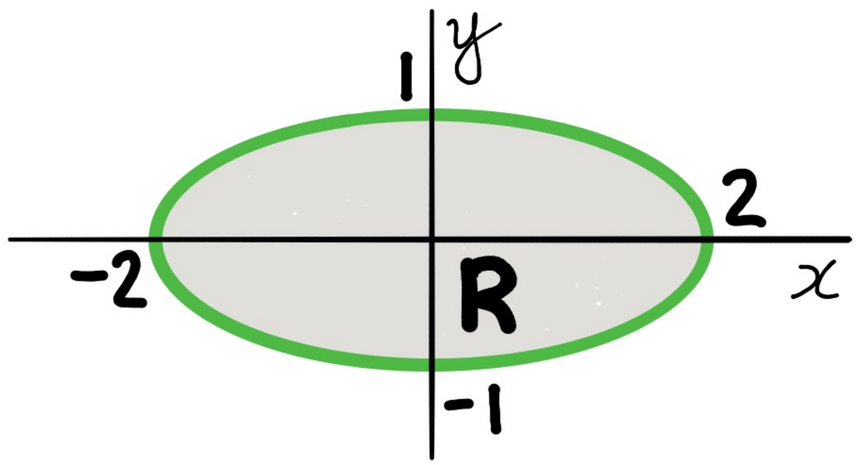


$$\oint_C -4y^3\,dx + x^3\,dy = \iint_R \frac{\partial x^3}{\partial x} - \frac{\partial(-4y^3)}{\partial y}\,dA$$

$$= \iint_R 3x^2 + 12y^2\,dx\,dy$$

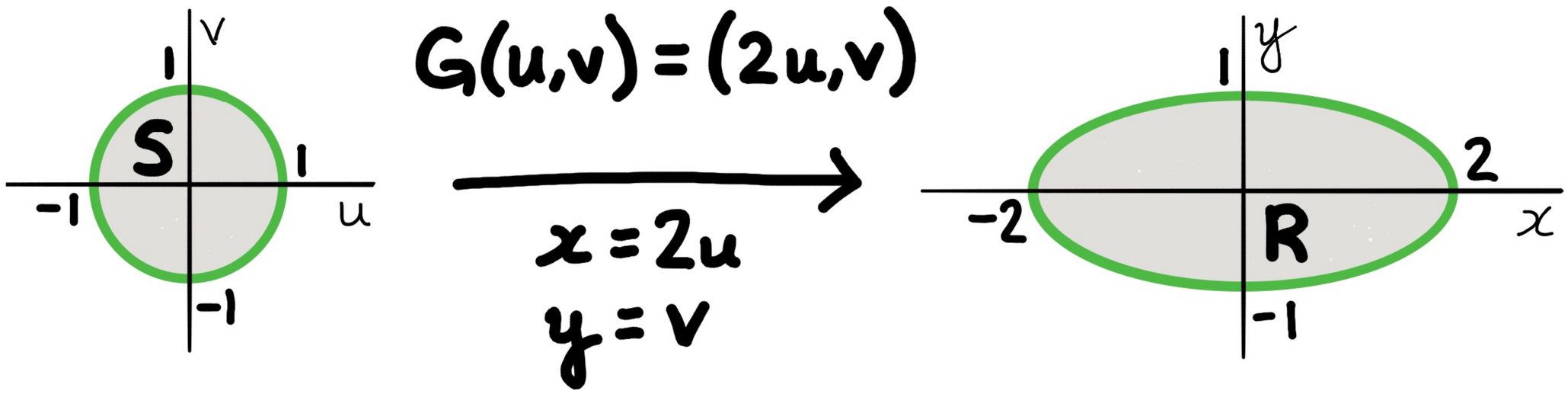
Solution:

$$\oint_C -4y^3\,dx + x^3\,dy = \iint_R 3x^2 + 12y^2\;dx\,dy$$



Solution:

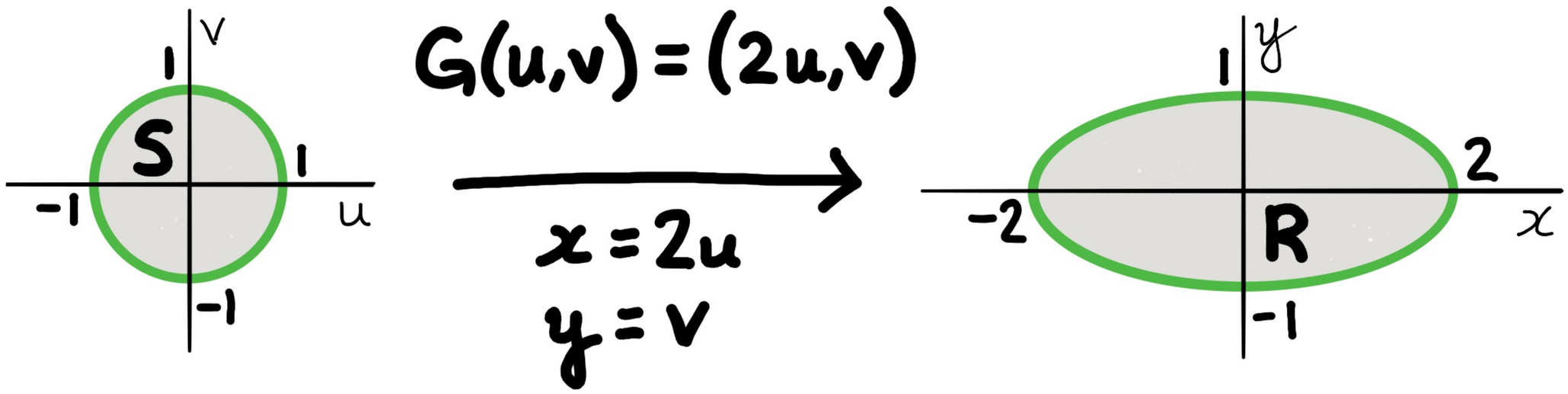
$$\oint_C -4y^3\,dx + x^3\,dy = \iint_R 3x^2 + 12y^2\;dx\,dy$$



Solution:

$$\oint_C -4y^3\,dx + x^3\,dy = \iint_R 3x^2 + 12y^2 \; dx\,dy$$

$$= \iint_S \left(3(2u)^2 + 12v^2\right)\left|\begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix}\right| du\,dv$$

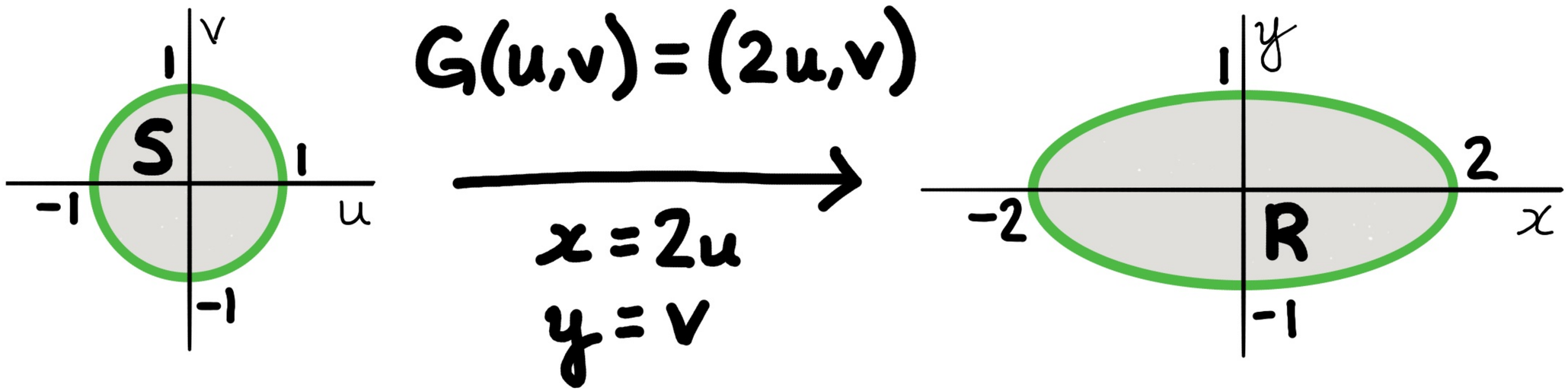


Solution:

$$\oint_C -4y^3\,dx + x^3\,dy = \iint_R 3x^2 + 12y^2 \; dx\,dy$$

$$= \iint_S \left(3(2u)^2 + 12v^2\right)\left|\begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix}\right| du\,dv$$

$$= 24\iint_S u^2 + v^2 \, du\,dv$$

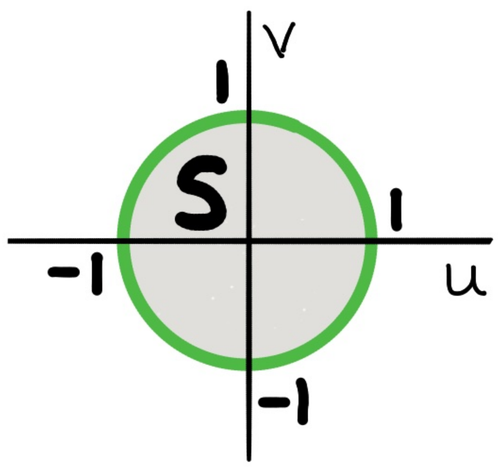


Solution:

$$\oint_C -4y^3\,dx + x^3\,dy = \iint_R 3x^2 + 12y^2 \; dx\,dy$$

$$= \iint_S \left(3(2u)^2 + 12v^2\right)\left|\begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix}\right| du\,dv$$

$$= 24\iint_S u^2 + v^2 \, du\,dv$$



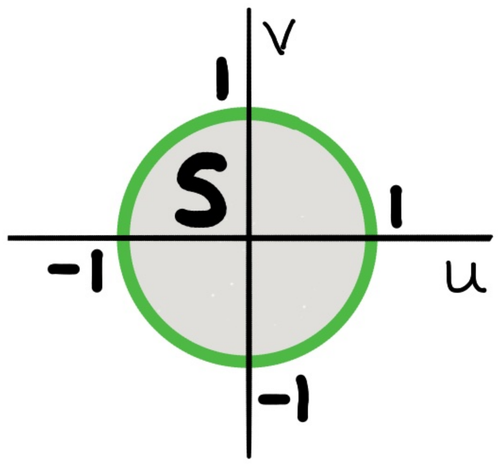
Solution:

$$\oint_C -4y^3\,dx + x^3\,dy = \iint_R 3x^2 + 12y^2 \; dx\,dy$$

$$= \iint_S \left(3(2u)^2 + 12v^2\right)\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} du\,dv$$

$$= 24\iint_S u^2 + v^2 \, du\,dv$$

$$= 24\int_0^{2\pi}\int_0^1 r^3 \, dr\,d\theta$$



Solution:

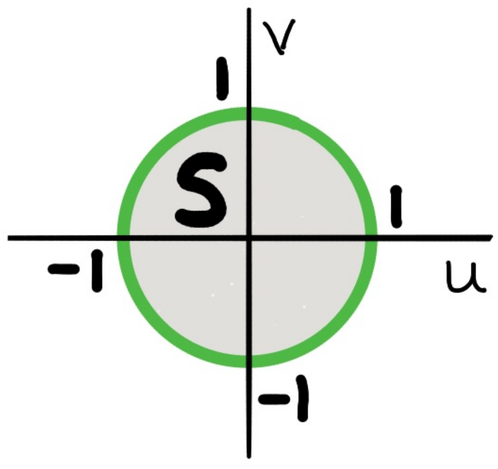
$$\oint_C -4y^3\,dx + x^3\,dy = \iint_R 3x^2 + 12y^2 \; dx\,dy$$

$$= \iint_S \left(3(2u)^2 + 12v^2\right)\left|\begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix}\right| du\,dv$$

$$= 24 \iint_S u^2 + v^2 \, du\,dv$$

$$= 24 \int_0^{2\pi} \int_0^1 r^3 \, dr\,d\theta$$

$$= 24 \cdot 2\pi \cdot \left[\tfrac{1}{4} r^4\right]_0^1$$



Solution:

$$\oint_C -4y^3\,dx + x^3\,dy = \iint_R 3x^2 + 12y^2 \; dx\,dy$$

$$= \iint_S \left(3(2u)^2 + 12v^2\right)\left|\begin{matrix} 2 & 0 \\ 0 & 1 \end{matrix}\right| du\,dv$$

$$= 24\iint_S u^2 + v^2 \, du\,dv$$

$$= 24\int_0^{2\pi}\int_0^1 r^3 \, dr\,d\theta$$

$$= 24 \cdot 2\pi \cdot \left[\tfrac{1}{4}r^4\right]_0^1$$

$$= 12\pi.$$