Twenty-nine

Ⅰ Line integrals

Ⅱ Idea of line integrals

Ⅲ Four useful lemmas

Ⅳ Independence of path

Ⅴ Independence of parametrization

Ⅰ Line integrals

Computing line integrals

C parametrized by $x = x(t)$, $y = y(t)$, $a \le t \le b$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

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- $\displaystyle\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$

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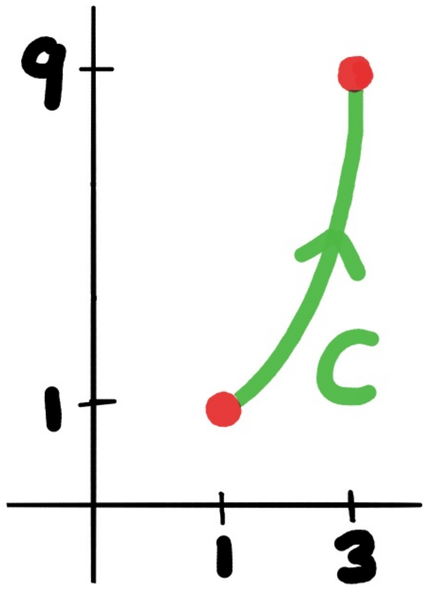
Example: C given by $x(t)=t$, $y(t)=t^2$, $1 \le t \le 3$.

Find $\displaystyle\int_C \frac{1}{\sqrt{4y+1}}\,ds$.



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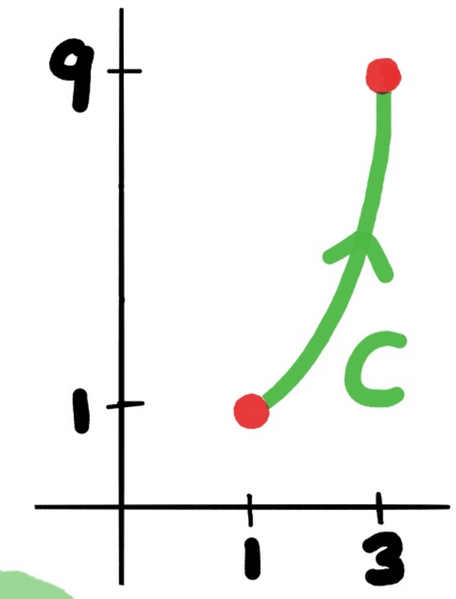

Solution:

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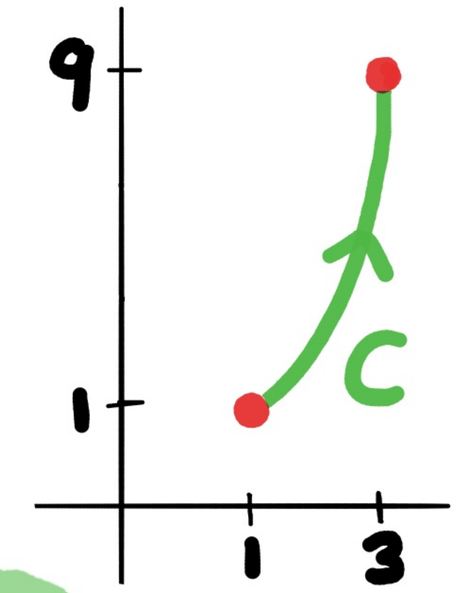

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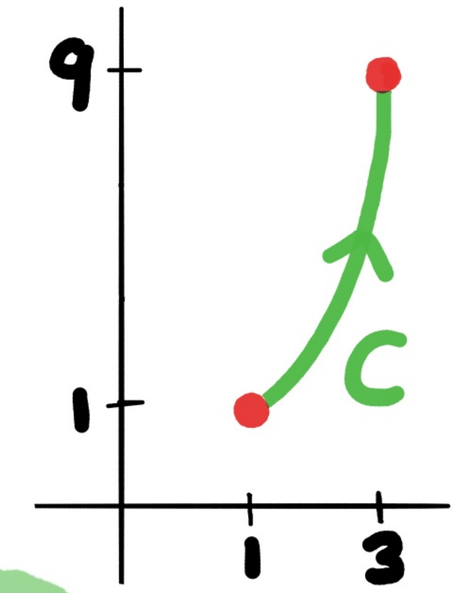

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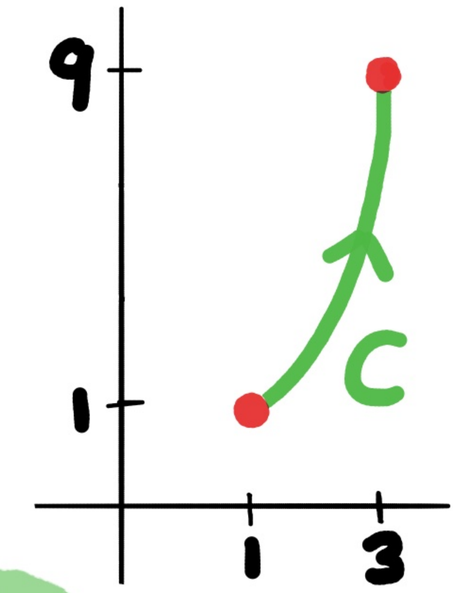

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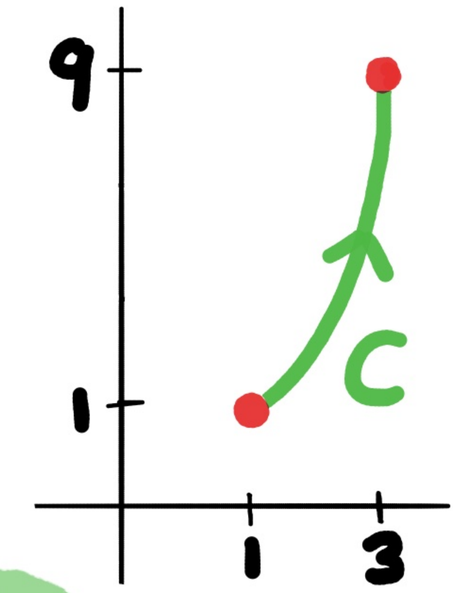

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$$= \int_1^3 \frac{1}{\sqrt{4t^2+1}}\sqrt{1^2+(2t)^2}\,dt$$

$$= \int_1^3 dt = 2$$

Definition:

$$\int_C \left(f(x,y)\,dx + g(x,y)\,dy \right)$$

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Consequently:

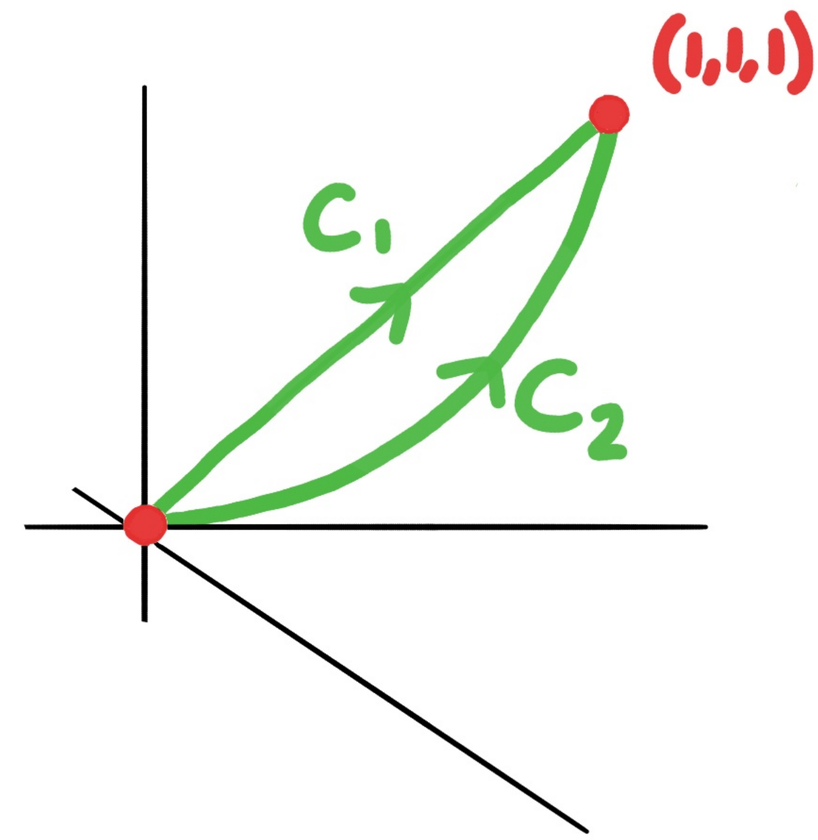
$$\int_C f(x,y)\,dx + g(x,y)\,dy$$

$$= \int_a^b \left(f(x(t),y(t))\,x'(t) + g(x(t),y(t))\,y'(t) \right) dt$$

Example: C_1 given by $x=t,\ y=t,\ z=t,\ 0\le t\le 1.$

C_2 given by $x=t^3,\ y=t^2,\ z=t,\ 0\le t\le 1.$

Find $\int_{C_i} yz\,dx + xz\,dy + xy\,dz$ for $i=1,2.$



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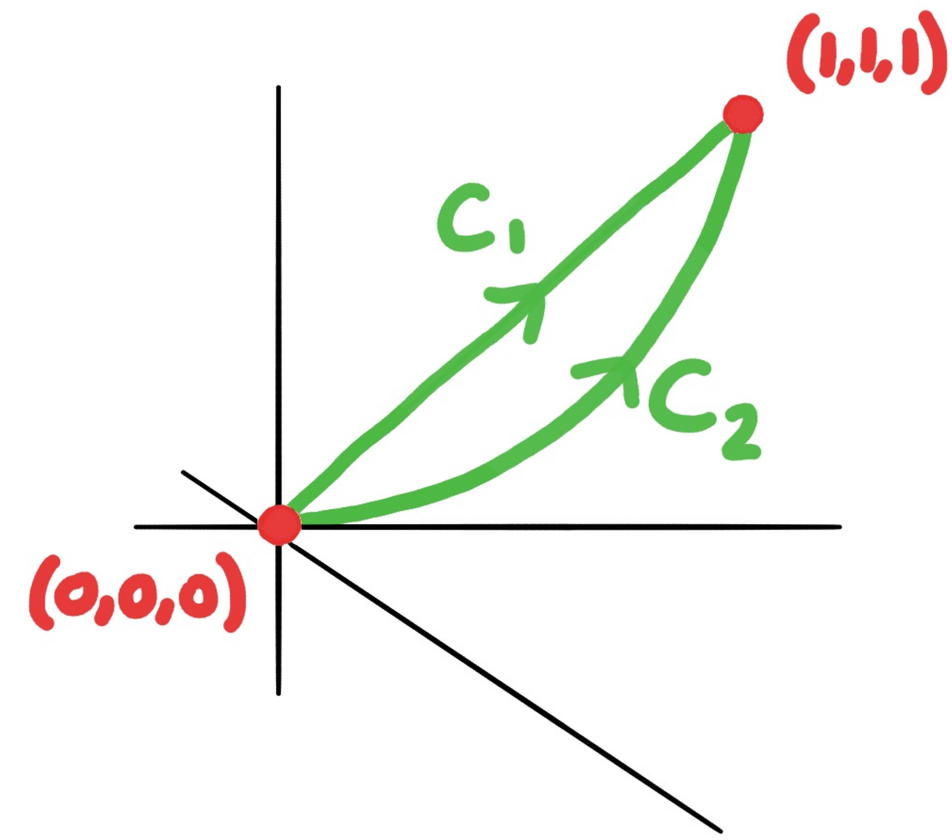

Solution:

$$\int_{C_1} yz\,dx + xz\,dy + xy\,dz$$

$$= \int_0^1 t\cdot t\cdot 1 + t\cdot t\cdot 1 + t\cdot t\cdot 1 \; dt$$

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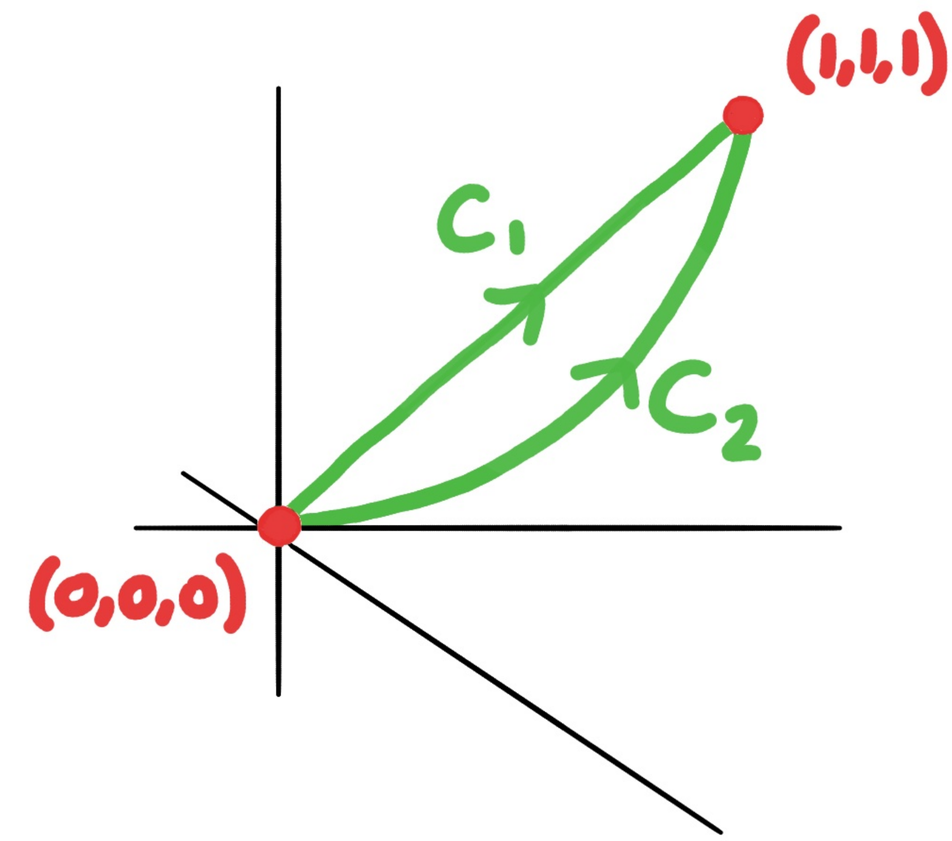

Solution:

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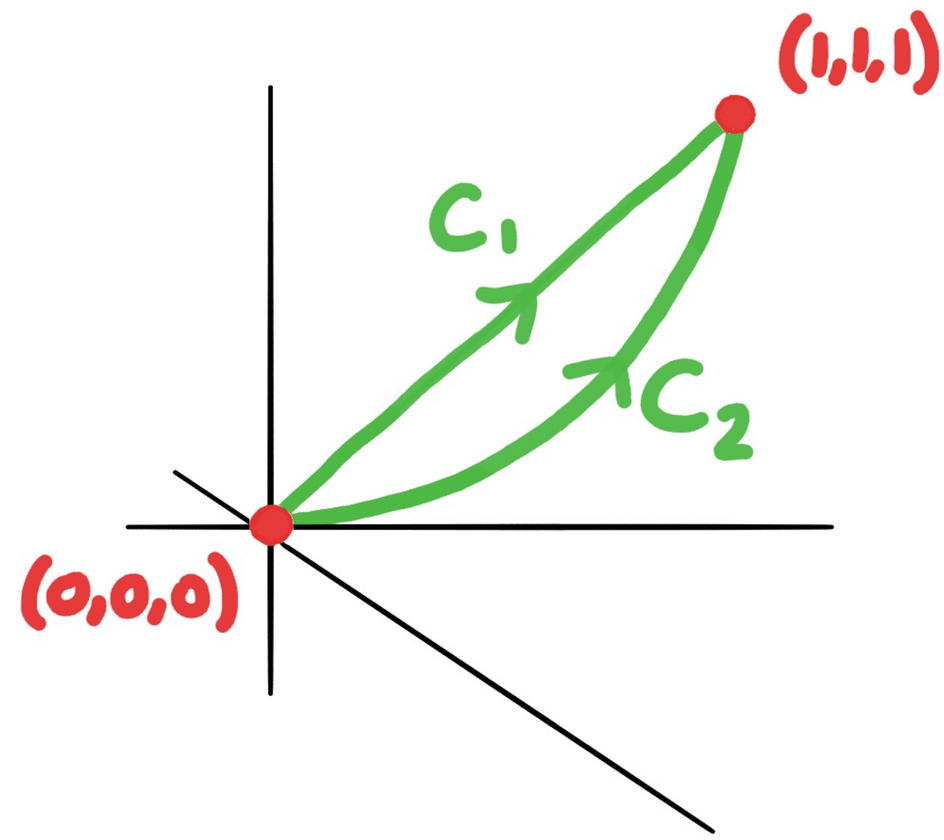

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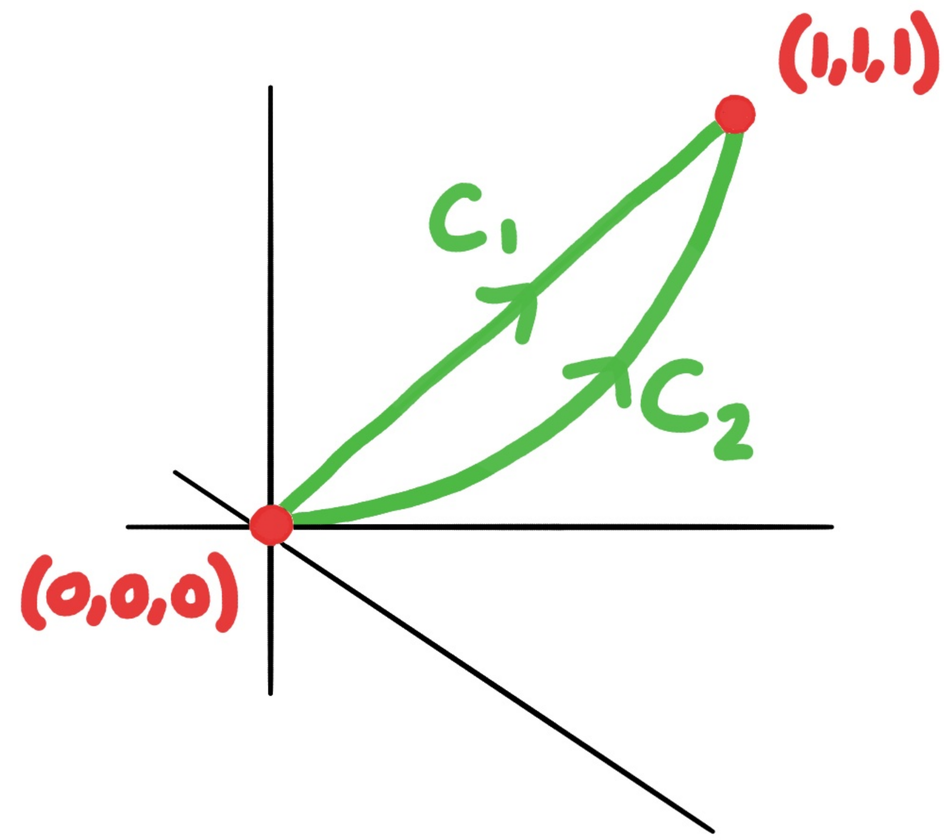

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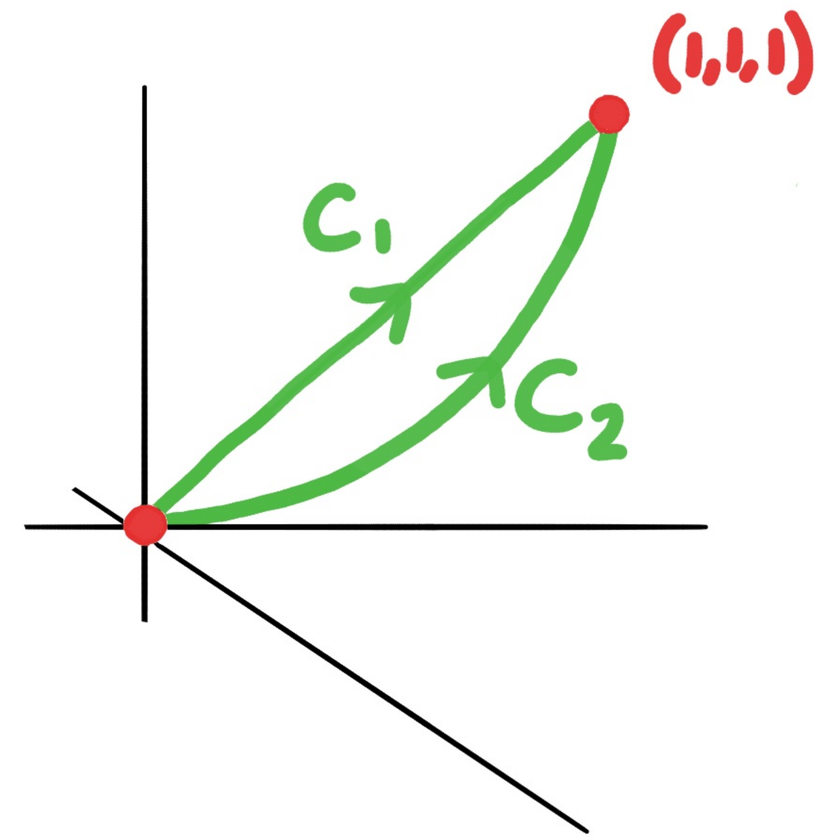

Solution:

$$\int_{C_2} yz\,dx + xz\,dy + xy\,dz$$

$$= \int_0^1 t^2\cdot t\cdot 3t^2 + t^3\cdot t\cdot 2t + t^3\cdot t^2\cdot 1\ dt$$

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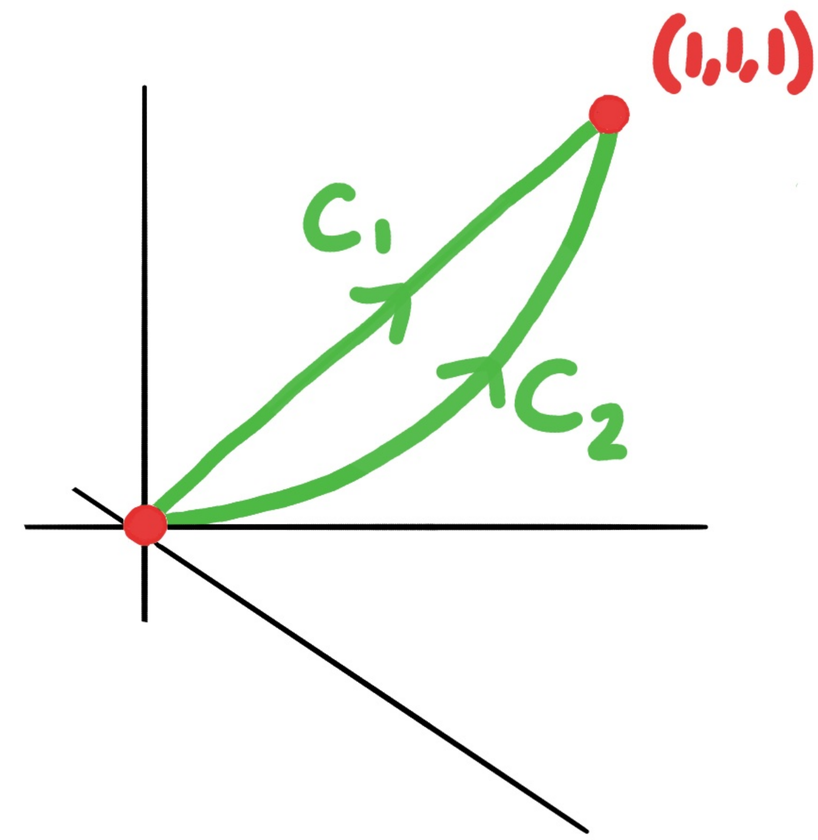

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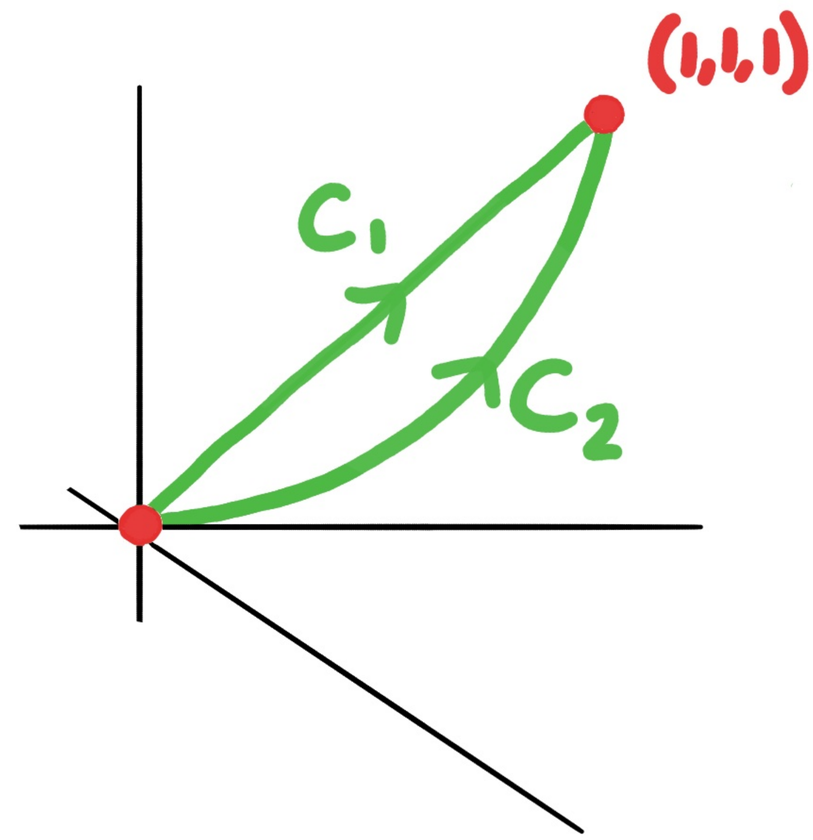

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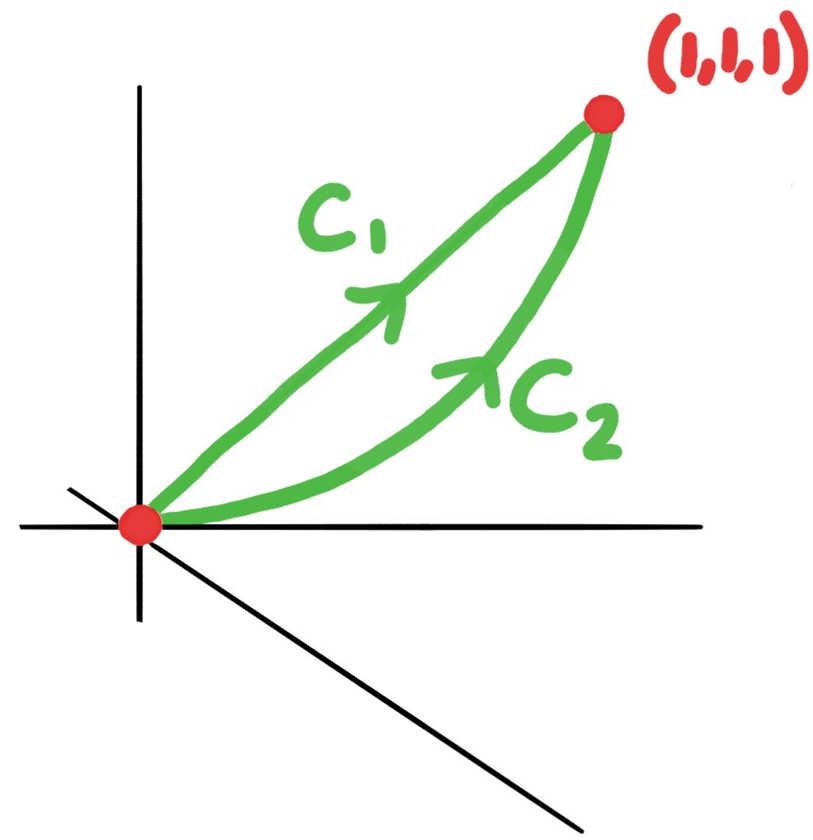

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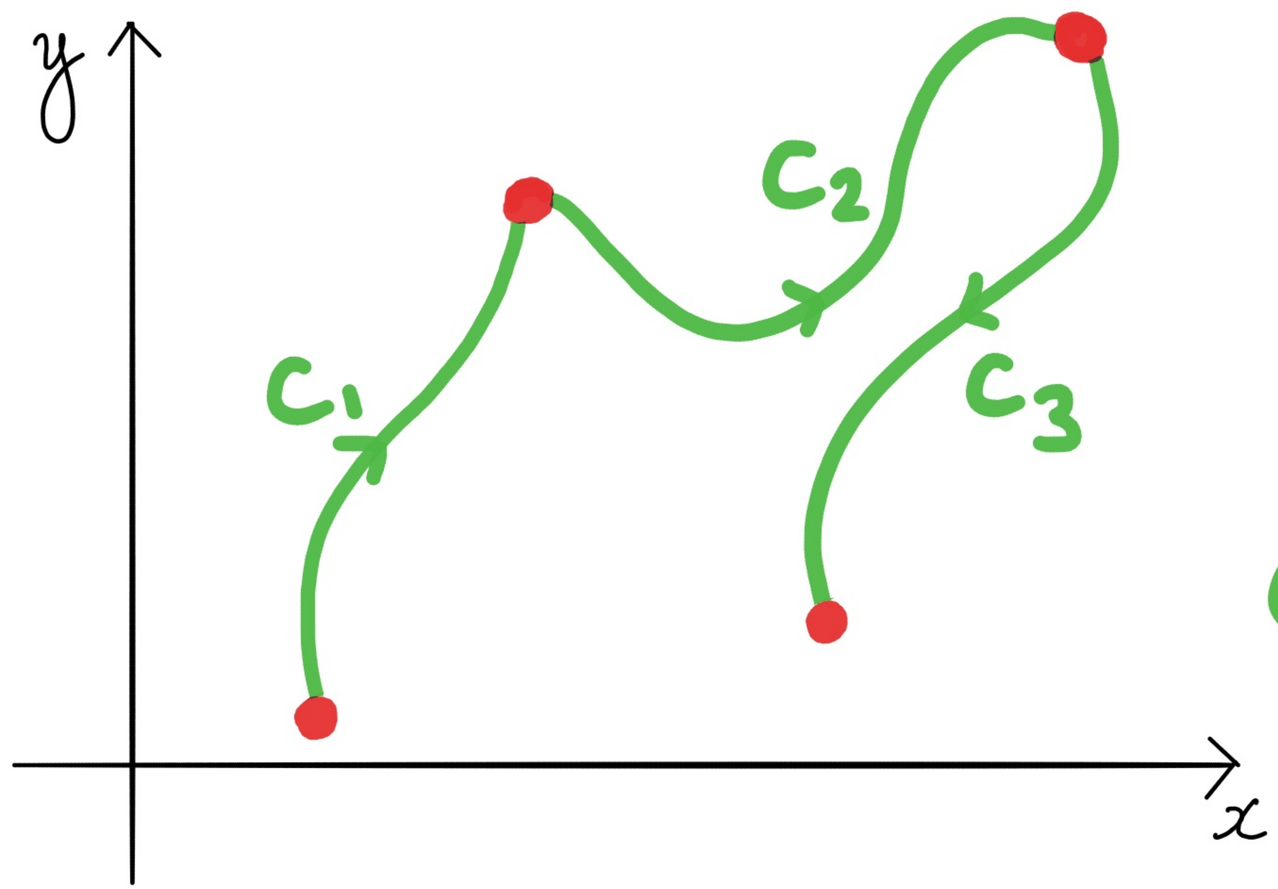
Integration along piecewise paths



$C = C_1 \cup C_2 \cup C_3$

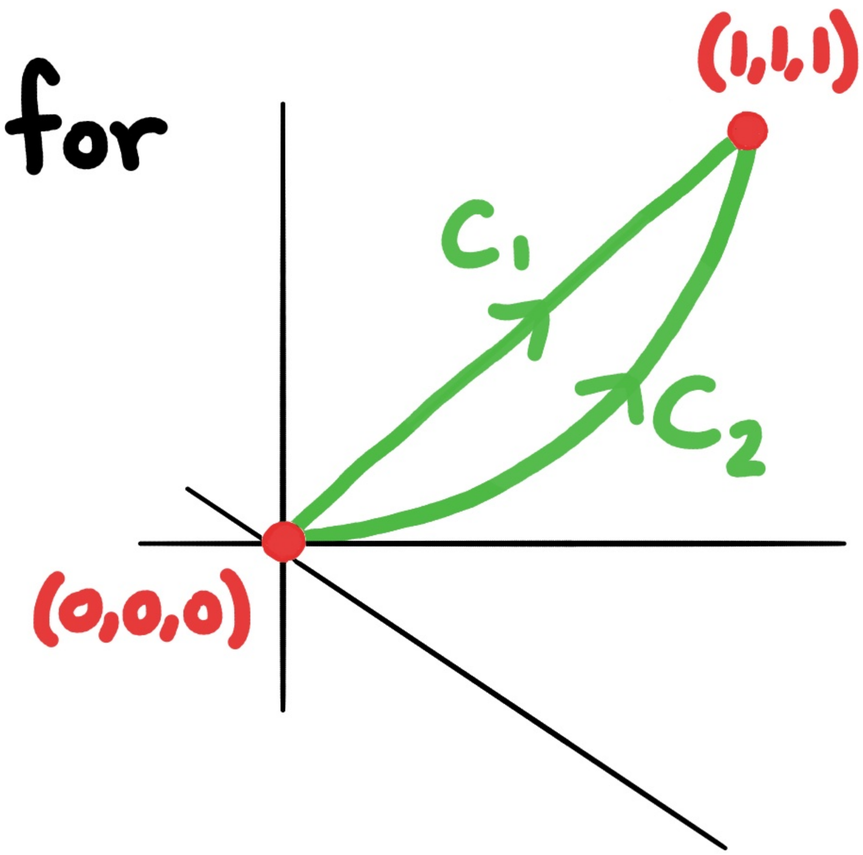
$$\int_C f(x,y)\,d* = \int_{C_1} f(x,y)\,d* + \int_{C_2} f(x,y)\,d* + \int_{C_3} f(x,y)\,d*$$

(whatever $*$ is)

Example: C_1 given by $x=t,\ y=t,\ z=t,\ 0\le t\le 1.$

C_2 given by $x=t^3,\ y=t^2,\ z=t,\ 0\le t\le 1.$

Find $\int_C yz\,dx + xz\,dy + xy\,dz$ for the simple closed curve $C = C_1 \cup -C_2$.



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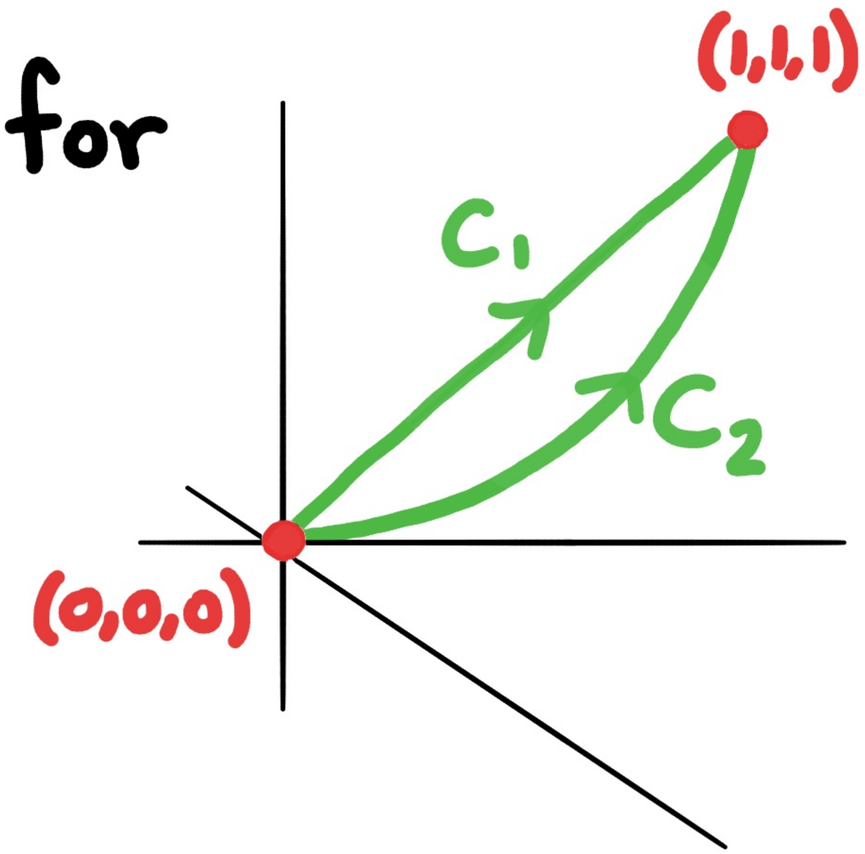


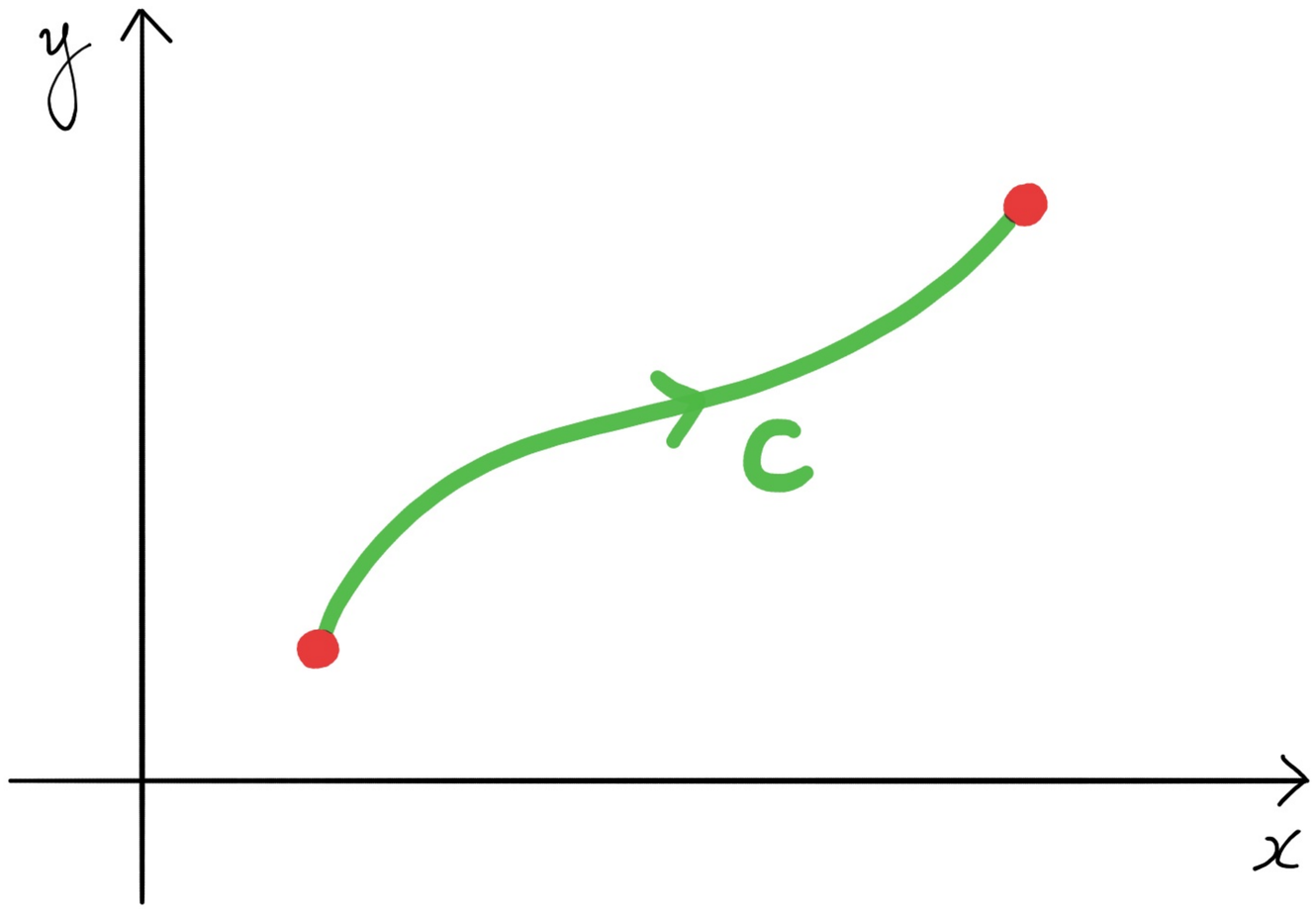
Solution:

$$\int_C yz\,dx + xz\,dy + xy\,dz$$

$$= \int_{C_1} yz\,dx + xz\,dy + xy\,dz + \int_{-C_2} yz\,dx + xz\,dy + xy\,dz$$

II Idea of line integrals

Idea of $\int_C ds$



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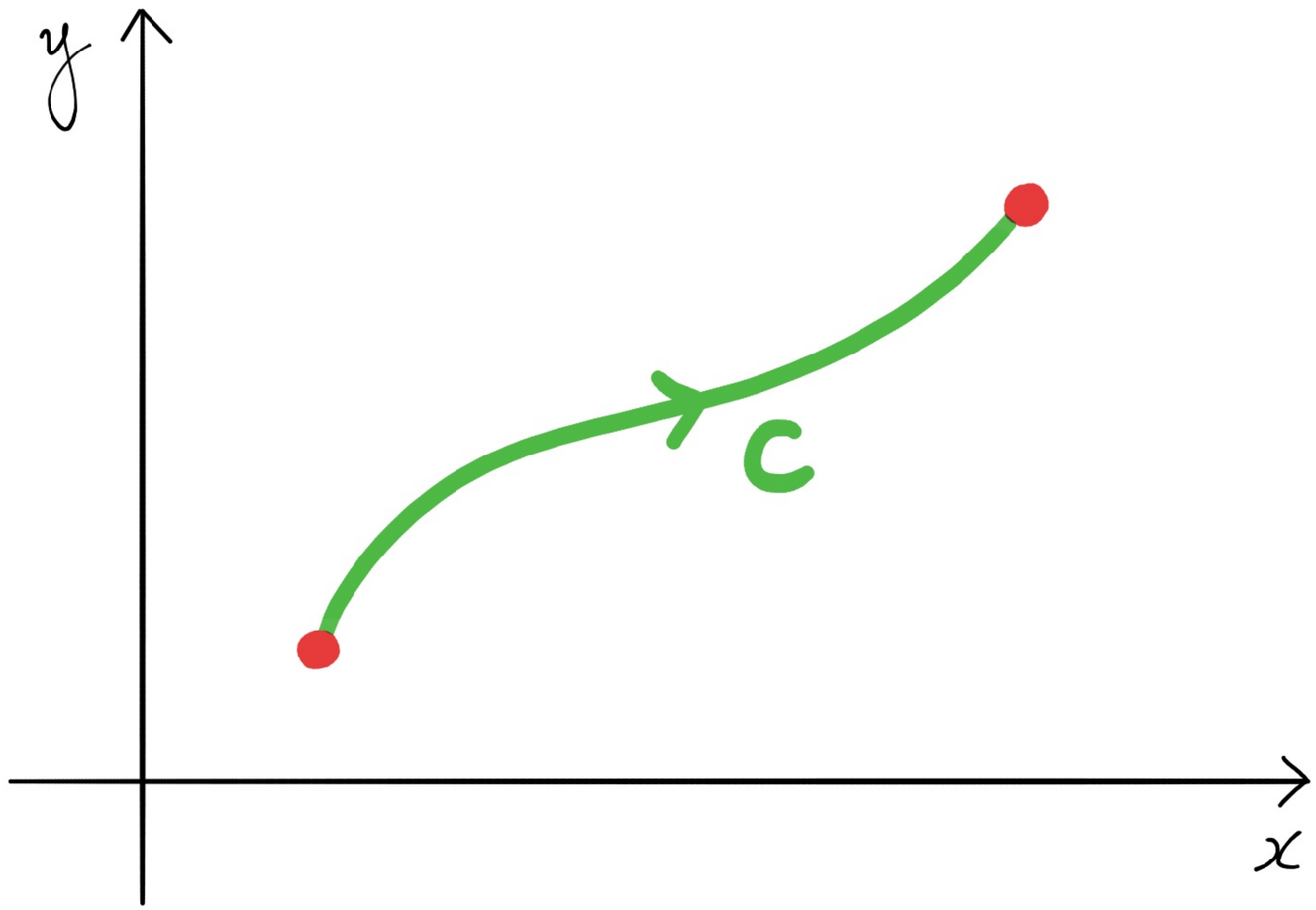
Idea of $\int_C ds$



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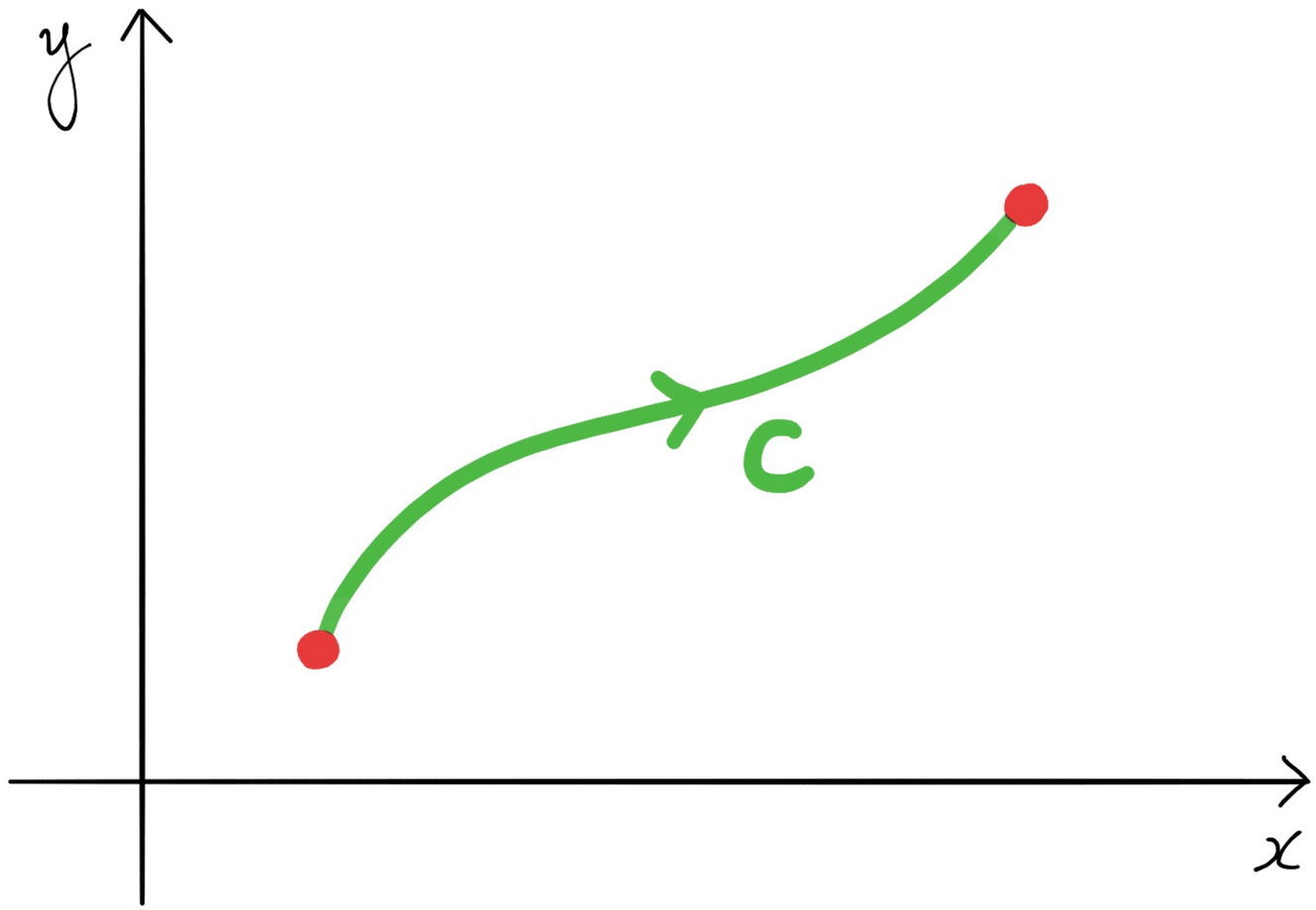
Idea of $\int_C ds$



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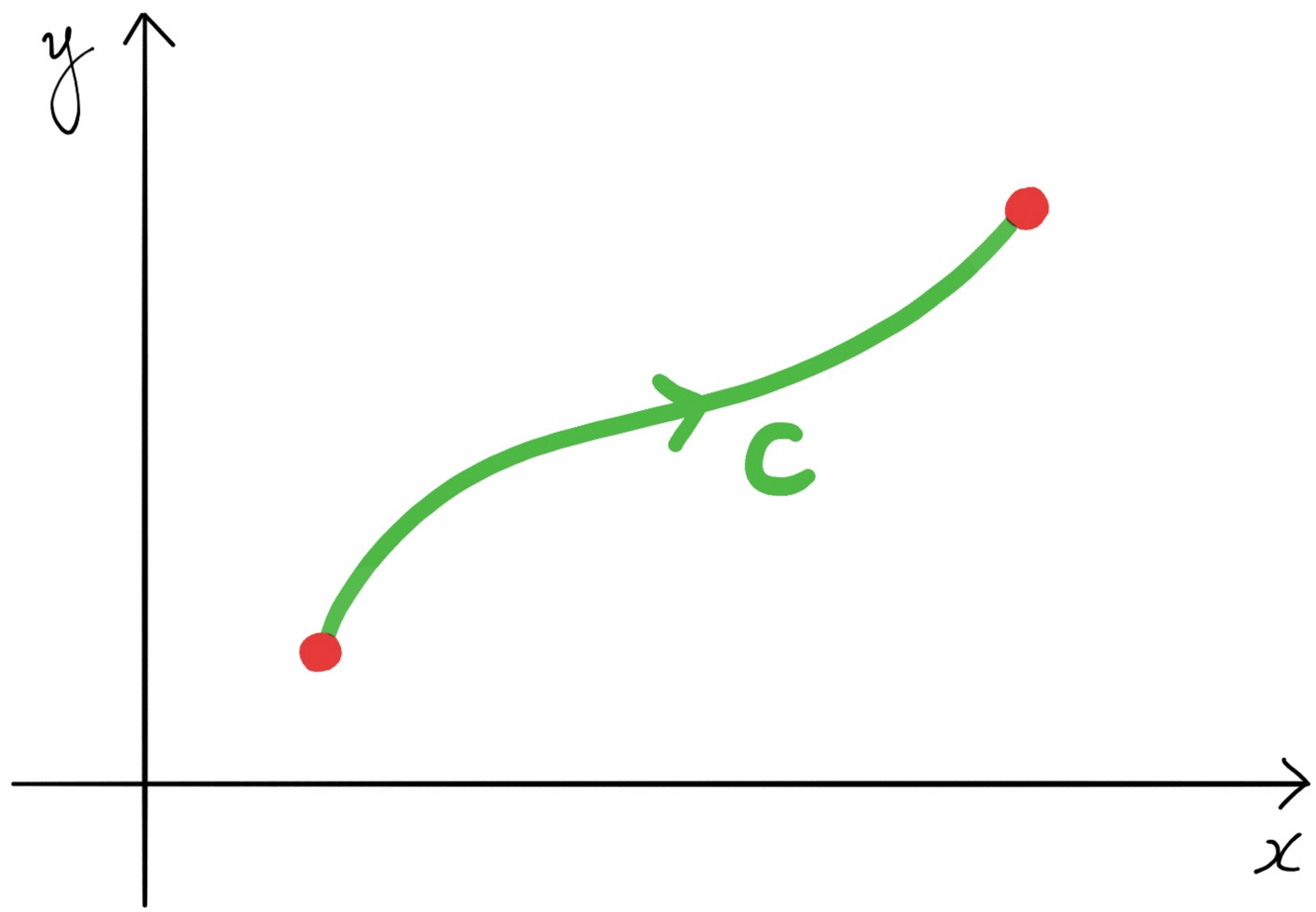
Idea of $\int_C ds$



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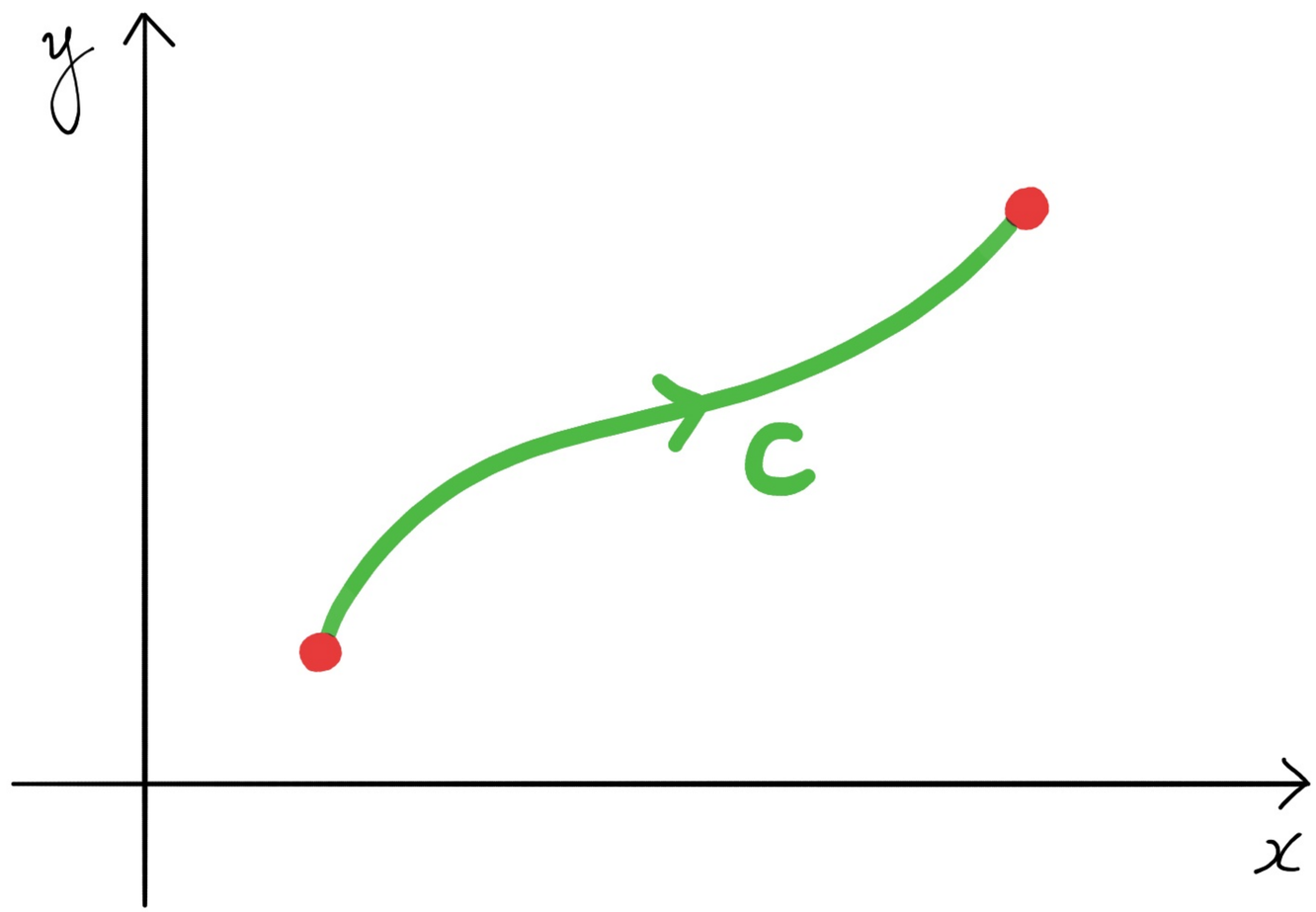
Idea of $\int_C ds$



$\underline{C}$: $a \le t \le b$

$x = x(t)$

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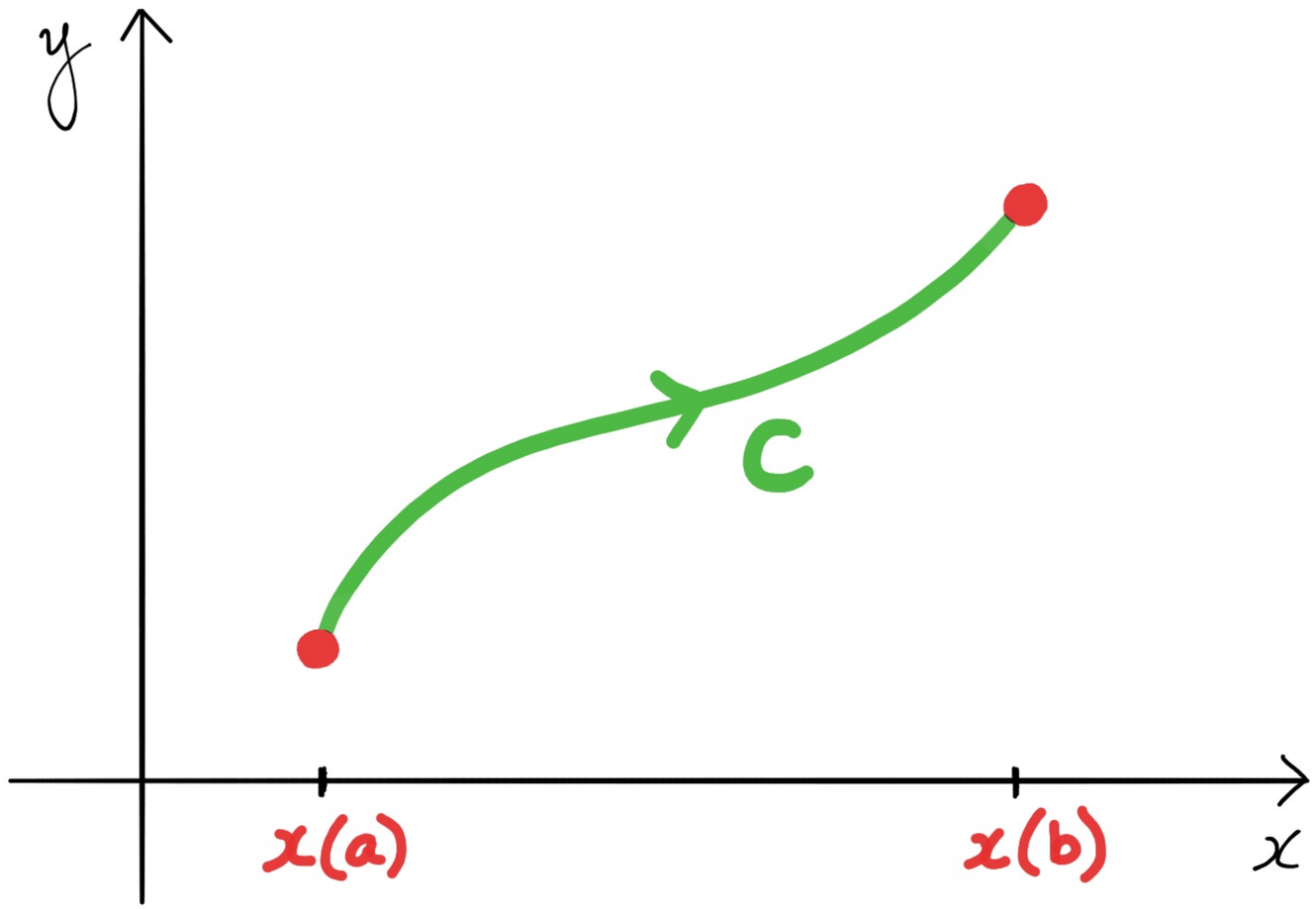
- $\int_C ds = \int_a^b \|(x'(t), y'(t))\| \, dt = \text{Arclength}(C)$
- $\int_C ds = \int_{-C} ds$

Idea of $\int_C dx$



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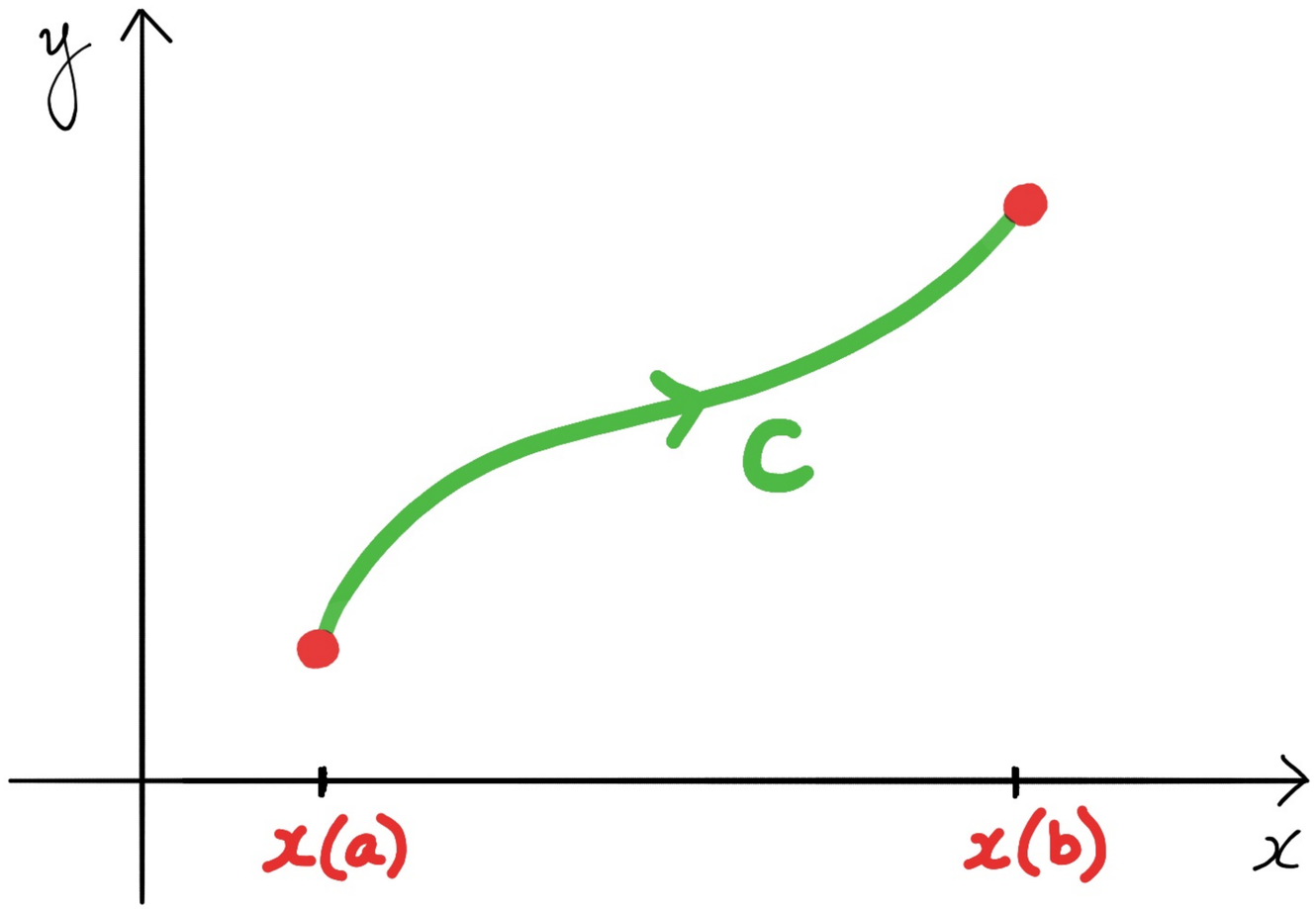
Idea of $\int_C dx$



$\underline{C}$: $a \le t \le b$

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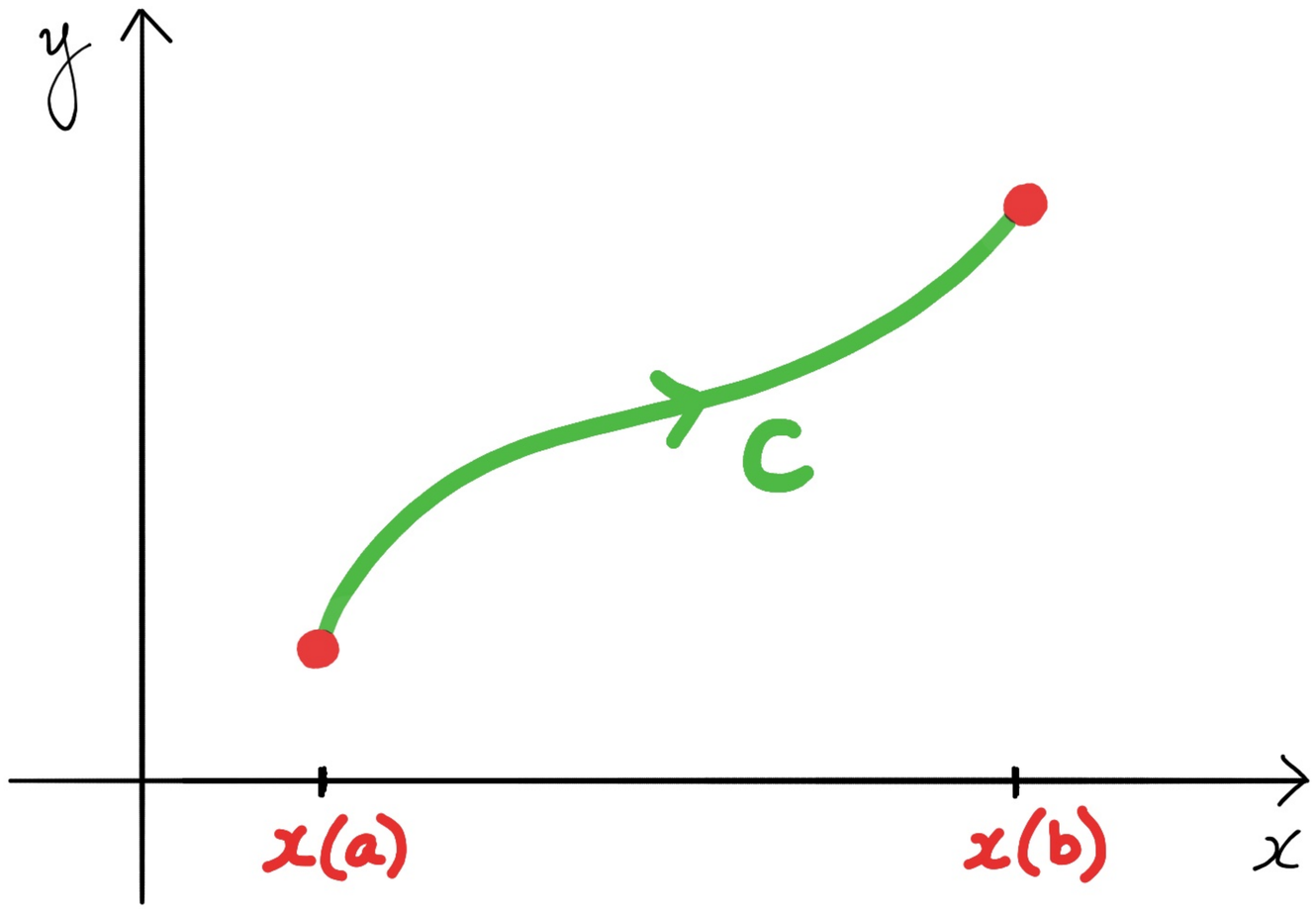
- $\int_C dx =$

Idea of $\int_C dx$



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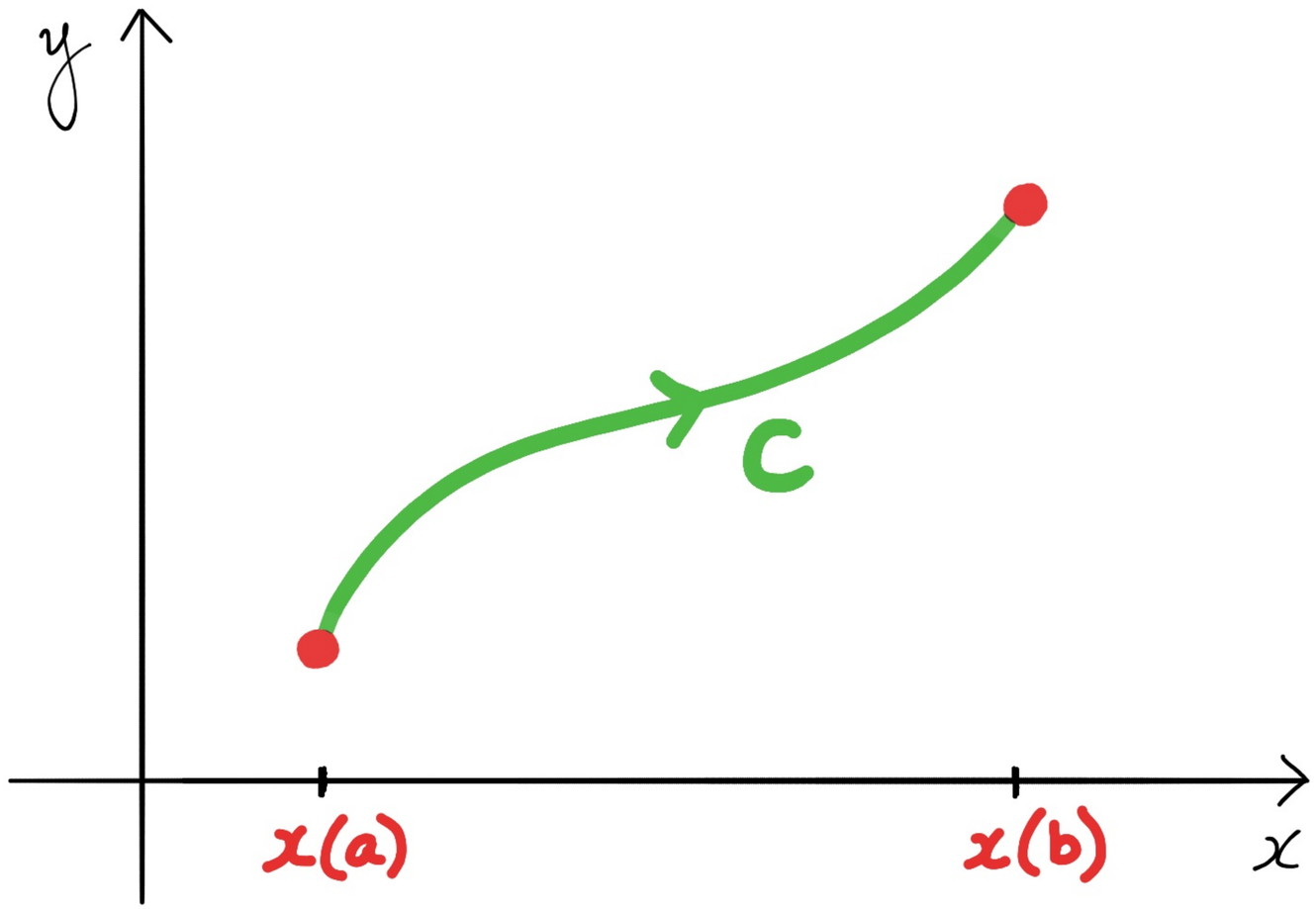
Idea of $\int_C dx$



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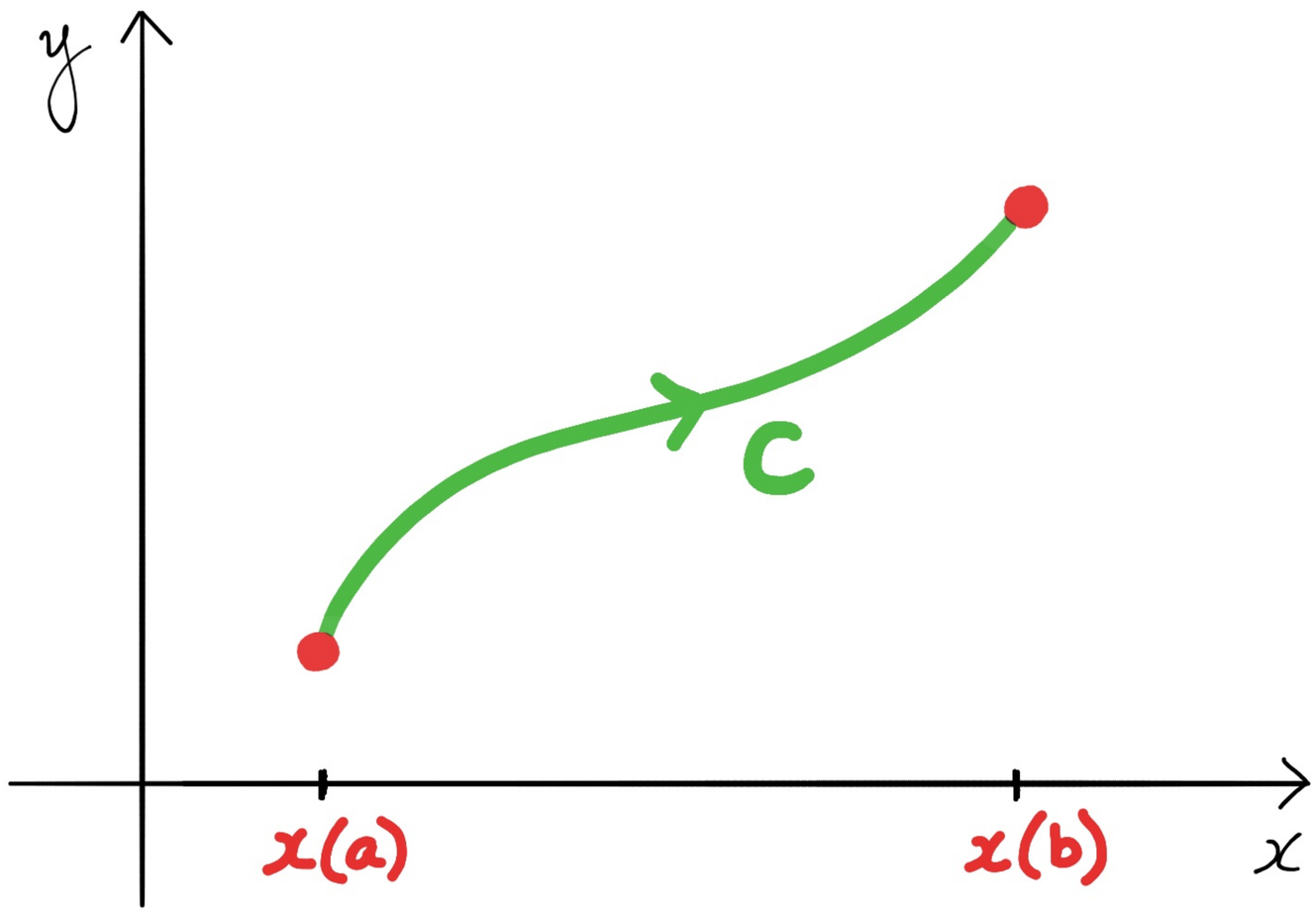
Idea of $\int_C dx$



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$x = x(t)$

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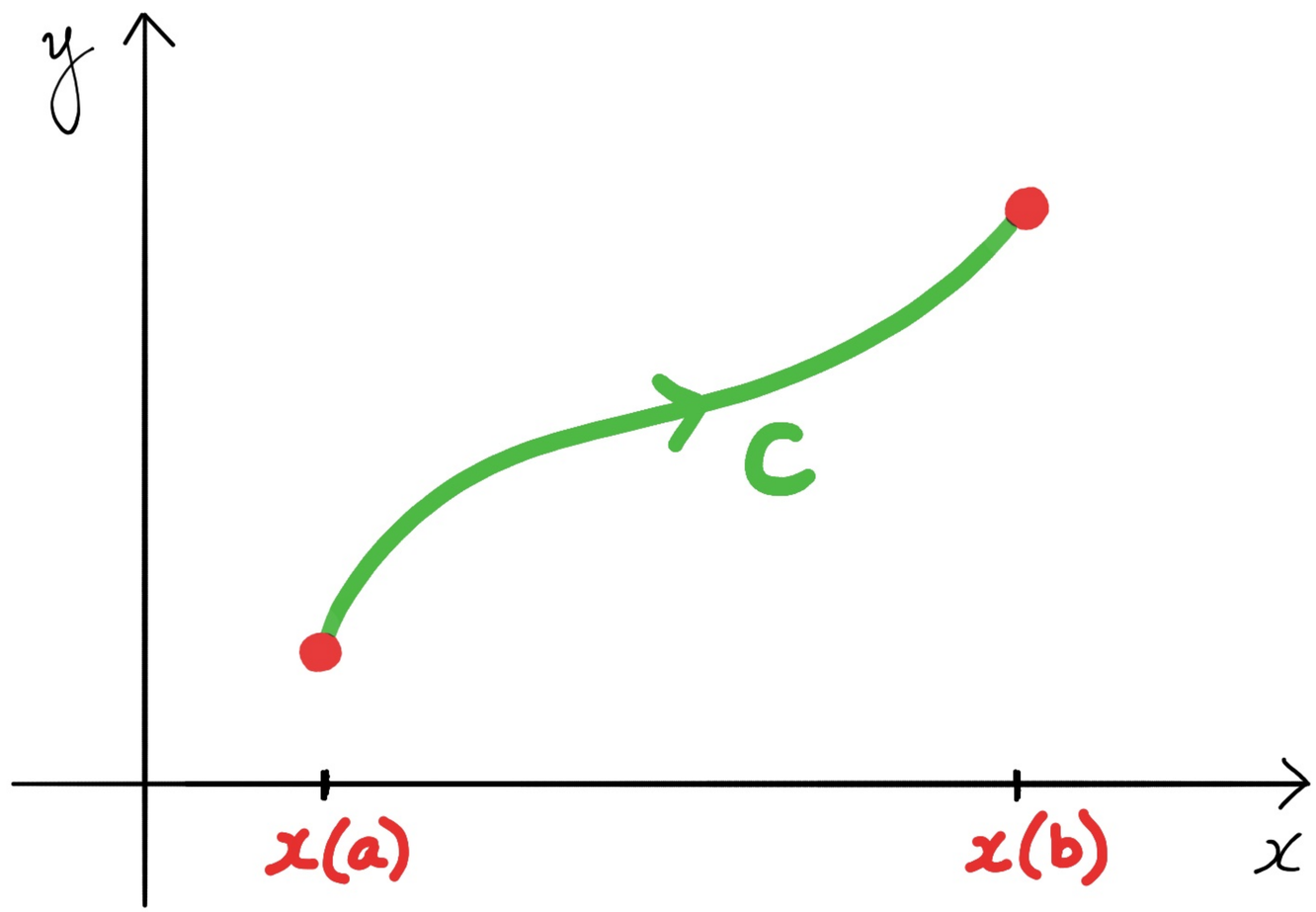
- $$\int_C dx = \int_a^b x'(t)\,dt = \Big[x(t)\Big]_a^b = x(b) - x(a)$$

Idea of $\int_C dx$



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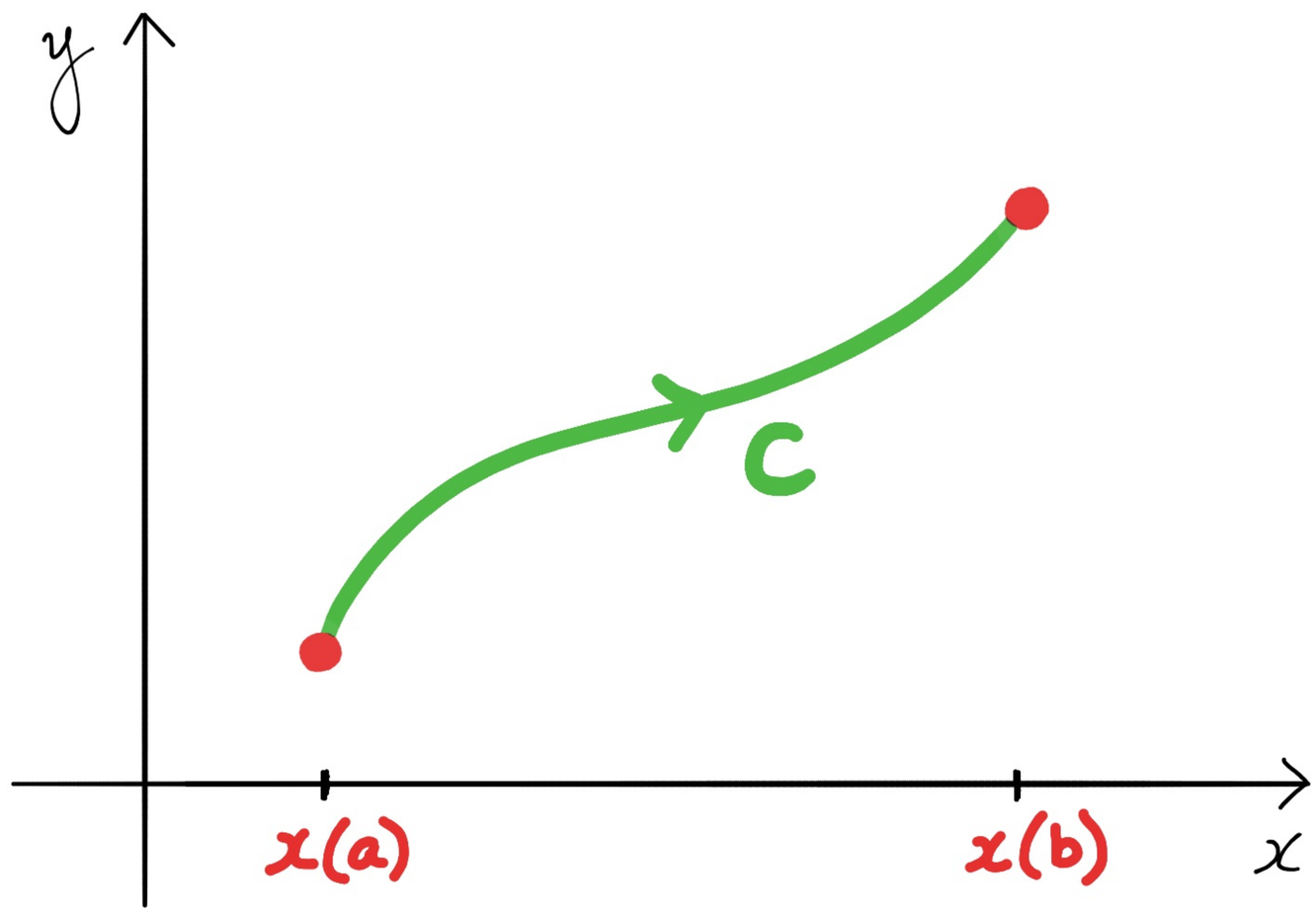
- $$\int_C dx = \int_a^b x'(t)\,dt = \Big[x(t)\Big]_a^b = \underbrace{x(b) - x(a)}_{\text{change in } x\text{-coordinate}}$$

Idea of $\int_C dx$



$\underline{C}$: $a \le t \le b$

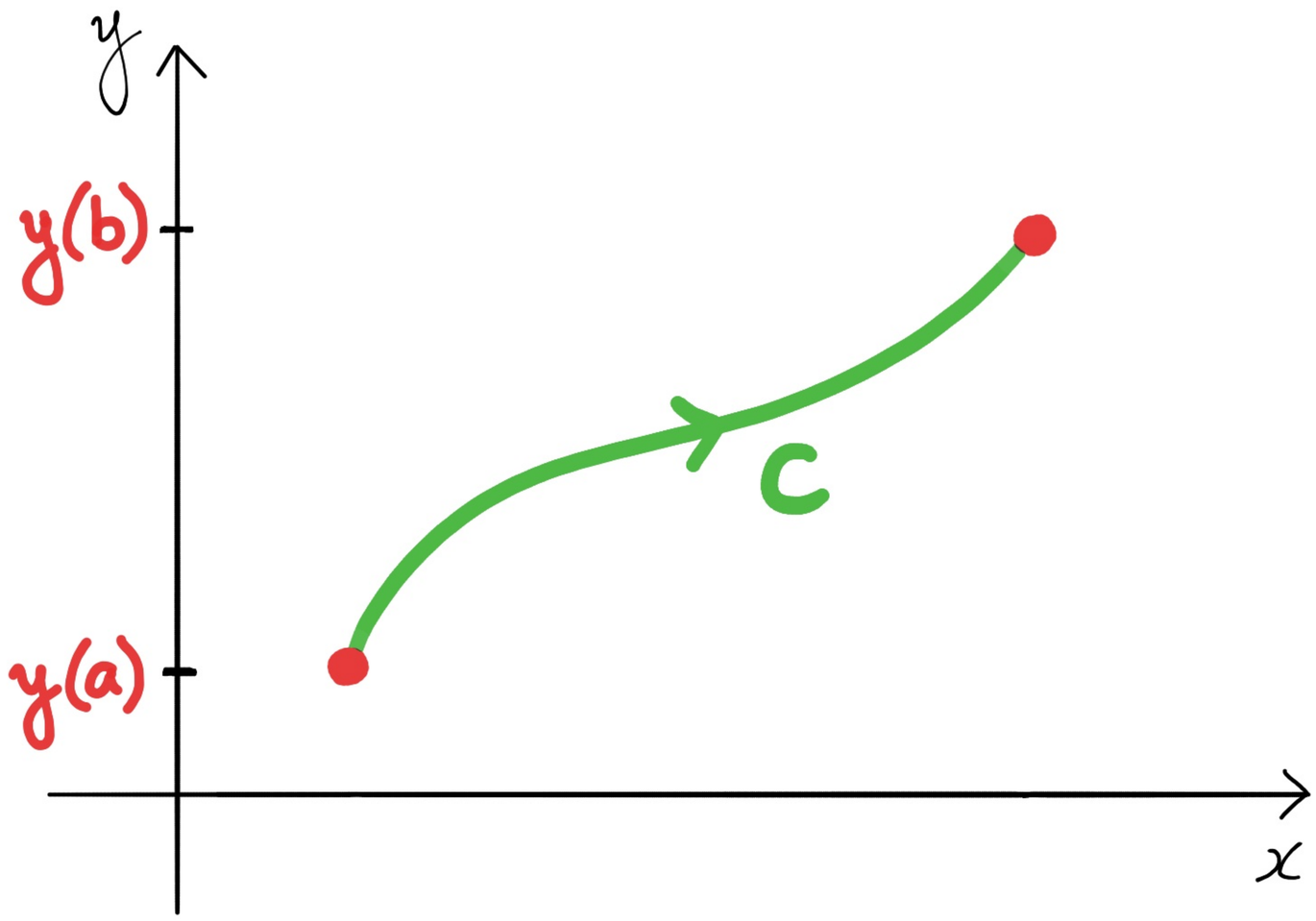
$x = x(t)$

$y = y(t)$

- $$\int_C dx = \int_a^b x'(t)\,dt = \Big[x(t)\Big]_a^b = \underbrace{x(b) - x(a)}_{\text{change in } x\text{-coordinate}}$$

- $$\int_{-C} dx = -\int_C dx$$

Idea of $\int_C dy$



$\underline{C}$: $a \le t \le b$

$x = x(t)$

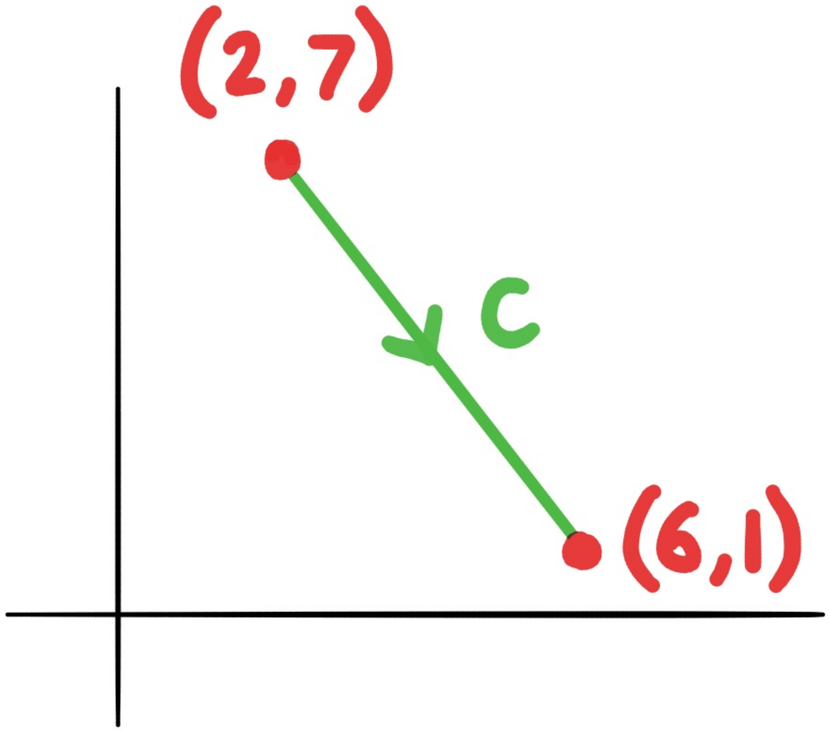
$y = y(t)$

- $\int_C dy = \int_a^b y'(t)\,dt = \left[y(t)\right]_a^b = \underbrace{y(b) - y(a)}_{\text{change in } y\text{-coordinate}}$
- $\int_{-C} dy = -\int_C dy$

Example:

C parametrized by

$$x = 2 + 4t$$
$$y = 7 - 6t$$
$$0 \le t \le 1.$$



Find $\int_C ds =$

$\int_C dx =$

$\int_C dy =$

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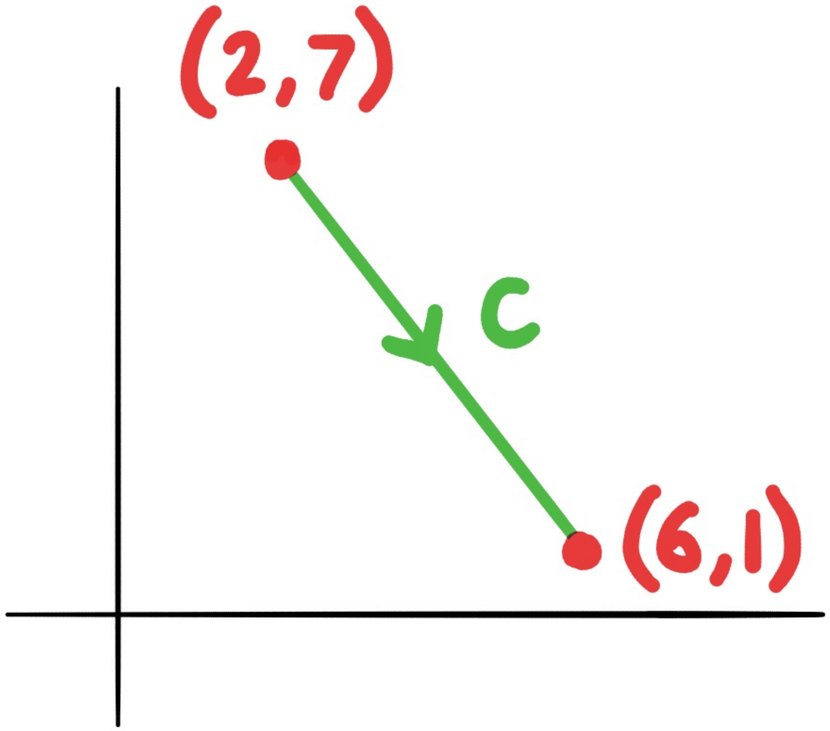


$$\int_C ds = \sqrt{4^2 + 6^2} = 2\sqrt{13}$$

$$\int_C dx = 6 - 2 = 4$$

$$\int_C dy = 1 - 7 = -6$$

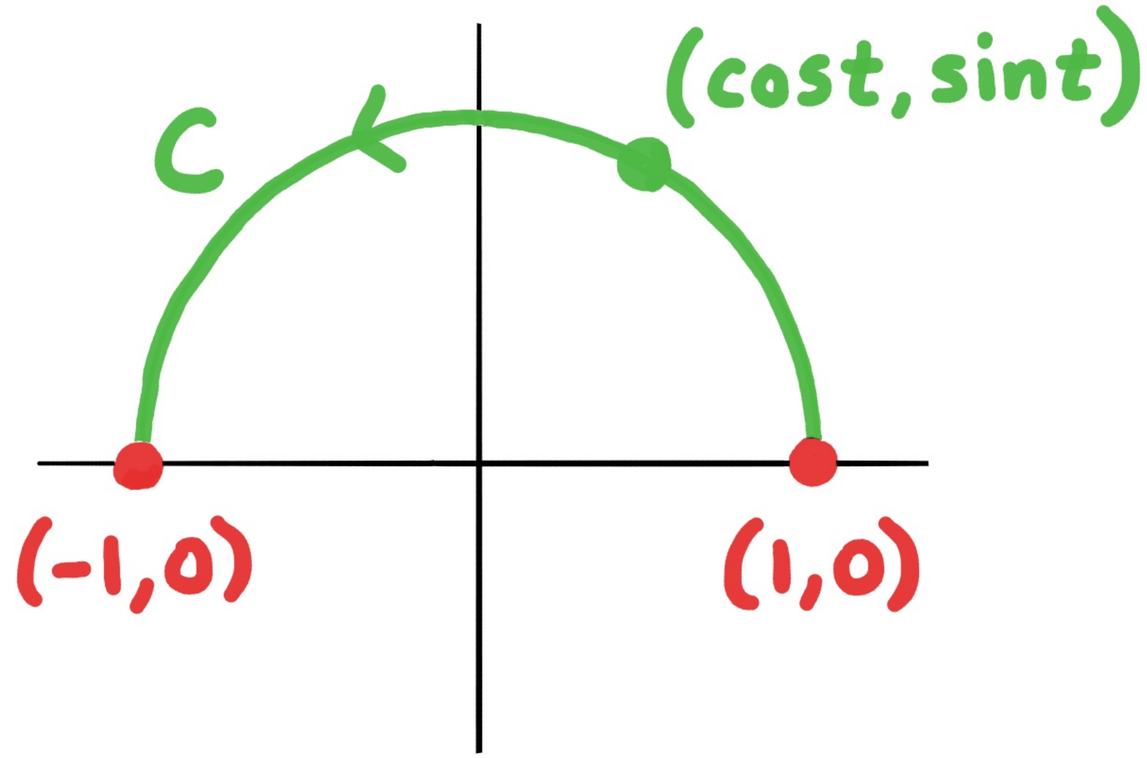
Example:

C parametrized by

$x = \cos t,$

$y = \sin t,$

$0 \le t \le \pi.$



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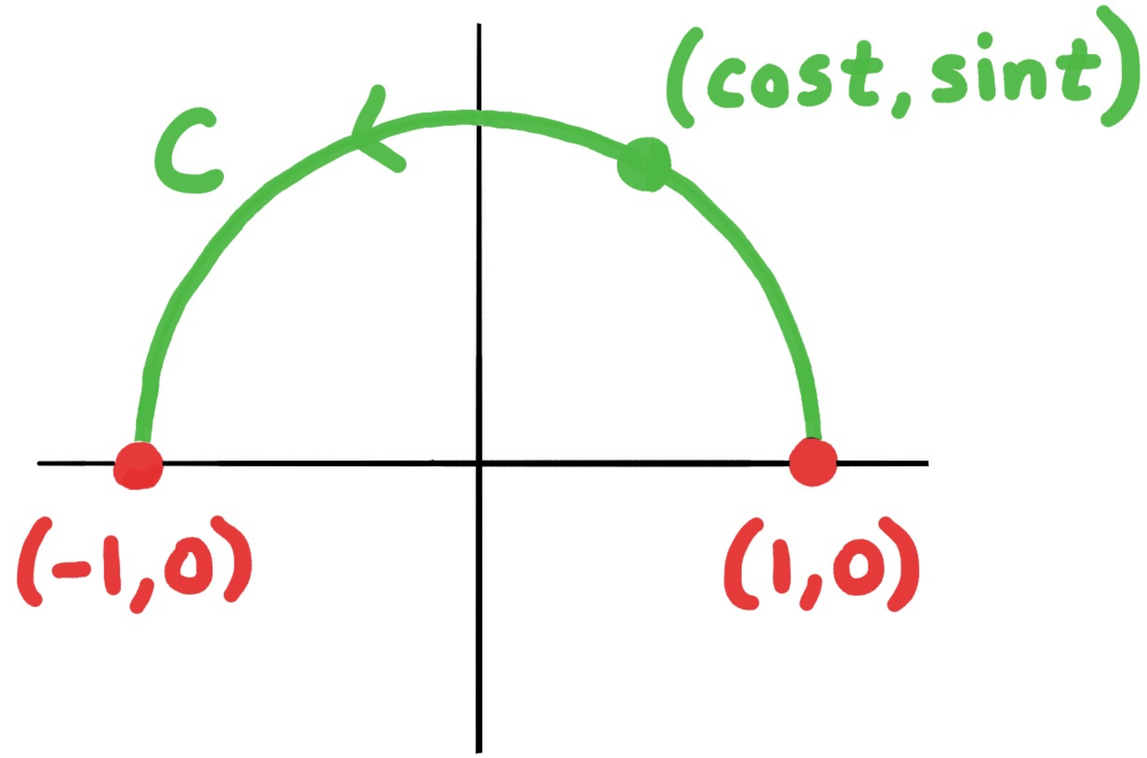
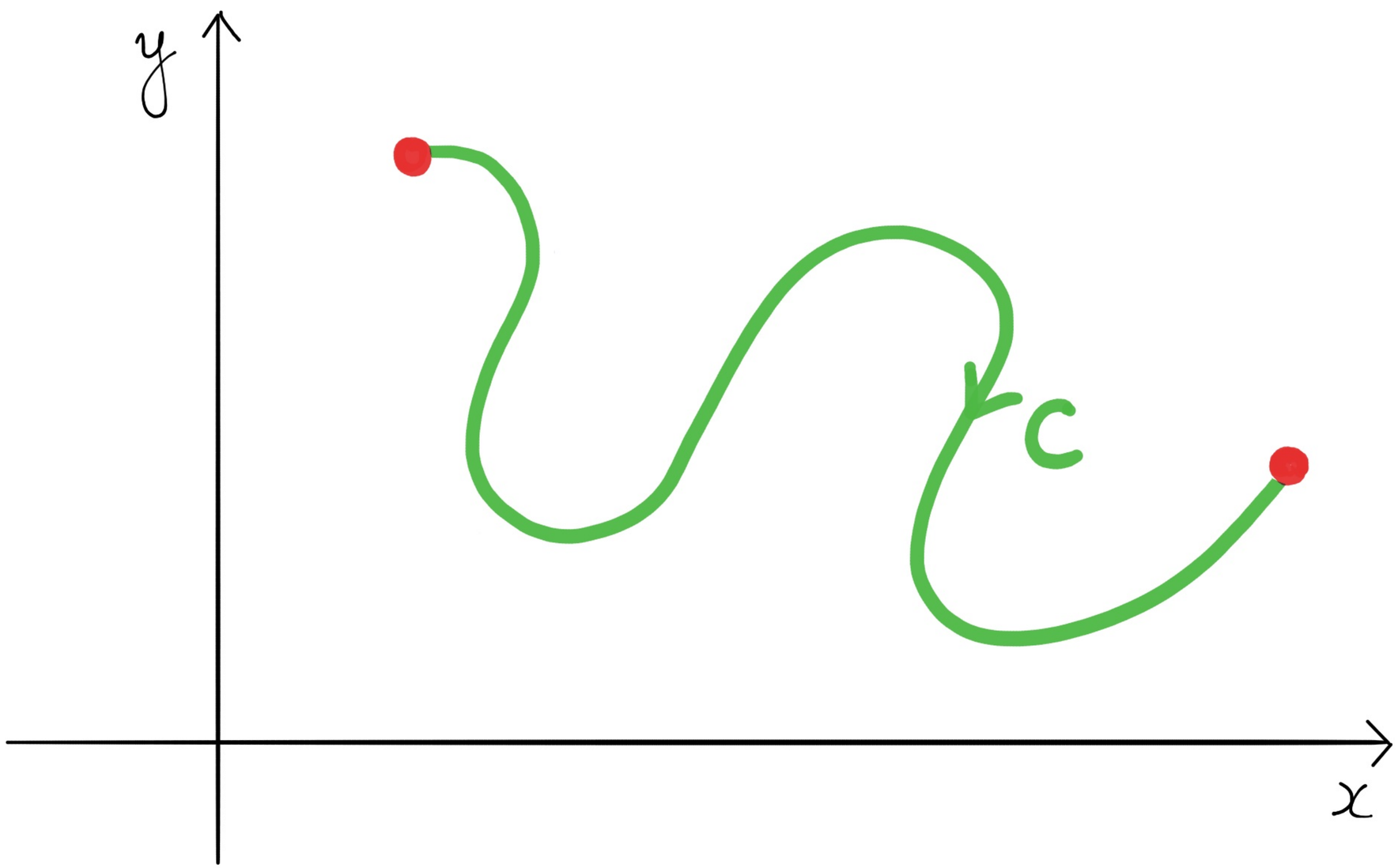


$$\int_C ds = \pi$$

$$\int_C dx = -2$$

$$\int_C dy = 0$$

$\int_C f(x,y)\,ds$ is arclength weighted by $f(x,y)$



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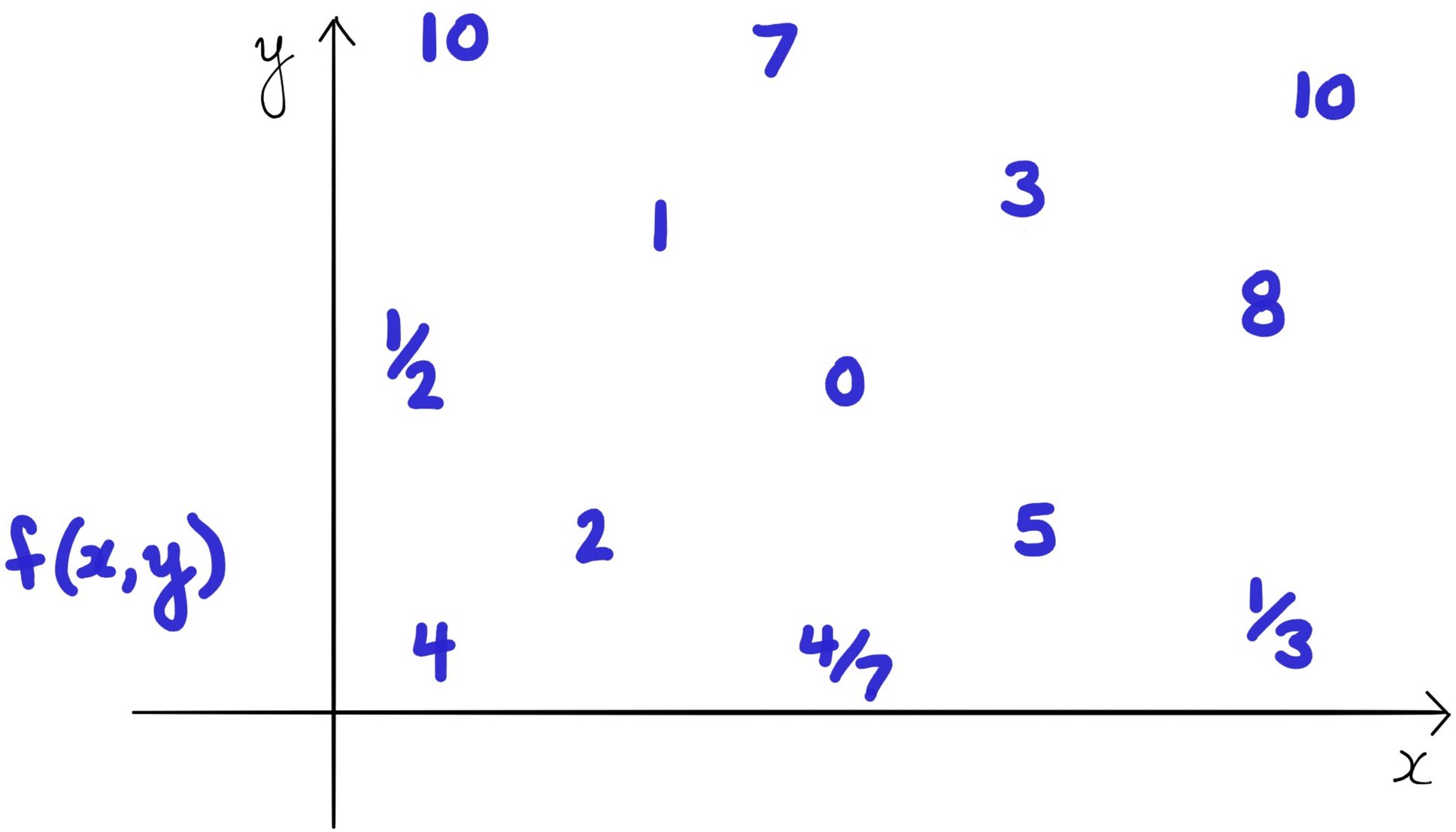

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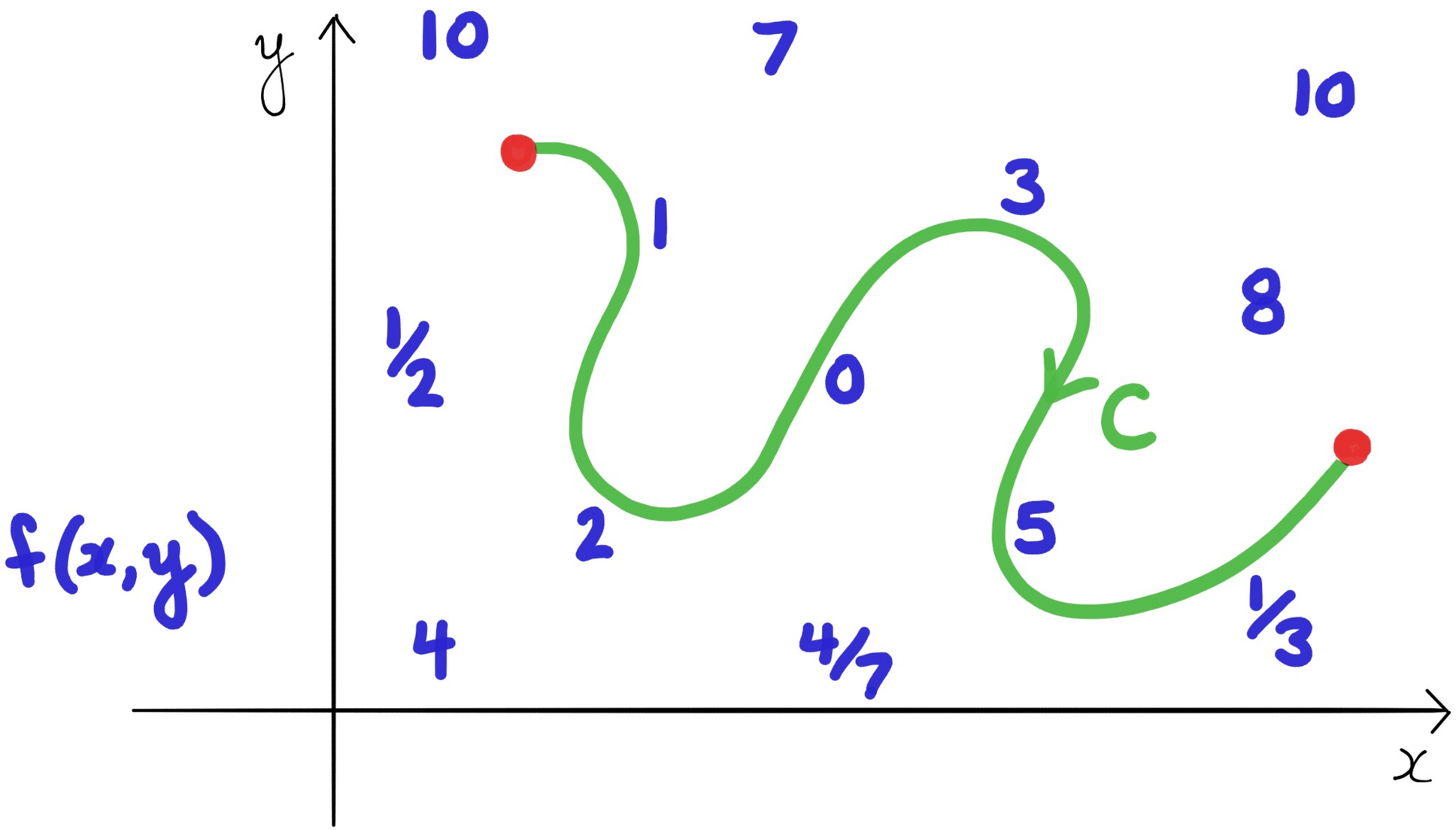

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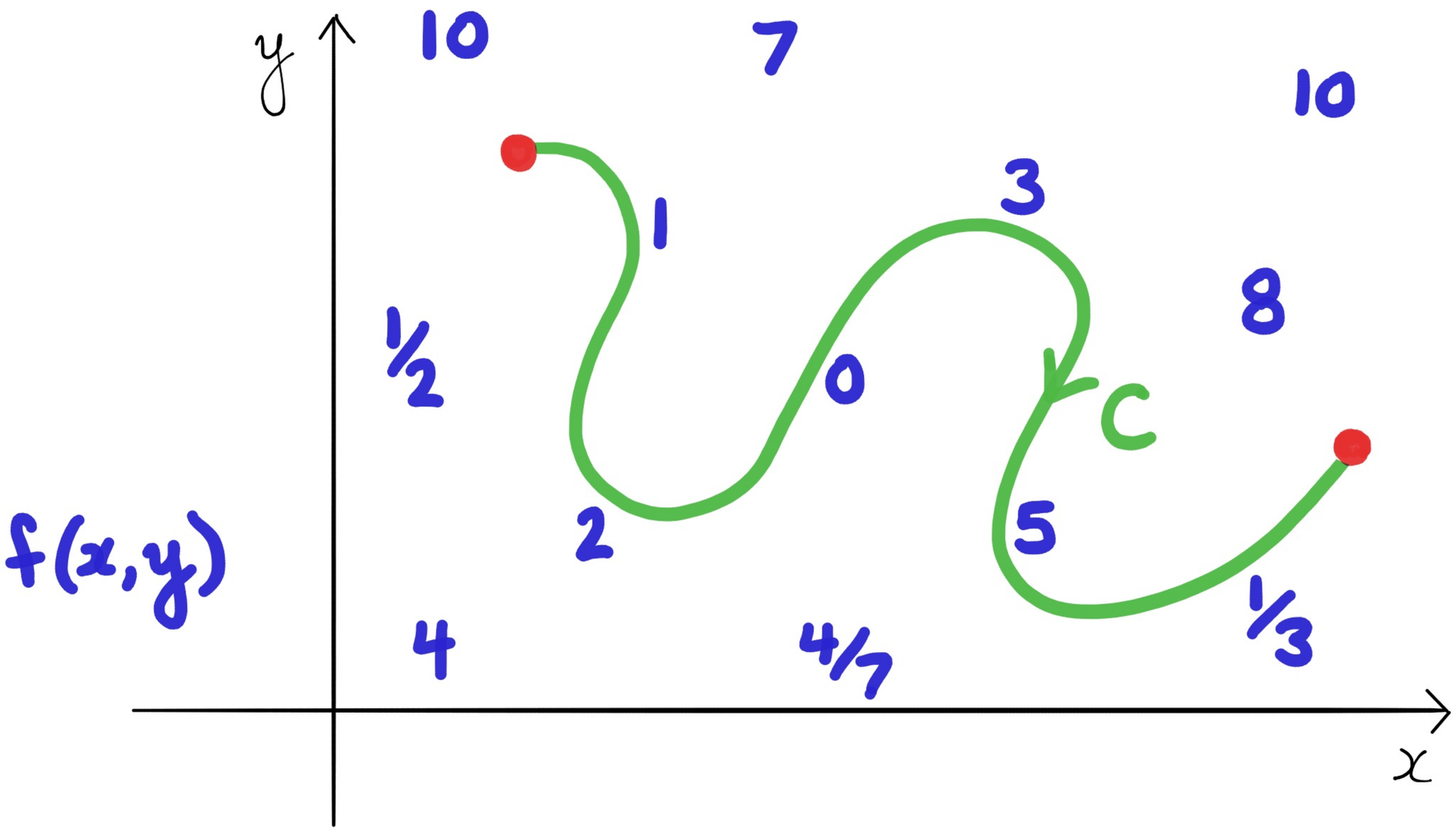


- $\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$

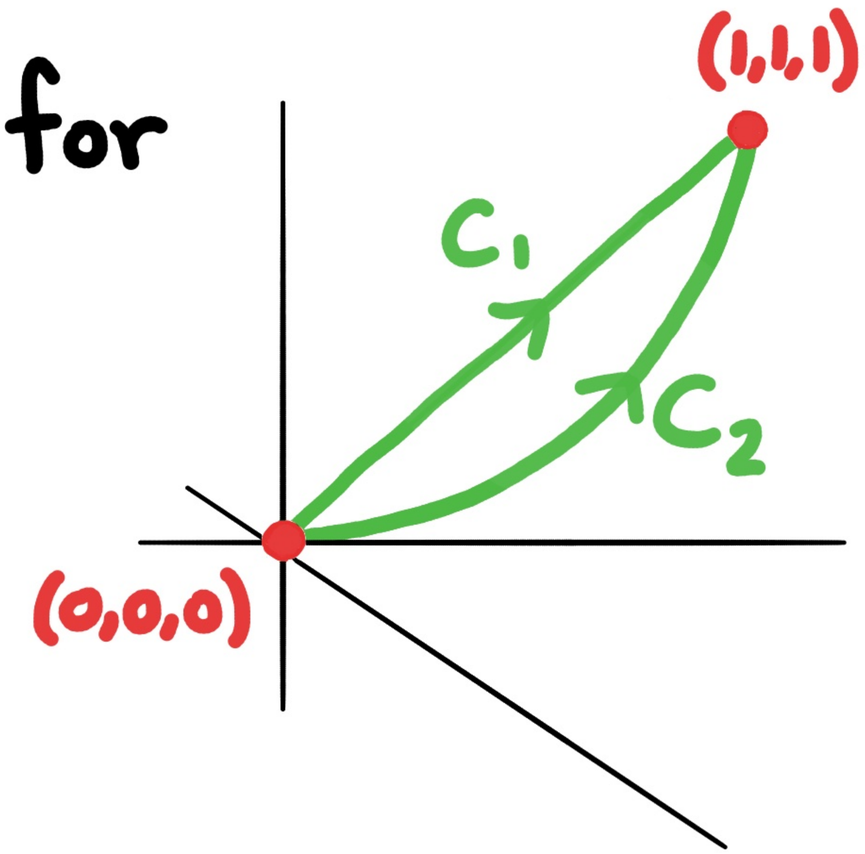
Conversely,

$$\int_{-C} f(x,y)\,dx + g(x,y)\,dy = -\int_{C} f(x,y)\,dx + g(x,y)\,dy$$

Example: C_1 given by $x=t,\ y=t,\ z=t,\ 0\le t\le 1.$

C_2 given by $x=t^3,\ y=t^2,\ z=t,\ 0\le t\le 1.$

Find $\int_C yz\,dx + xz\,dy + xy\,dz$ for the simple closed curve $C = C_1 \cup -C_2$.



Solution:

$$\int_C yz\,dx + xz\,dy + xy\,dz$$

$$= \int_{C_1} yz\,dx + xz\,dy + xy\,dz + \int_{-C_2} yz\,dx + xz\,dy + xy\,dz$$

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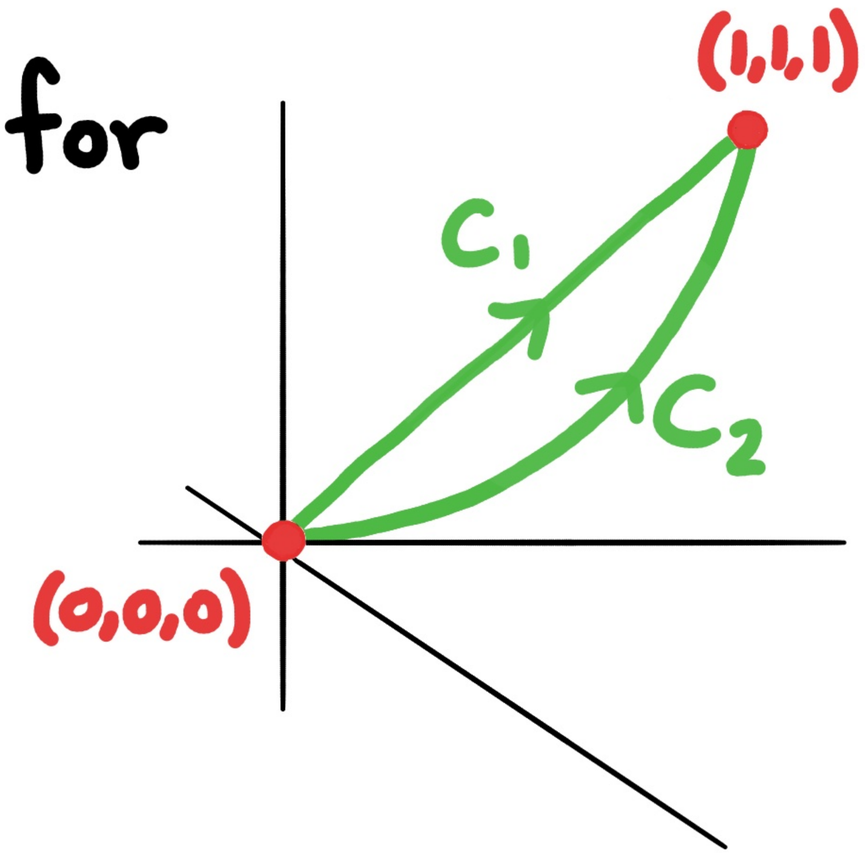

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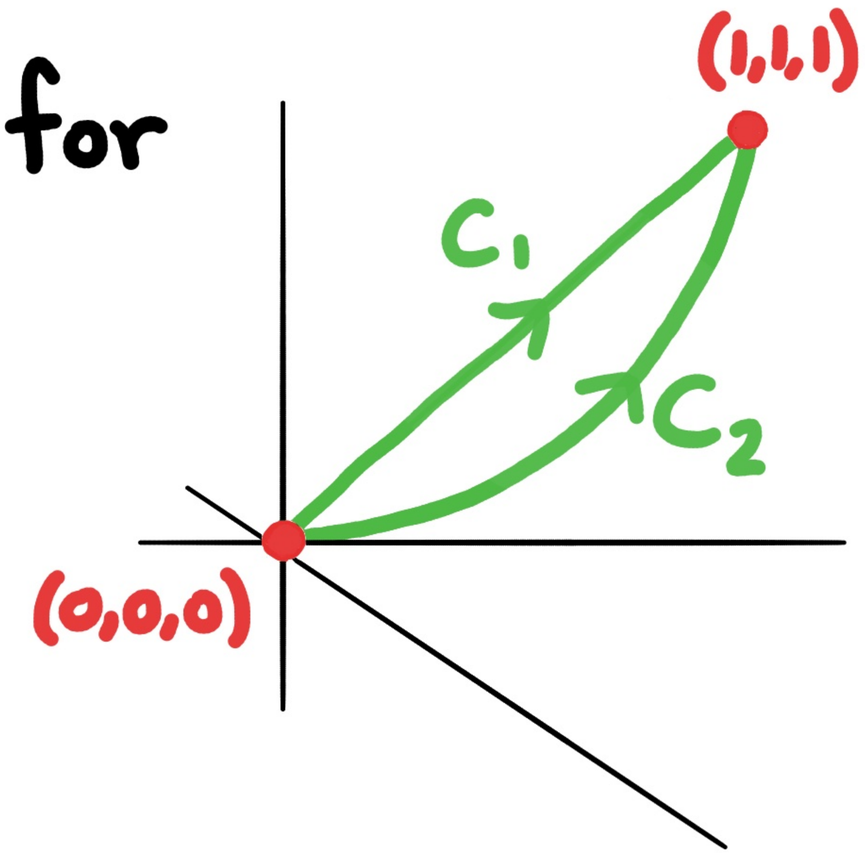

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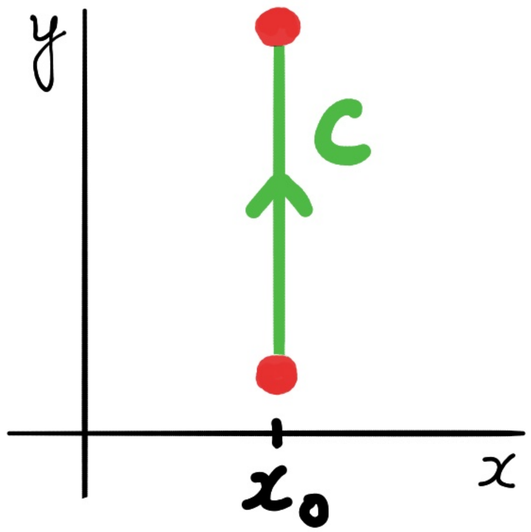
$$= \int_{C_1} yz\,dx + xz\,dy + xy\,dz - \int_{C_2} yz\,dx + xz\,dy + xy\,dz$$

$$= 1 - 1 = 0.$$

III Four useful lemmas

Lemma:

If C is a vertical straight line in $\mathbb{R}^2$, then for any $f:\mathbb{R}^2 \to \mathbb{R}$, $\int_C f(x,y)\,dx = 0$.



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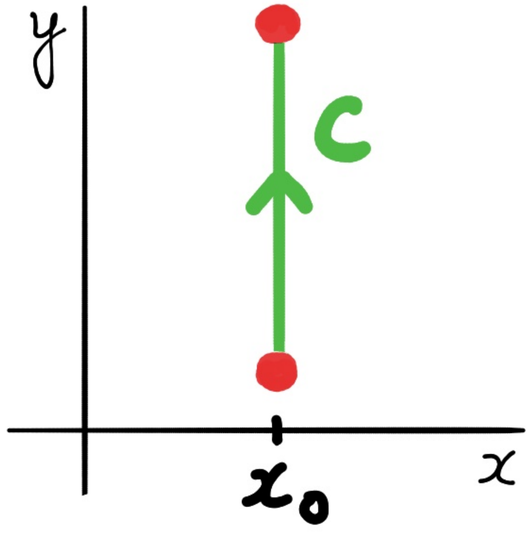

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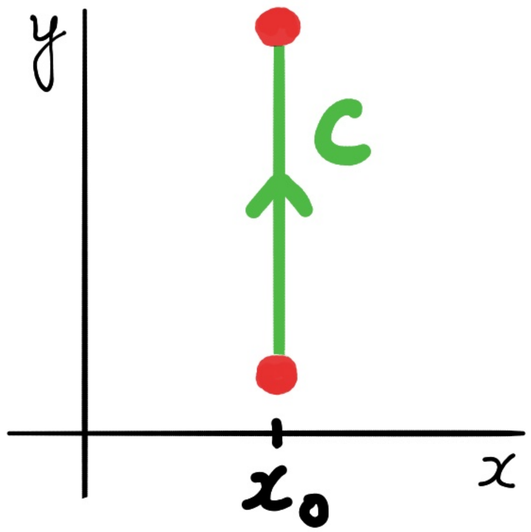

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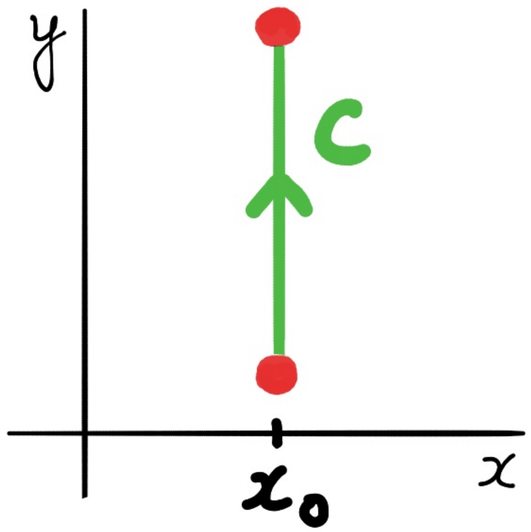

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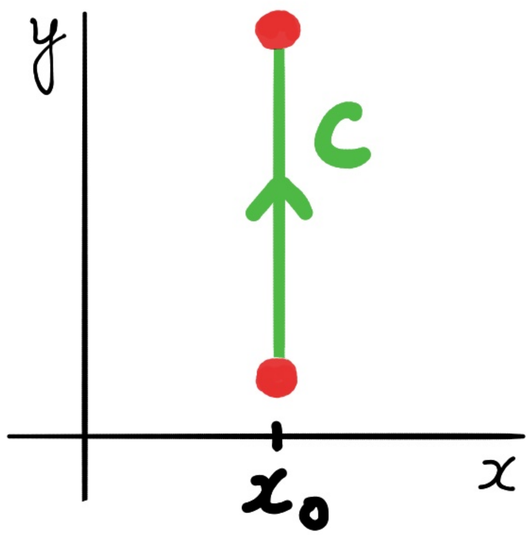

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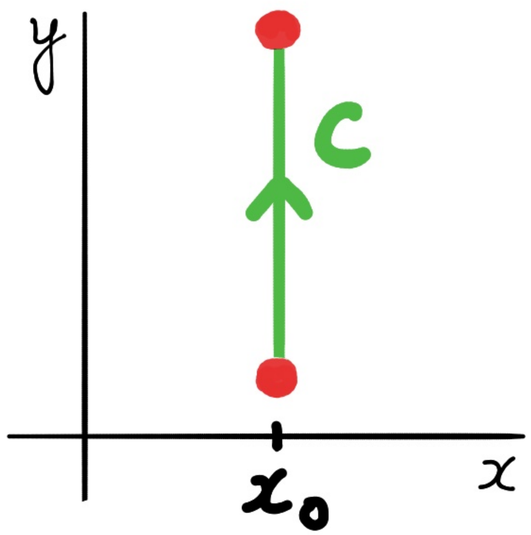

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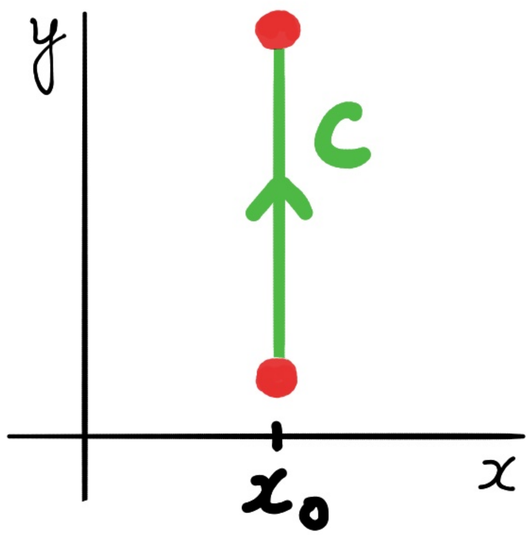

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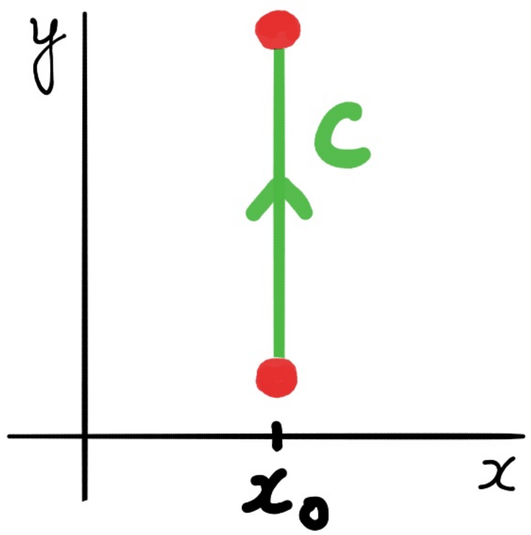

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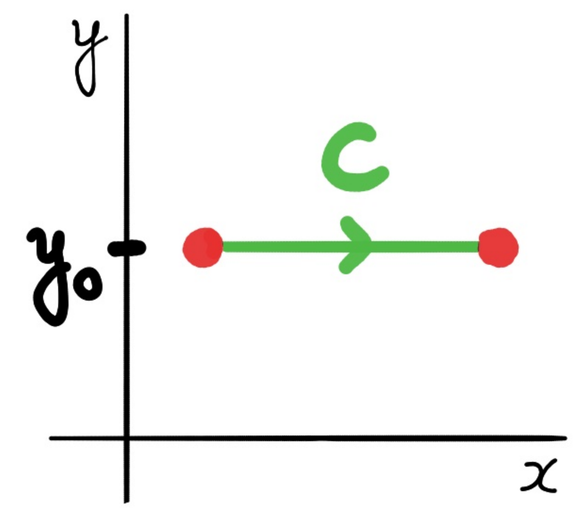
END

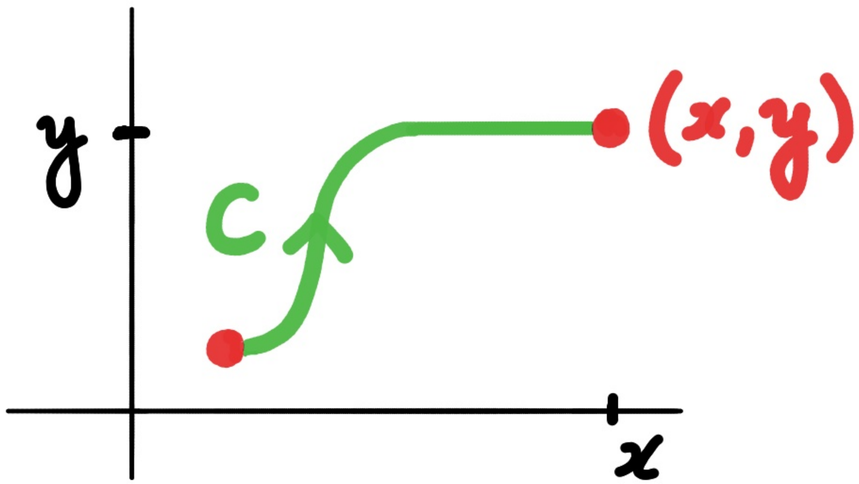
Lemma:

If C is a horizontal straight line in $\mathbb{R}^2$, then for any $f:\mathbb{R}^2 \to \mathbb{R}$, $\int_C f(x,y)\,dy = 0$.



Lemma:

If C is a curve with a horizontal end, and terminal point (x,y), then



$$\frac{\partial}{\partial x}\left(\int_C f(x,y)\,dx + g(x,y)\,dy\right) = f(x,y).$$

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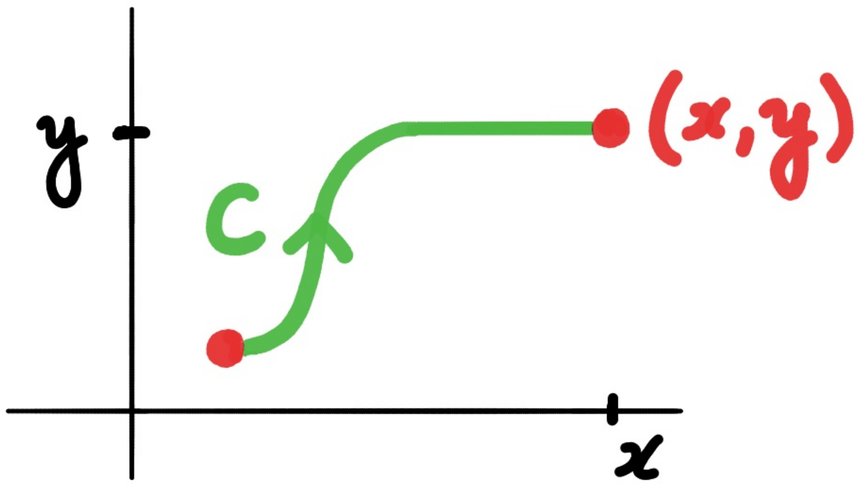

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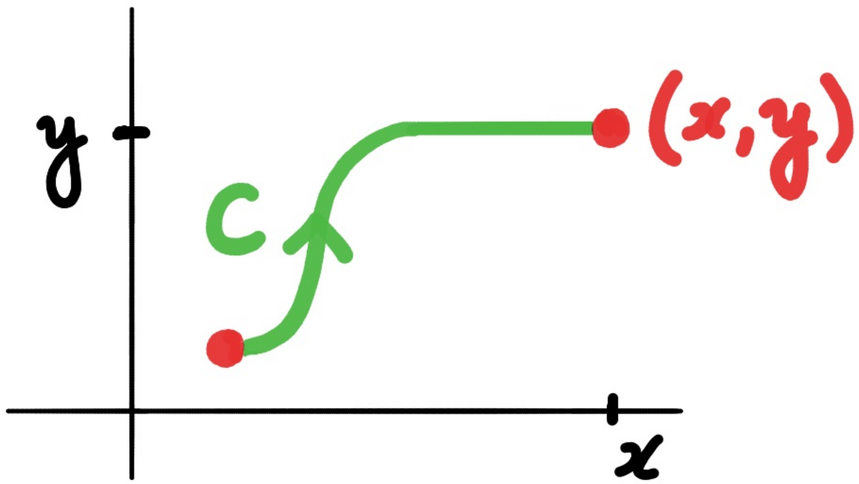

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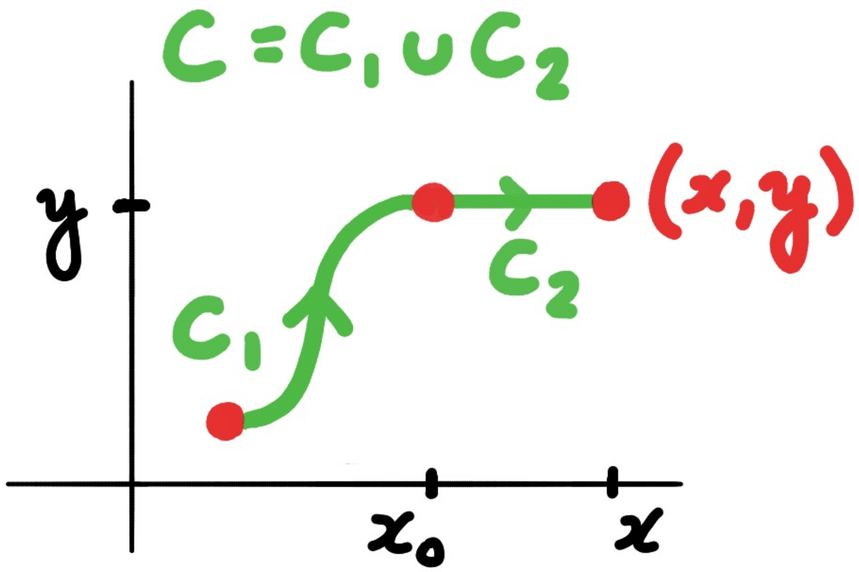

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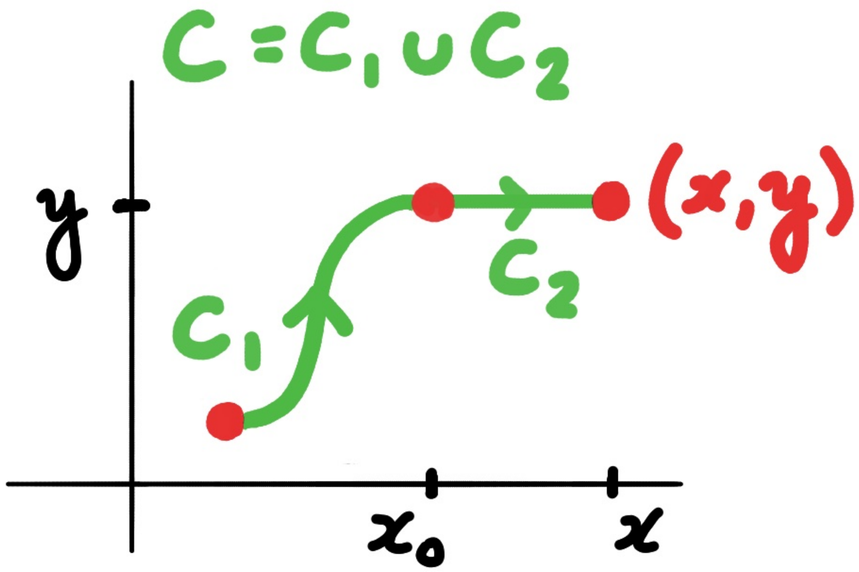

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proof:

$$\frac{\partial}{\partial x}\left(\int_C f(x,y)\,dx + g(x,y)\,dy\right) =$$

$$\frac{\partial}{\partial x}\left(\int_{C_1} f(x,y)\,dx + g(x,y)\,dy\right) + \frac{\partial}{\partial x}\left(\int_{C_2} f(x,y)\,dx + g(x,y)\,dy\right)$$

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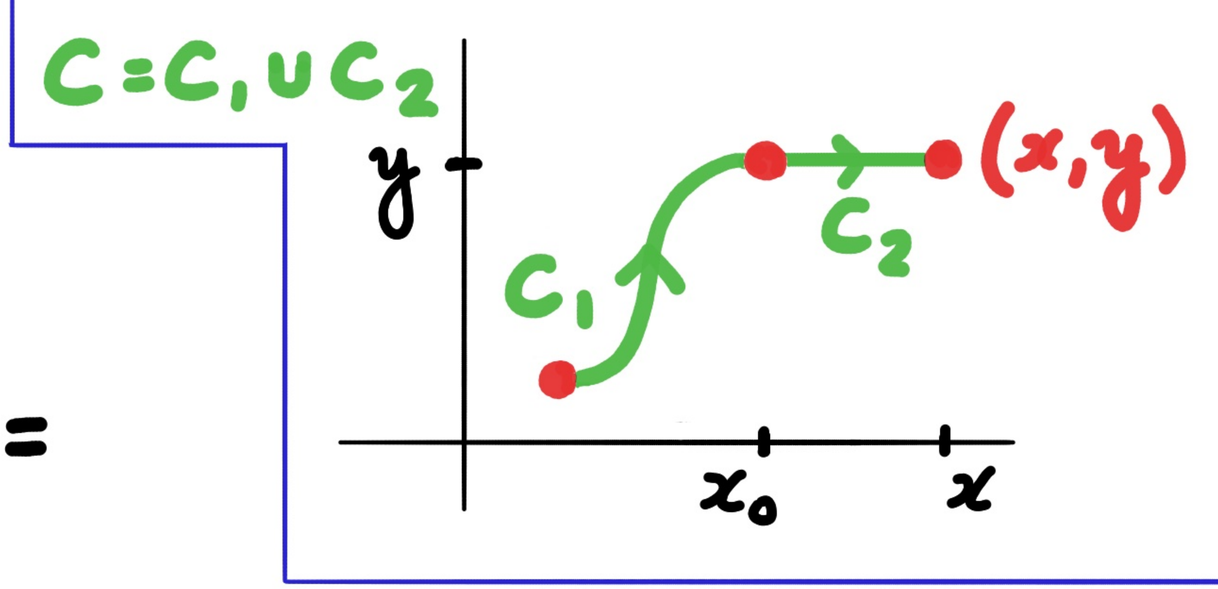

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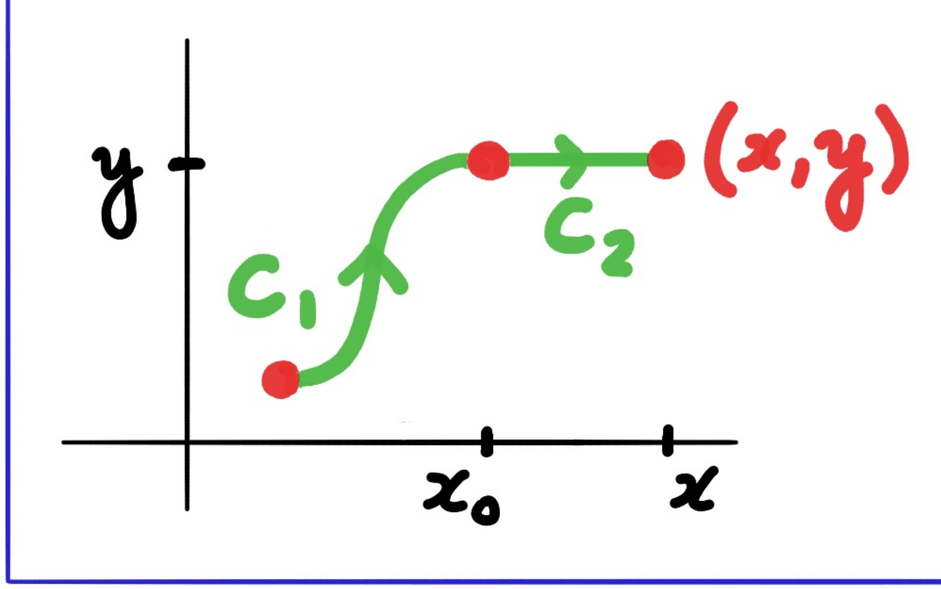

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$$= 0 +$$

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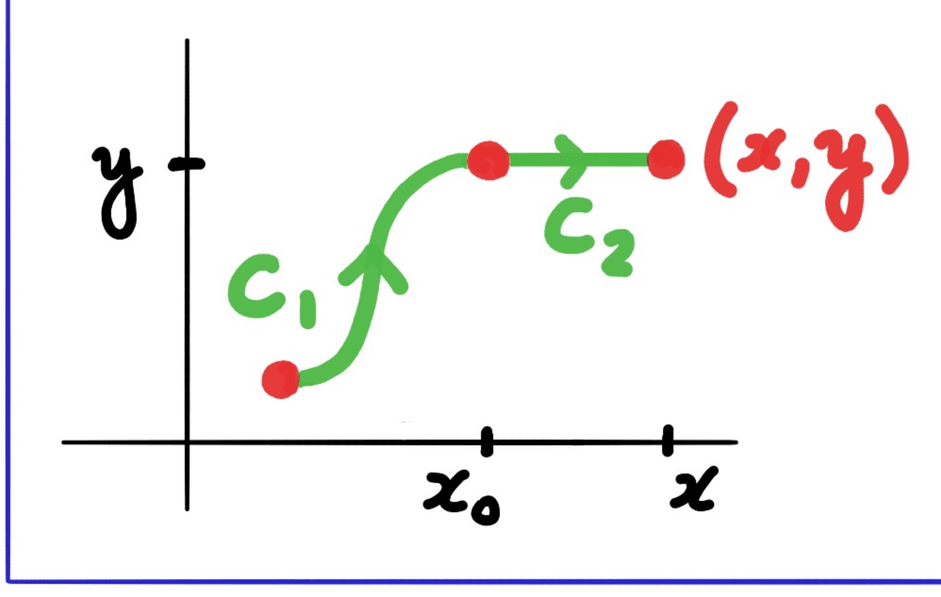

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$$= 0 + \frac{\partial}{\partial x}\left(\int_{C_2} f(x,y)\,dx\right) + \frac{\partial}{\partial x}\left(\int_{C_2} g(x,y)\,dy\right)$$

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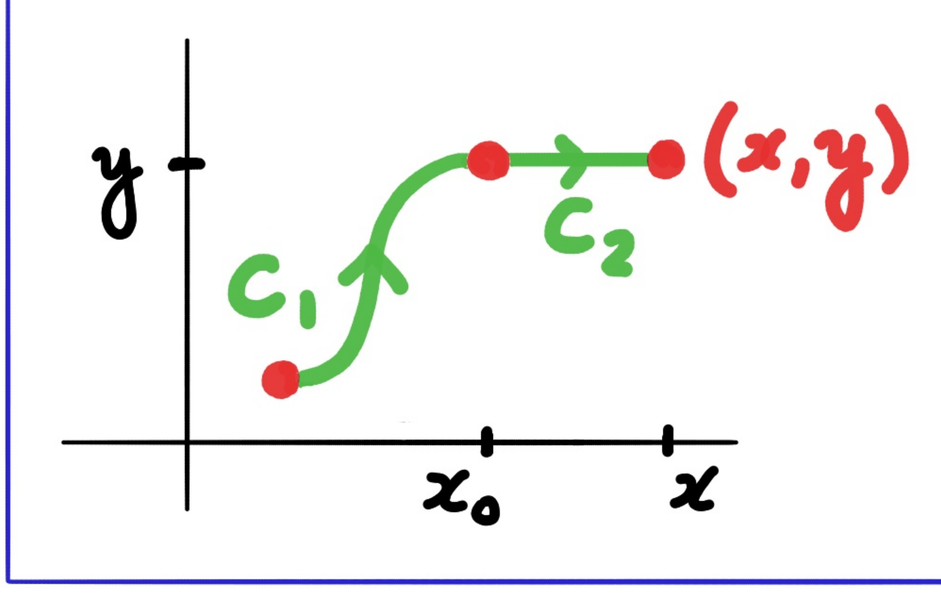

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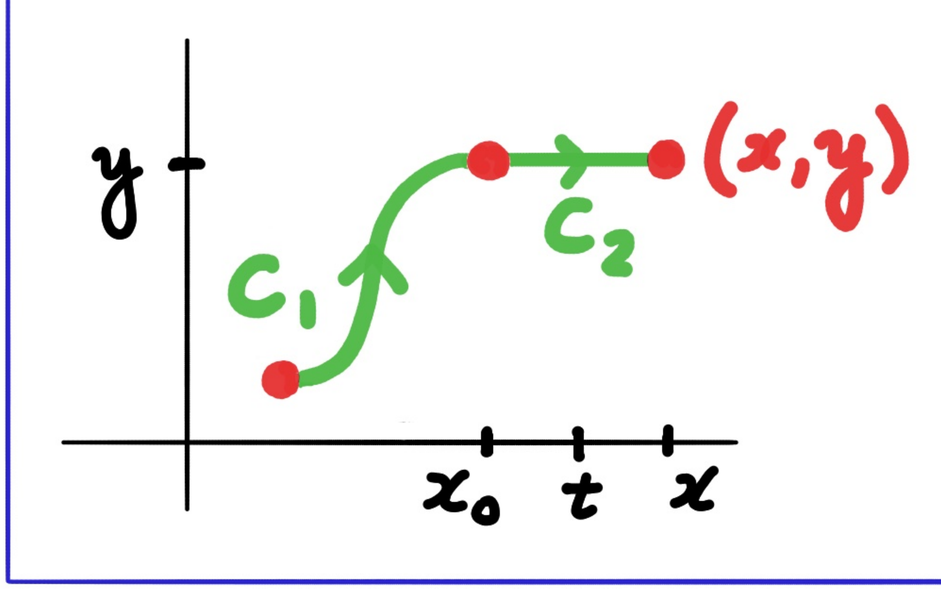

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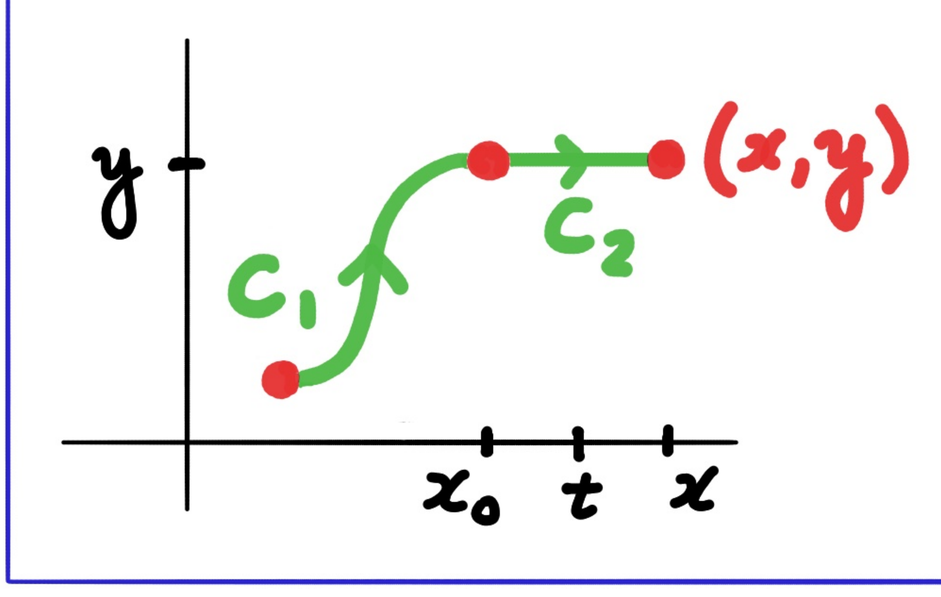
$$\boxed{\begin{array}{ll} \underline{C_2:} & x_0 \le t \le x \\ & x = t \\ & y = y \end{array}}$$

proof:



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$$= 0 + \frac{\partial}{\partial x}\left(\int_{C_2} f(x,y)\,dx\right) + \frac{\partial}{\partial x}\left(\int_{C_2} g(x,y)\,dy\right)$$

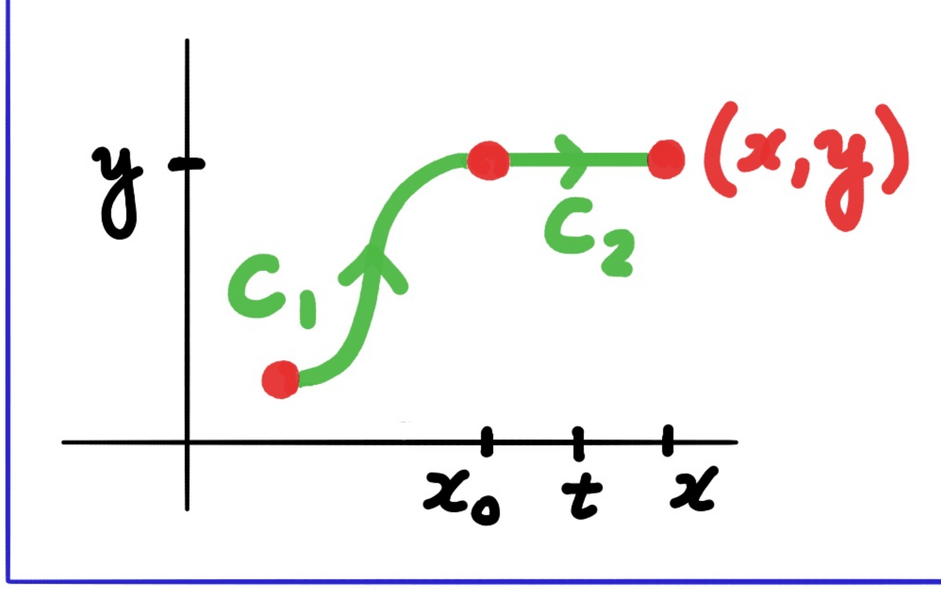
$$= \frac{\partial}{\partial x}\left(\int_{x_0}^{x} f(t,y)\,dt\right) + 0$$

$\underline{C_2:}$ $x_0 \le t \le x$

$x = t$

$y = y$

proof:



$$\frac{\partial}{\partial x}\left(\int_C f(x,y)\,dx + g(x,y)\,dy\right) =$$

$$= \frac{\partial}{\partial x}\left(\int_{C_1} f(x,y)\,dx + g(x,y)\,dy\right) + \frac{\partial}{\partial x}\left(\int_{C_2} f(x,y)\,dx + g(x,y)\,dy\right)$$

$$= 0 + \frac{\partial}{\partial x}\left(\int_{C_2} f(x,y)\,dx\right) + \frac{\partial}{\partial x}\left(\int_{C_2} g(x,y)\,dy\right)$$

$$= \frac{\partial}{\partial x}\left(\int_{x_0}^{x} f(t,y)\,dt\right) + 0$$

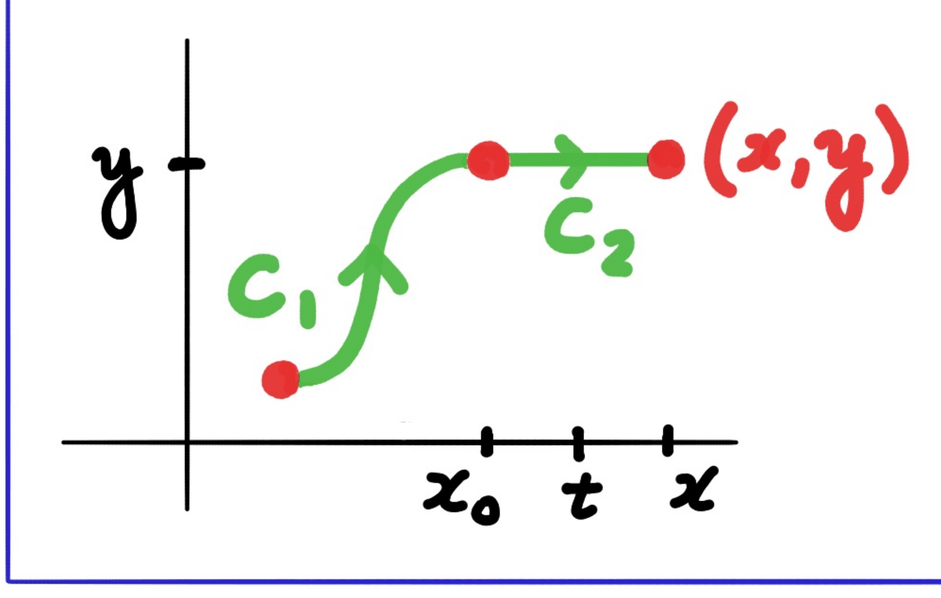
$$= f(x,y).$$

$\underline{C_2:}$ $x_0 \le t \le x$

$x = t$

$y = y$

proof:



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$$= \frac{\partial}{\partial x}\left(\int_{x_0}^{x} f(t,y)\,dt\right) + 0$$

$$= f(x,y).$$

$\underline{C_2:}$ $x_0 \le t \le x$

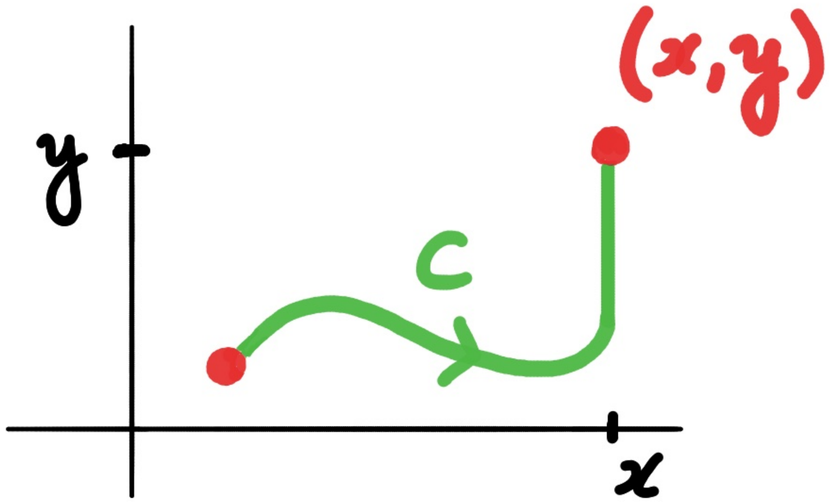
$x = t$

$y = y$

$\boxed{\text{END}}$

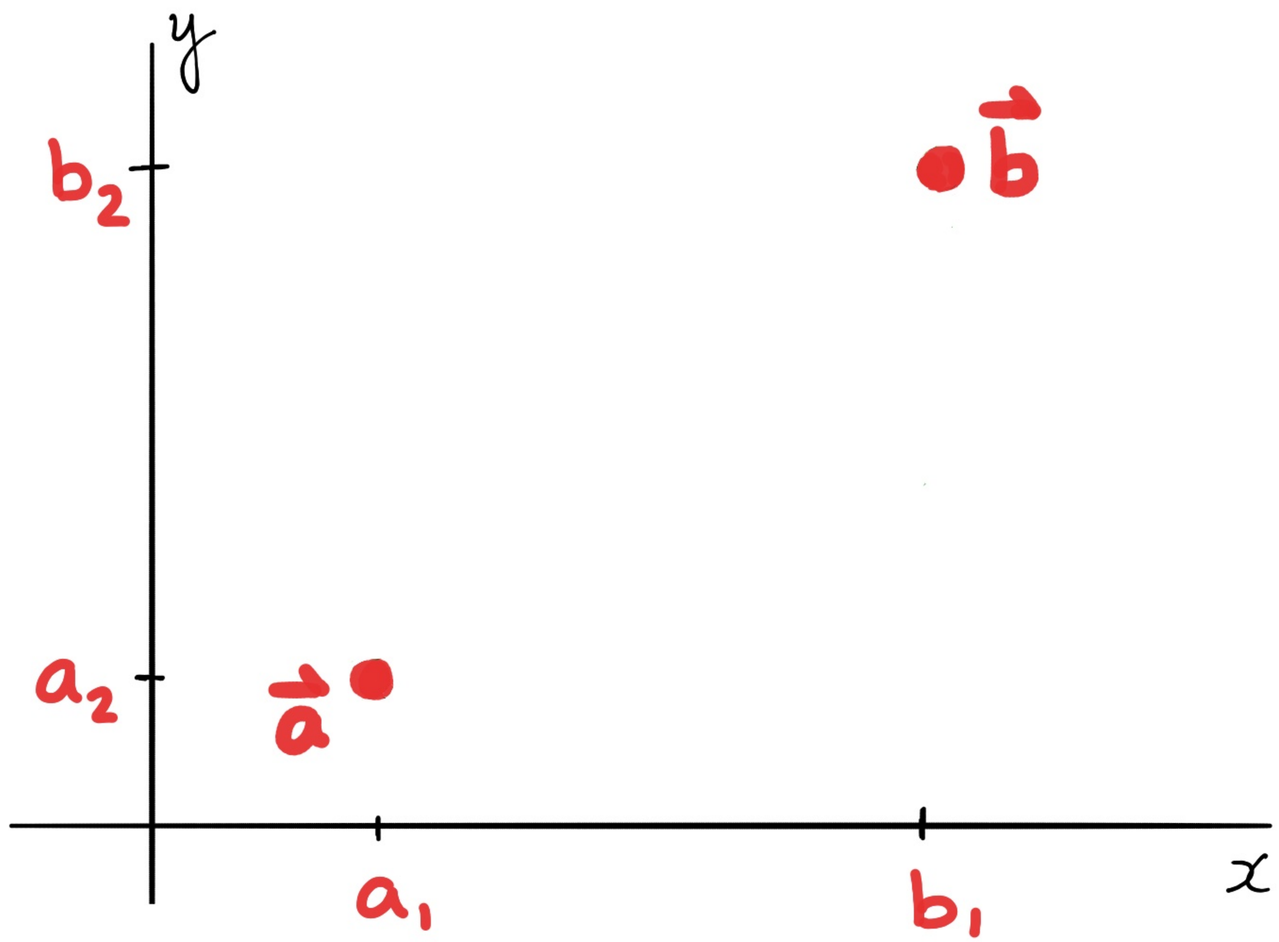
Lemma:

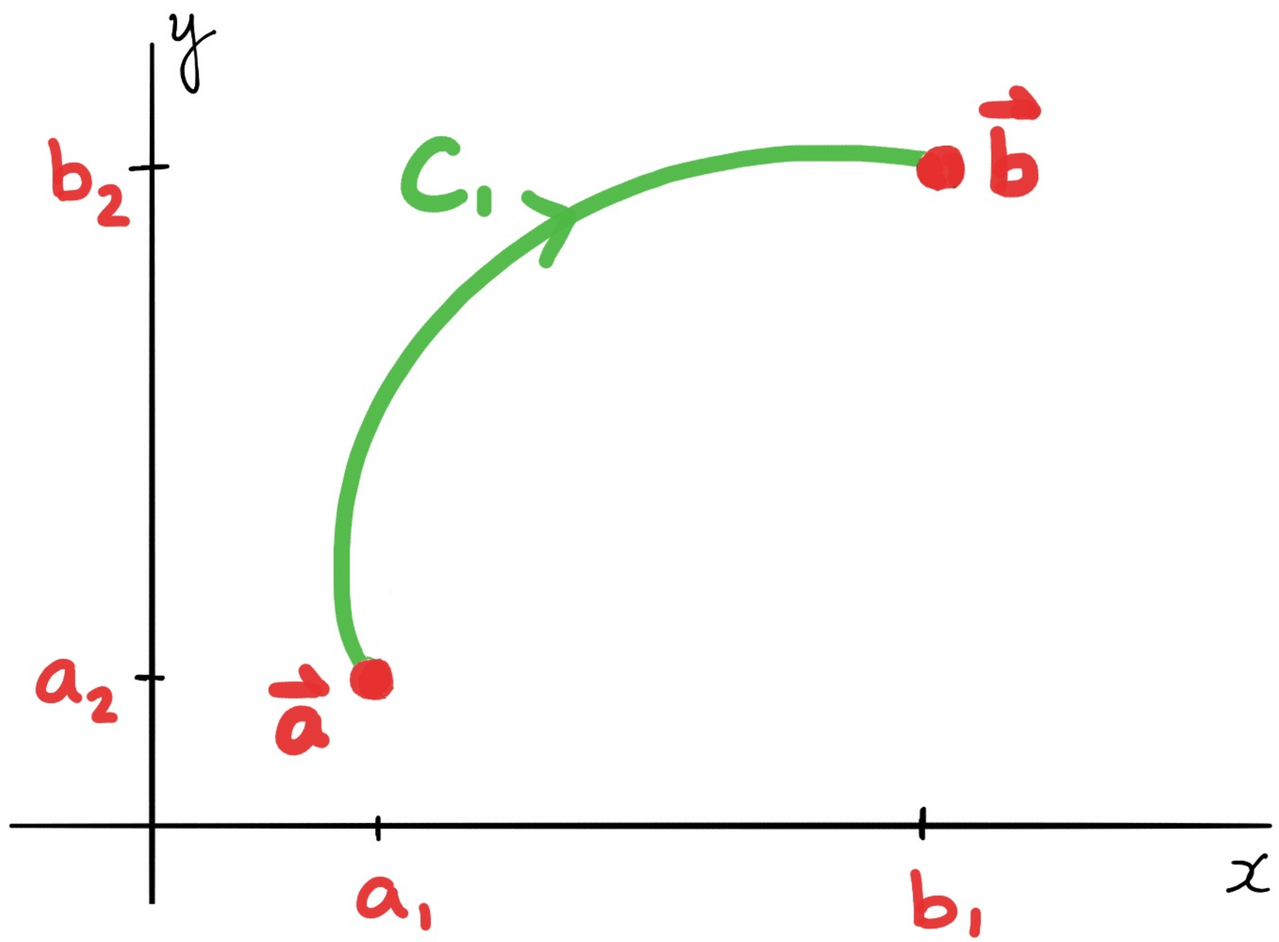
If C is a curve with a vertical end, and terminal point (x,y), then



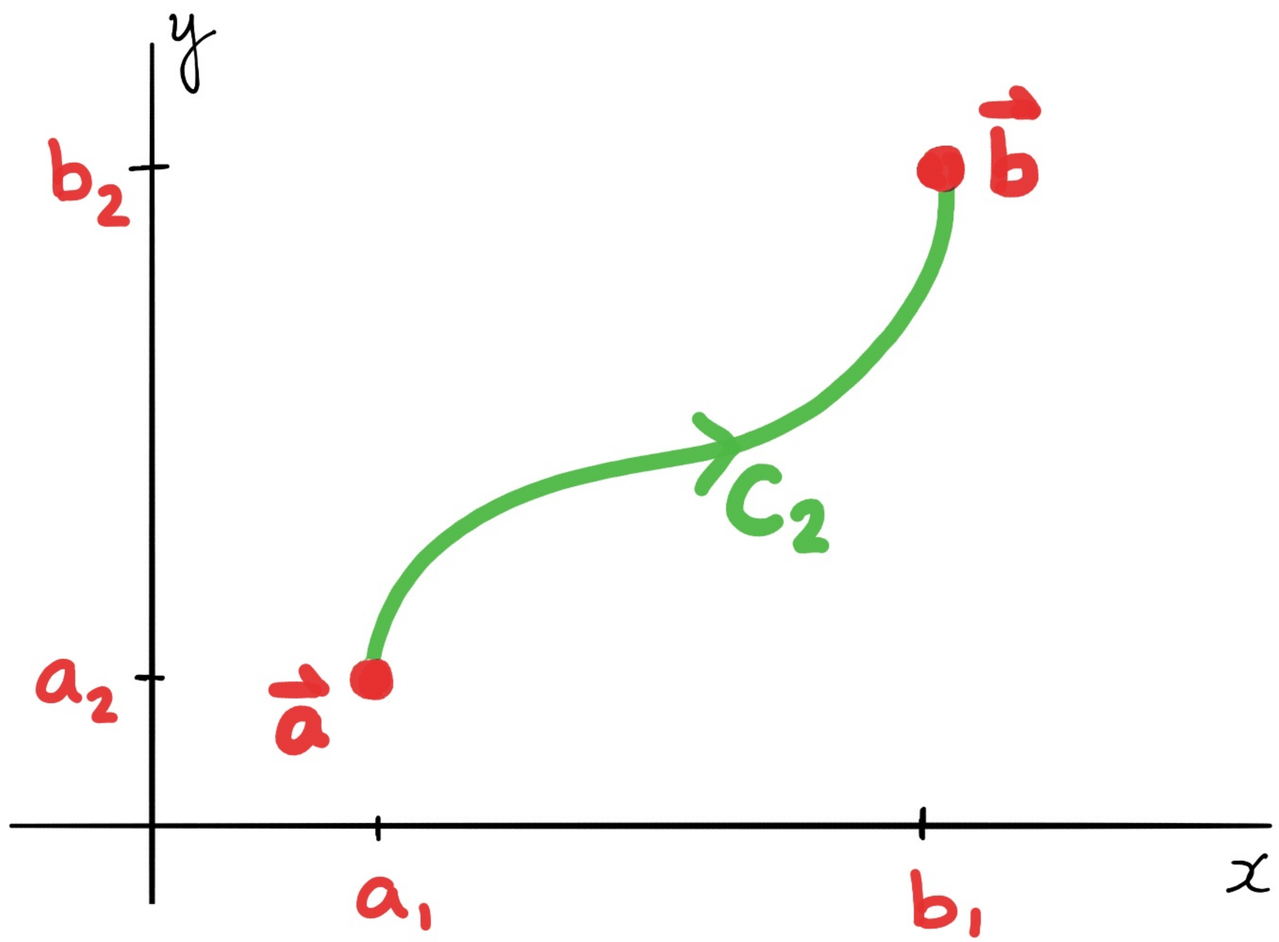
$$\frac{\partial}{\partial y}\left(\int_C f(x,y)\,dx + g(x,y)\,dy\right) = g(x,y).$$

Ⅳ Independence of path

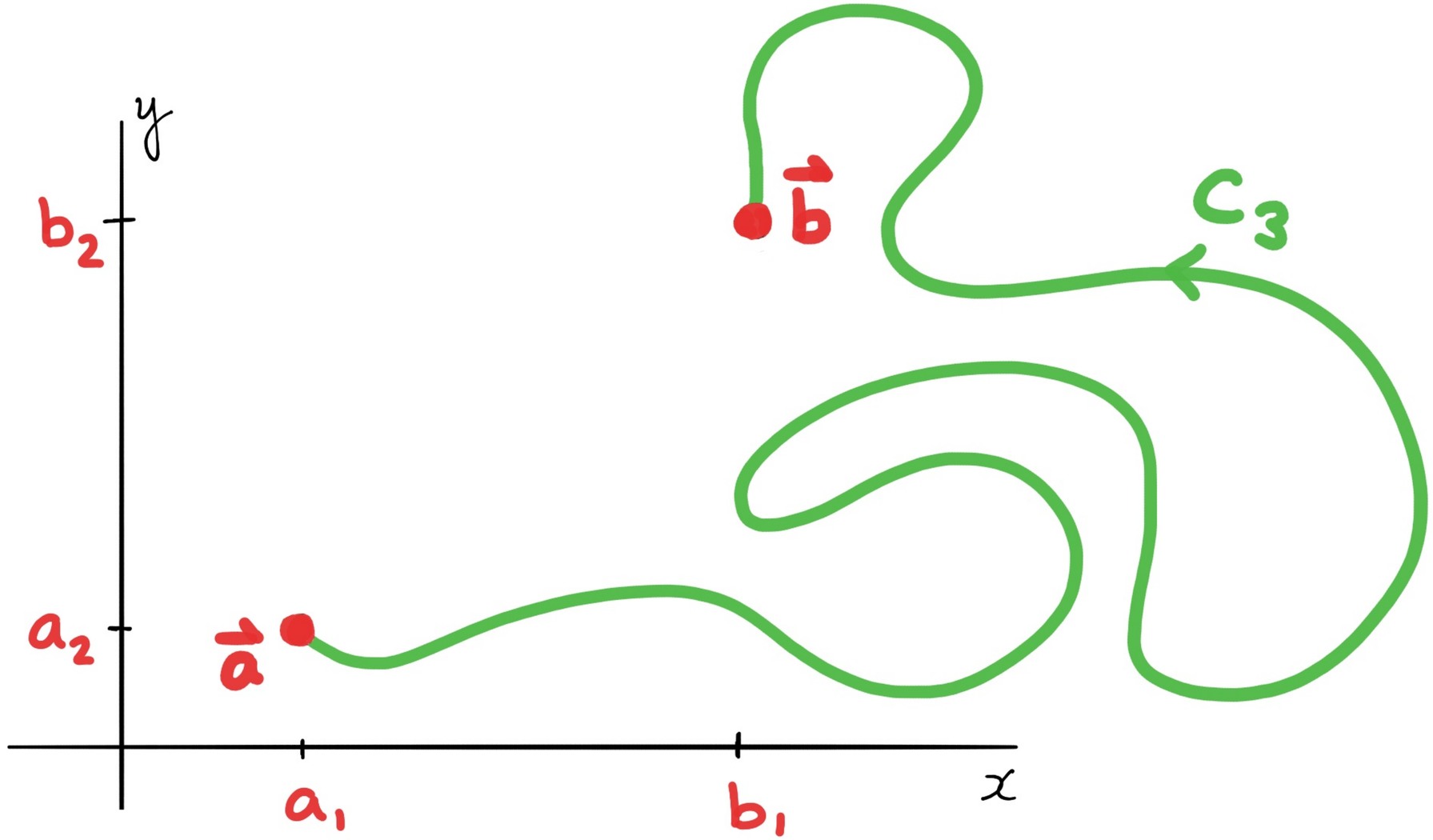

y
b_2
$\vec{b}$
a_2
$\vec{a}$
a_1
b_1
x



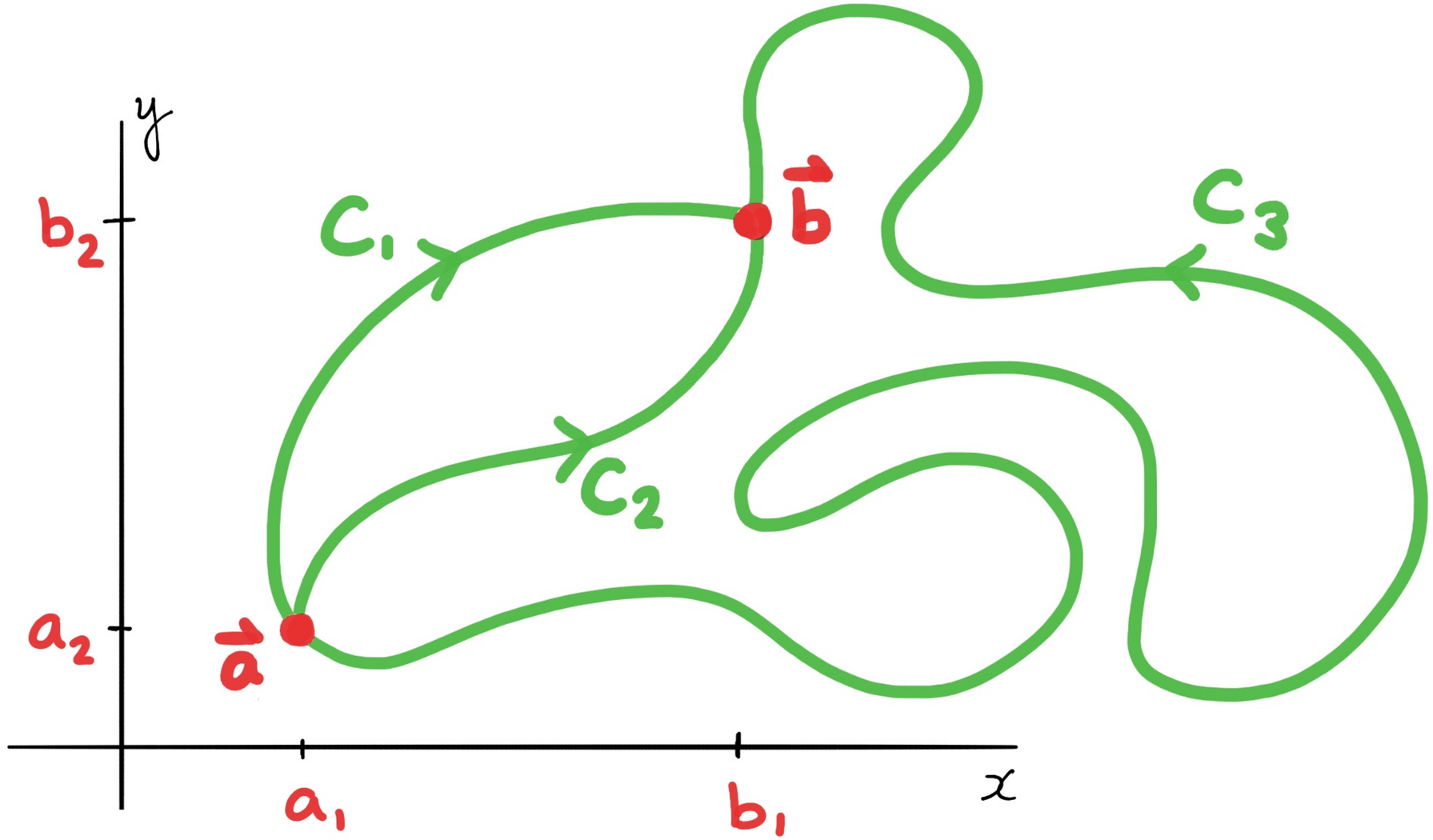
$$\int_{C_1} dx = b_1 - a_1$$



$$\int_{C_2} dx = b_1 - a_1$$



$$\int_{c_3} dx = b_1 - a_1$$



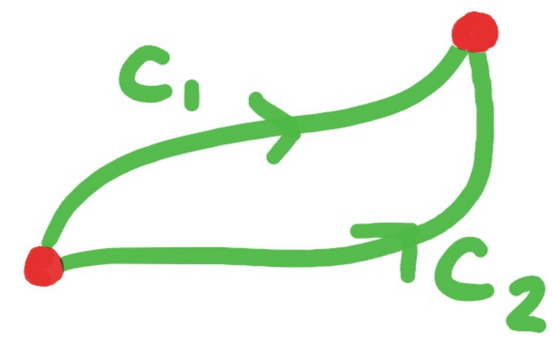
$$\int_{C_1} dx = b_1 - a_1 \qquad \int_{C_2} dx = b_1 - a_1 \qquad \int_{C_3} dx = b_1 - a_1$$

If C is <u>any</u> path with initial point $\vec{a}$ and terminal point $\vec{b}$, then $\int_C dx = b_1 - a_1$.

Independence of path

If

$$\int_{C_1} f(x,y)\,dx + g(x,y)\,dy$$



$$= \int_{C_2} f(x,y)\,dx + g(x,y)\,dy$$

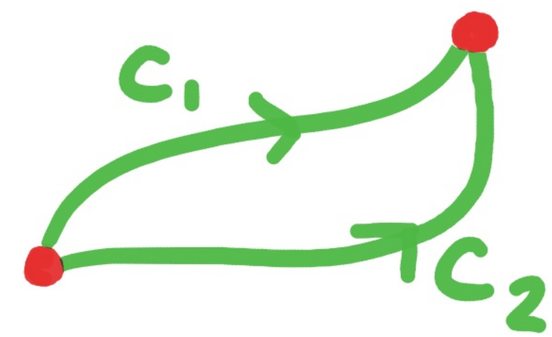
whenever C_1 and C_2 have the same initial points and the same terminal points, we say $\int_C f(x,y)\,dx + g(x,y)\,dy$ is independent of path.

Independence of path

If

$$\int_{C_1} f(x,y)\,dx + g(x,y)\,dy$$

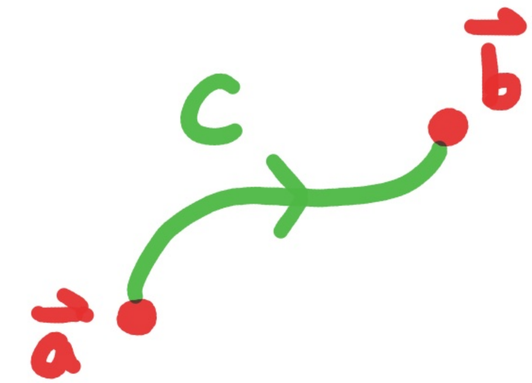
$$= \int_{C_2} f(x,y)\,dx + g(x,y)\,dy$$



whenever C_1 and C_2 have the same initial points and the same terminal points, we say $\int_C f(x,y)\,dx + g(x,y)\,dy$ is independent of path.

<u>Example:</u> $\int_C dx$ is independent of path.

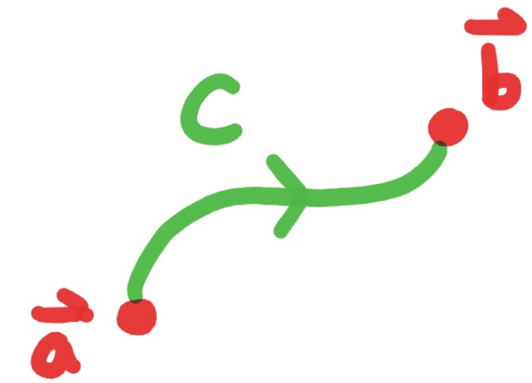
Independence of path



If

- C is a curve with initial point $\vec{a}$ and terminal point $\vec{b}$, and

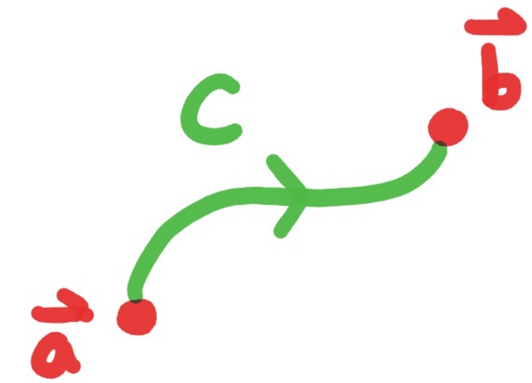
Independence of path



If

- C is a curve with initial point $\vec{a}$ and terminal point $\vec{b}$, and
- $\int_C f(x,y)\,dx + g(x,y)\,dy$ is independent of path,

Independence of path



If

- C is a curve with initial point $\vec{a}$ and terminal point $\vec{b}$, and
- $\int_C f(x,y)\,dx + g(x,y)\,dy$ is independent of path,

then we write

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \int_{\vec{a}}^{\vec{b}} f(x,y)\,dx + g(x,y)\,dy$$

Example: We'll see later that

$\int_C yz\,dx + xz\,dy + xy\,dz$ is independent of path.

Example: We'll see later that

$\int_C yz\,dx + xz\,dy + xy\,dz$ is independent of path.

So

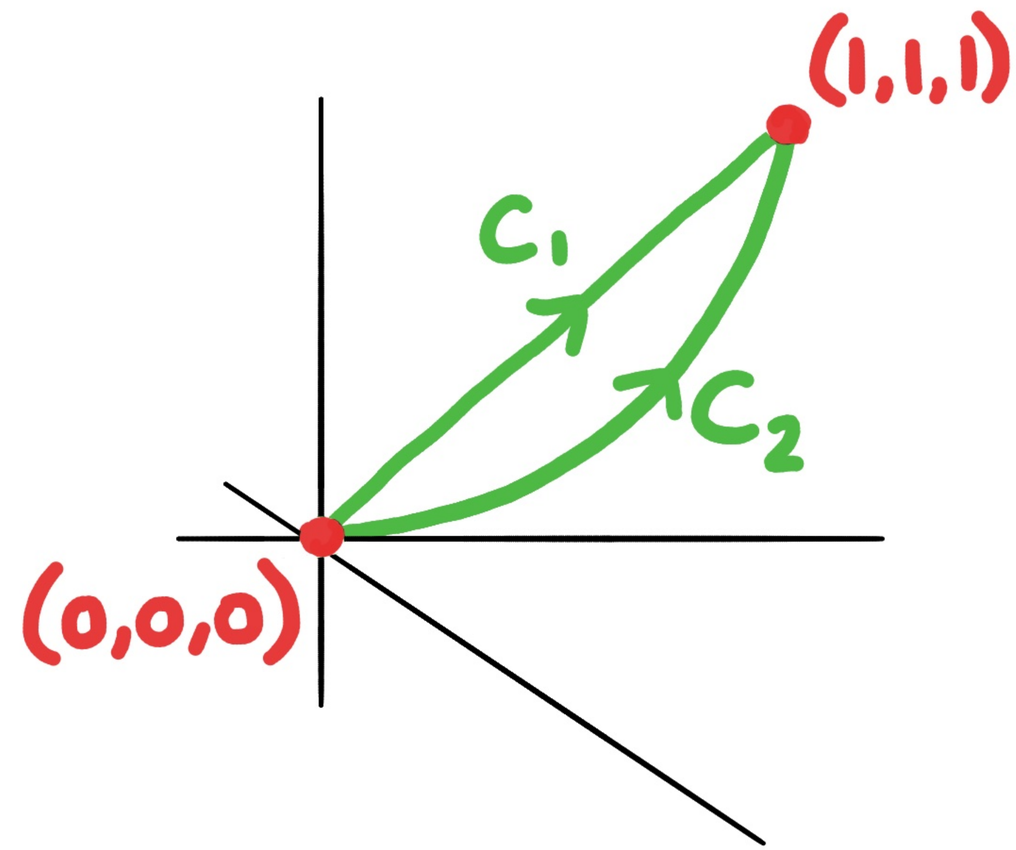
$$\int_{C_i} yz\,dx + xz\,dy + xy\,dz$$

$$=$$

$$\int_{(0,0,0)}^{(1,1,1)} yz\,dx + xz\,dy + xy\,dz$$

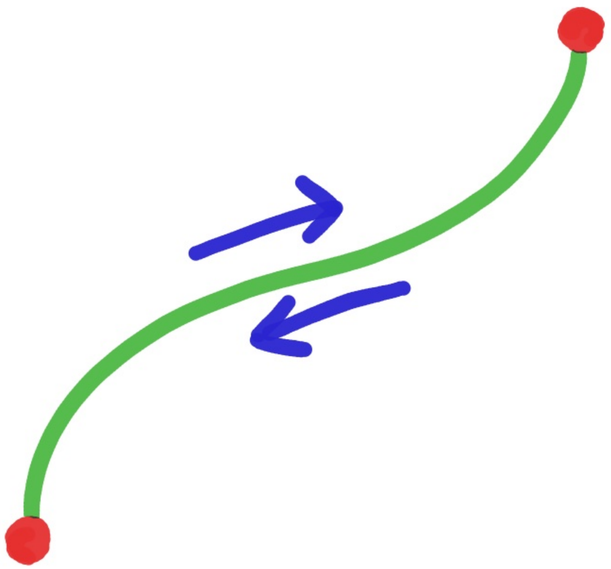
$$=$$

$$1$$

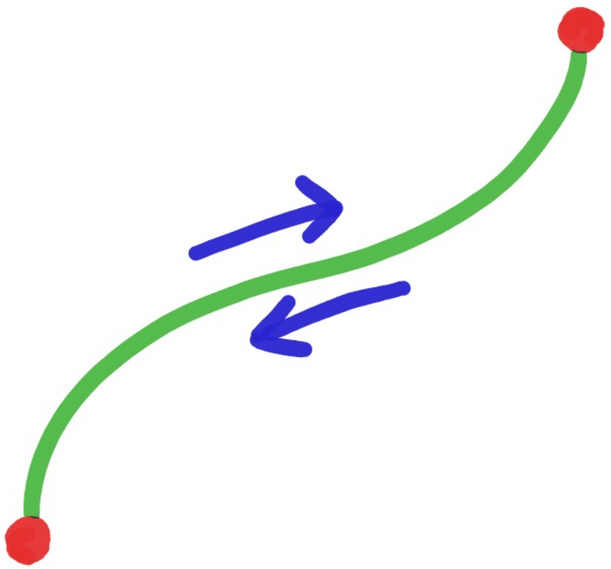


Ⓥ Independence of parametrization

Any curve has two choices for orientation: forwards or backwards.



Any curve has two choices for orientation: forwards or backwards.



As long as orientation is unchanged, line integrals are independent of parametrization. (Using change of variables.)