Three

Ⅰ Vectors in a plane

Ⅱ Dot product

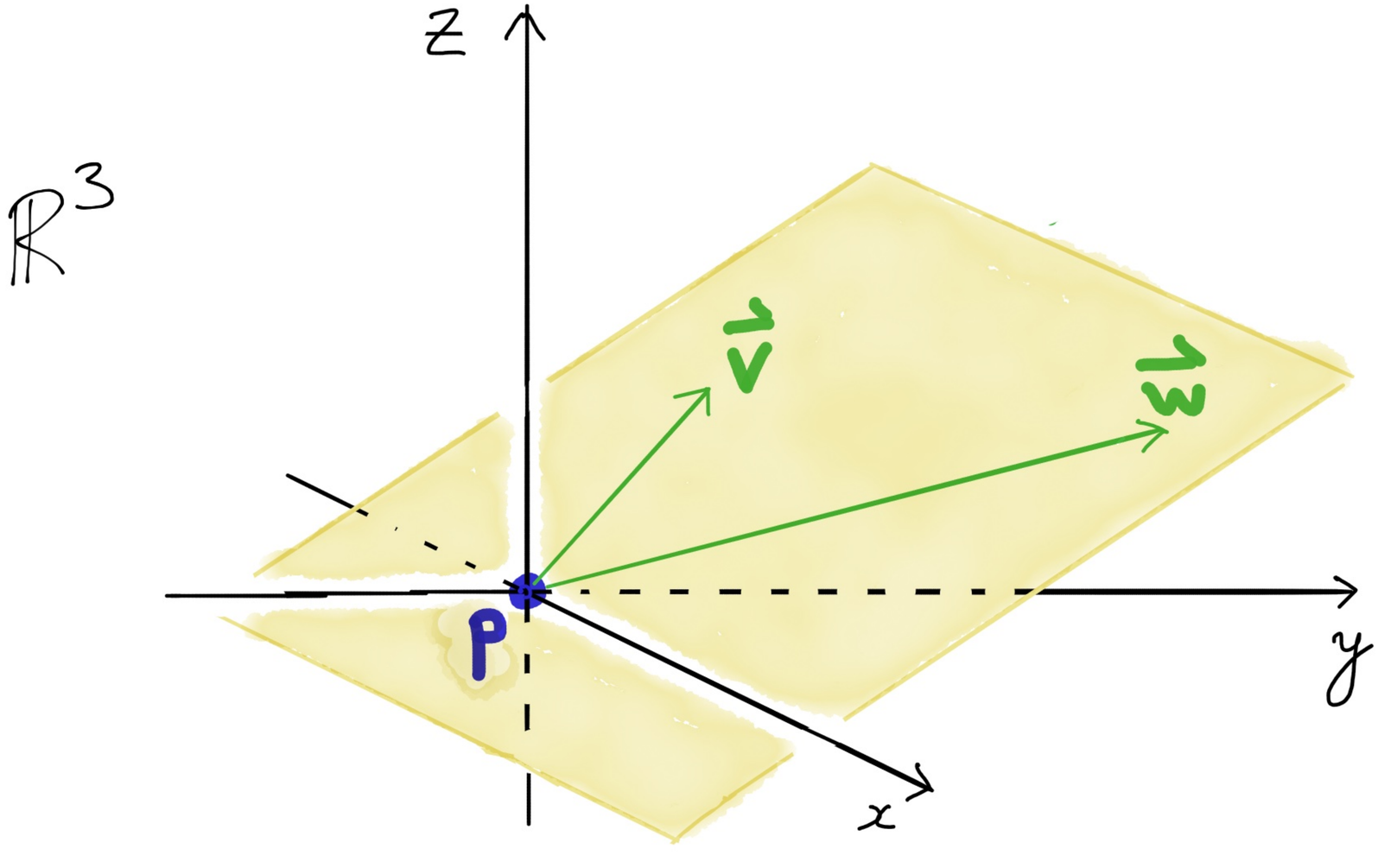
Ⅲ Geometry of dot product

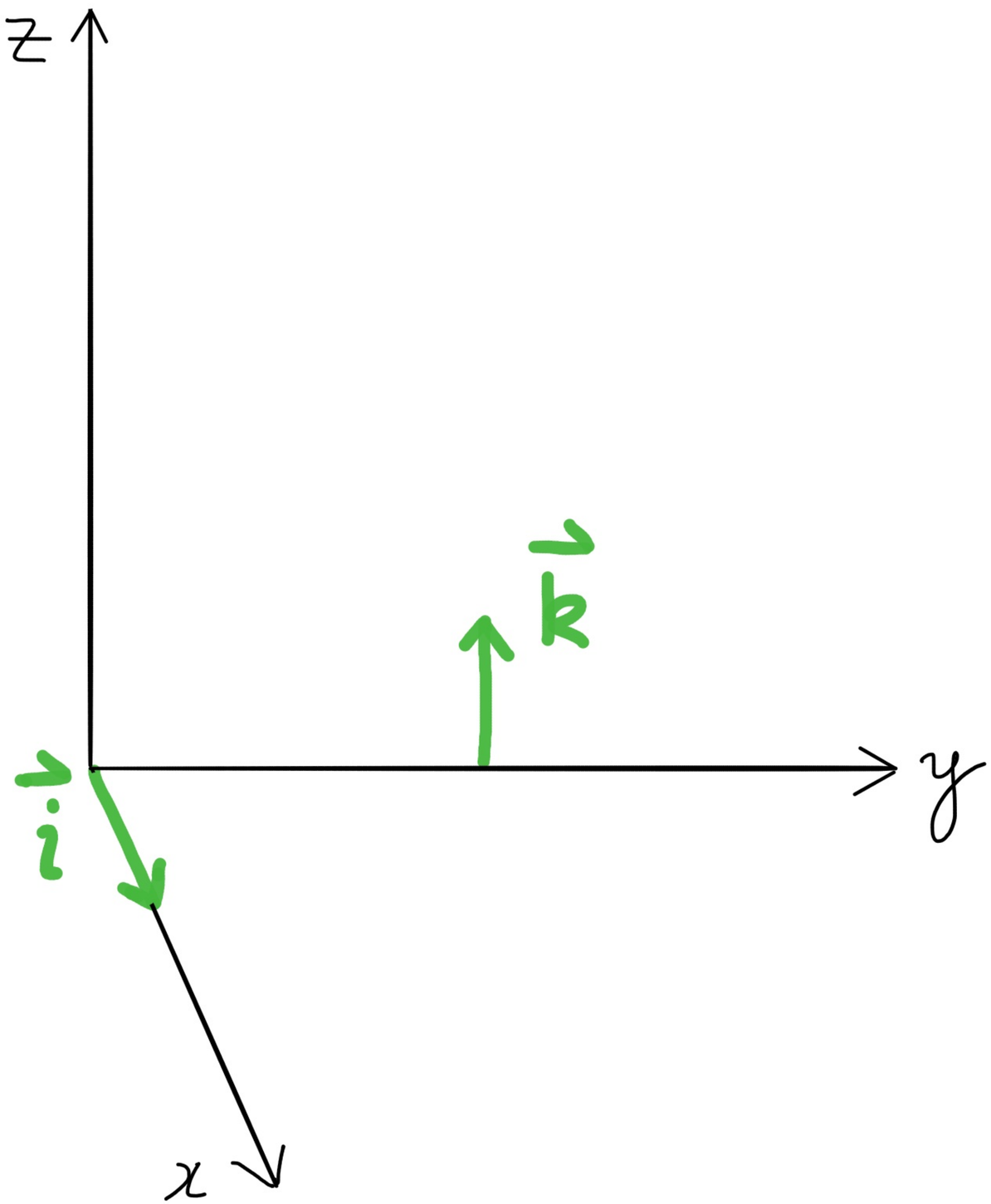
Ⅳ Equations of planes

Ⅴ Column vectors

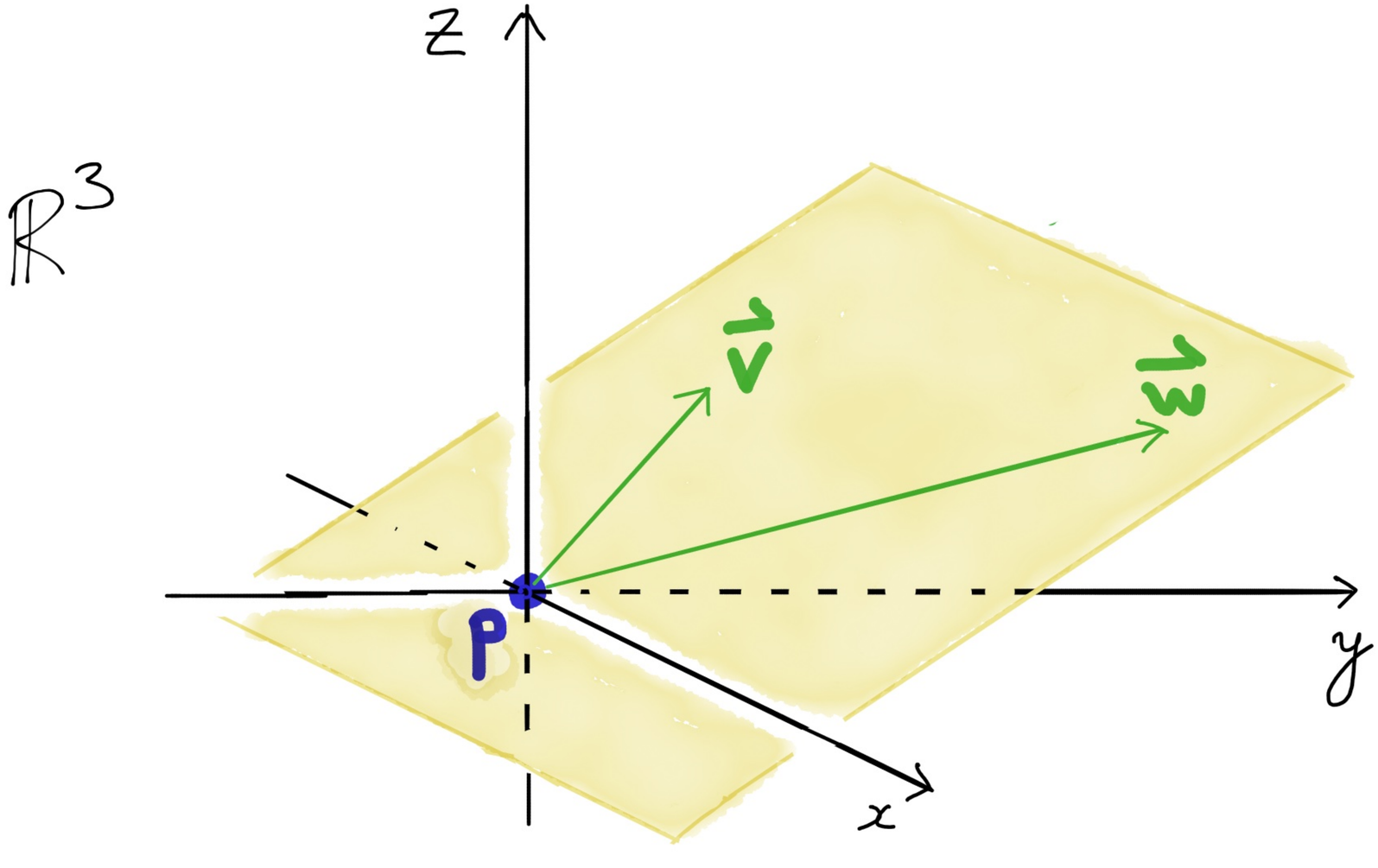
Ⅰ Vectors in a plane

If $\vec{v}, \vec{w} \in \mathbb{R}^n$ are based at the same point, then $\vec{v}$ and $\vec{w}$ are contained in a plane.

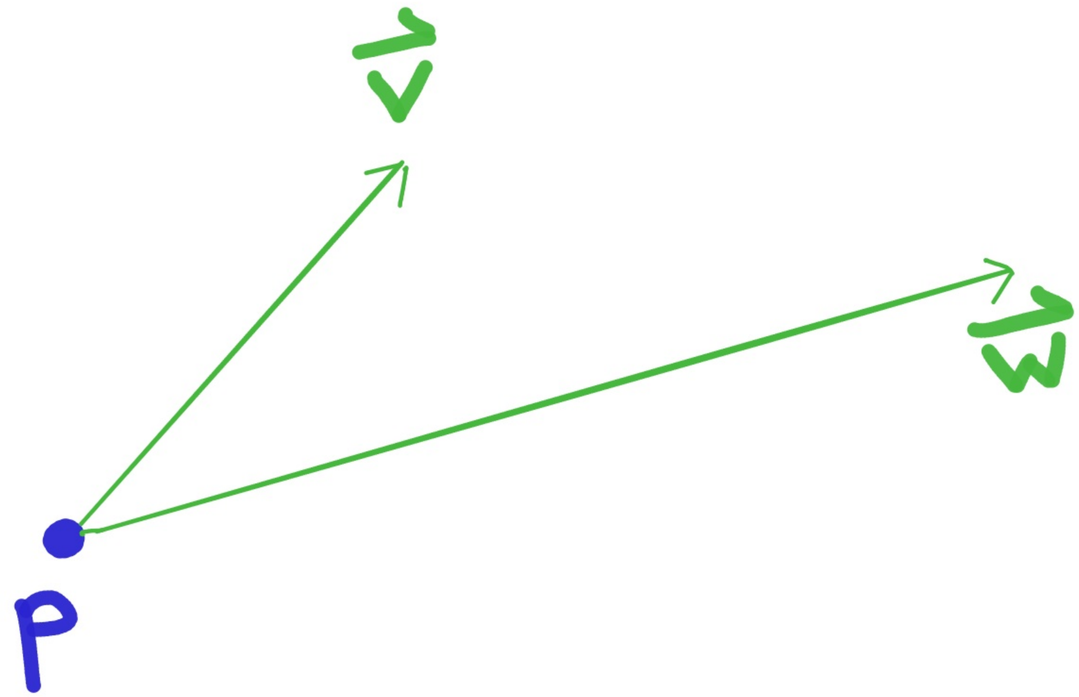



z
k
y
i
x

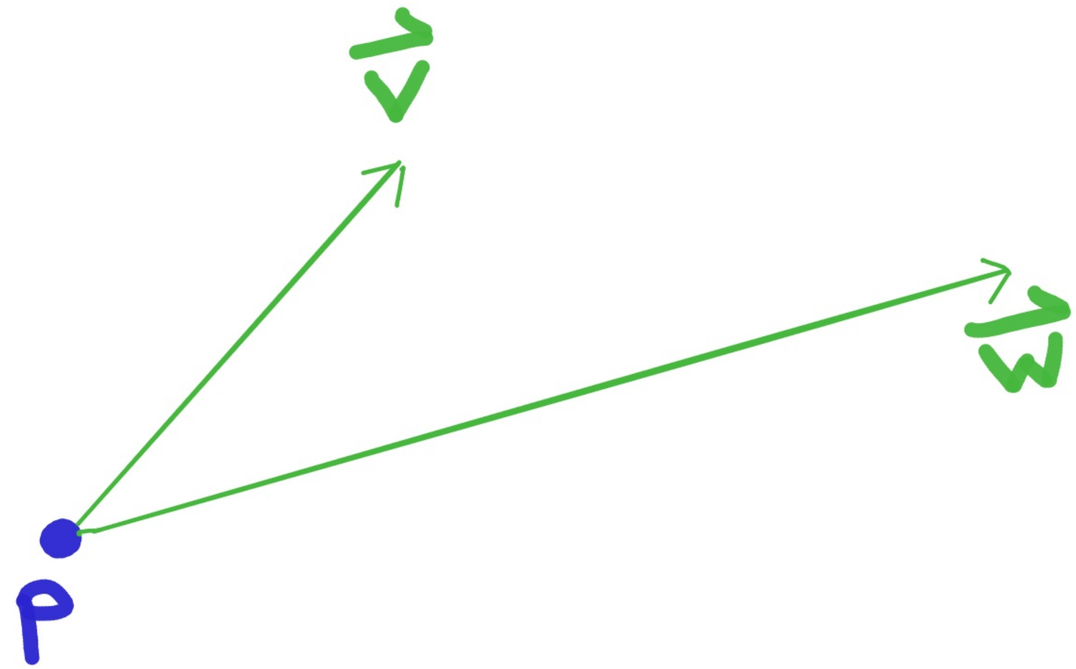
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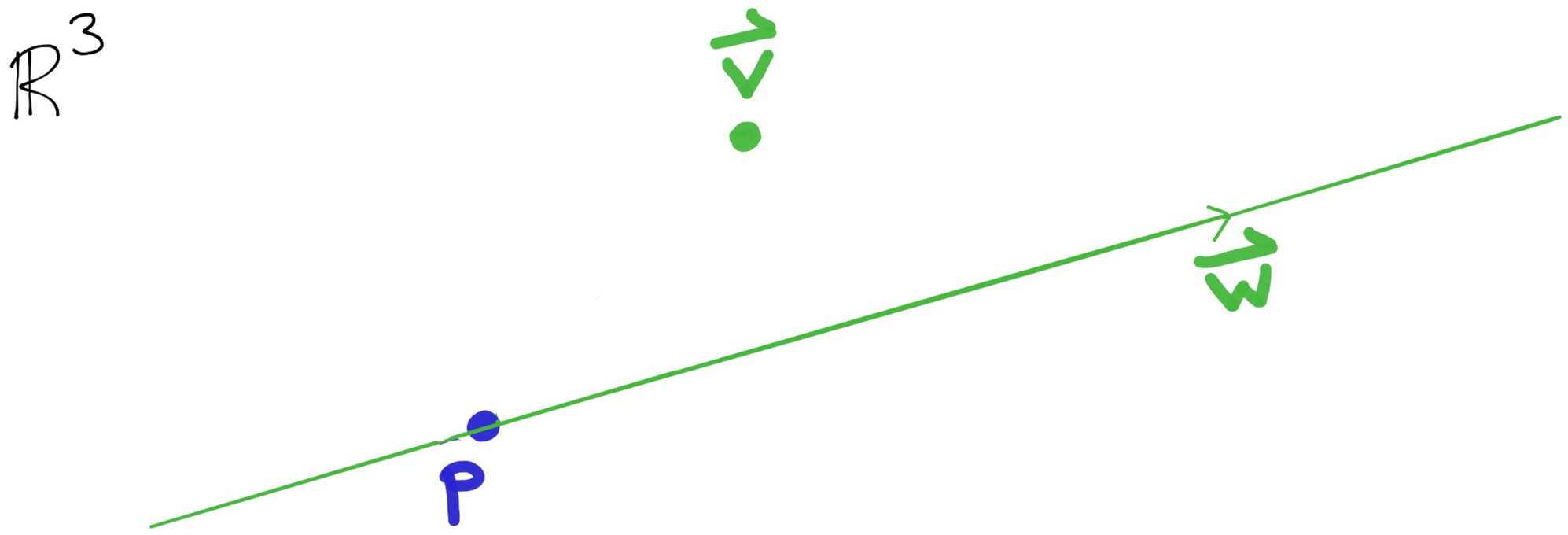
$\mathbb{R}^3$



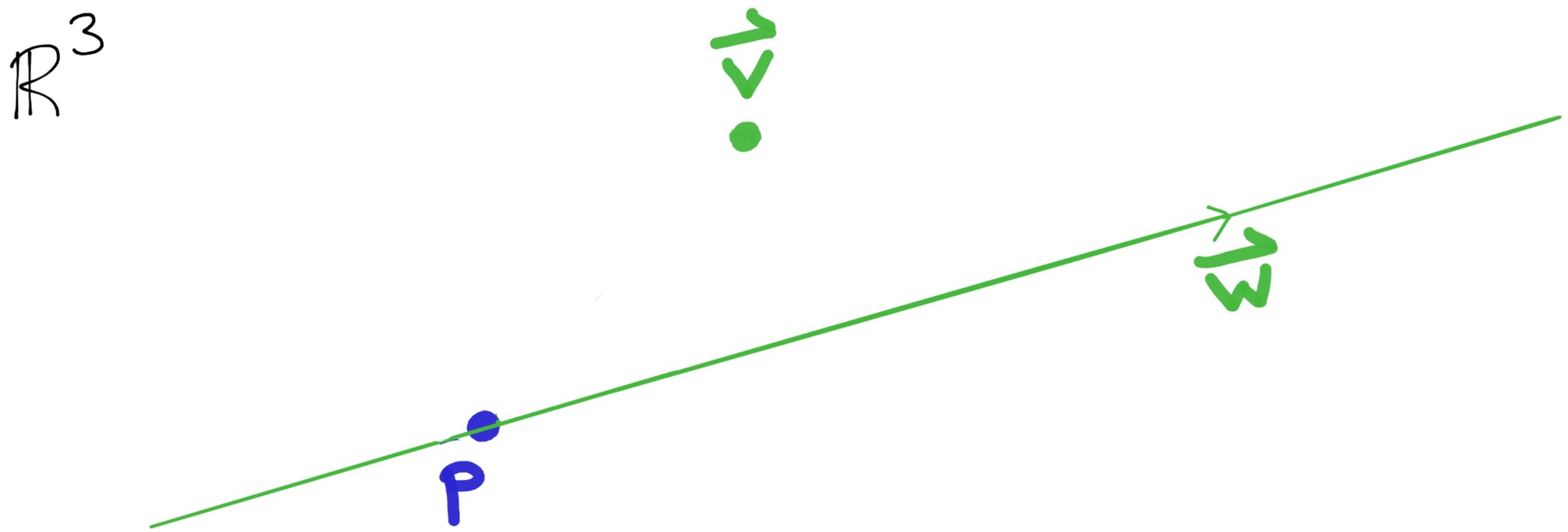
$\mathbb{R}^3$



① Extend $\vec{w}$ to a line. Make $\vec{v}$ a point.

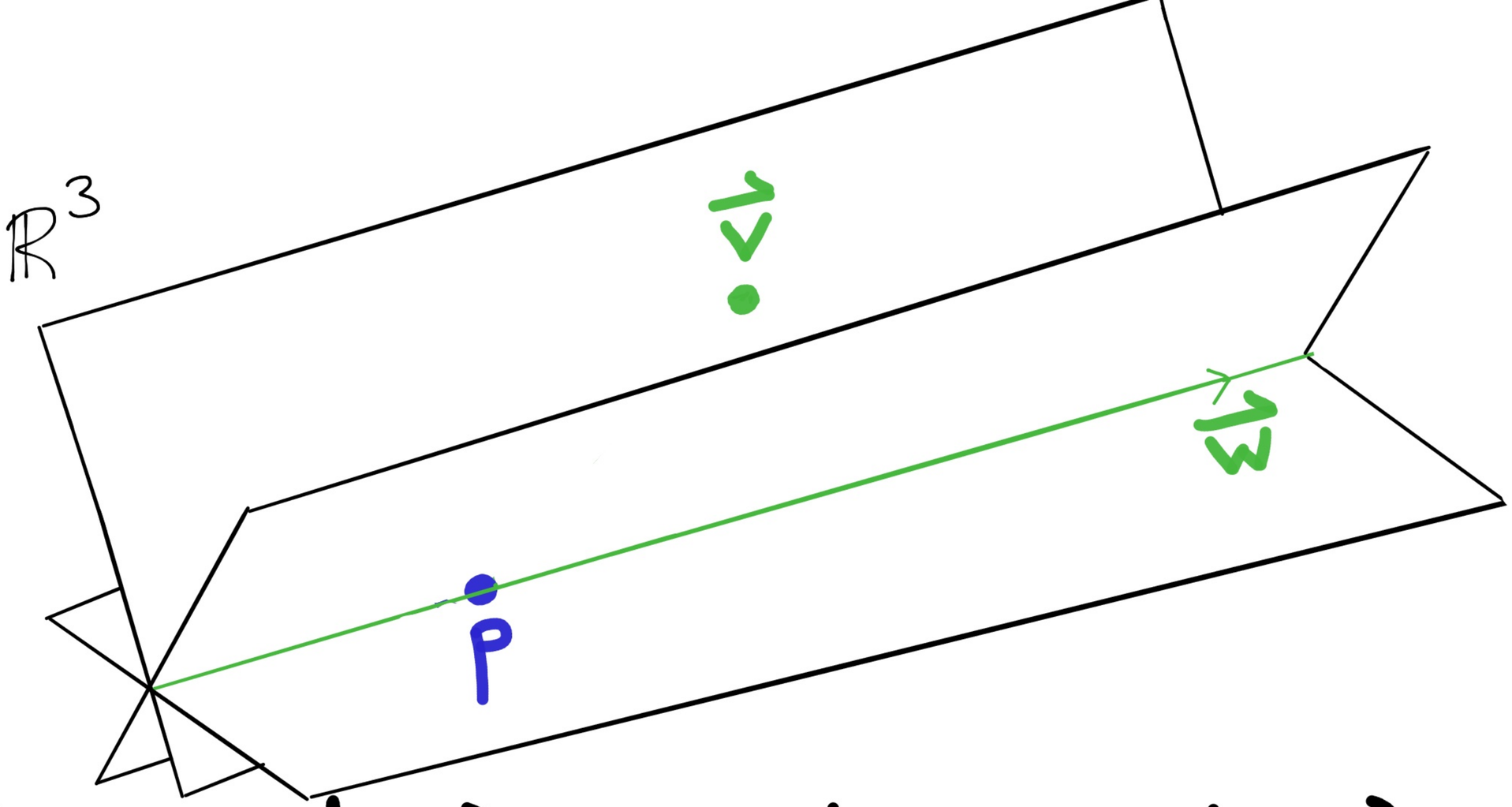


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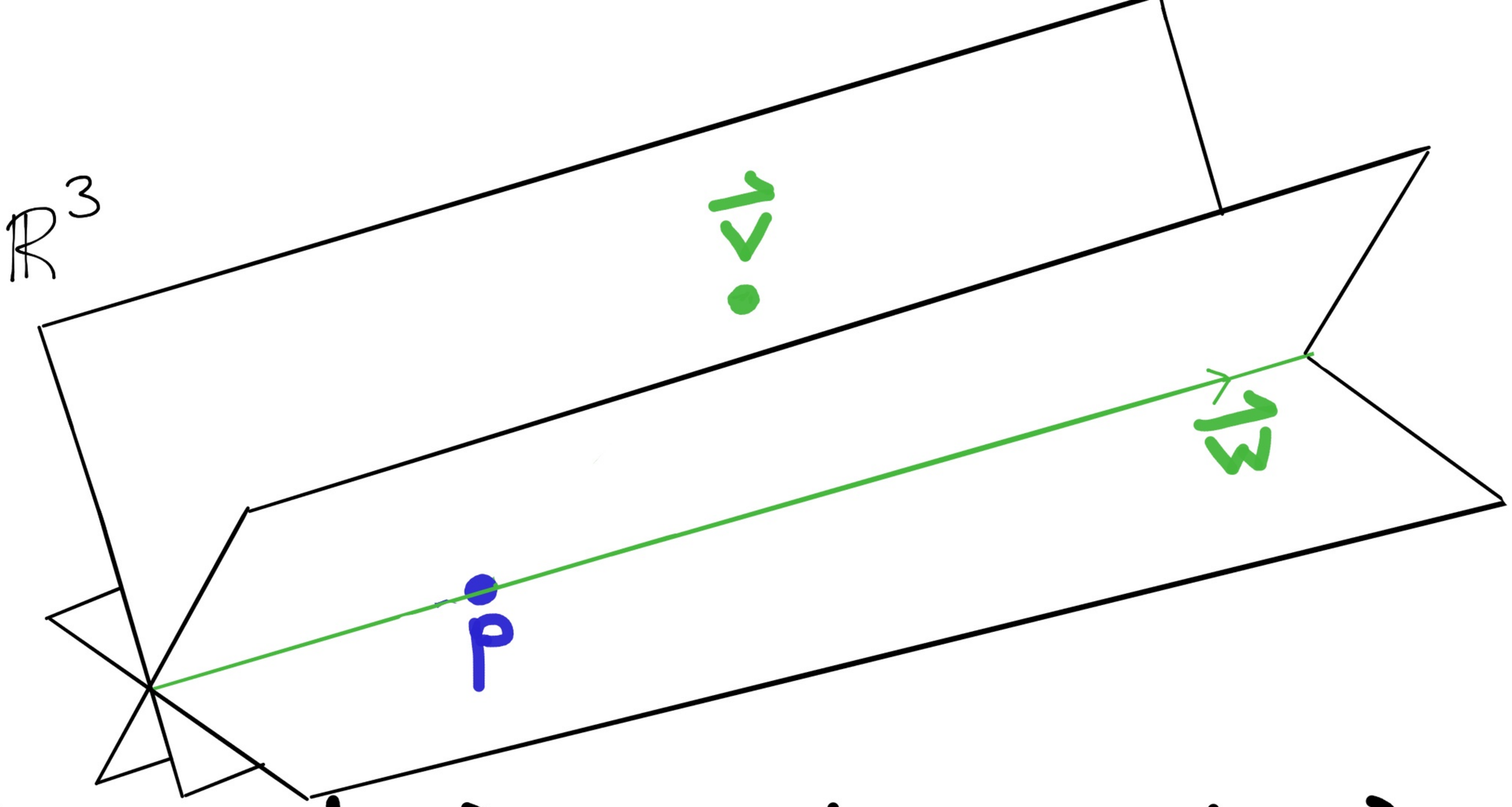
① Extend $\vec{w}$ to a line. Make $\vec{v}$ a point.

② There are a circle's worth of planes containing the line.



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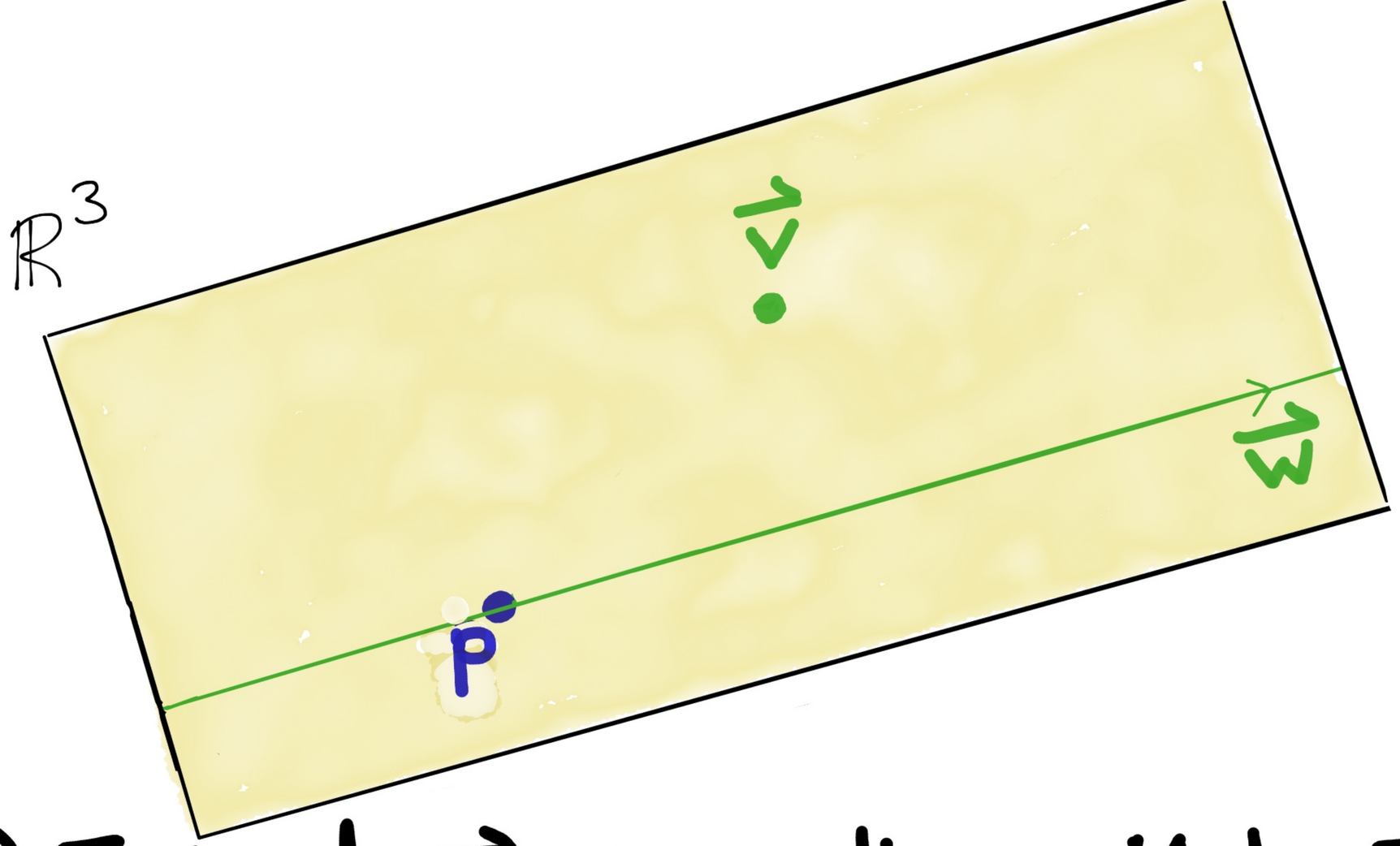
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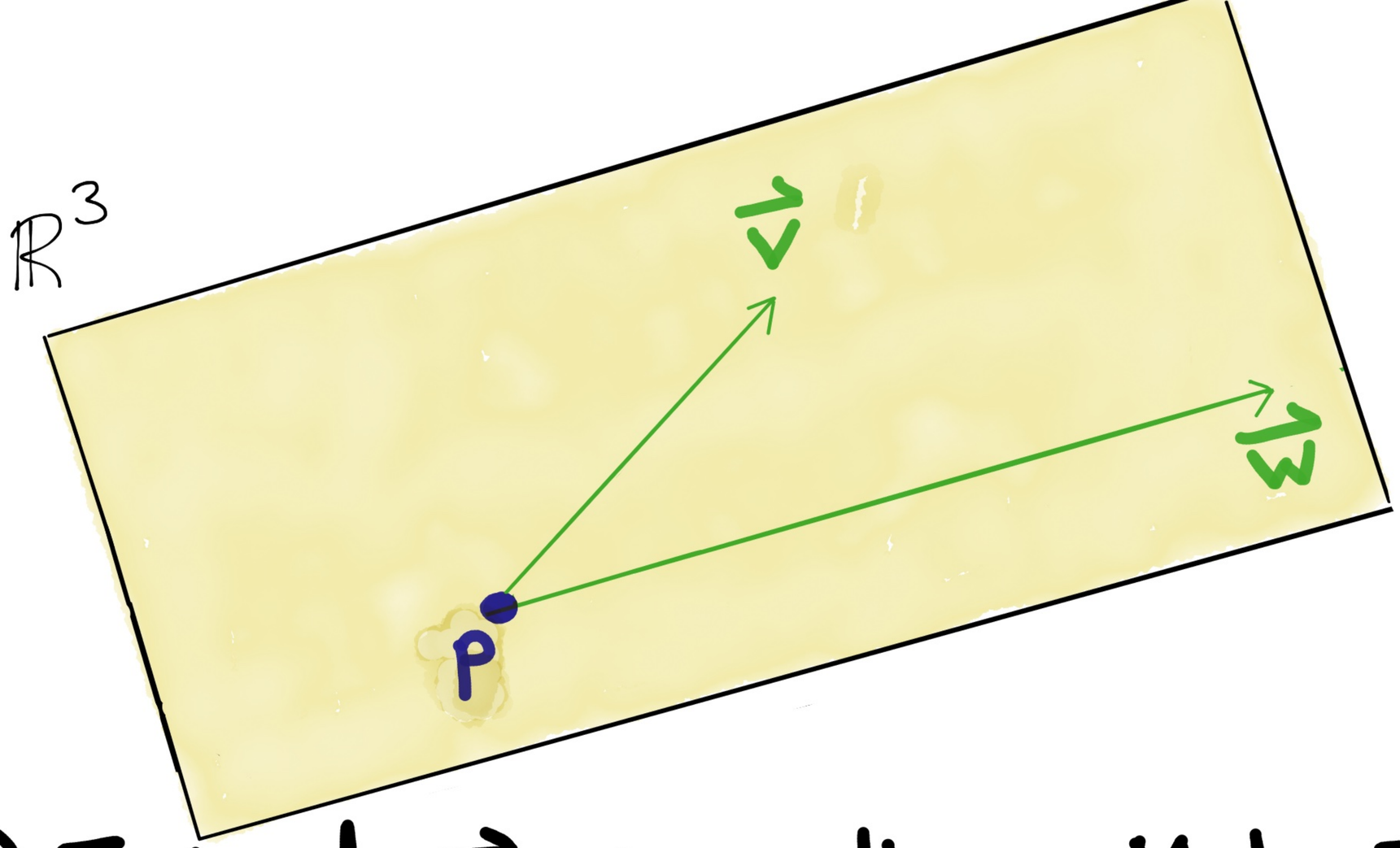
③ One of the planes contains $\vec{v}$.



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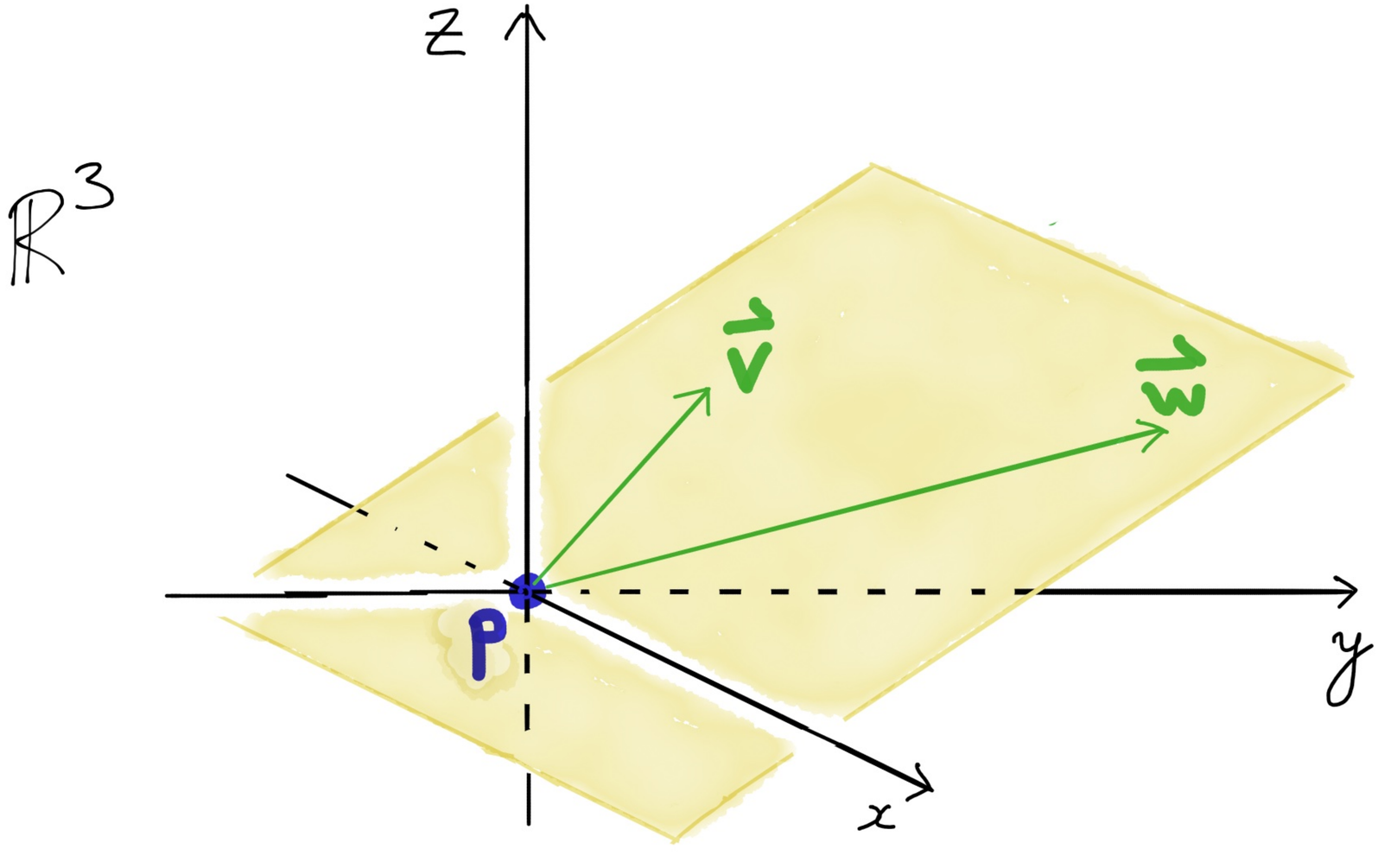


① Extend $\vec{w}$ to a line. Make $\vec{v}$ a point.

② There are a circle's worth of planes containing the line.

③ One of the planes contains $\vec{v}$.

If $\vec{v}, \vec{w} \in \mathbb{R}^n$ are based at the same point, then $\vec{v}$ and $\vec{w}$ are contained in a plane.



② Dot product

Dot product

$$\langle a, b, c \rangle \cdot \langle x, y, z \rangle = ax + by + cz$$

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$$(a\vec{i} + b\vec{j} + c\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k}) = ax + by + cz$$

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$$(a\vec{i} + b\vec{j}) \cdot (x\vec{i} + y\vec{j}) = ax + by$$

$\langle 1, 3, -5 \rangle \cdot \langle 2, 0, 1 \rangle$

$\langle -2, 7 \rangle \cdot \langle 3, -2 \rangle$

$$\langle 1,3,-5\rangle \cdot \langle 2,0,1\rangle = 1\cdot 2 + 3\cdot 0 + (-5)\cdot 1$$

$$= 2 + 0 - 5$$

$$= -3$$

$$\langle -2,7\rangle \cdot \langle 3,-2\rangle = (-2)\cdot 3 + 7\cdot(-2)$$

$$= -6 - 14$$

$$= -20$$

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If $\vec{v} = \langle 1,3,-5\rangle$ and $\vec{w} = \langle 2,0,1\rangle$, then $\vec{v}\cdot\vec{w} = -3$.

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---

dot product commutes !!

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

Lemma: If $\vec{v} \in \mathbb{R}^n$, $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$.

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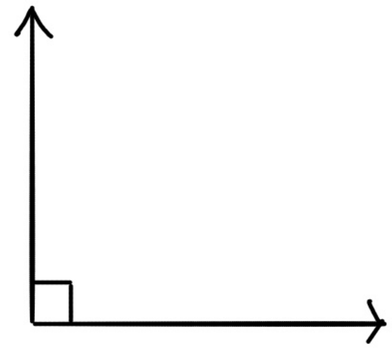
$$= x^2+y^2+z^2$$

$$= \langle x, y, z \rangle \cdot \langle x, y, z \rangle$$

$$= \vec{v} \cdot \vec{v}$$

END

III Geometry of dot product



perpendicular

$\vec{v} \perp \vec{w}$

orthogonal

orthonormal

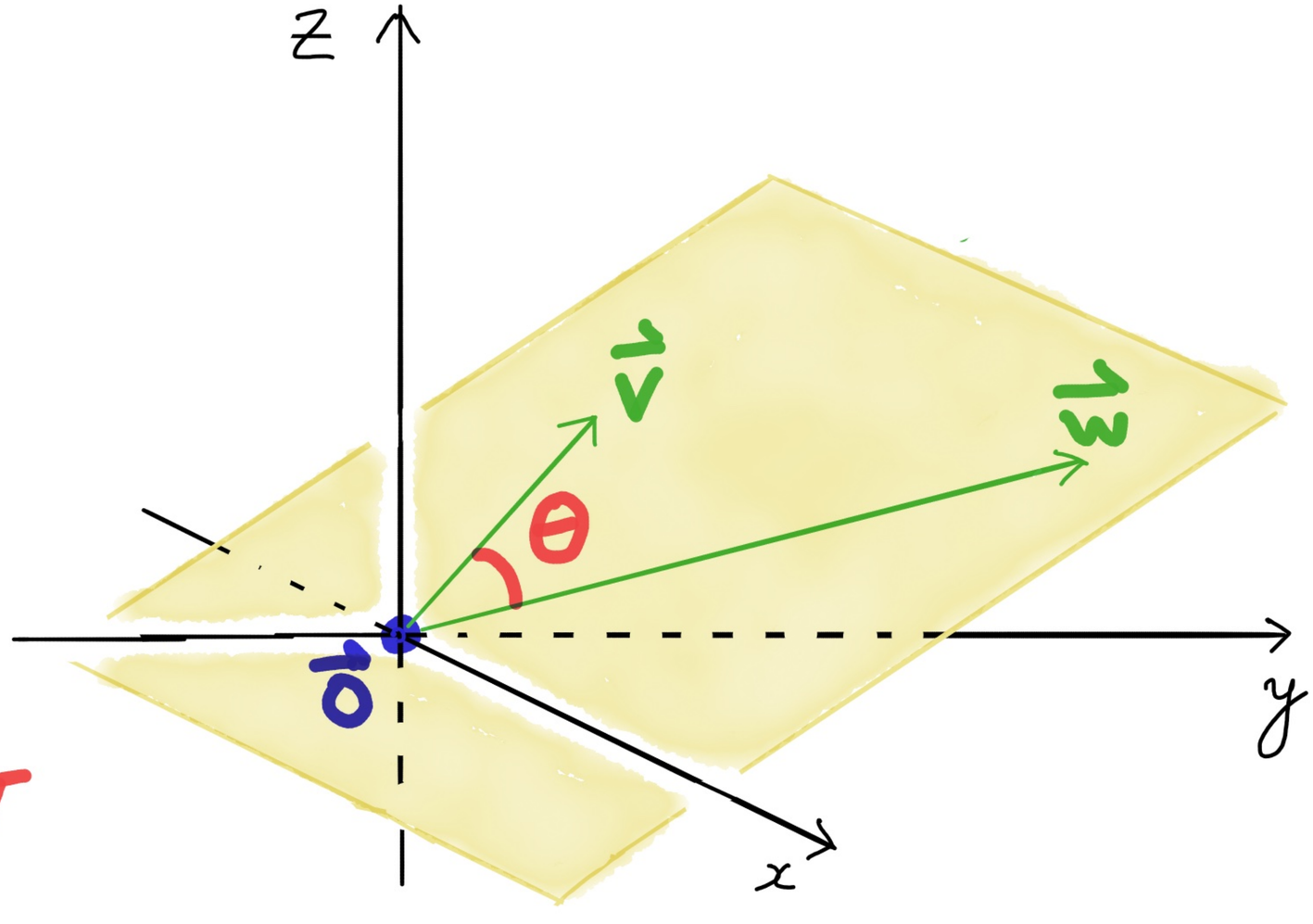
normal

90°

$\pi/2$

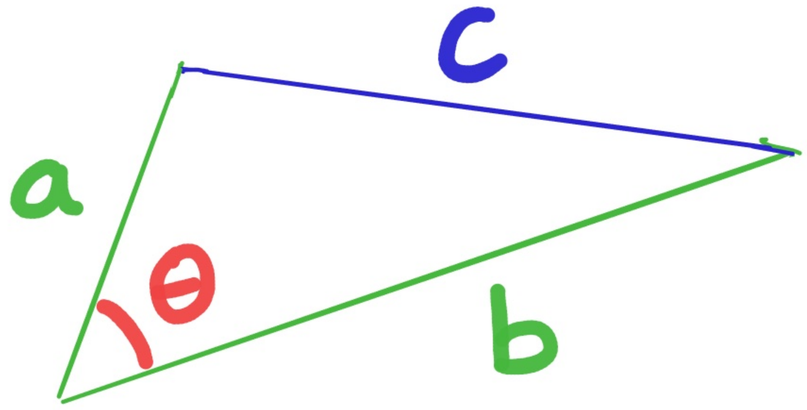
right angle

Theorem: Let θ be the angle between $\vec{v}, \vec{w} \in \mathbb{R}^n$.
Then $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$.



$0 \leq \theta \leq \pi$

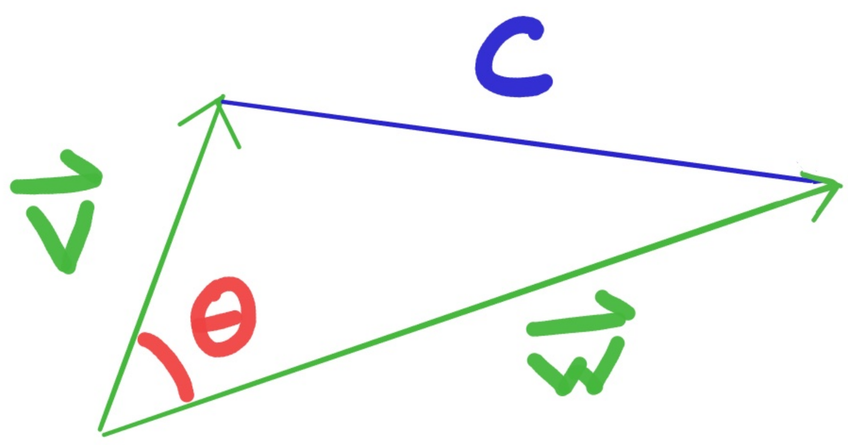
proof:



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

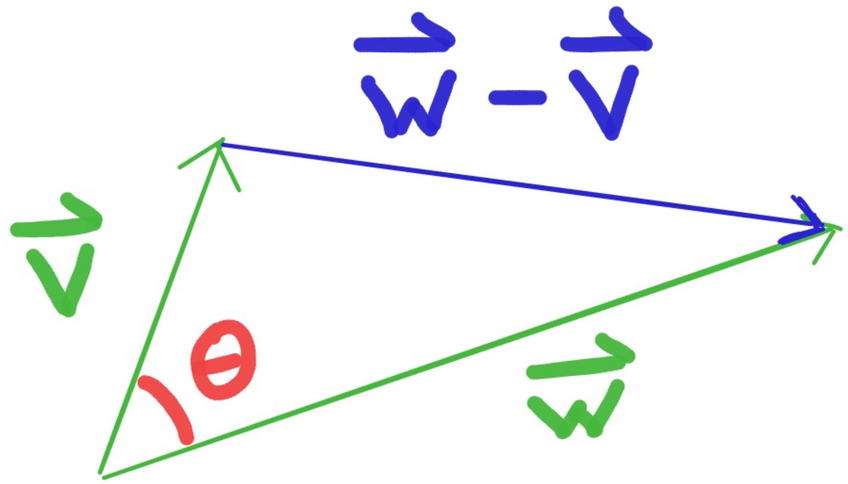
proof:



Law of cosines:

$$c^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\,\|\vec{w}\|\cos\theta$$

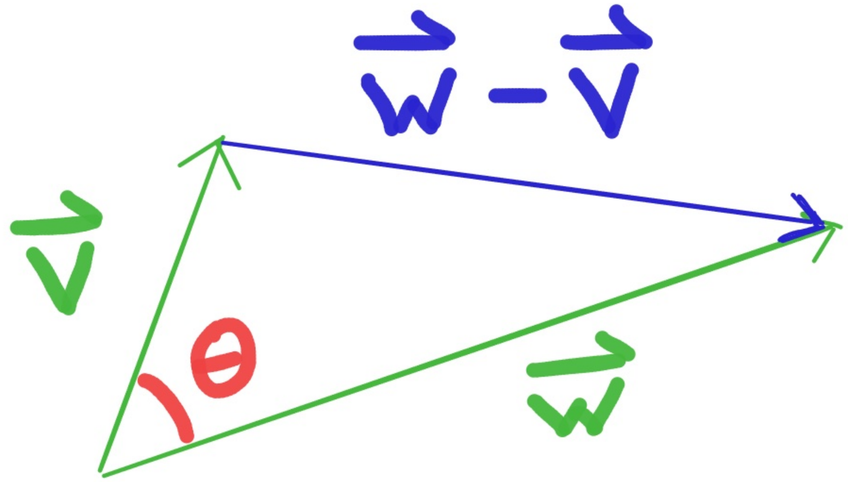
proof:



Law of cosines:

$$\|\vec{w}-\vec{v}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\,\|\vec{w}\|\cos\theta$$

proof:

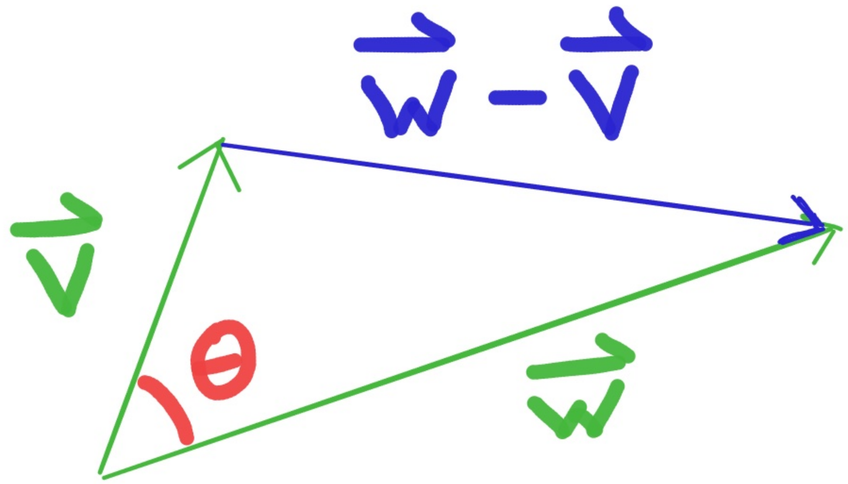


Law of cosines:

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$$(\vec{w}-\vec{v})\cdot(\vec{w}-\vec{v}) = \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{w} - 2\|\vec{v}\|\,\|\vec{w}\|\cos\theta$$

proof:



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⋮ things cancel!

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$$

END

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$



1.) If $\vec{v}, \vec{w} \in \mathbb{R}^n$ are unit vectors and $\theta = \pi/4$, what is $\vec{v} \cdot \vec{w}$?

2.) If $\vec{v} \perp \vec{w}$, what is $\vec{v} \cdot \vec{w}$?

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$



1.) If $\vec{v}, \vec{w} \in \mathbb{R}^n$ are unit vectors and $\theta = \pi/4$, what is $\vec{v} \cdot \vec{w}$?

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\frac{\pi}{4} = 1 \cdot 1 \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

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$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\frac{\pi}{2} = \|\vec{v}\| \, \|\vec{w}\| \cdot 0 = 0$$

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$



Lemma: $\vec{v} \cdot \vec{w} = 0$ exactly when $\vec{v} \perp \vec{w}$.

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$



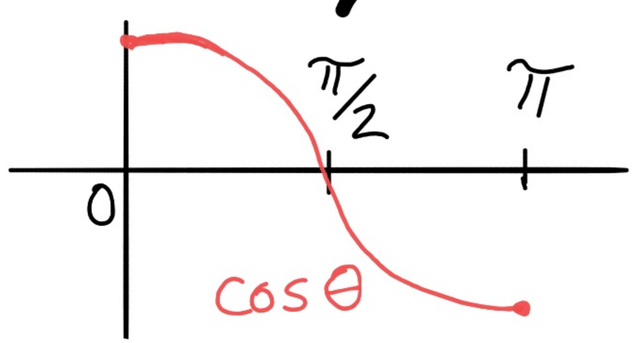
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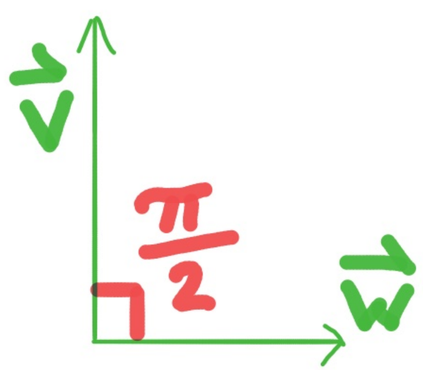
proof:

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$

equals 0 exactly when $\cos\theta = 0$,

so $\theta = \frac{\pi}{2}$.





END

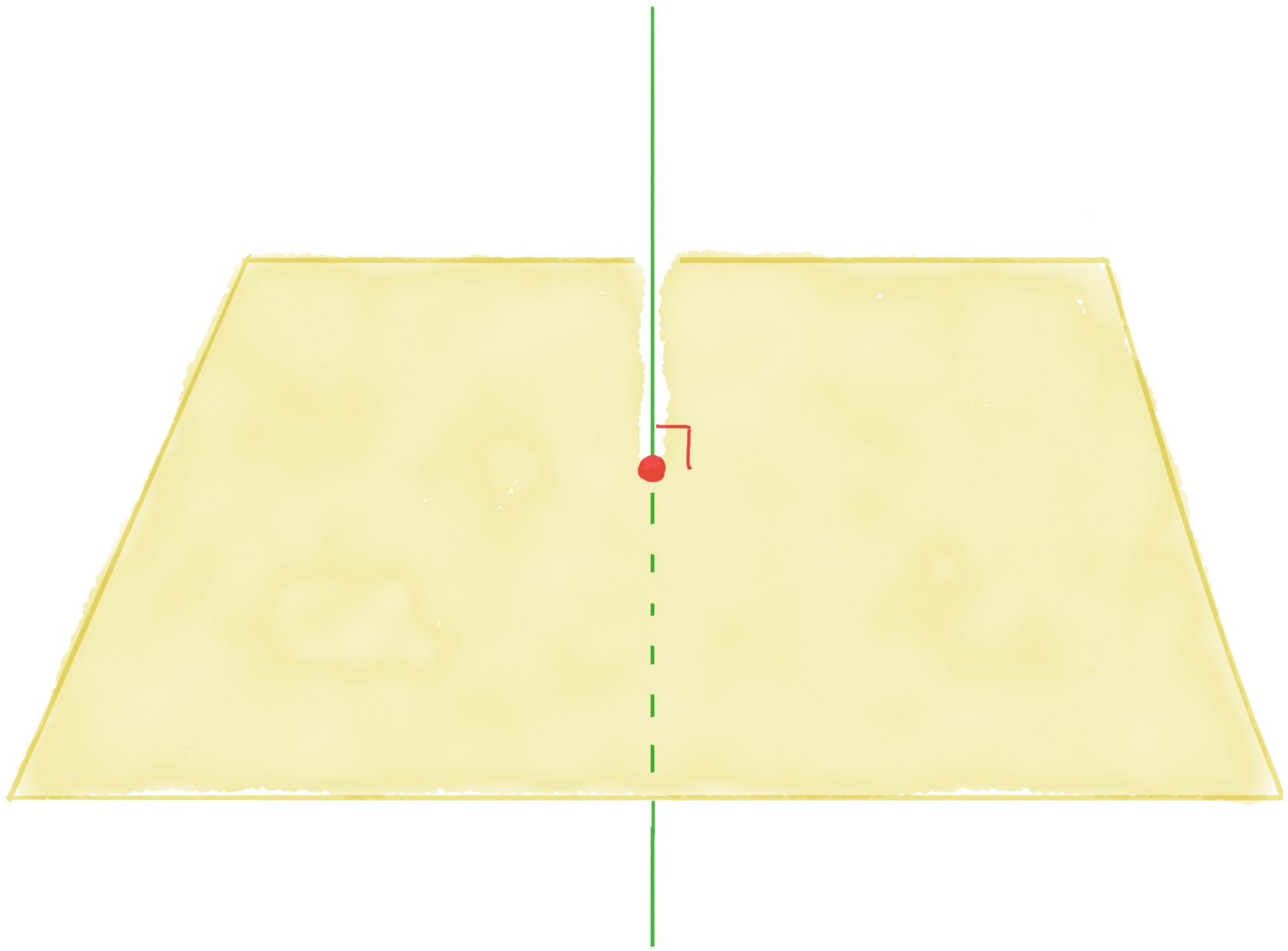
Question: Are $\langle 2,0,1 \rangle$ and $\langle 1,7,1 \rangle$ perpendicular?

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Solution: No, because

$$\langle 2,0,1\rangle \cdot \langle 1,7,1\rangle = 3 \neq 0$$

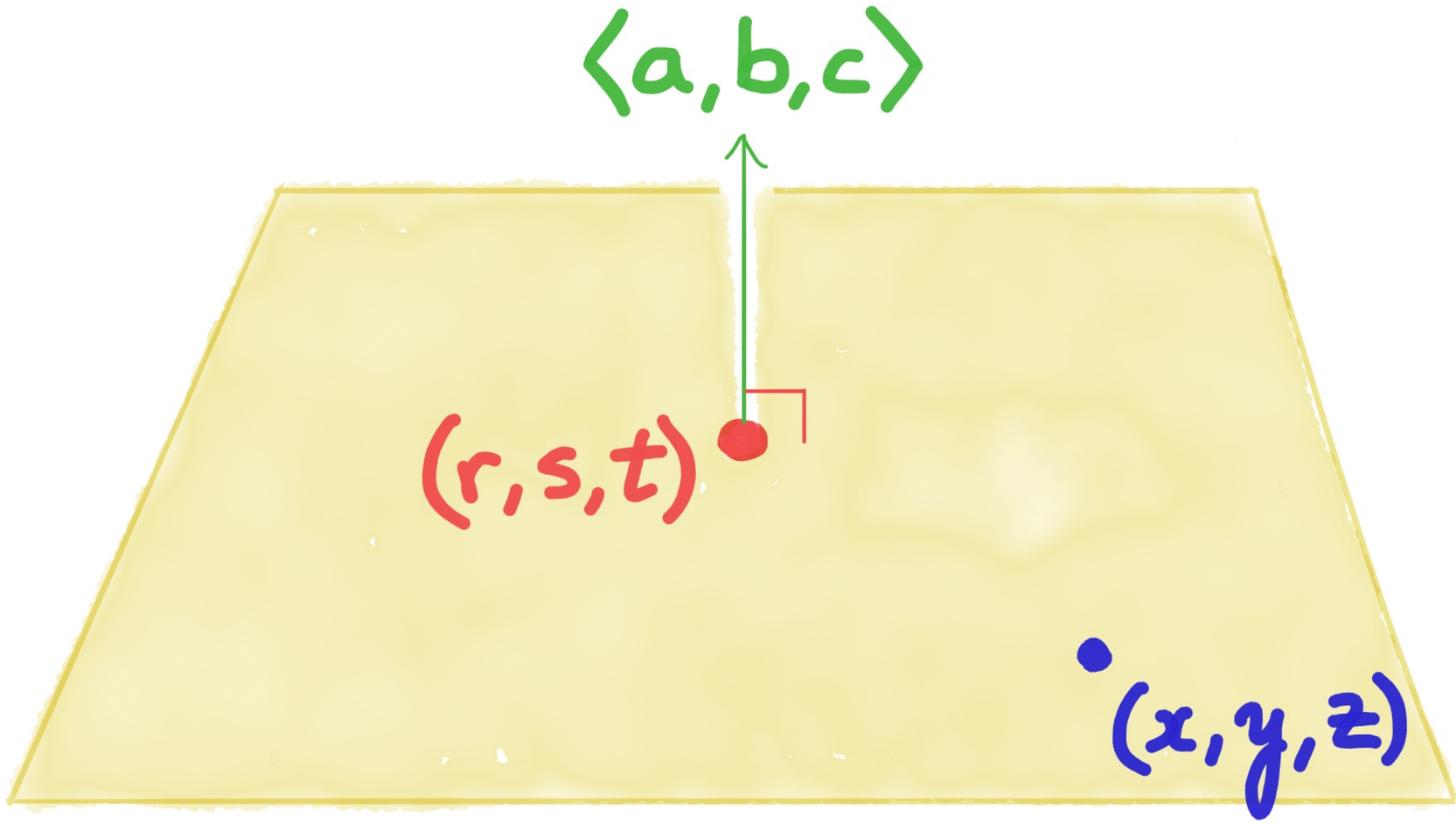
Ⅳ Equations of planes



Any plane in $\mathbb{R}^3$ has a line that is perpendicular to it.

Proposition: In $\mathbb{R}^3$, the points (x,y,z) in the plane containing (r,s,t) and perpendicular to $\langle a,b,c \rangle$ are the solutions (x,y,z) of the equation

$$a(x-r)+b(y-s)+c(z-t)=0$$



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$$a(x-r)+b(y-s)+c(z-t)=0$$

Question: What's the equation of the plane containing $(1,2,3)$ and perpendicular to $\langle 4,5,6\rangle$?

Answer: $4(x-1)+5(y-2)+6(z-3)=0$ is the equation of the plane containing $(1,2,3)$ and perpendicular to $\langle 4,5,6\rangle$.

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Question:

Is $(1,3,4)$ in the plane above?

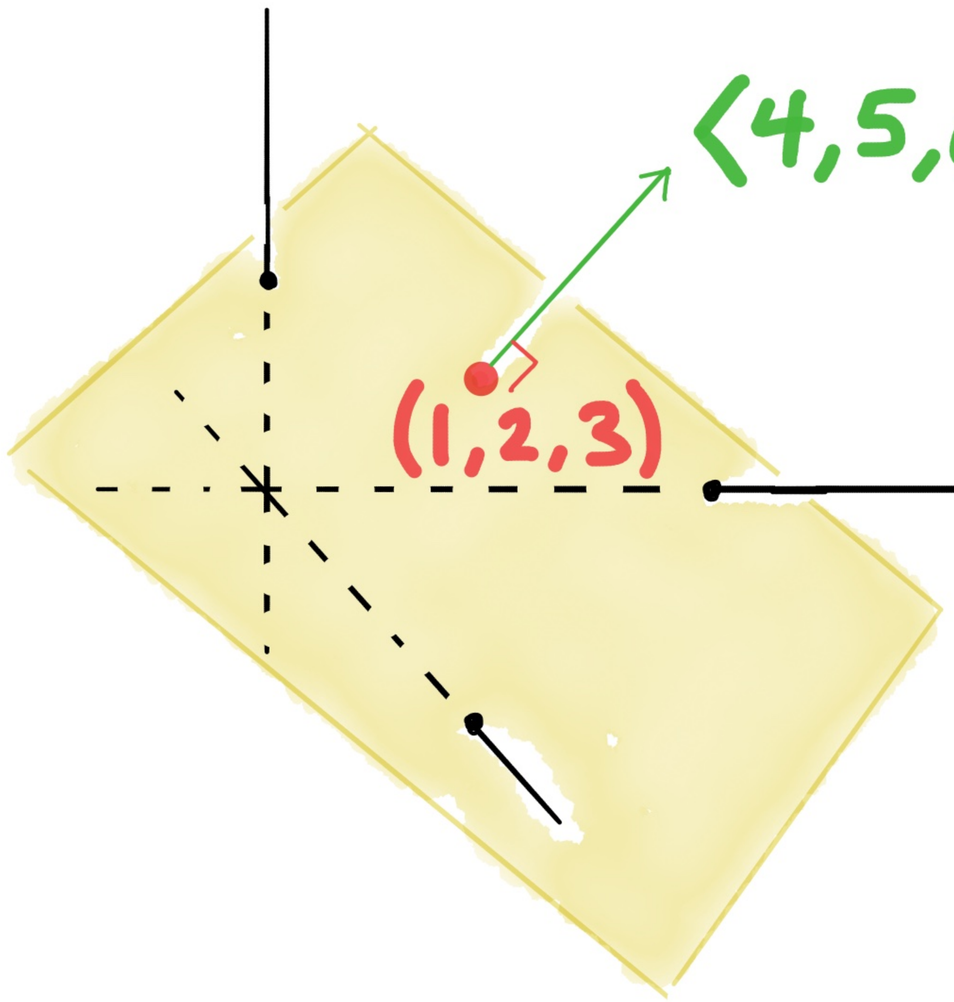
Answer: $4(x-1)+5(y-2)+6(z-3)=0$ is the equation of the plane containing $(1,2,3)$ and perpendicular to $\langle 4,5,6\rangle$.

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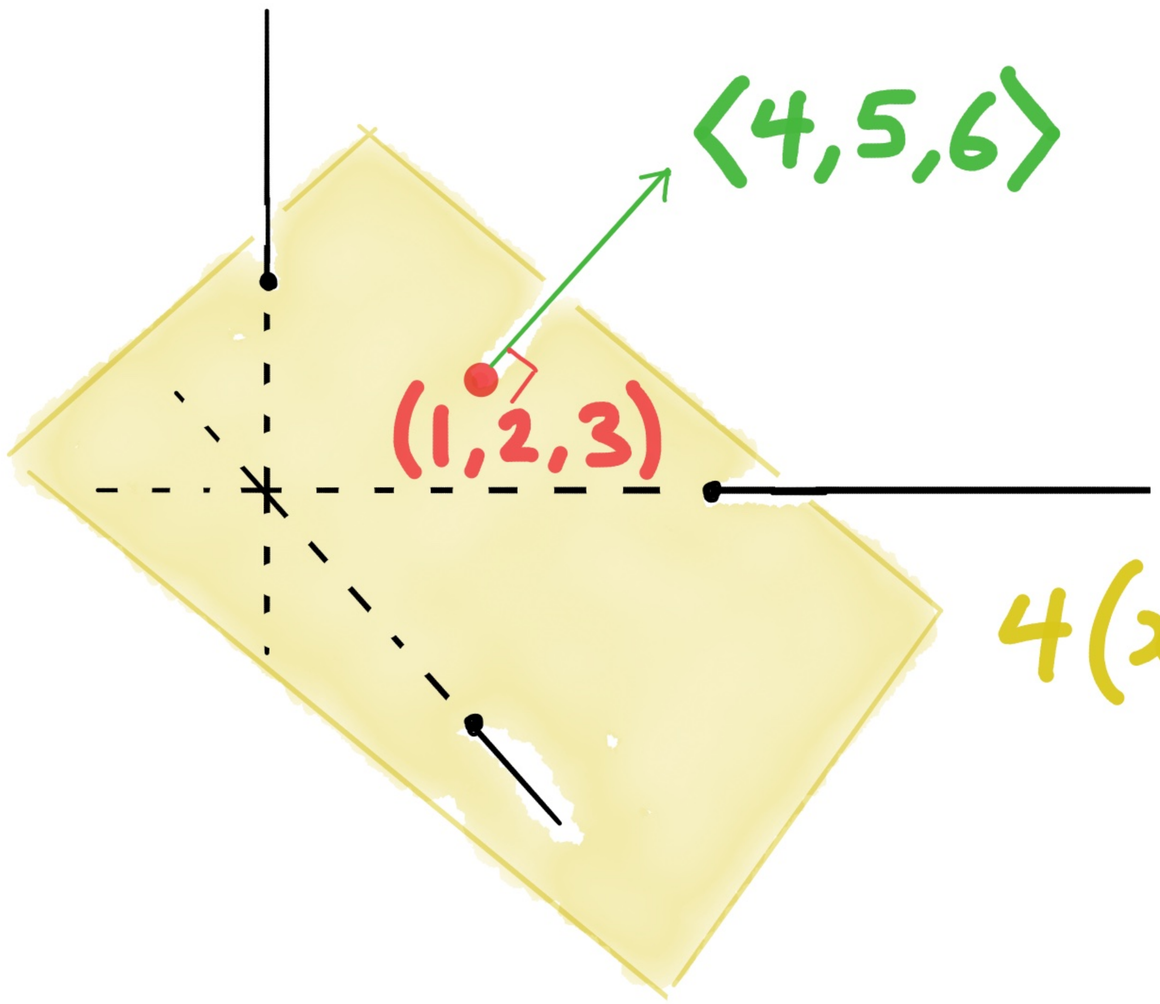
Is $(1,3,4)$ in the plane above?

Answer: No, because

$$4(1-1)+5(3-2)+6(4-3)=11\neq 0.$$



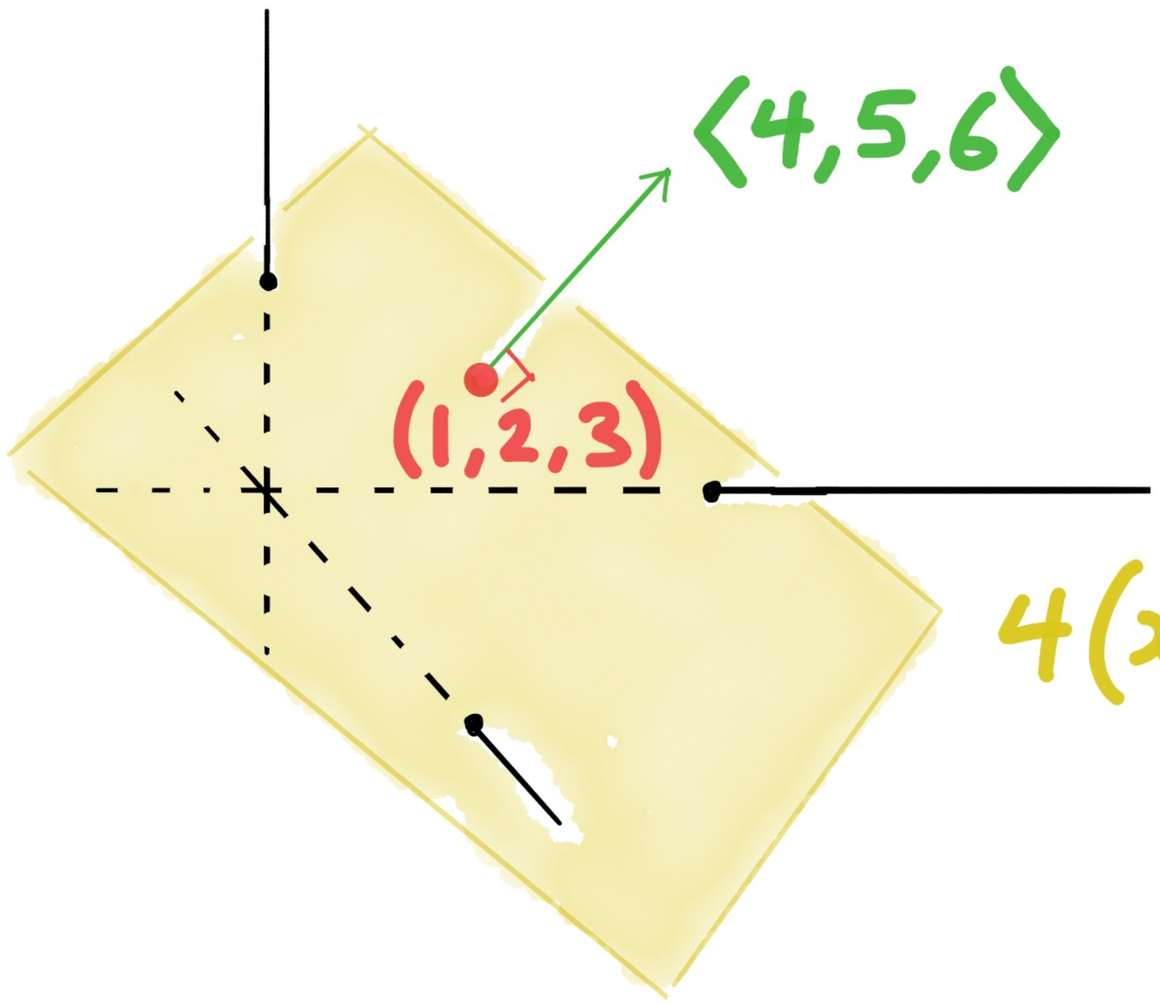
$$4(x-1)+5(y-2)+6(z-3)=0$$



$$4(x-1)+5(y-2)+6(z-3)=0$$

distribute

$$4x-4+5y-10+6z-18=0$$



$$4(x-1)+5(y-2)+6(z-3)=0$$

distribute

$$4x-4+5y-10+6z-18=0$$

rearrange terms

$$4x+5y+6z=32$$

If $a, b, c, d \in \mathbb{R}$ then

$$ax + by + cz = d$$

is an equation of a plane in $\mathbb{R}^3$.

(assuming $\langle a, b, c \rangle \neq \vec{0}$)

If $a,b,c,d \in \mathbb{R}$ then

$$ax+by+cz=d$$

is an equation of a plane in $\mathbb{R}^3$.

(assuming $\langle a,b,c \rangle \neq \vec{0}$)

$3x+5y+7z=32$

$z=0$

$y=-3$

$-x+y=2$

$\sqrt{2}x-3y+z=0$

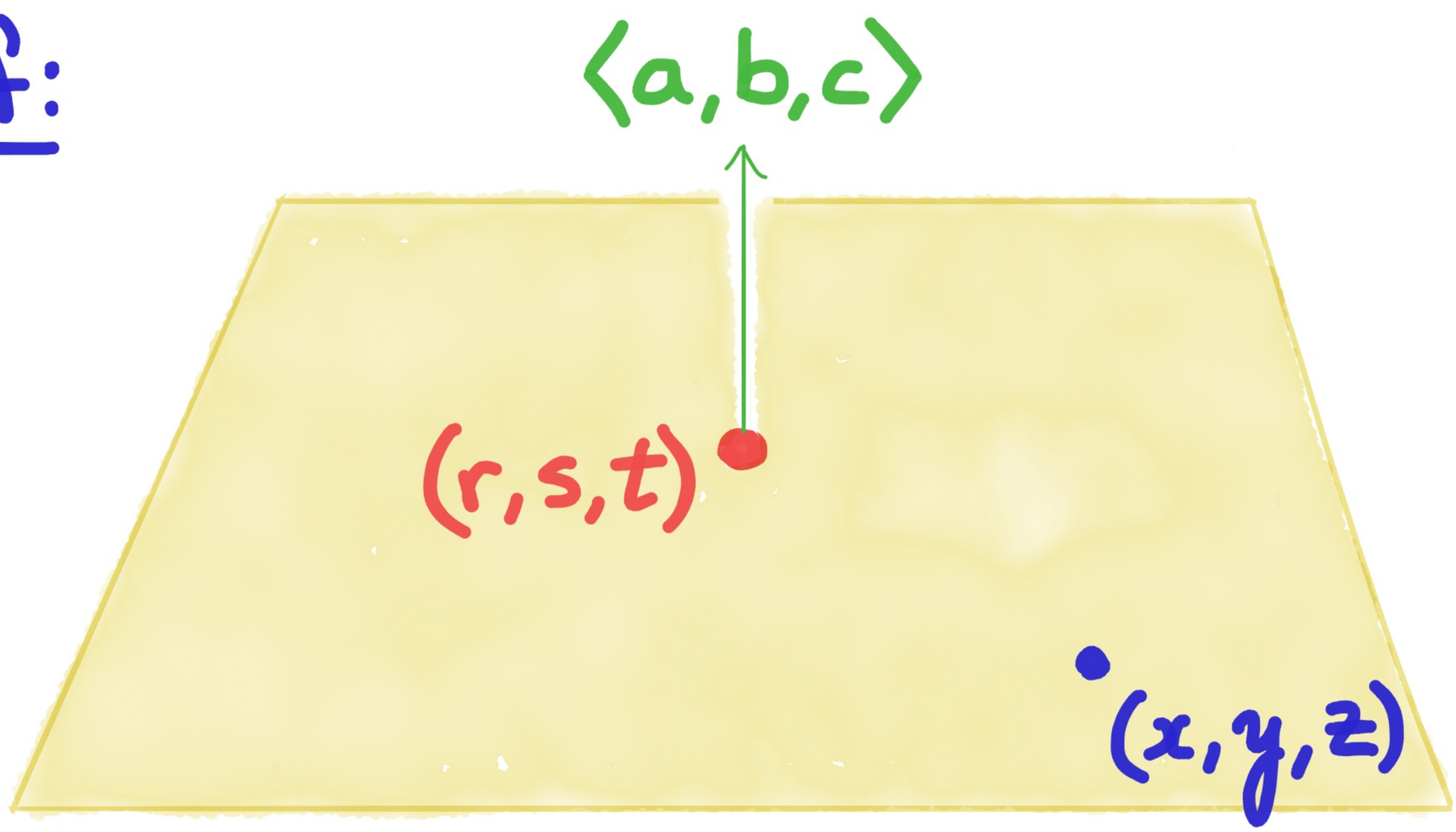
$x=2$

$-7x+7z=-8$

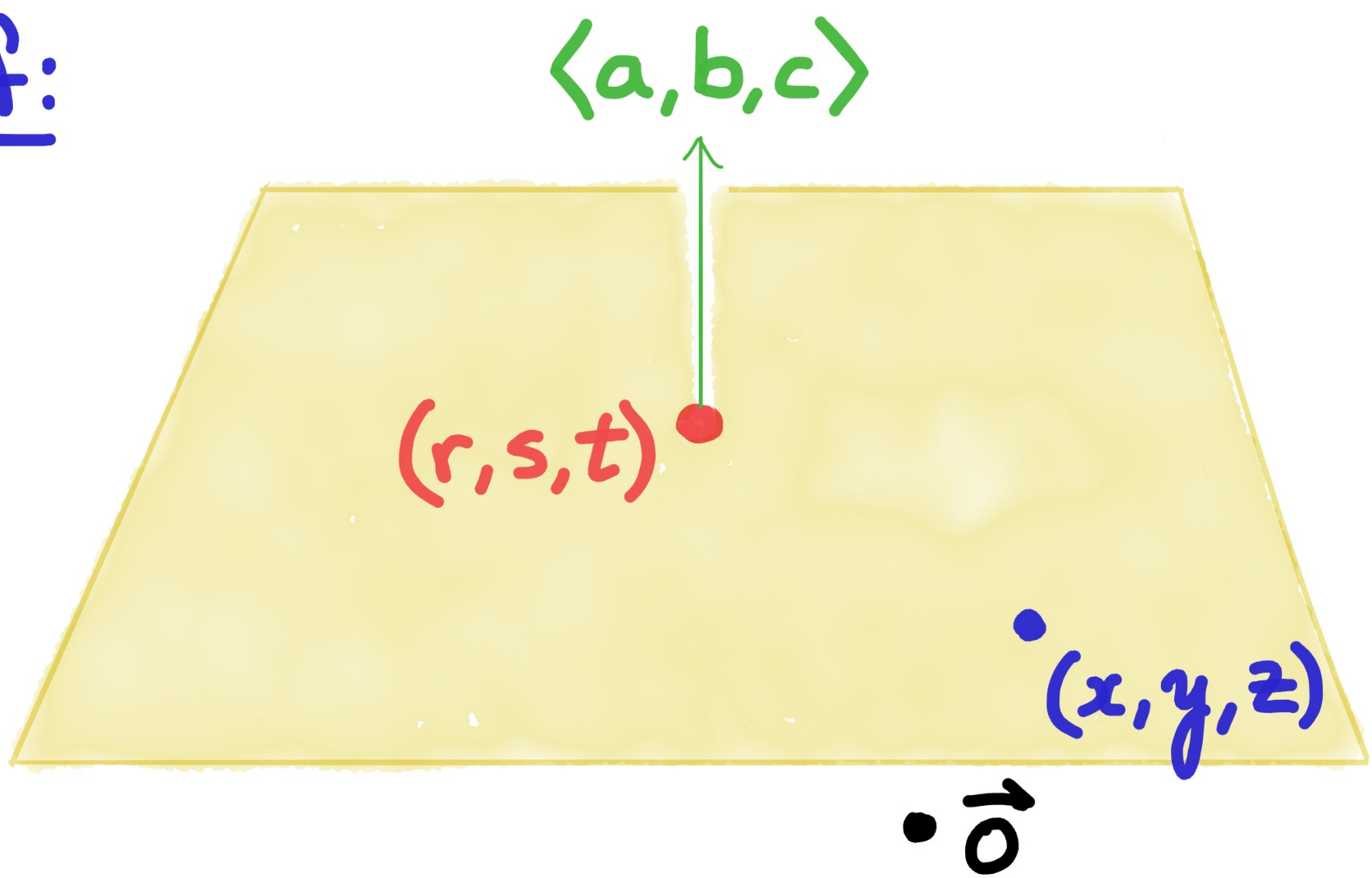
Proposition: In $\mathbb{R}^3$, the points (x,y,z) in the plane containing (r,s,t) and perpendicular to $\langle a,b,c \rangle$ are the solutions (x,y,z) of the equation

$$a(x-r)+b(y-s)+c(z-t)=0$$

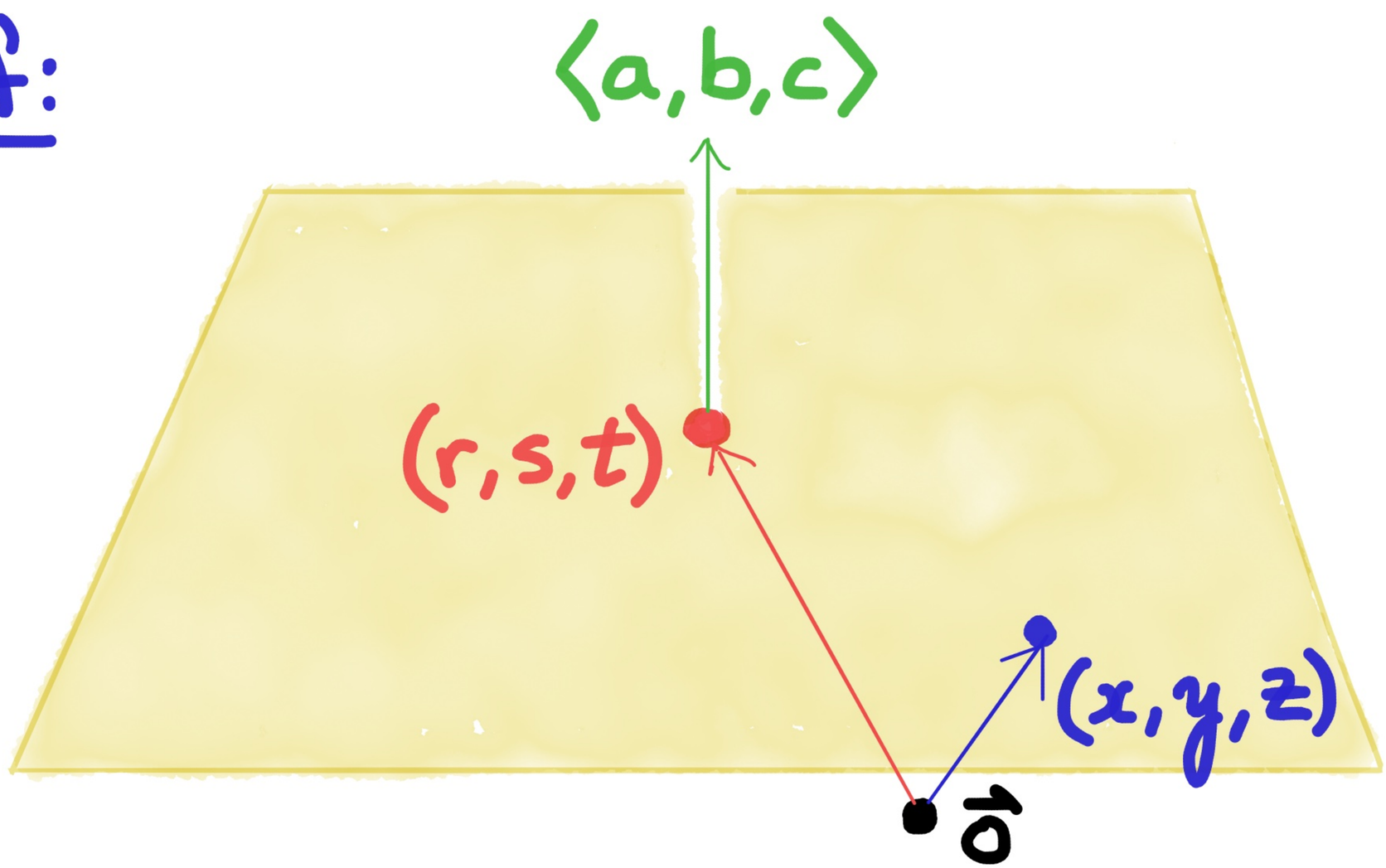
proof:



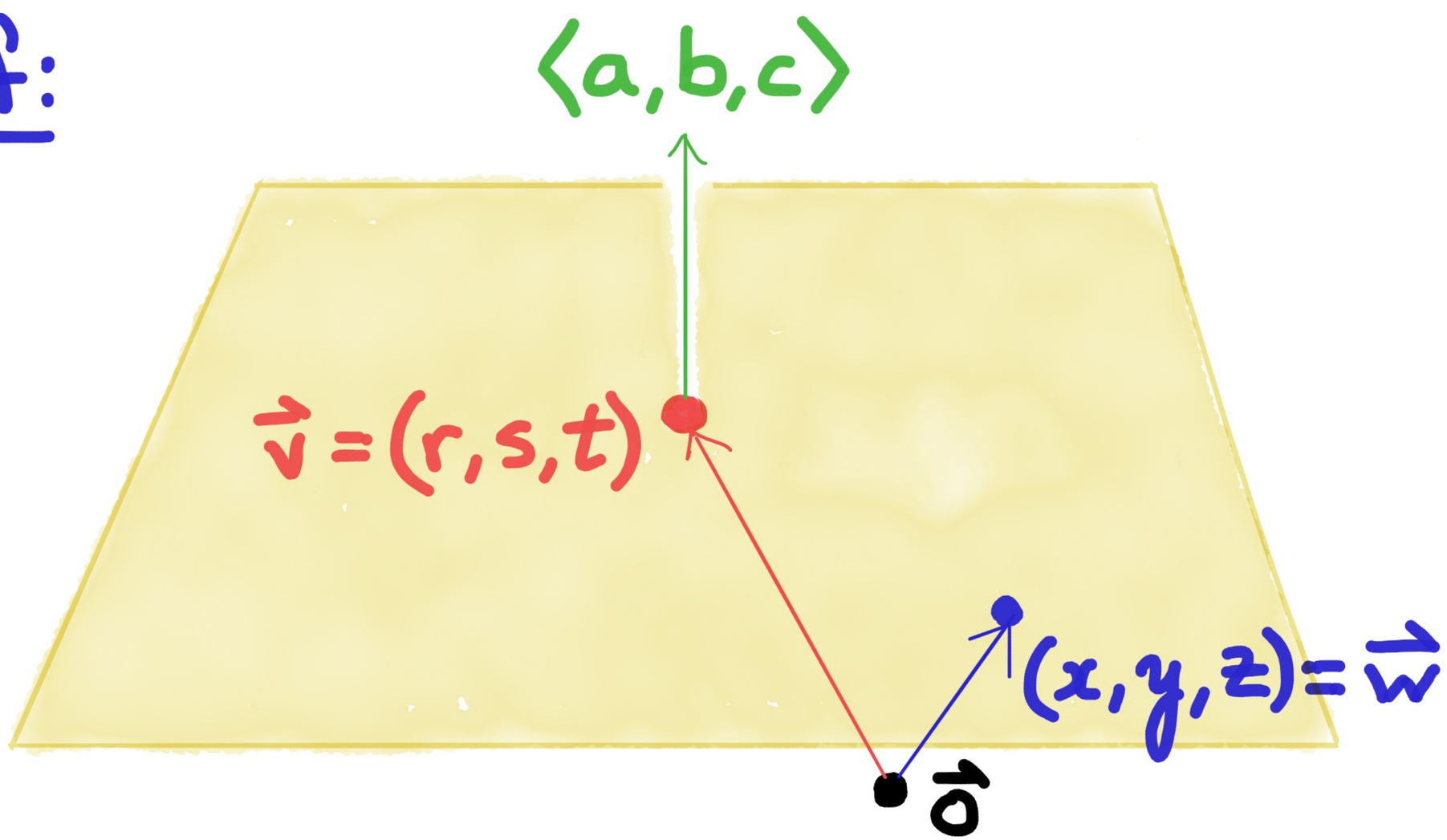
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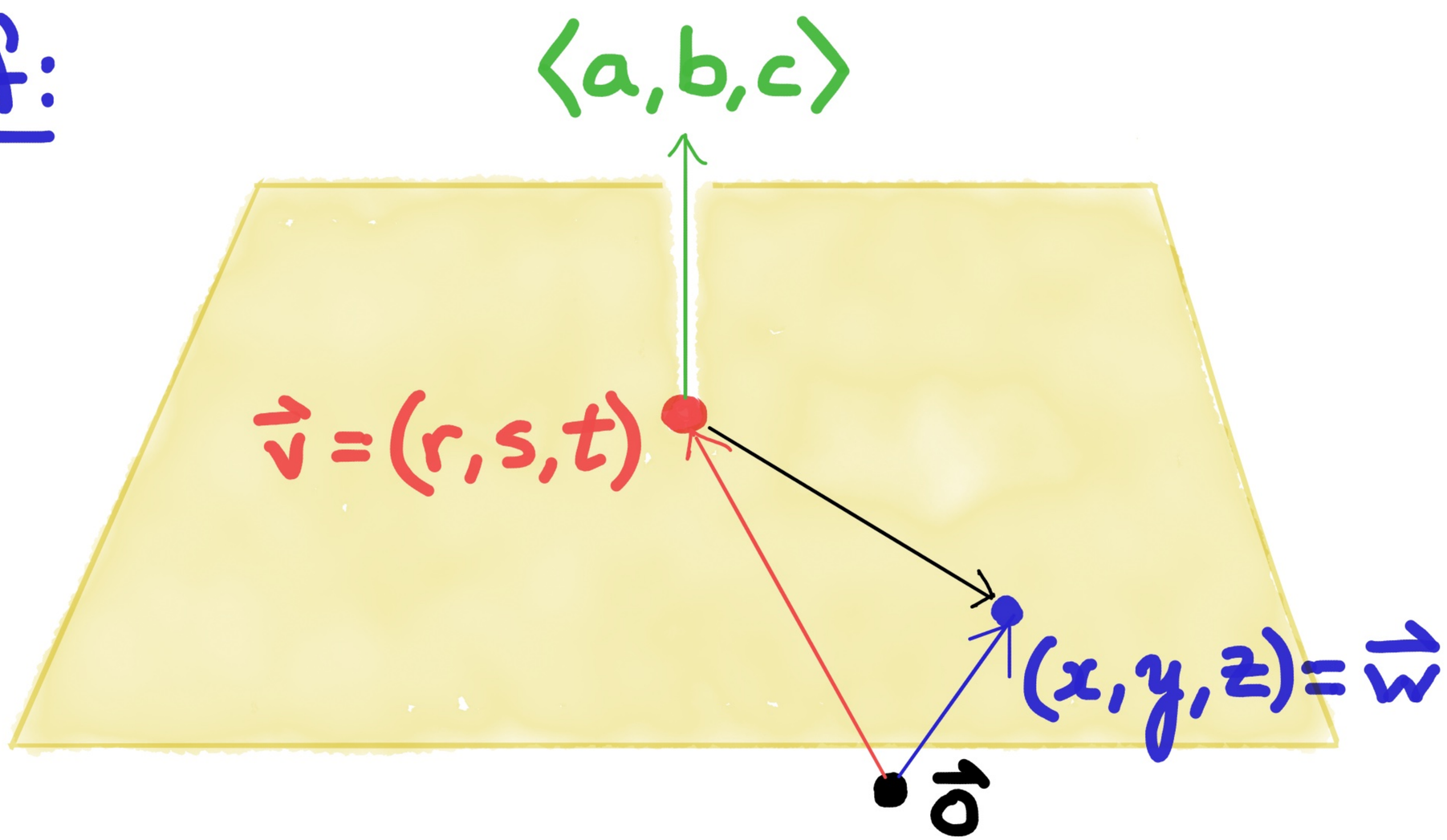
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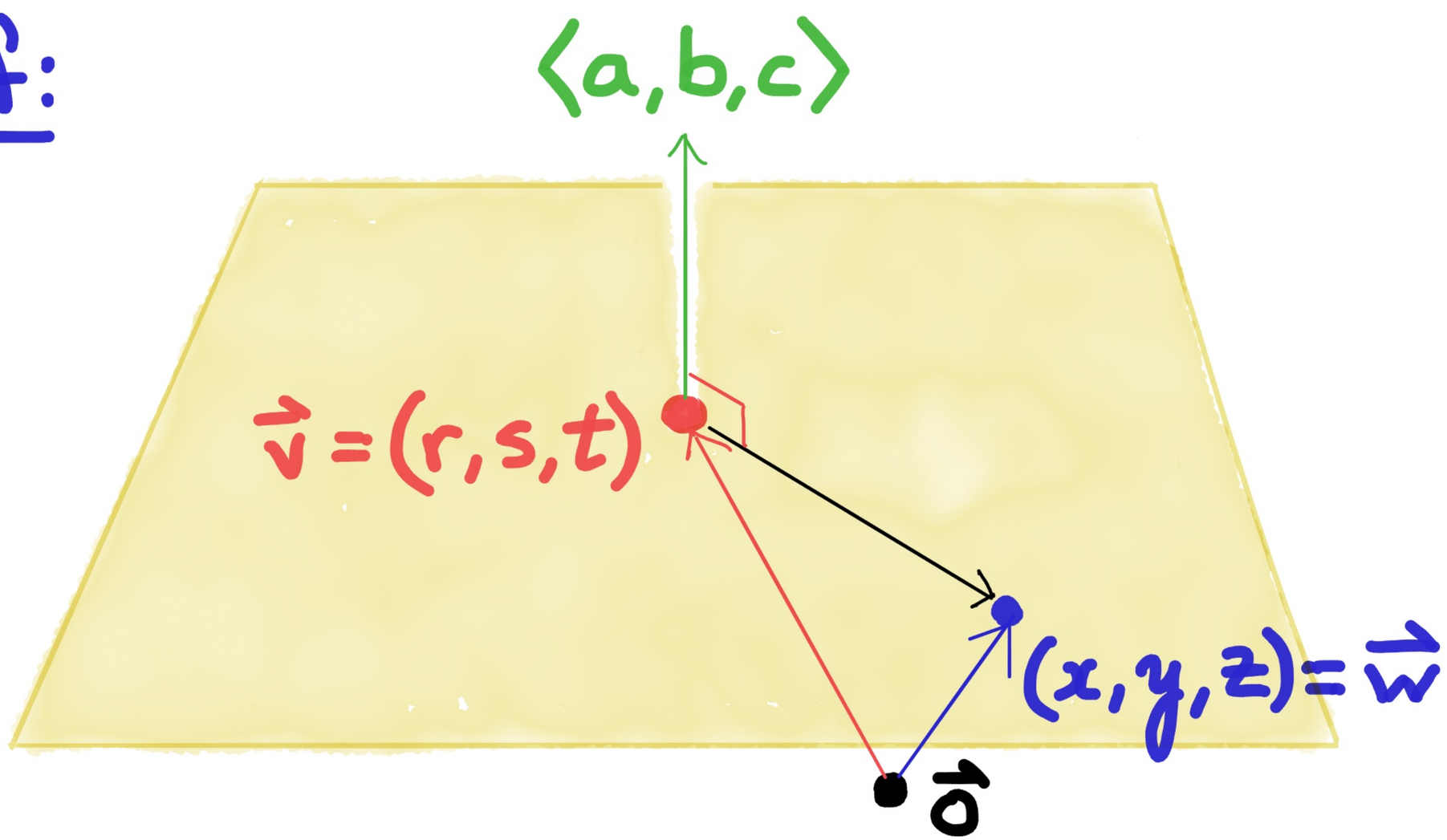
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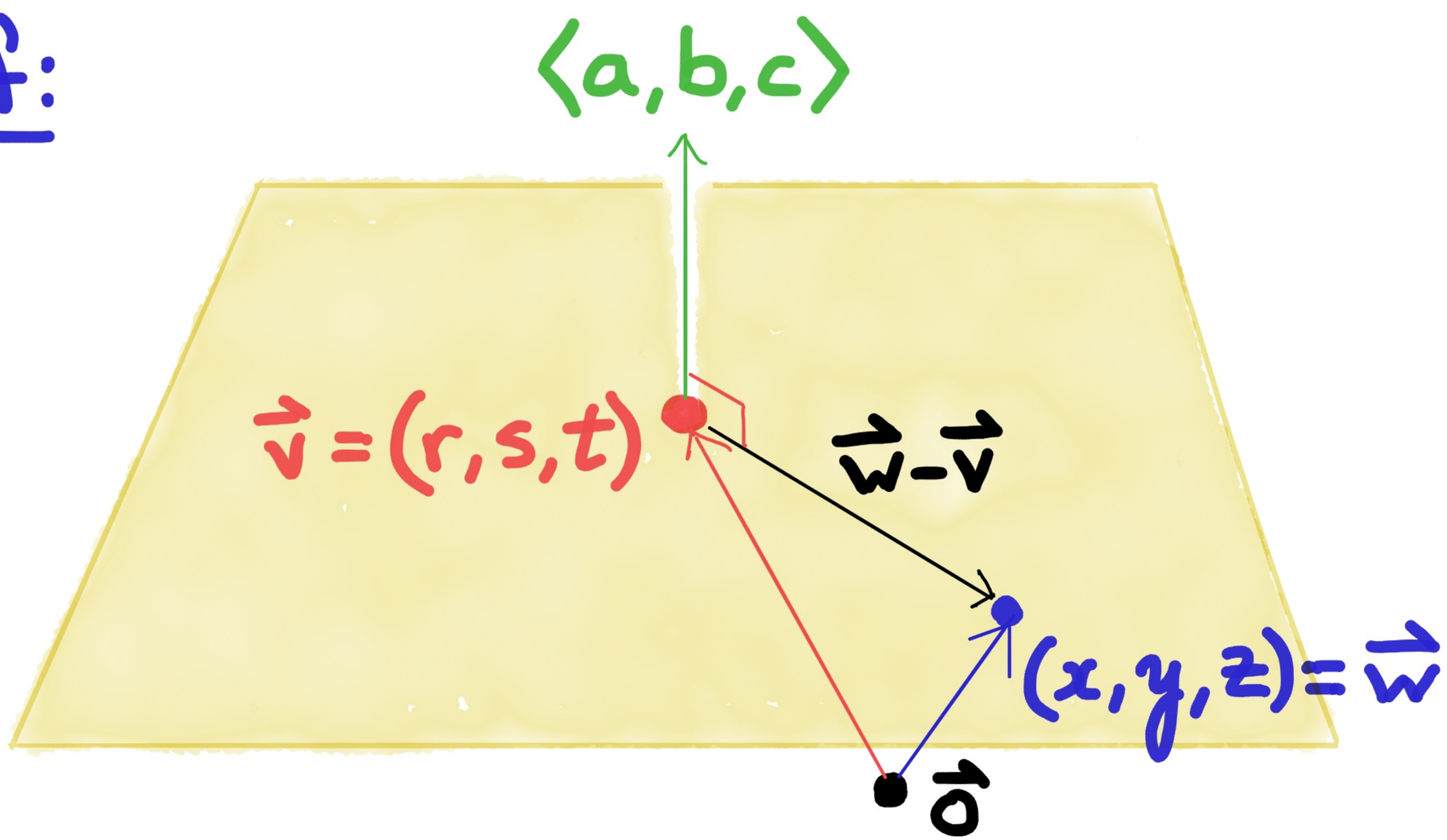
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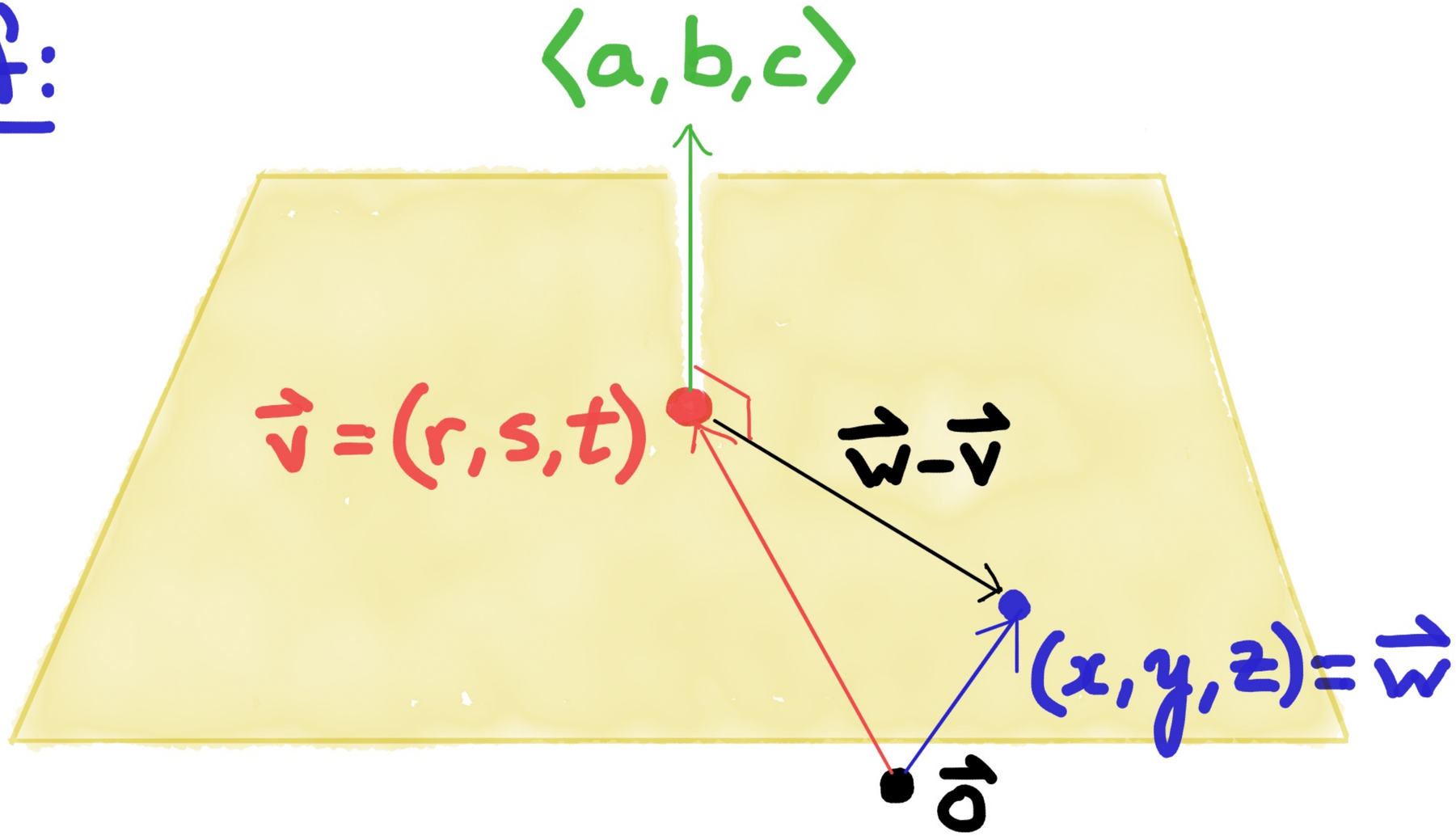
proof:



proof:

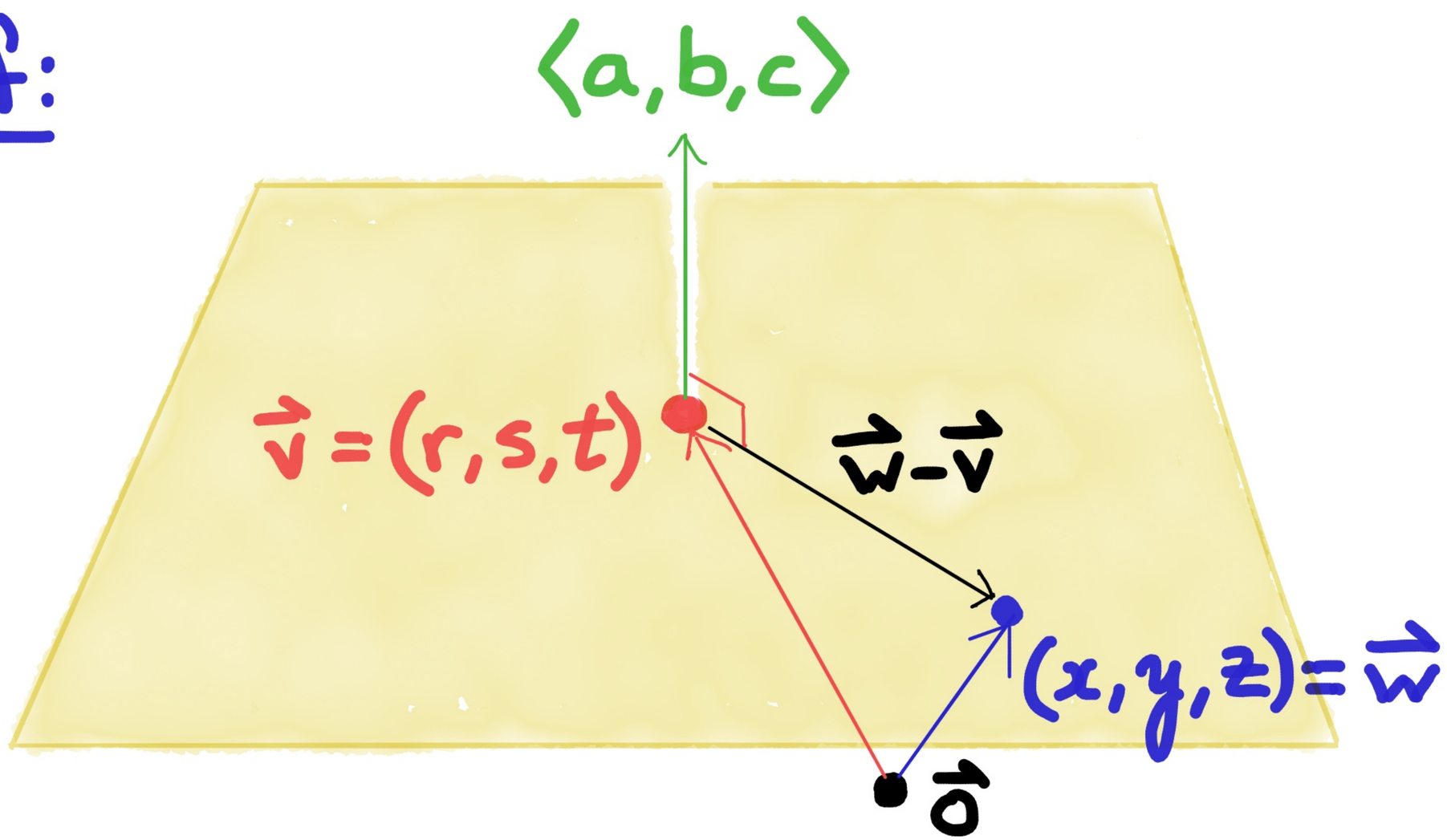


proof:



Since $\langle a,b,c\rangle \perp (\vec{w}-\vec{v})$, $\langle a,b,c\rangle \cdot (\vec{w}-\vec{v}) = 0$.

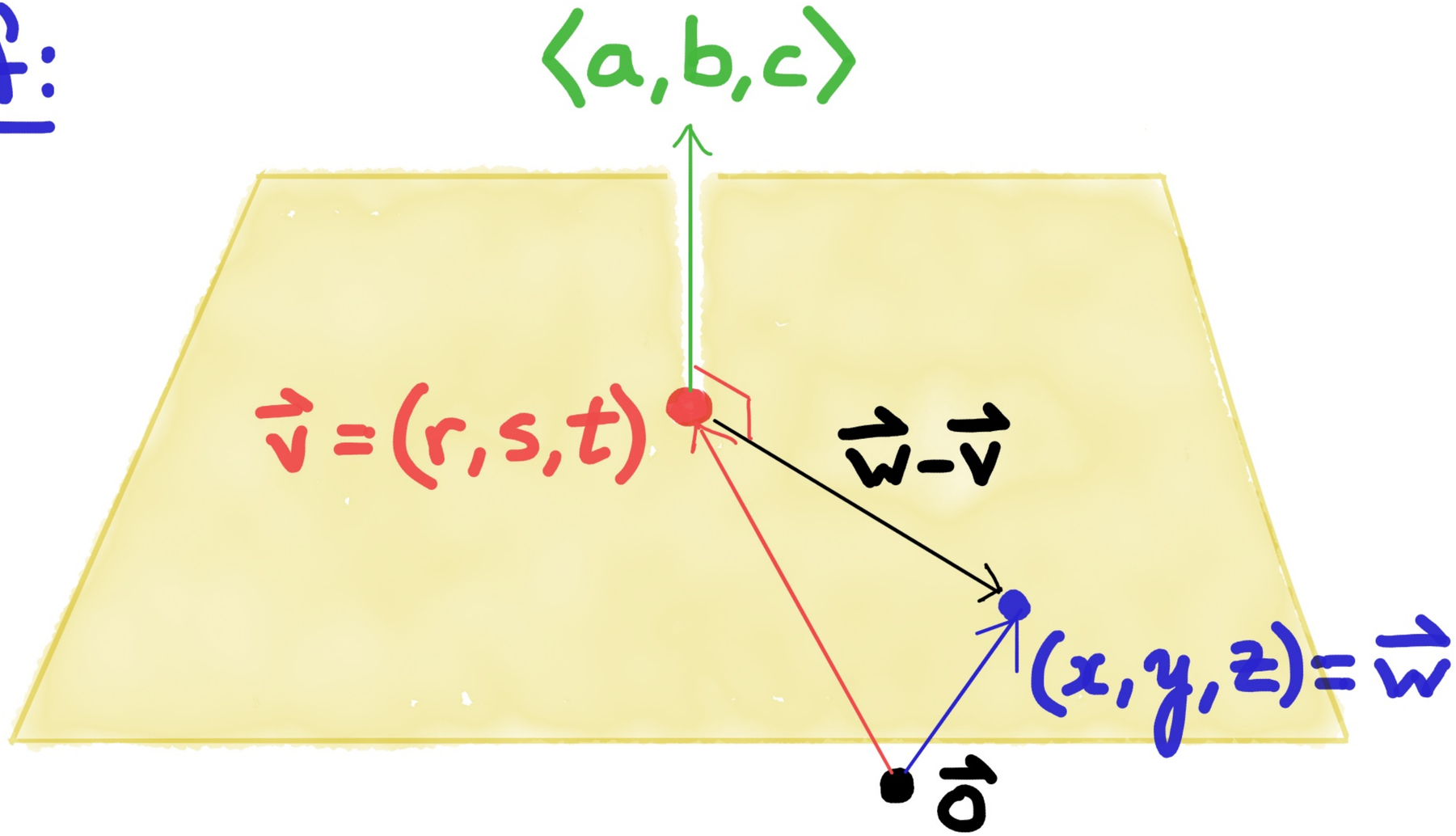
proof:



Since $\langle a,b,c\rangle \perp (\vec{w}-\vec{v})$, $\langle a,b,c\rangle \cdot (\vec{w}-\vec{v}) = 0$.

That is, $\langle a,b,c\rangle \cdot \langle x-r, y-s, z-t\rangle = 0$.

proof:

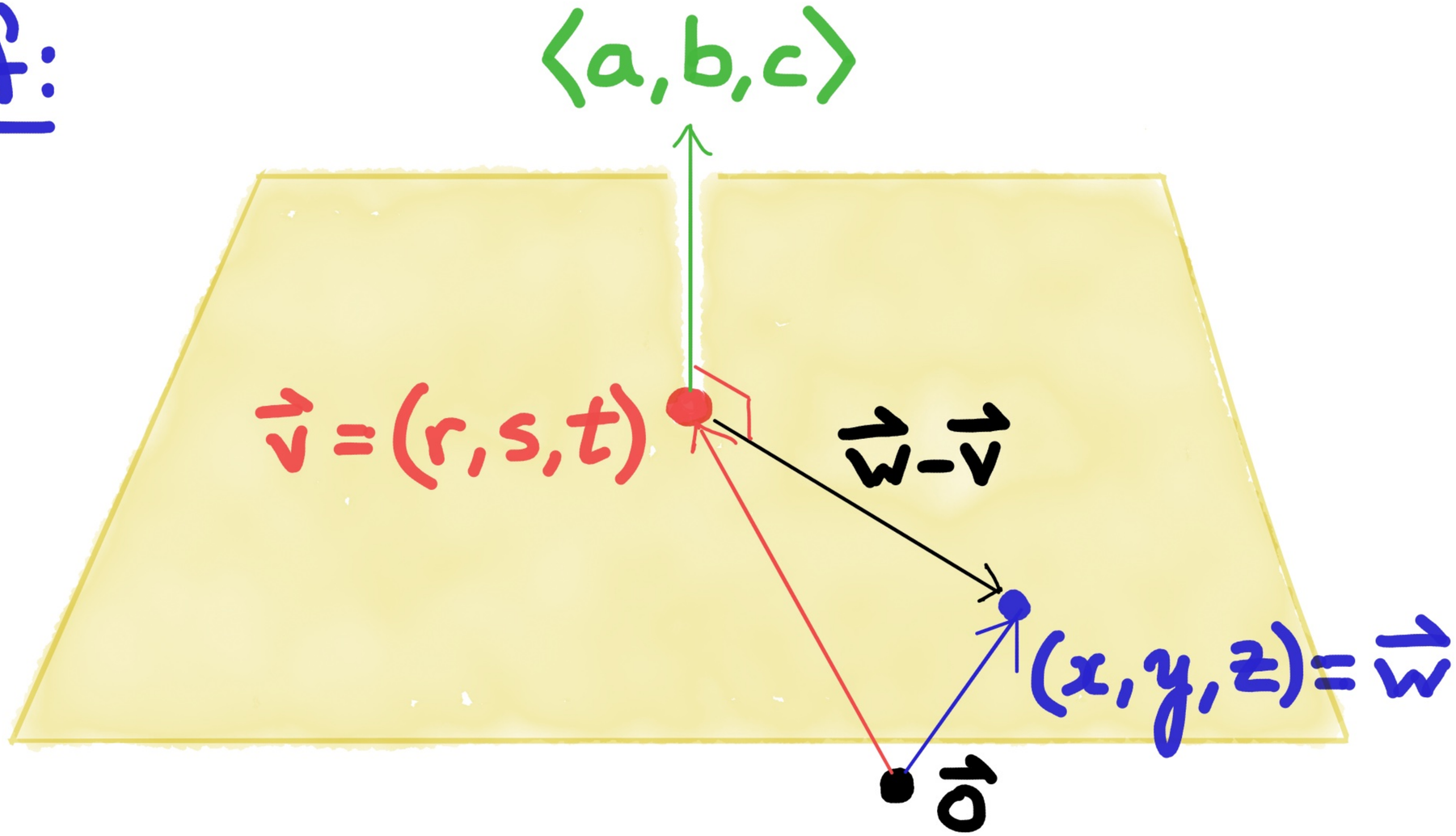


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That is, $\langle a,b,c\rangle \cdot \langle x-r, y-s, z-t\rangle = 0$.

That is, $a(x-r)+b(y-s)+c(z-t)=0$.

proof:

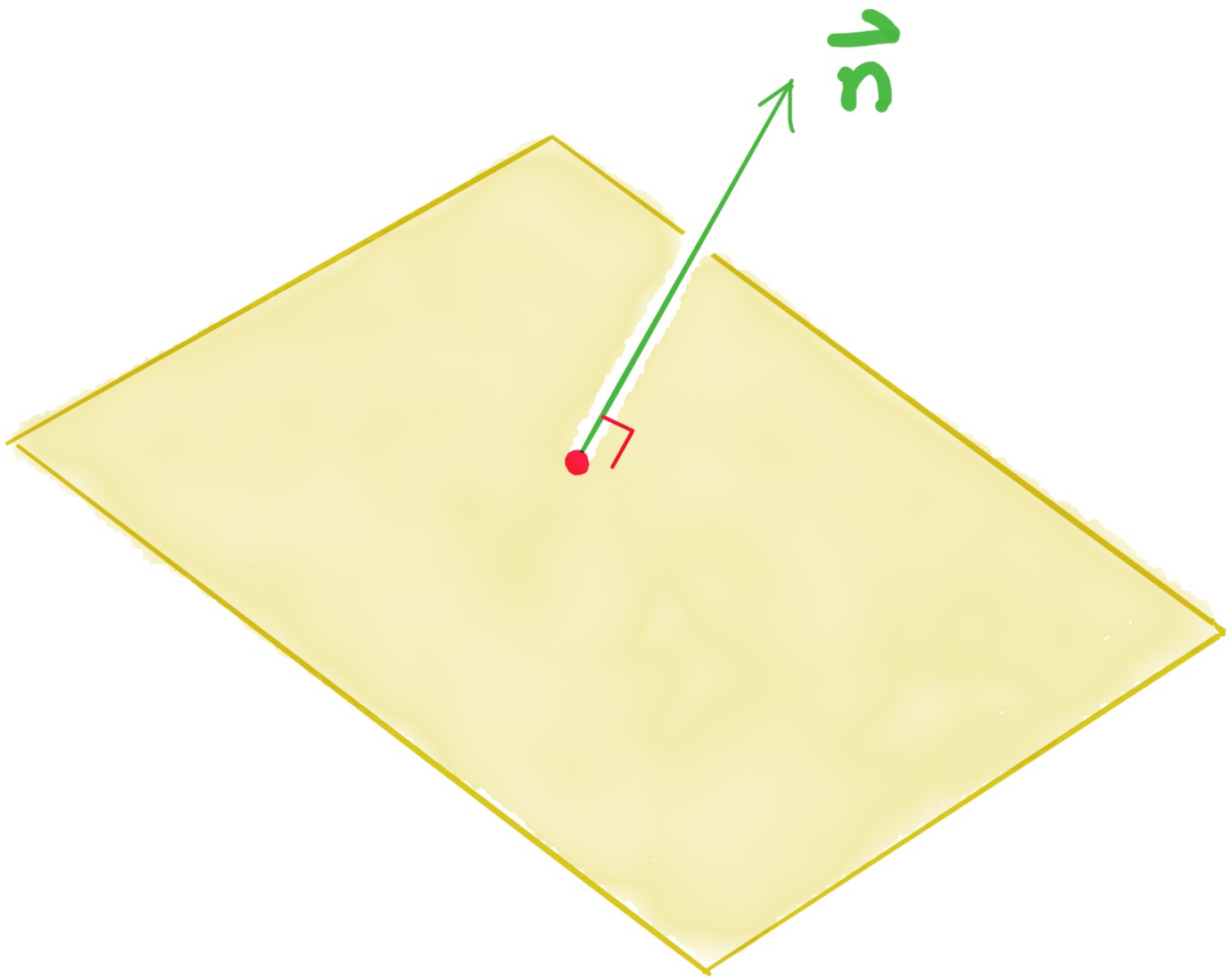


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That is, $a(x-r)+b(y-s)+c(z-t)=0$.

END



A vector orthogonal to a plane is called a **normal vector**.

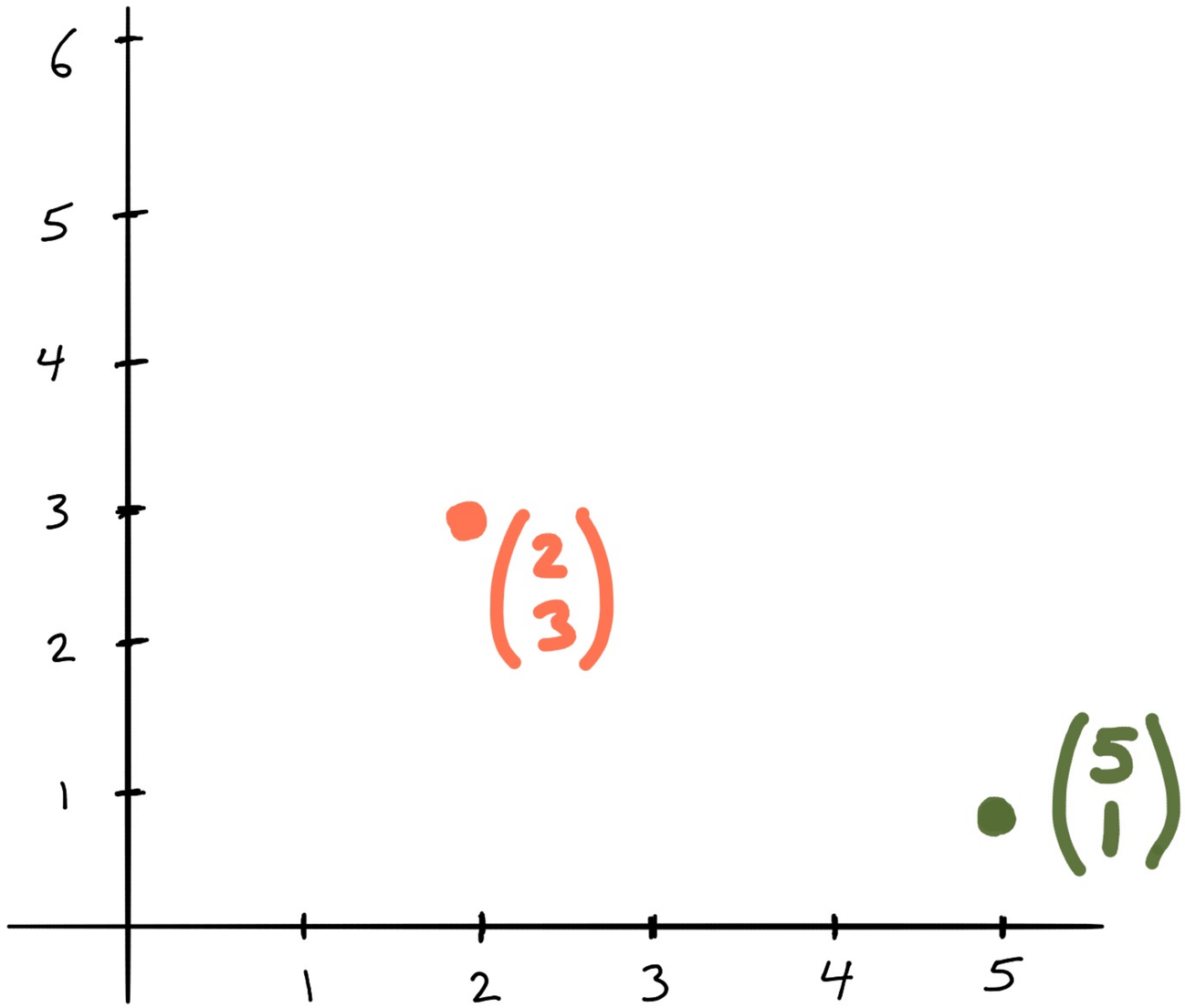


Column vectors

Sometimes we write vectors as columns instead of rows.

$$\vec{v} = \langle v_1, v_2, v_3 \rangle = (v_1, v_2, v_3) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$\mathbb{R}^2$


6
5
4
3
2
1
1
2
3
4
5
$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
$\begin{pmatrix} 5 \\ 1 \end{pmatrix}$

Row on left, column on right

It means dot product.

$$(2,0,1)\begin{pmatrix}3\\7\\-2\end{pmatrix} =$$

$$(3,-4)\begin{pmatrix}1\\4\end{pmatrix} =$$

Row on left, column on right

It means dot product.

$$(2,0,1)\begin{pmatrix}3\\7\\-2\end{pmatrix} = 2\cdot 3+0\cdot 7+1\cdot(-2)=4$$

$$(3,-4)\begin{pmatrix}1\\4\end{pmatrix} = 3\cdot 1+(-4)\cdot 4=-13$$