Thirty

I. Line integrals and vector fields

II. Line integrals and gradients

III. Theorem about line integrals, vector fields, and gradients

I Line integrals and vector fields

C, a curve in $\mathbb{R}^3$.

$$\int_C x^2 y \, dx + (2x - z) \, dy + x y^3 z \, dz$$

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Write the line integral corresponding to C and the vector field

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If C, a curve in $\mathbb{R}^n$, and

$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$, $\vec{F} = (F_1, \ldots, F_n)$,

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$f: \mathbb{R}^n \to \mathbb{R}$

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II Line integrals and gradients

Theorem:

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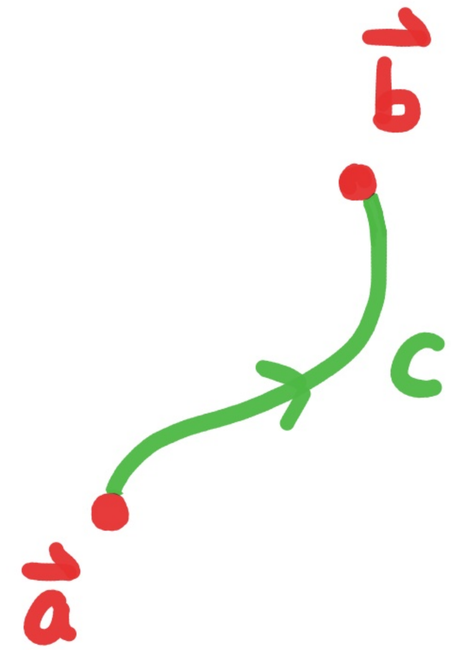

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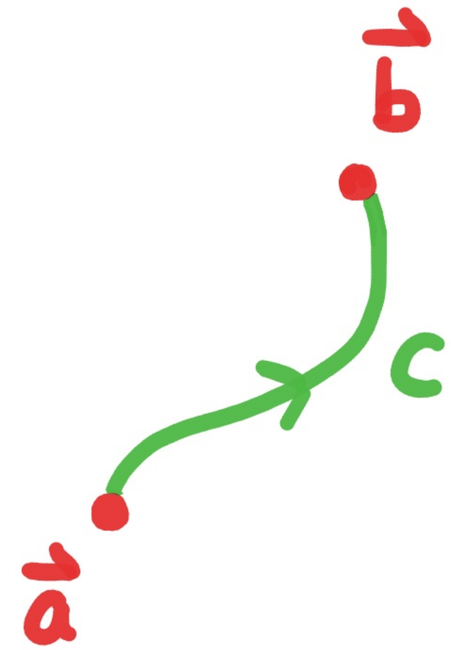

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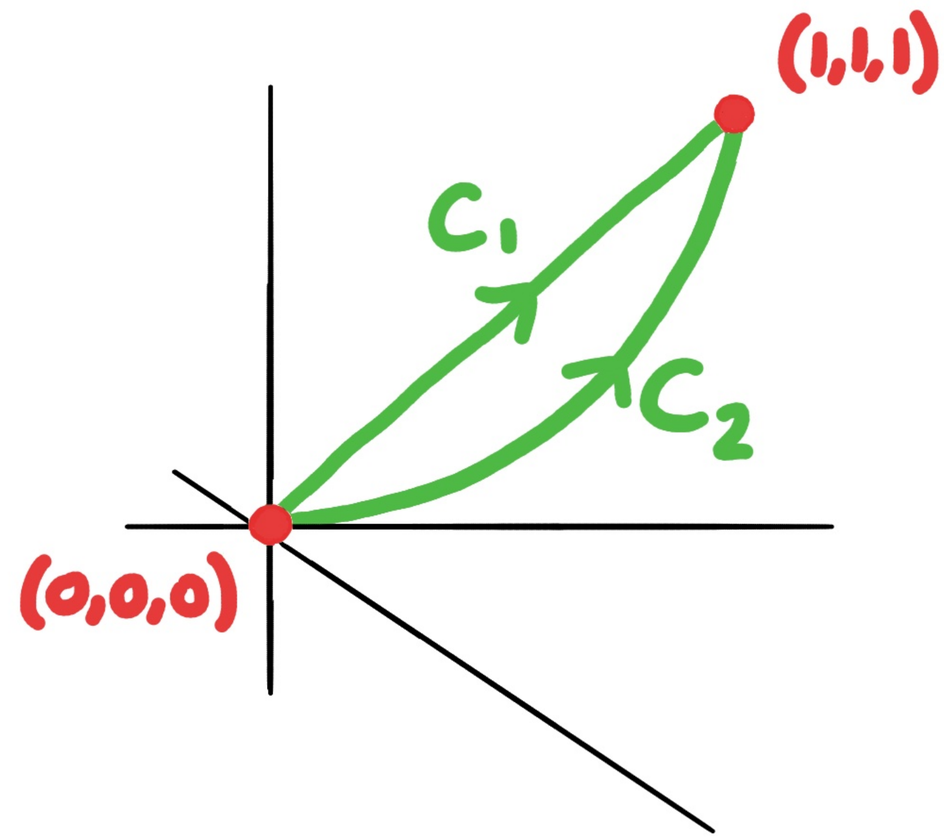
$$\int_C \nabla f \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$$

Example: C_1 given by $x=t,\ y=t,\ z=t,\ 0\le t\le 1.$

C_2 given by $x=t^3,\ y=t^2,\ z=t,\ 0\le t\le 1.$

Find $\int_{C_i} yz\,dx + xz\,dy + xy\,dz$ for $i=1,2.$



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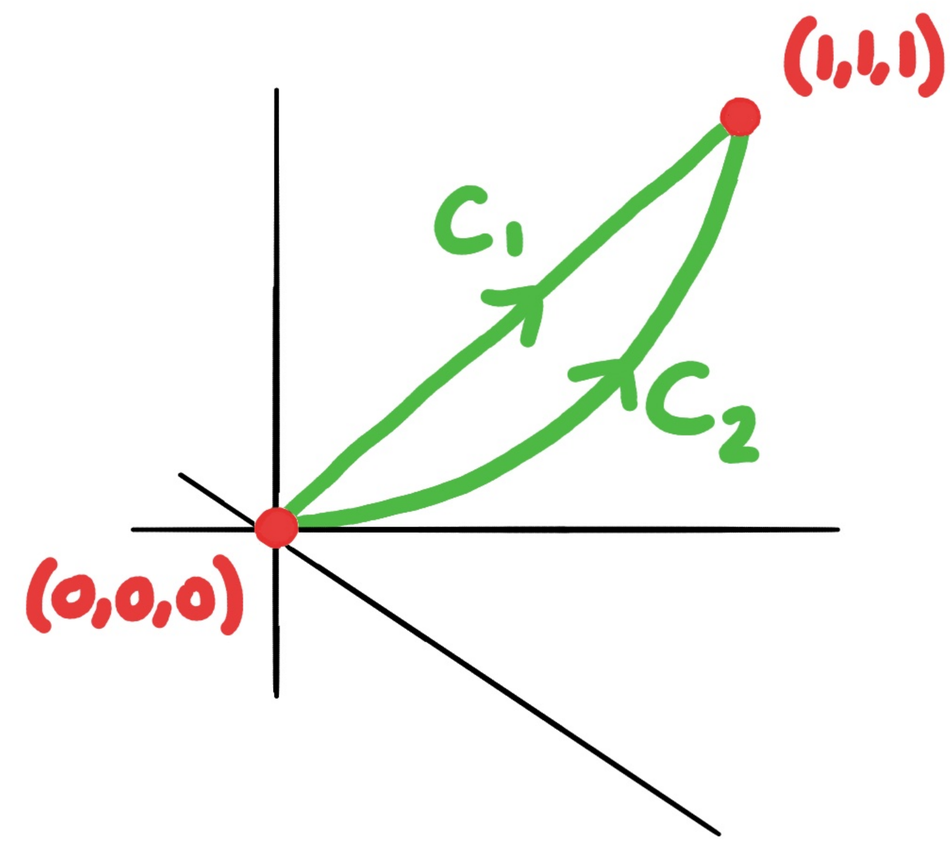

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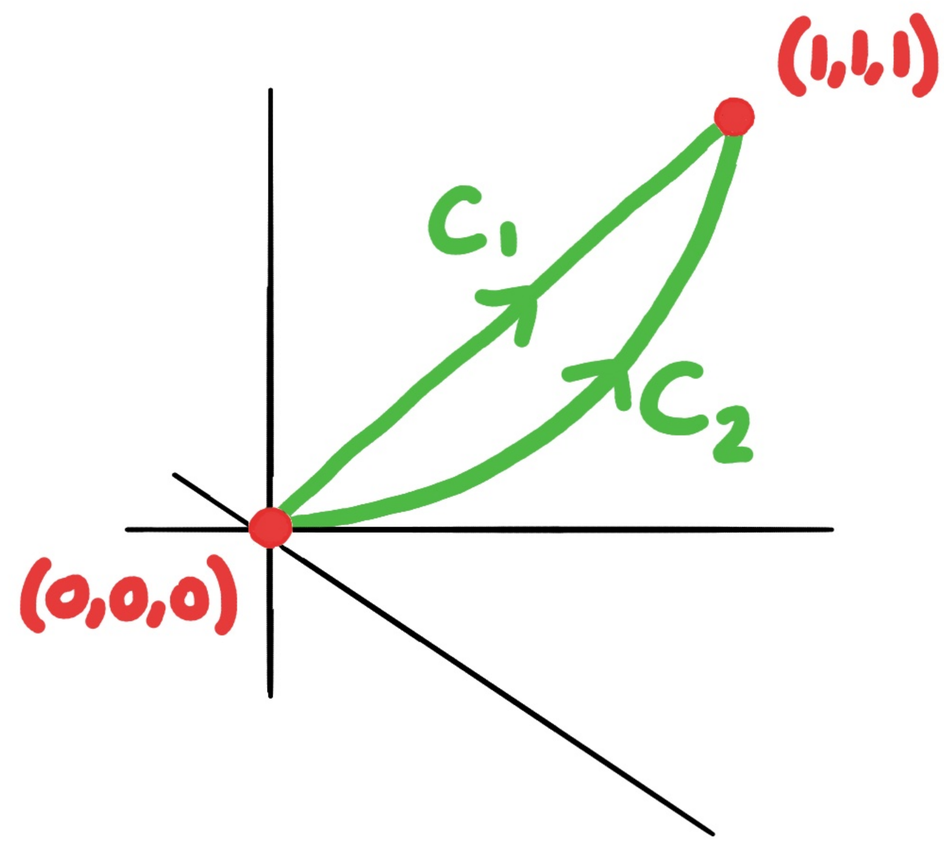

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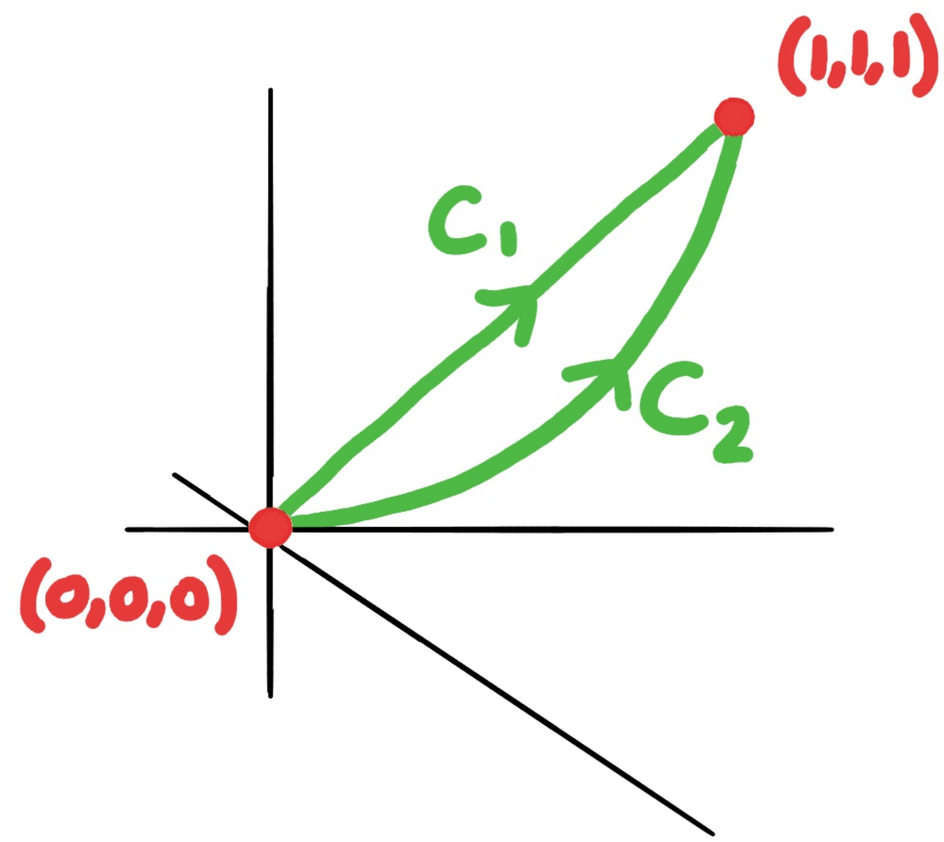

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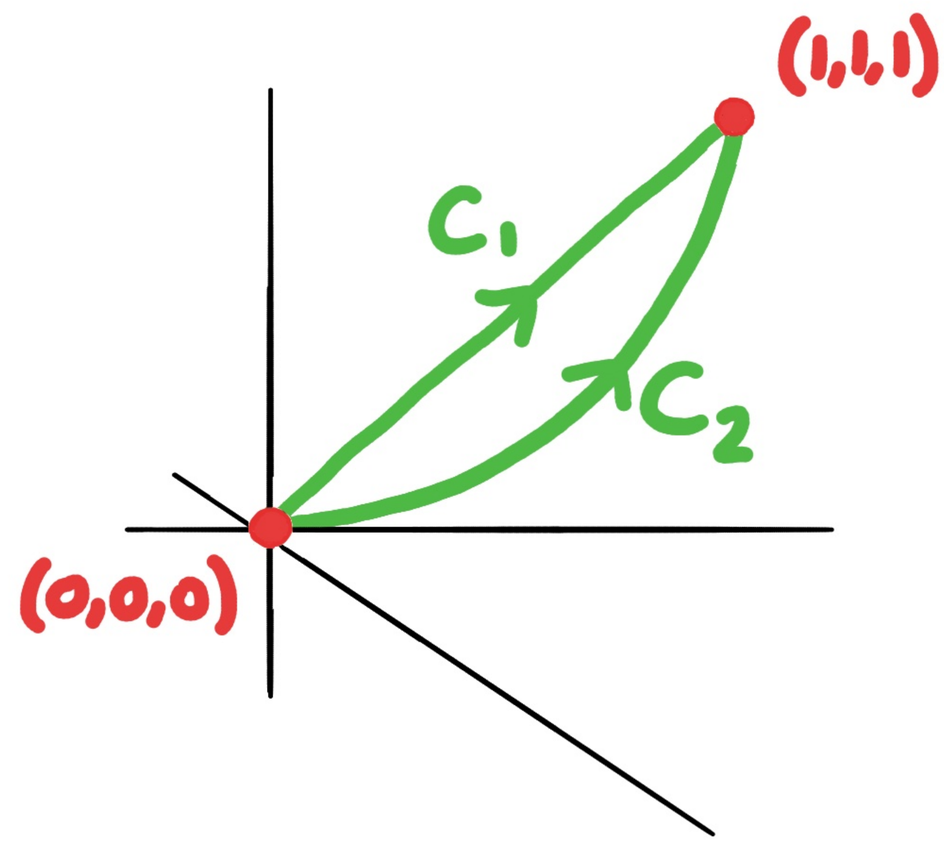

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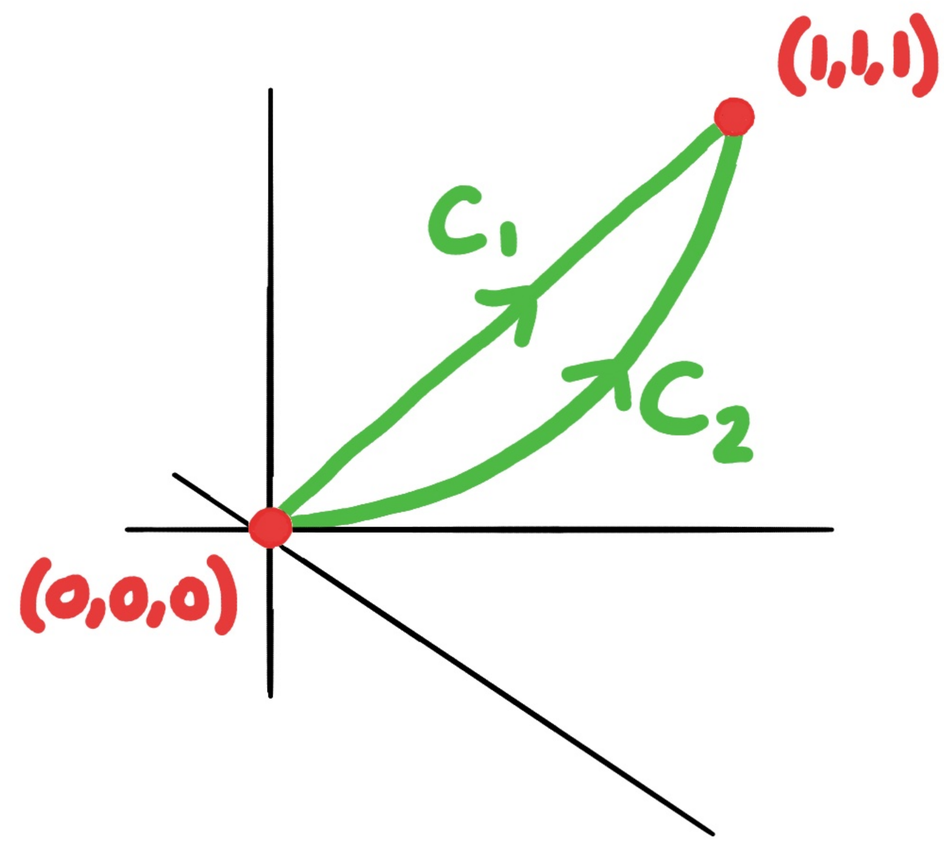

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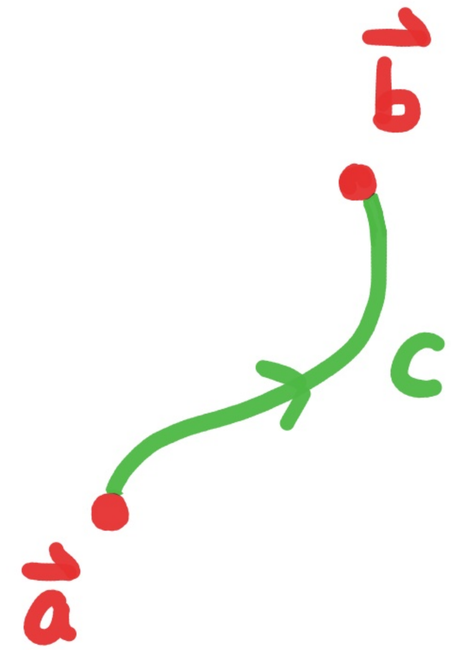

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$\boxed{\text{END}}$

Corollary:

For any $f: \mathbb{R}^n \to \mathbb{R}$, and any curve C in $\mathbb{R}^n$ with initial point $\vec{a} \in \mathbb{R}^n$ and terminal point $\vec{b} \in \mathbb{R}^n$,

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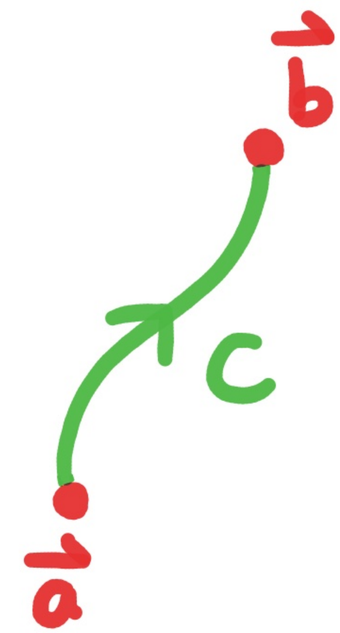
is independent of path.



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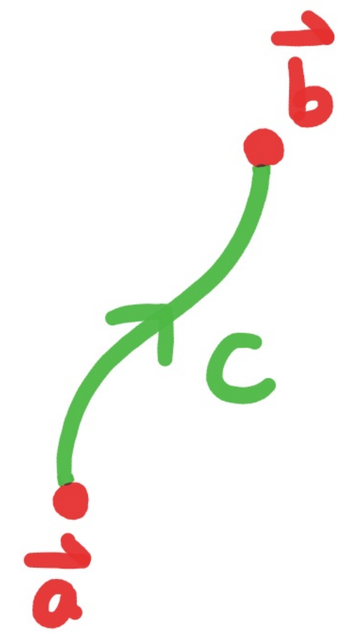
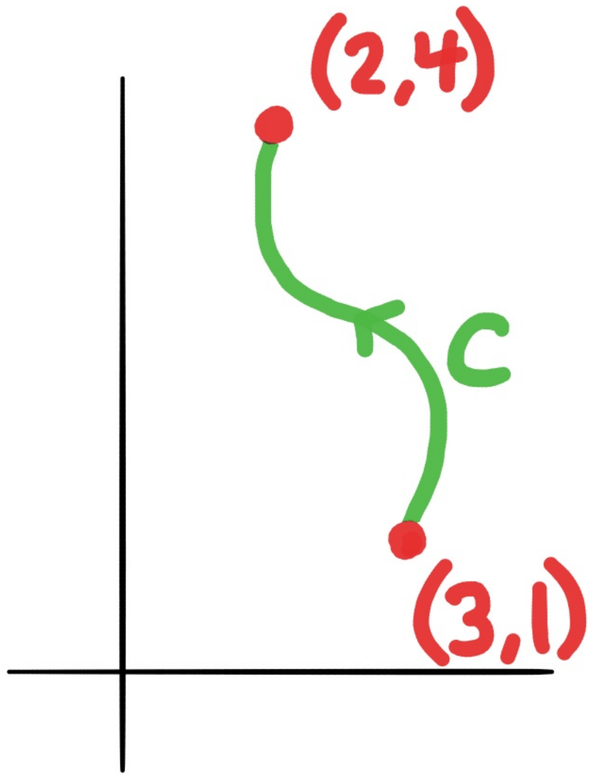


We write $\displaystyle\int_{\vec{a}}^{\vec{b}} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i$ for $\displaystyle\int_C \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i$.

Problem:

Let C be a curve in $\mathbb{R}^2$ with initial point $(3,1)$ and terminal point $(2,4)$. Find

$$\int_C 2xy\,dx + x^2\,dy$$



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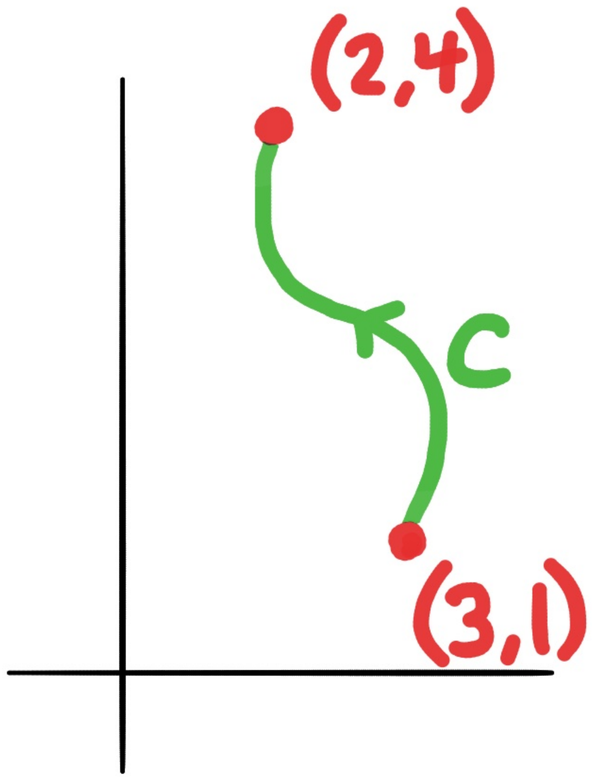

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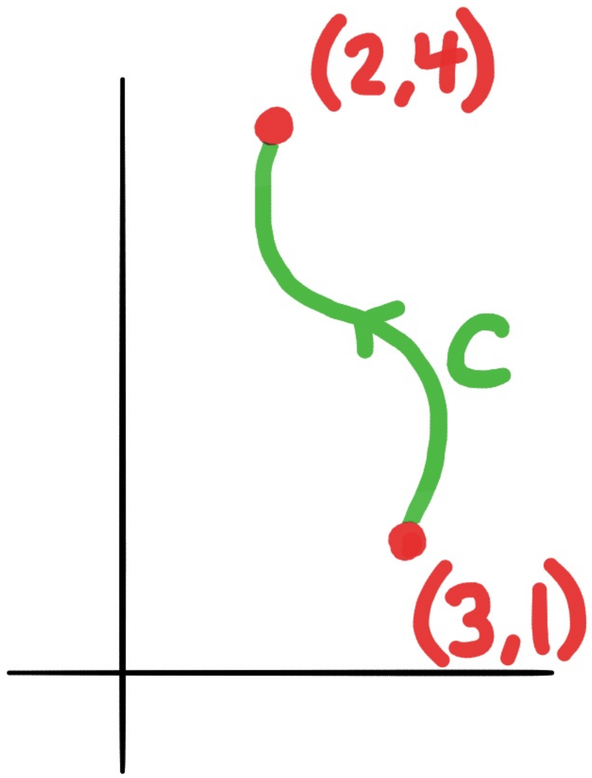

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So $$\int_C 2xy\,dx + x^2\,dy = \Big[x^2 y\Big]_{(3,1)}^{(2,4)} = 2^2 \cdot 4 - 3^2 \cdot 1 = 7.$$

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$$\int_{(0,\pi)}^{(1,0)} e^x \cos y \, dx - e^x \sin y \, dy = \left[e^x \cos y \right]_{(0,\pi)}^{(1,0)}$$

Problem:

Find $\int_{(0,\pi)}^{(1,0)} e^x \cos y \, dx - e^x \sin y \, dy$.

Solution:

Integral notation suggests $(e^x \cos y, -e^x \sin y)$ is conservative.

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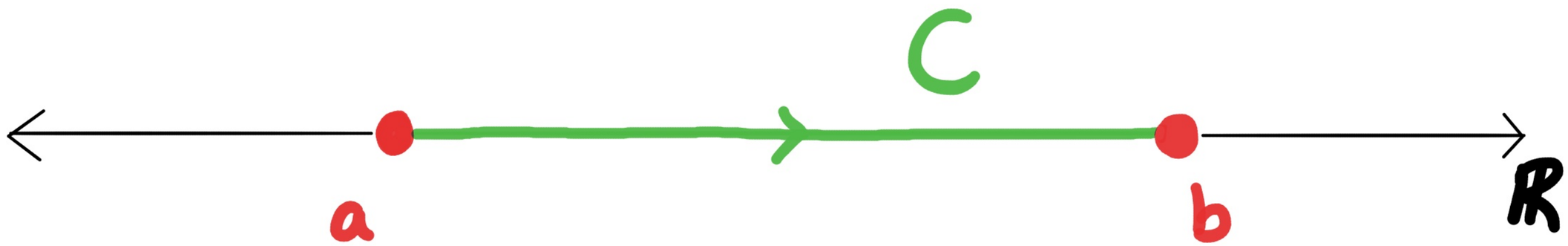
$$= e + 1.$$

$$\left[f(x)\right]_a^b = \int_a^b f'(x)\,dx$$



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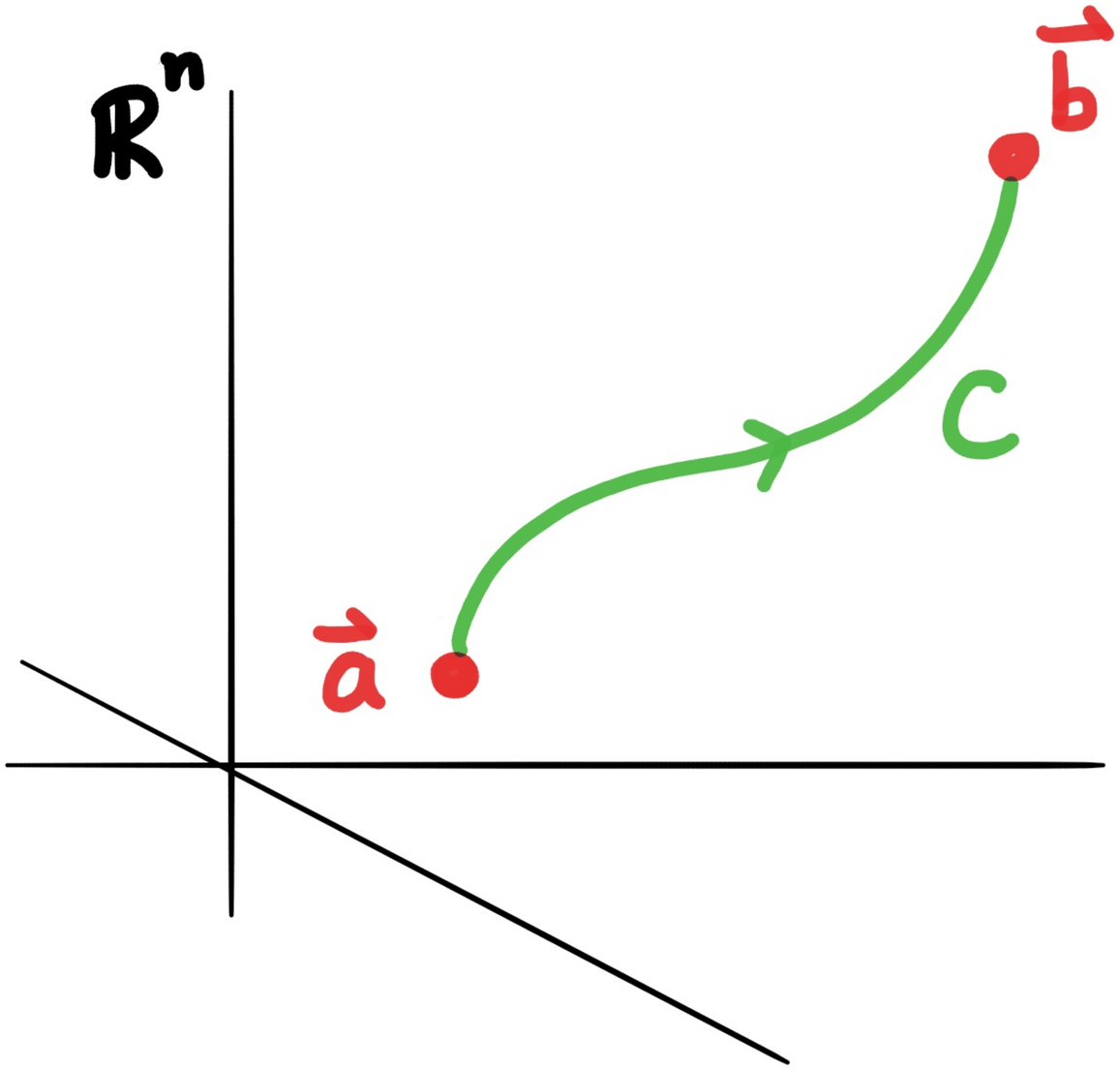


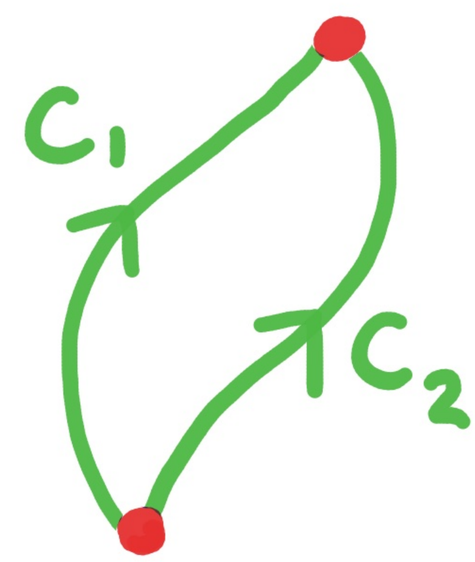
$$[f]_{\vec{a}}^{\vec{b}} = \int_C \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

III Theorem about line integrals, vector fields, and gradients

$\int_C \sum_{i=1}^{n} F_i \, dx_i$ is independent of path if

$$\int_{C_1} \sum_{i=1}^{n} F_i \, dx_i = \int_{C_2} \sum_{i=1}^{n} F_i \, dx_i$$

whenever C_1 and C_2 have the same initial point and the same terminal point.



If C is a simple closed curve, it's common to write $\oint_C \sum_{i=1}^{n} F_i \, dx_i$ instead of $\int_C \sum_{i=1}^{n} F_i \, dx_i$.

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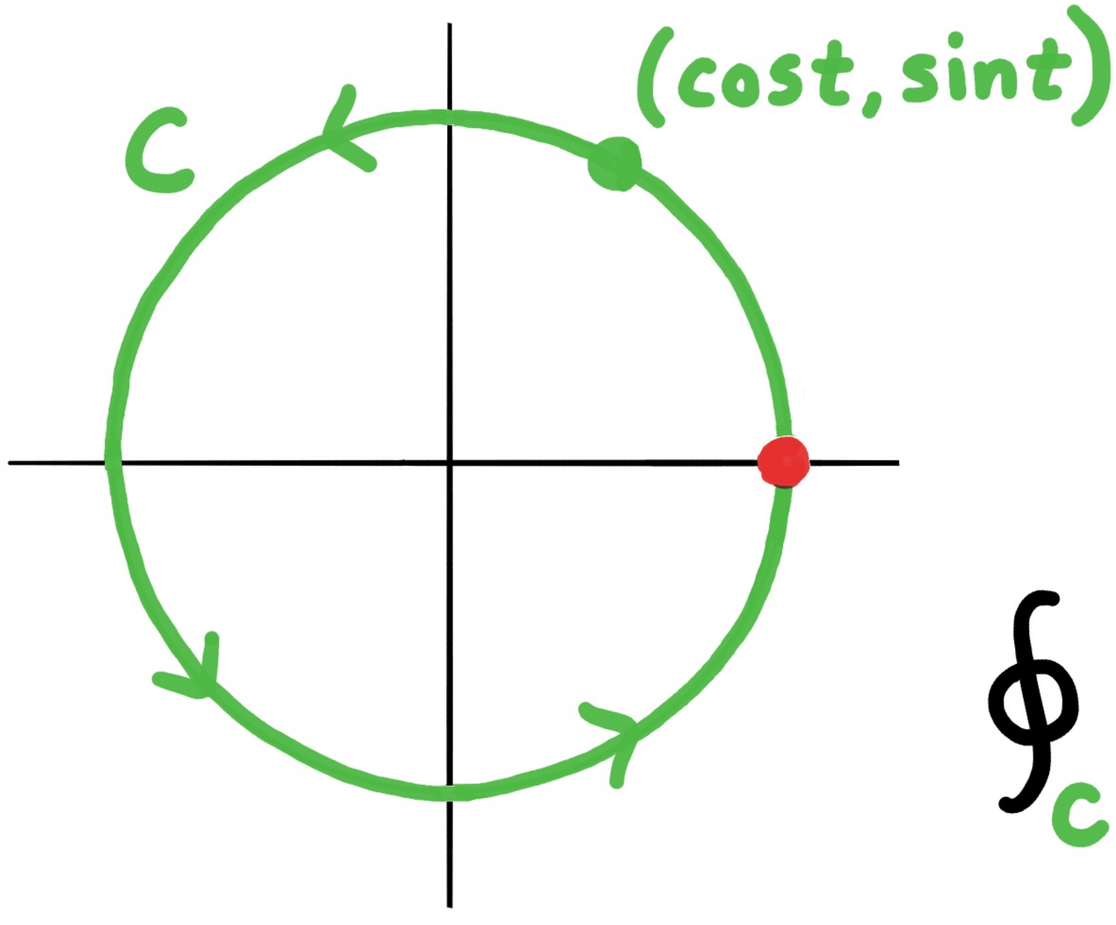
Example:

C param. by
$x = \cos t$,
$y = \sin t$,
$0 \le t \le 2\pi$.



$$\oint_C x^2 dx - xy \, dy$$

Theorem:

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$, $\vec{F} = (F_1, \ldots, F_n)$.

The following are equivalent:

① $\vec{F} = \nabla f$.

② $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$ for $1 \le i, j \le n$.

③ $\oint_C \sum_{i=1}^{n} F_i \, dx_i = 0$ for simple closed curves C.

④ $\int_C \sum_{i=1}^{n} F_i \, dx_i$ is independent of path.

$$\vec{F} = \nabla f$$

1

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4 $\displaystyle\int_C \sum_{i=1}^{n} F_i \, dx_i$ ind. of path

<u>proof:</u> (① ⇒ ②)

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Suppose $\vec{F} = \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right).$

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Note that $F_i = \frac{\partial f}{\partial x_i}$.

<u>proof:</u> (① ⇒ ②)

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$\boxed{\text{END}}$

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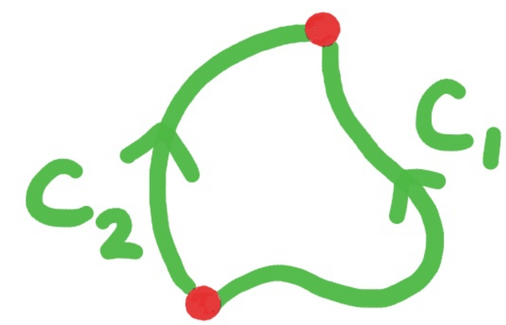
Suppose C_1 and C_2 have the same initial and terminal points.



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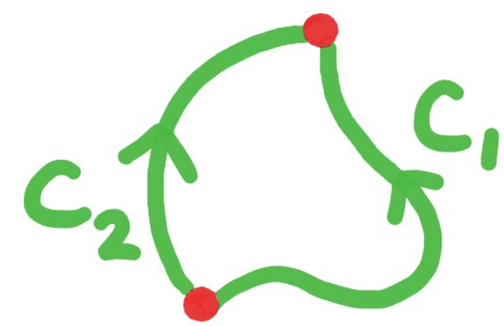

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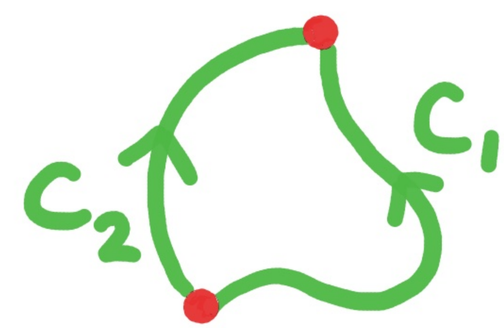

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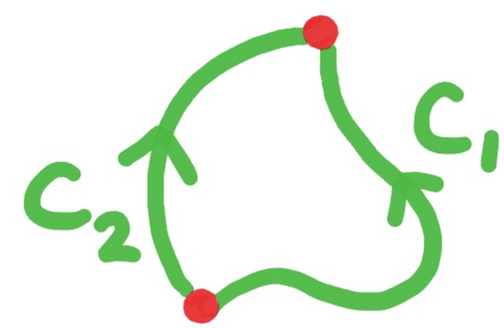

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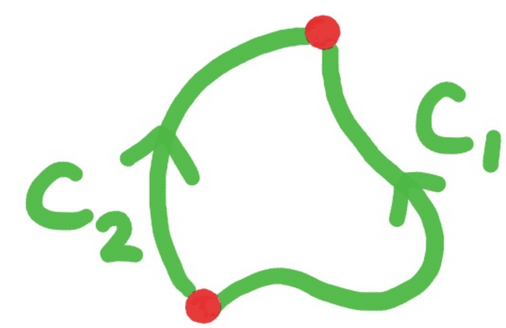

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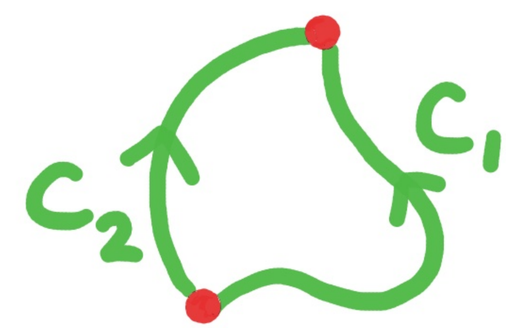

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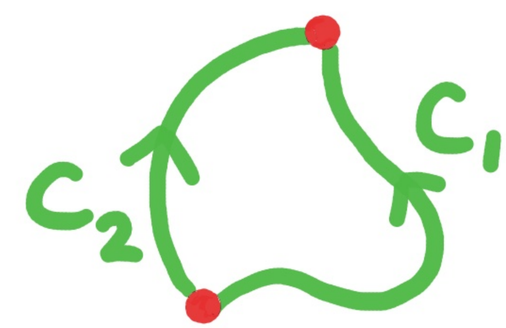

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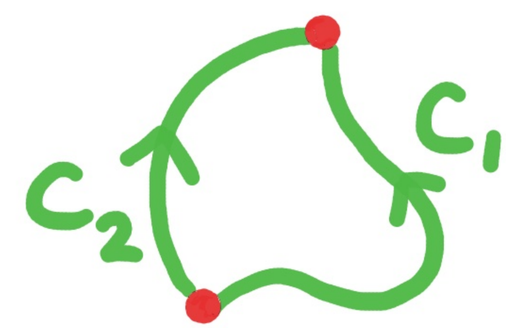

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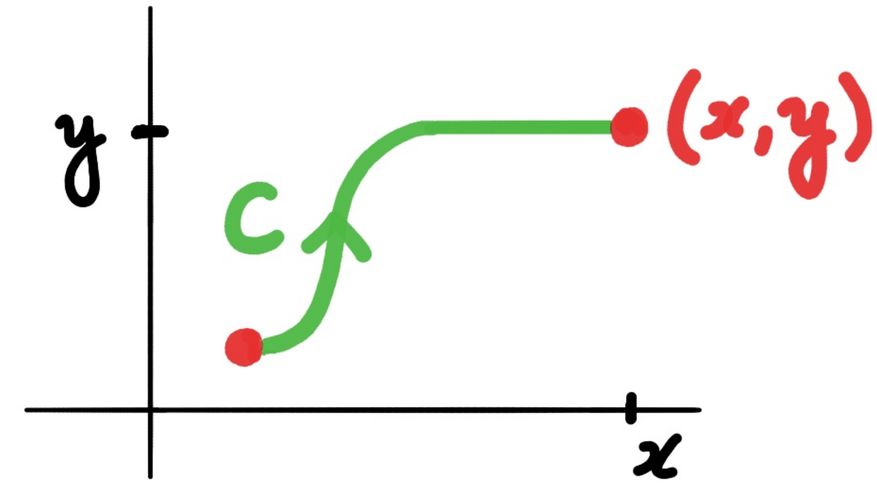
Lemma:

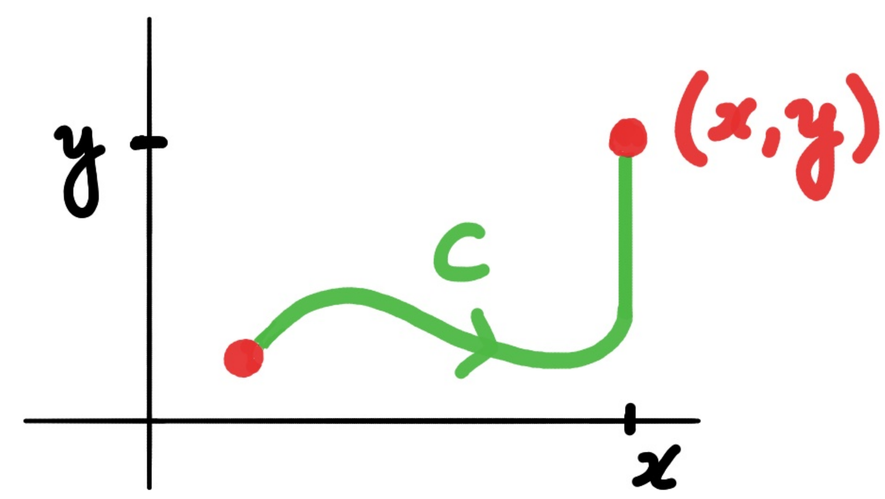
If C is a curve with a horizontal end, and terminal point (x,y), then



$$\frac{\partial}{\partial x}\left(\int_C f(x,y)\,dx + g(x,y)\,dy\right) = f(x,y).$$

Lemma:

If C is a curve with a vertical end, and terminal point (x,y), then



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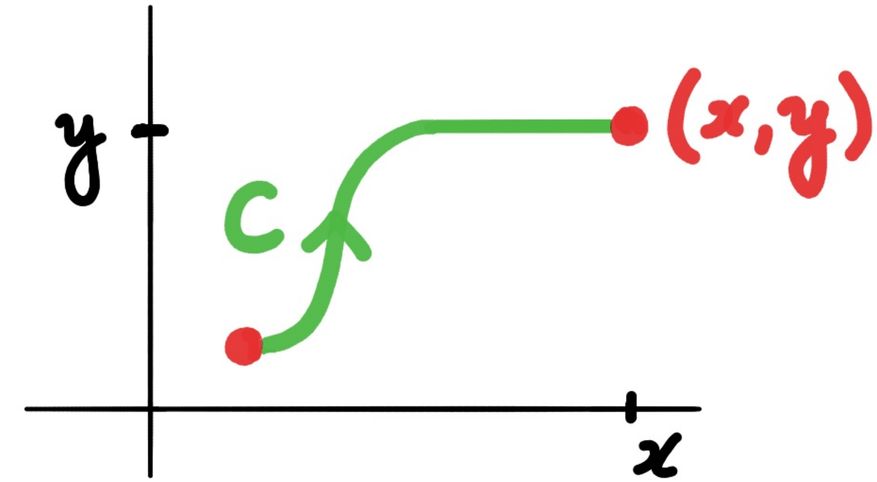

$$\frac{\partial}{\partial x}\left(\int_C F_1(x,y)\,dx + F_2(x,y)\,dy\right) = F_1(x,y).$$

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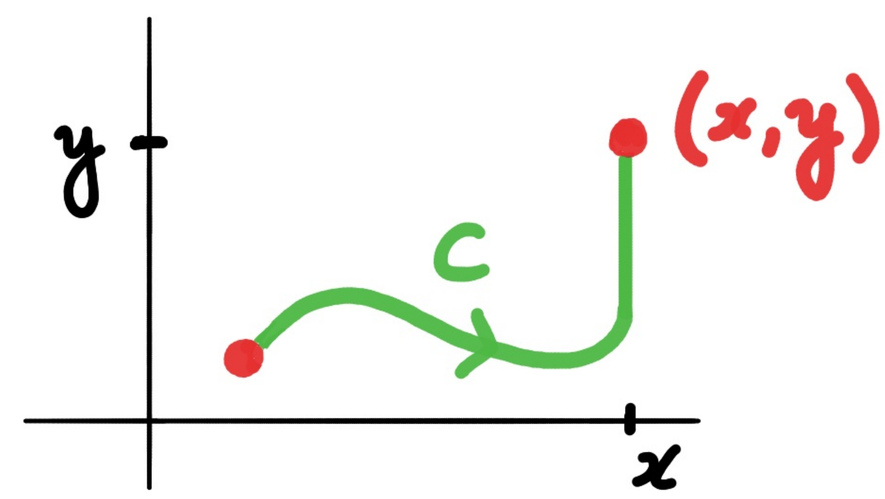

$$\frac{\partial}{\partial y}\left(\int_C F_1(x,y)\,dx + F_2(x,y)\,dy\right) = F_2(x,y).$$

Lemma:

If C is a curve with a horizontal end, and terminal point (x,y), then

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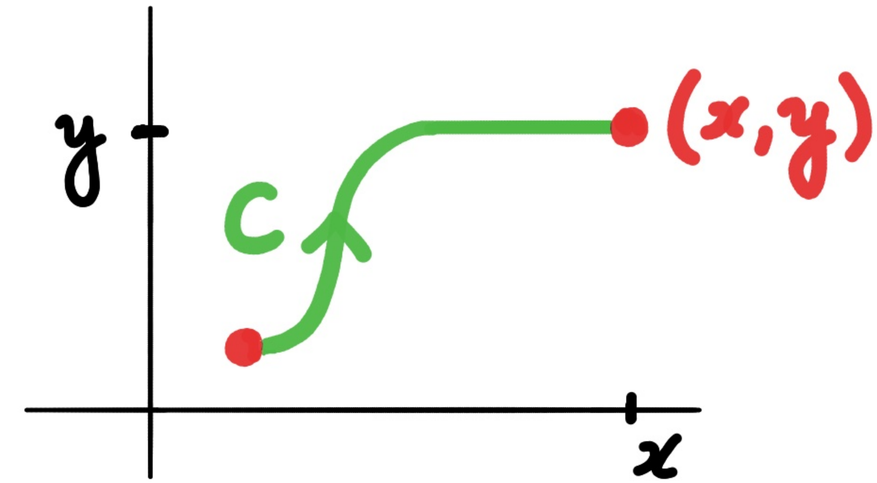
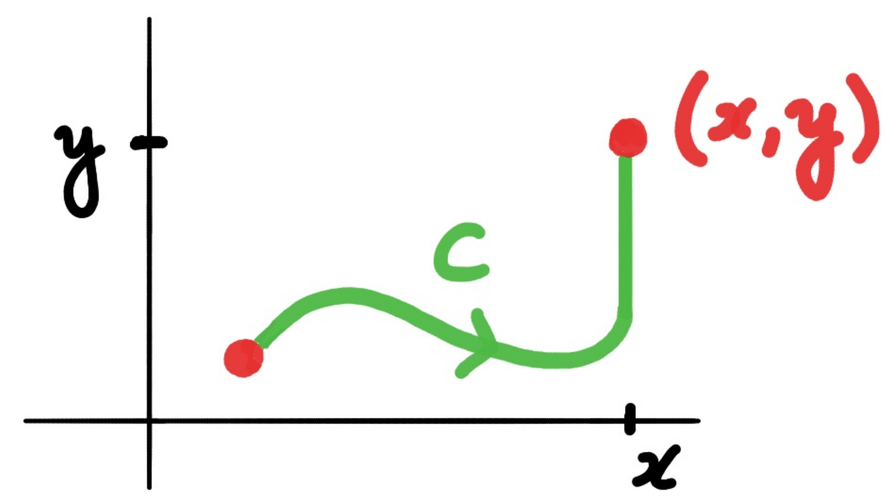


Lemma:

If C is a curve with a vertical end, and terminal point (x,y), then

$$\frac{\partial}{\partial y}\left(\int_C F_1\,dx + F_2\,dy\right) = F_2.$$



<u>proof:</u> (④ ⇒ ①) (For $n=2$.)

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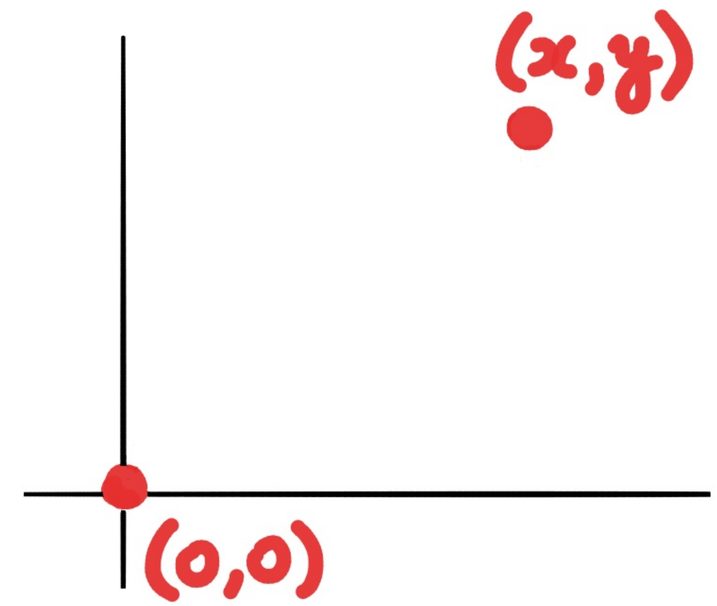
proof: (④ ⇒ ①) (For $n=2$.)

Suppose $\int_C F_1\,dx + F_2\,dy$ is independent of path.

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be $f(x,y) = \int_{(0,0)}^{(x,y)} F_1\,dx + F_2\,dy$.



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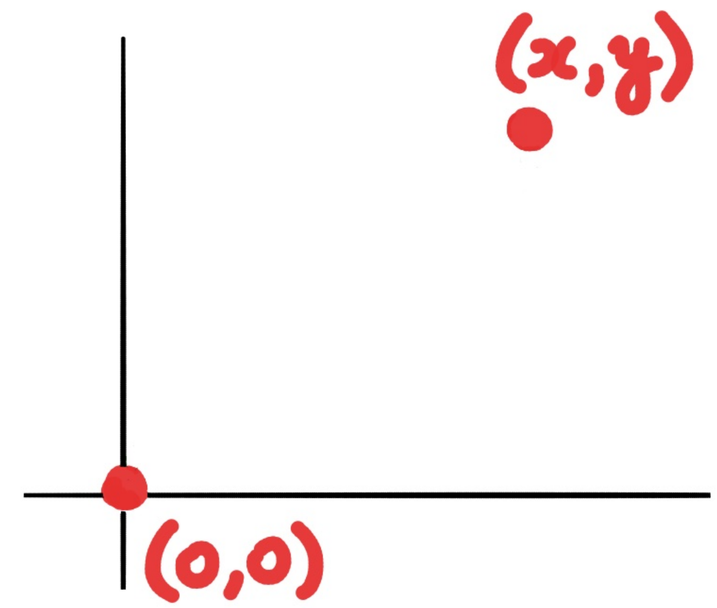

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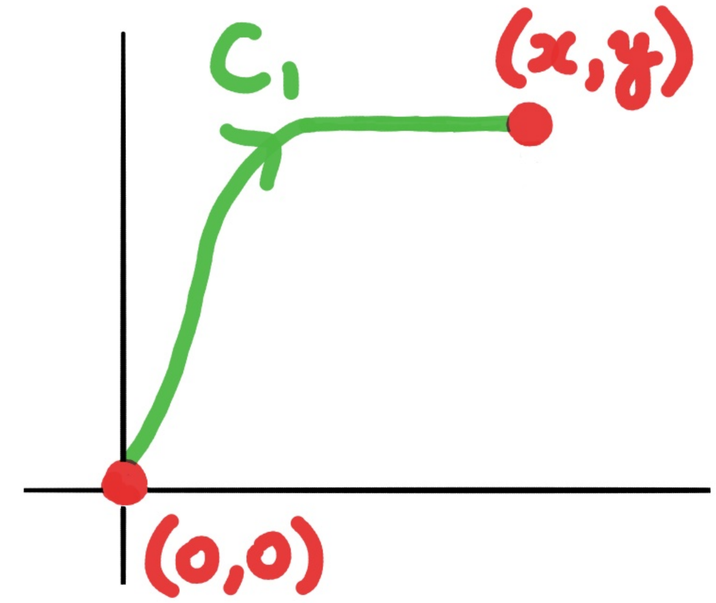

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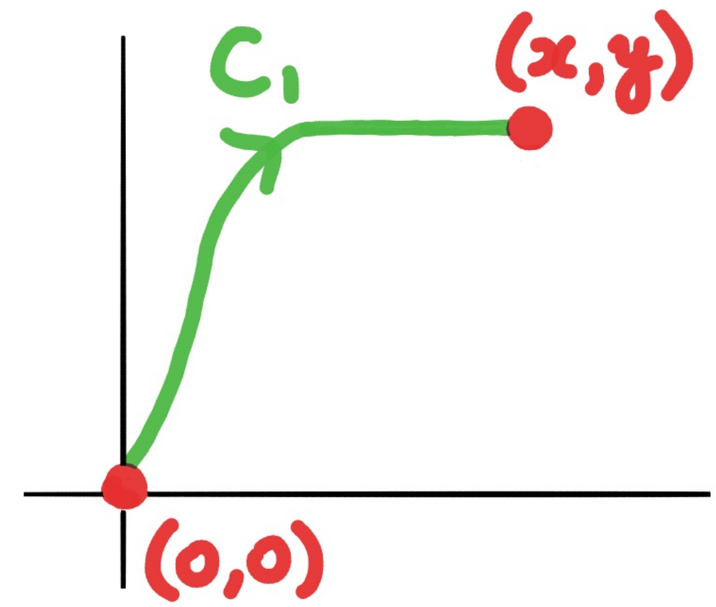

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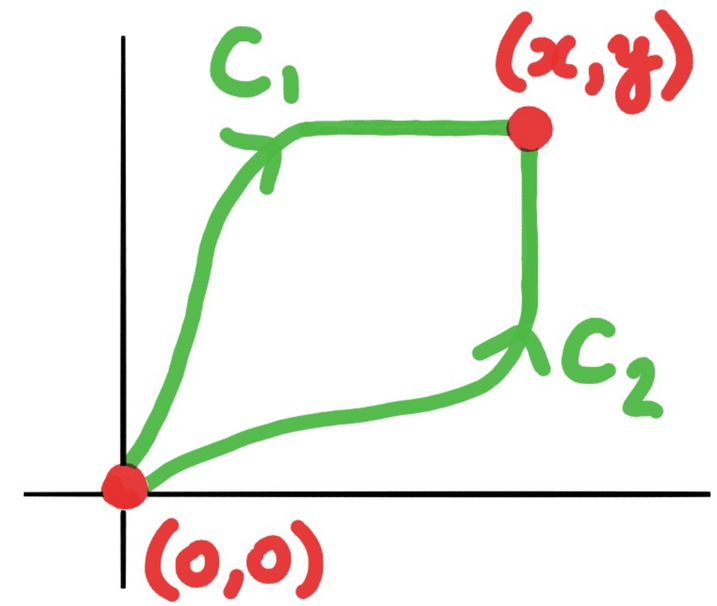

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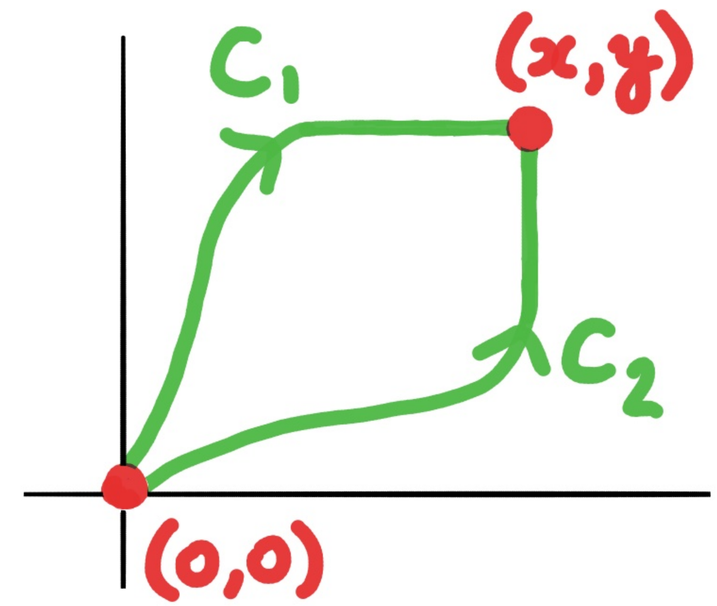

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$$= F_1(x,y).$$

Similarly, using C_2,

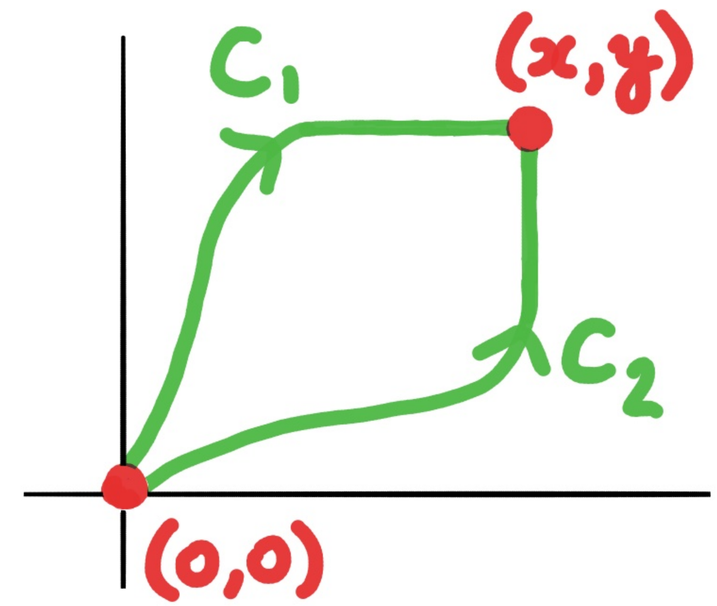
$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

proof: (④ ⇒ ①) (For $n=2$.)

Suppose $\int_C F_1\,dx + F_2\,dy$ is independent of path.

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be $f(x,y) = \int_{(0,0)}^{(x,y)} F_1\,dx + F_2\,dy$.

Note,



$$\frac{\partial}{\partial x}\Big(f(x,y)\Big) = \frac{\partial}{\partial x}\left(\int_{(0,0)}^{(x,y)} F_1\,dx + F_2\,dy\right)$$

$$= \frac{\partial}{\partial x}\left(\int_{C_1} F_1\,dx + F_2\,dy\right)$$

$$= F_1(x,y).$$

Similarly, using C_2,

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (F_1, \quad)$$

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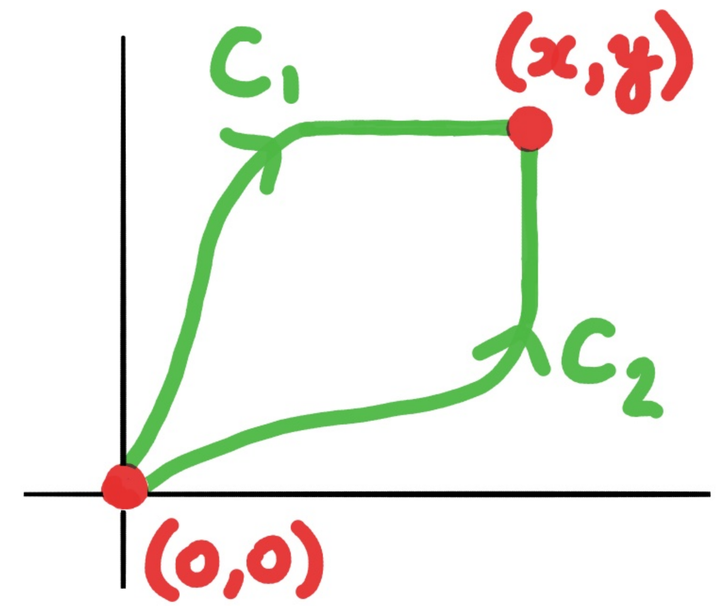

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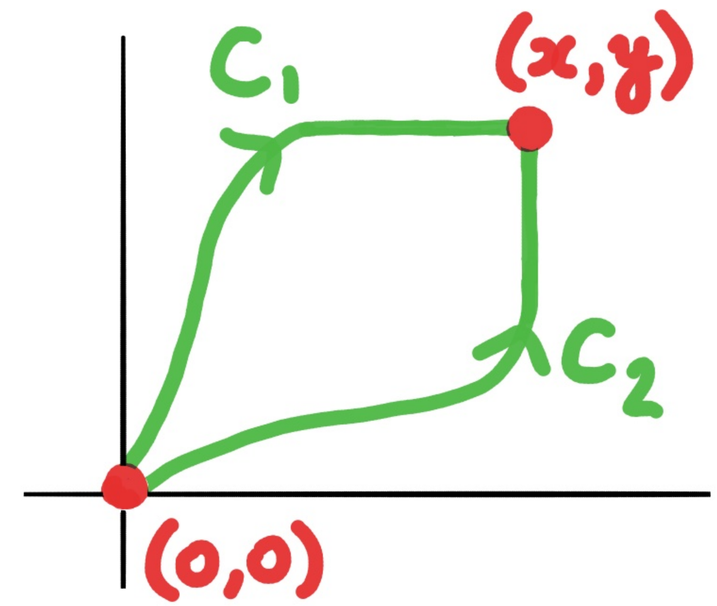

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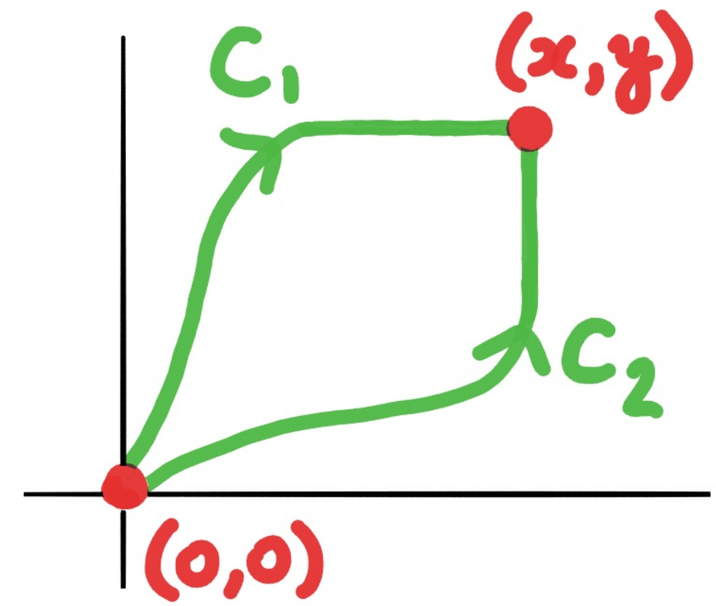

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$\boxed{\text{END}}$

$$\vec{F} = \nabla f$$

1

$$\int_C \sum_{i=1}^{n} F_i \, dx_i$$

ind. of path

4

2

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

3

$$\oint_C \sum_{i=1}^{n} F_i \, dx_i = 0$$

Theorem:

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$, $\vec{F} = (F_1, \ldots, F_n)$.

The following are equivalent:

① $\vec{F} = \nabla f$.

② $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$ for $1 \le i, j \le n$.

③ $\oint_C \sum_{i=1}^{n} F_i \, dx_i = 0$ for simple closed curves C.

④ $\int_C \sum_{i=1}^{n} F_i \, dx_i$ is independent of path.